Vertex-Minors and Circle Graphs

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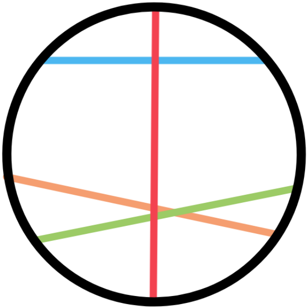
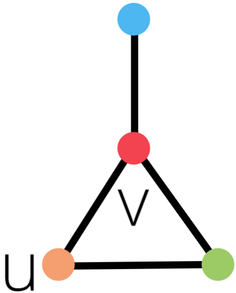
A **circle graph** is the intersection graph of chords on a circle.

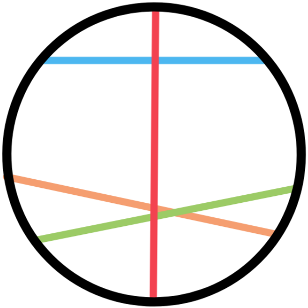


chord diagram



circle graph
G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.



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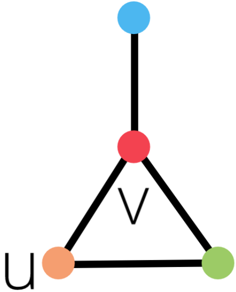

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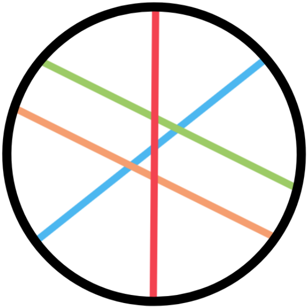
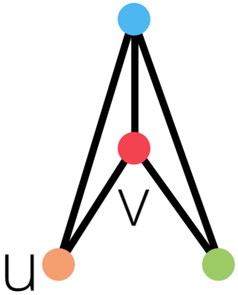


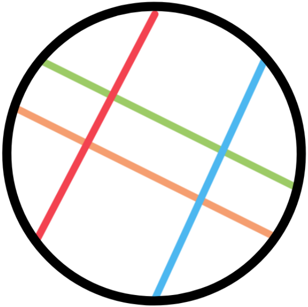
chord diagram



circle graph

$G * v$

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.



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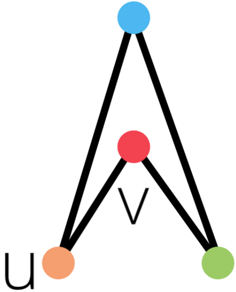
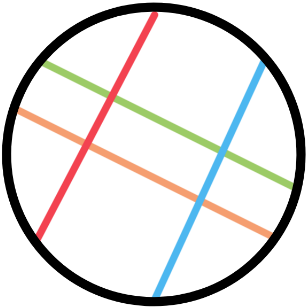


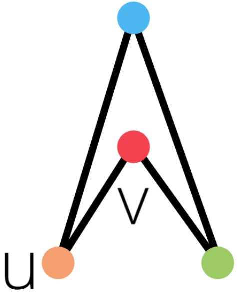
circle graph

$G * v * u$

Two graphs are **locally equivalent** if one can be obtained from the other by local complementations.



chord diagram



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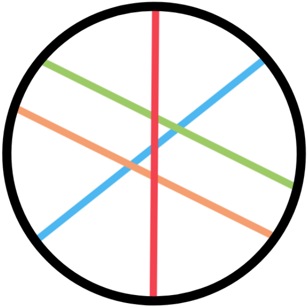

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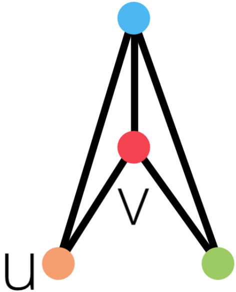

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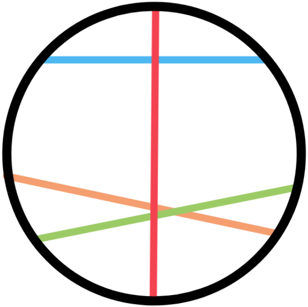
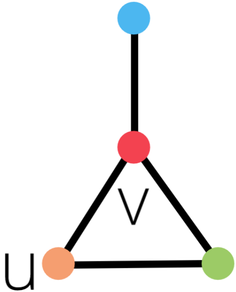


