Structural graph theory and monadic stability

Rose McCarty

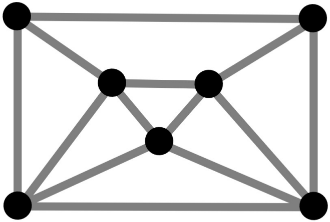
PRINCETON UNIVERSITY

March 19, 2023

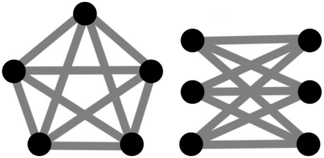
AMS Special Session on
Logic, Combinatorics, and Their Interactions

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



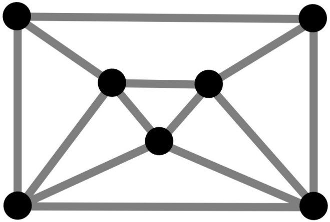
planar graph



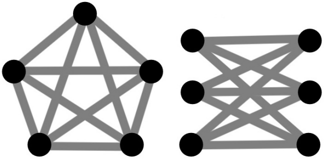
forbidden minors

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



planar graph

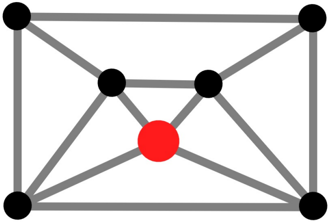


forbidden minors

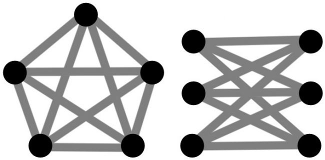
Minors are obtained by:

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



planar graph



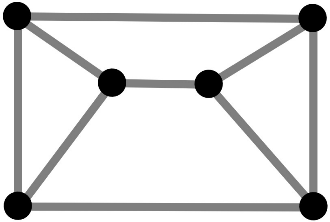
forbidden minors

Minors are obtained by:

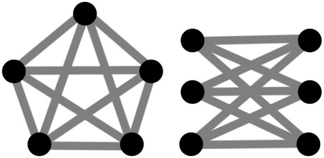
deleting vertices,

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



planar graph



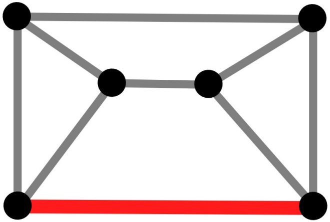
forbidden minors

Minors are obtained by:

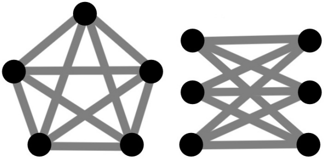
deleting vertices,

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



planar graph



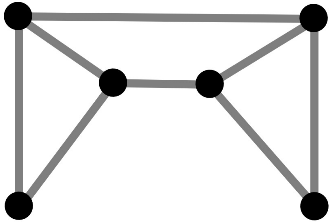
forbidden minors

Minors are obtained by:

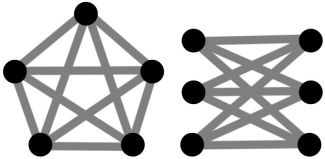
deleting vertices, deleting edges,

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 *or* $K_{3,3}$ *as a* **minor**.



planar graph



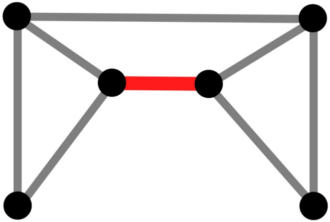
forbidden minors

Minors are obtained by:

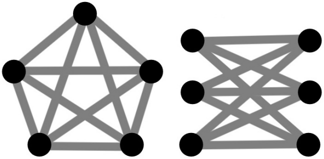
deleting vertices, deleting edges,

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor***.*



planar graph

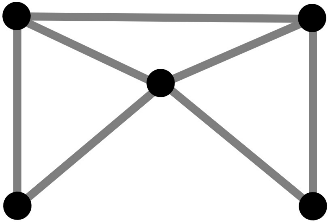


forbidden minors

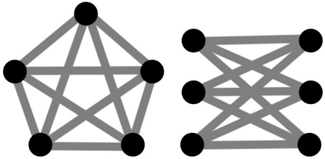
Minors are obtained by:
deleting vertices, deleting edges, and contracting edges.