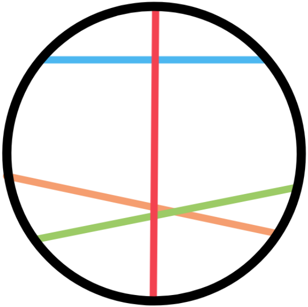
chord diagram



circle graph

$$G * v * u * u * v = G$$

A graph H is a **vertex-minor** of G if H can be obtained from a graph that is locally equivalent to G by deleting vertices.



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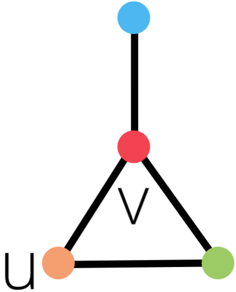

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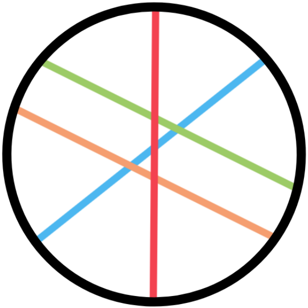

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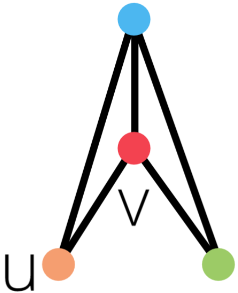

circle graph

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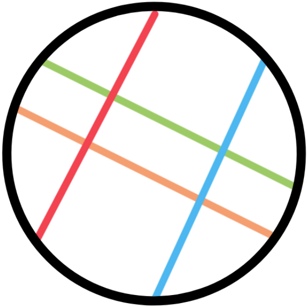

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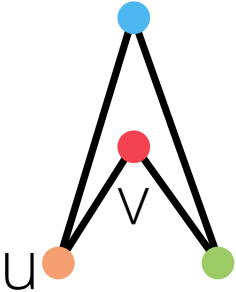

circle graph

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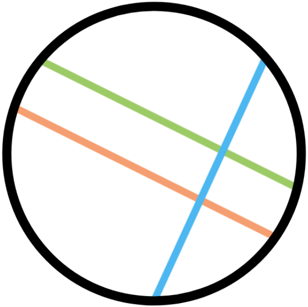
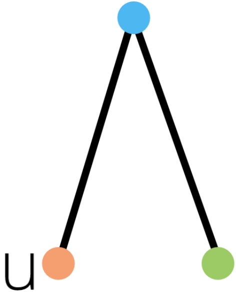


chord diagram



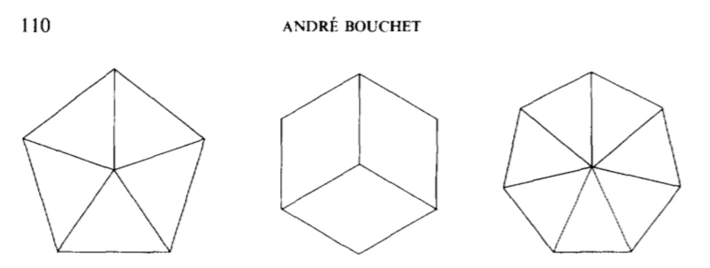
circle graph

$G * v * u - v$

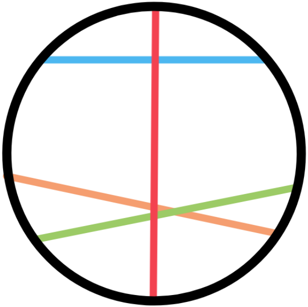
Can we describe the structure of graphs without a vertex-minor isomorphic to H?

Theorem (Bouchet, 94)

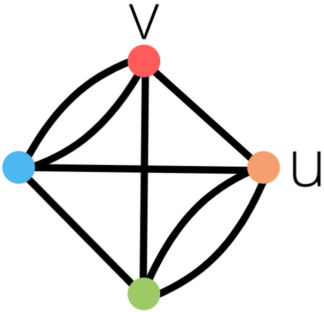
A graph is a circle graph if and only if it does not have one of the following as a vertex minor.



How can we represent the local equivalence class of a circle graph?



chord diagram



tour graph

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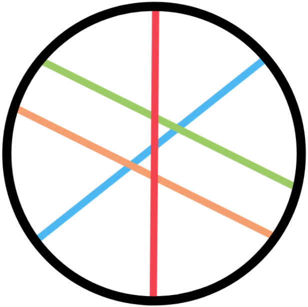
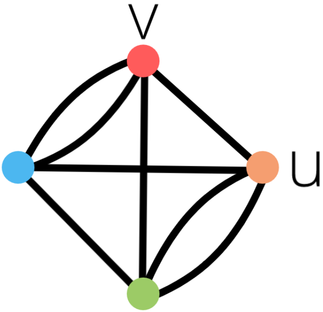
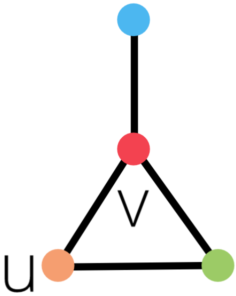


chord diagram

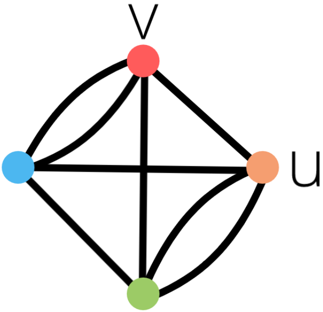


tour graph

Bouchet (94) gave "connectivity" conditions under which local equivalence classes of circle graphs are in bijection with 4-regular graphs.

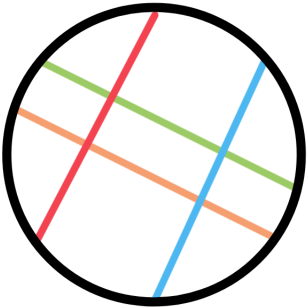


circle graph

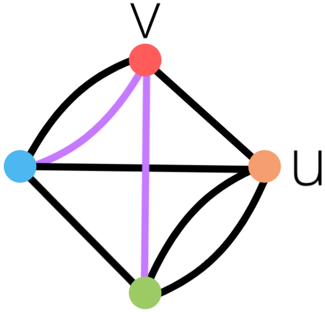


tour graph

What about deleting vertices?



chord diagram



tour graph

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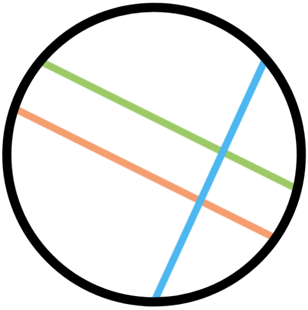

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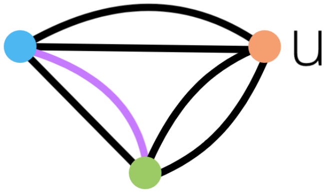


tour graph

Can we describe the structure of graphs G without a vertex-minor isomorphic to H?

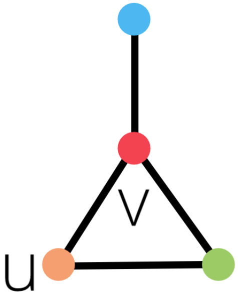
- If G is a circle graph, this is captured by **immersions** of its tour graph. (Robertson and Seymour; DeVos, McDonald, Mohar, Scheide, 13; Wollan, 15)
- What if H is a circle graph?

Then G can be "recursively decomposed" along "simple" vertex partitions.

The **adjacency matrix** of G is the $V(G) \times V(G)$ matrix over the binary field with (u, v)-entry 1 if $uv \in E(G)$ and 0 otherwise.

	u	v		
u	0	1	1	0
v	1	0	1	1
	1	1	0	0
	0	1	0	0

adjacency matrix A

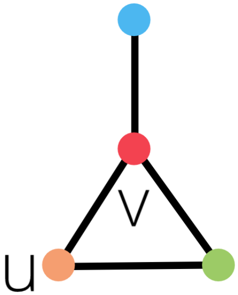


graph G

The **rank** of a vertex partition (X, Y) is the rank of the submatrix $A[X, Y]$.