Kuratowski's Theorem

A (finite) graph embeds in the plane iff it does not have K_5 or $K_{3,3}$ as a **minor**.



planar graph



forbidden minors

Minors are obtained by:
deleting vertices, deleting edges, and contracting edges.

Structure Theorem (Robertson & Seymour 2003)

A class of graphs excludes a **minor** *iff its graphs "decompose" into parts that "almost embed" in a surface of bounded genus.*

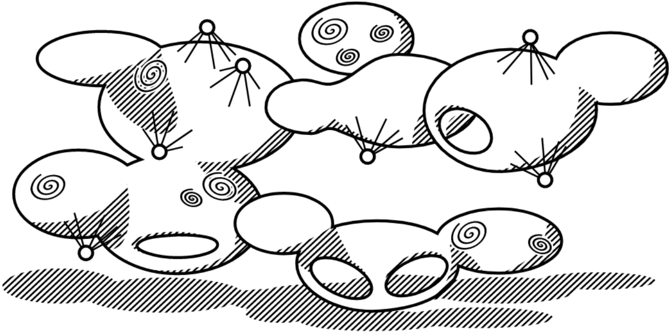


Figure by Felix Reidl

Structure Theorem (Robertson & Seymour 2003)

A class of graphs excludes a **minor** *iff its graphs "decompose" into parts that "almost embed" in a surface of bounded genus.*

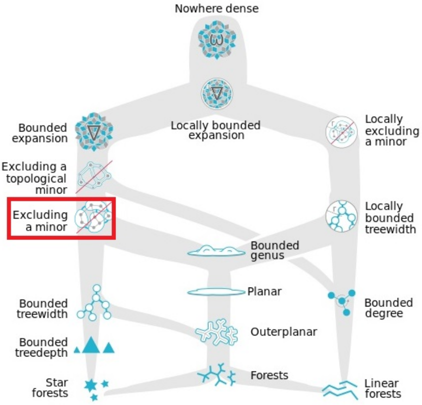


Figure by Felix Reidl

Structure Theorem (Robertson & Seymour 2003)

A class of graphs excludes a **minor** *iff its graphs "decompose" into parts that "almost embed" in a surface of bounded genus.*

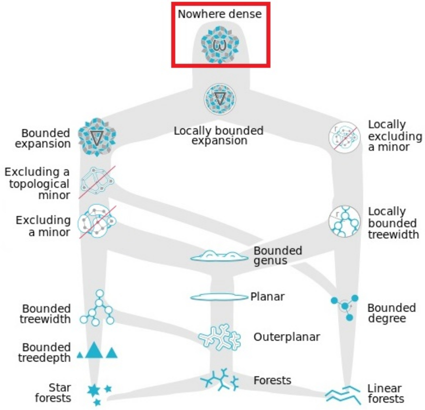
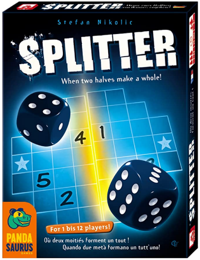


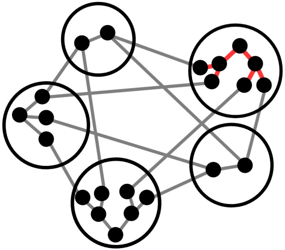
Figure by Felix Reidl

Theorem (Grohe, Kreutzer, & Siebertz 2017)

*A class of graphs excludes an r-**shallow minor** for each $r \in \mathbb{N}$ iff Splitter wins the radius-r splitter game for each $r \in \mathbb{N}$.*