	u	v		
u	0	1	1	0
v	1	0	1	1
	1	1	0	0
	0	1	0	0

rank(X, Y)



$X = \{u, v\}$

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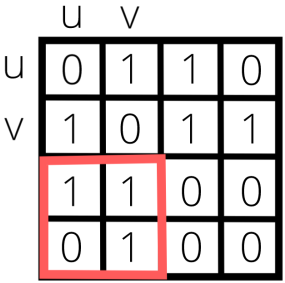
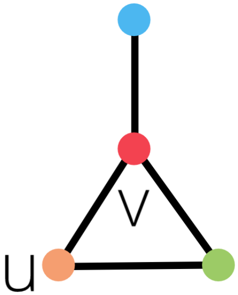


$= \textbf{rank}(Y, X)$



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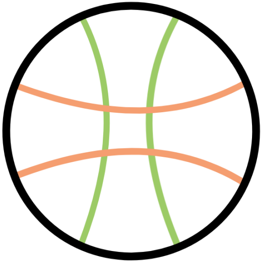

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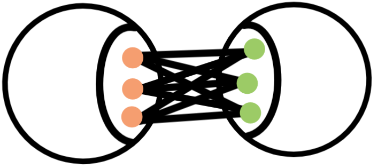
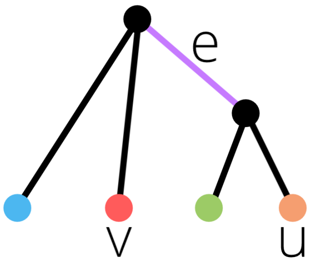
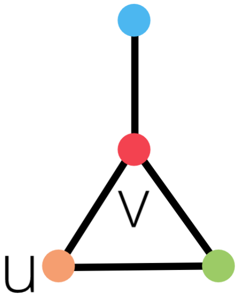


circle graph with $\text{rank}(X, Y) = 1$

The **rank-width** of G is the minimum over all subcubic trees T with Leaves$(T) = V(G)$, of the maximum of rank(X_e, Y_e) for $e \in E(T)$.

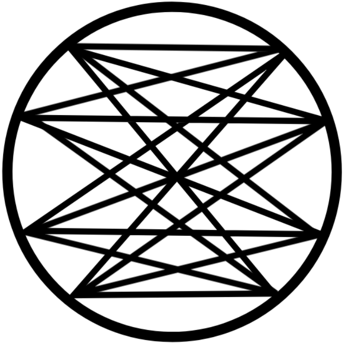




If H is a vertex-minor of G, then rw$(H) \leq$ rw(G). (Oum and Seymour, 06)

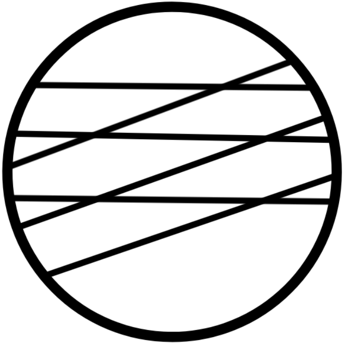
Theorem (Geelen, Kwon, McCarty, Wollan 19+)

For any circle graph H, there exists c_H so that every graph with no vertex-minor isomorphic to H has rank-width at most c_H.



Theorem (Kwon, McCarty, Oum, Wollan 19+)

For any path H, there exists c_H so that every graph with no vertex-minor isomorphic to H has **rank-depth** *at most c_H.*



Can we describe the structure of graphs G without a vertex-minor isomorphic to H?

- We may assume G has our favorite circle graph ξ as a vertex-minor.
- The way a vertex $v \in V(G) \setminus V(\xi)$ "attaches onto ξ" can be stored as a subset of the edges of the **tour graph** of ξ.
- So we work with 4-regular graphs with edges labelled in $\mathbb{Z}_2^k$.

Conjecture

For any proper vm-closed class of graphs, there exists a polynomial p such that each graph in the class with clique number ω has chromatic number at most $p(\omega)$.

Conjecture

For any proper vm-closed class of graphs, there is a polynomial time algorithm for max clique.

Conjecture

Graphs are well-quasi-ordered by vertex-minors.