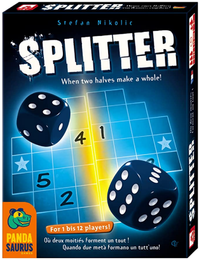
winning strategy



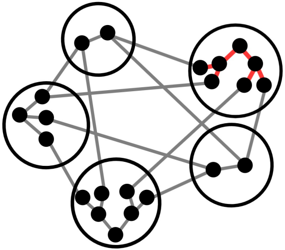
excluded **shallow minors**

Theorem (Grohe, Kreutzer, & Siebertz 2017)

*A class of graphs excludes an r-**shallow minor** for each $r \in \mathbb{N}$ iff Splitter wins the radius-r splitter game for each $r \in \mathbb{N}$.*



winning strategy



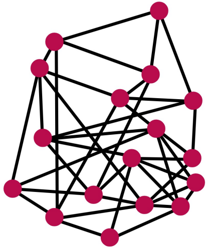
excluded **shallow minors**

Theorem (Adler & Adler 2014 + Dvořák 2018)

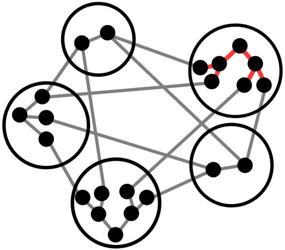
*iff **stable** in the sense of Shelah and excludes $K_{t,t}$ **subgraph**.*

Theorem (Grohe, Kreutzer, & Siebertz 2017)

A class of graphs excludes an r-**shallow minor** *for each* $r \in \mathbb{N}$ *iff Splitter wins the radius-*r *splitter game for each* $r \in \mathbb{N}$.



excluded $K_{t,t}$ **subgraph**



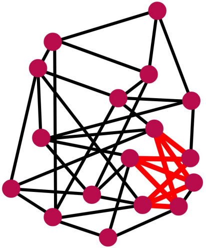
excluded **shallow minors**

Theorem (Adler & Adler 2014 + Dvořák 2018)

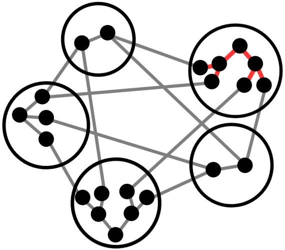
iff **stable** *in the sense of Shelah and excludes* $K_{t,t}$ **subgraph**.

Theorem (Grohe, Kreutzer, & Siebertz 2017)

*A class of graphs excludes an r-**shallow minor** for each $r \in \mathbb{N}$ iff Splitter wins the radius-r splitter game for each $r \in \mathbb{N}$.*



excluded $K_{t,t}$ **subgraph**



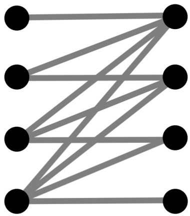
excluded **shallow minors**

Theorem (Adler & Adler 2014 + Dvořák 2018)

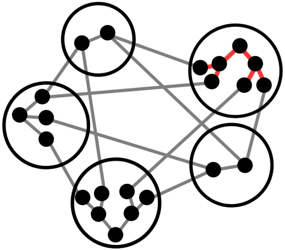
*iff **stable** in the sense of Shelah and excludes $K_{t,t}$ **subgraph**.*

Theorem (Grohe, Kreutzer, & Siebertz 2017)

*A class of graphs excludes an r-**shallow minor** for each $r \in \mathbb{N}$ iff Splitter wins the radius-r splitter game for each $r \in \mathbb{N}$.*



and **stable**



excluded **shallow minors**

Theorem (Adler & Adler 2014 + Dvořák 2018)

*iff **stable** in the sense of Shelah and excludes $K_{t,t}$ **subgraph**.*

View G as a model over the signature with a binary relation E.

View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

1) $xy \in E$ and $x = y$ are formulas.

View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

1) $xy \in E$ and $x = y$ are formulas.
2) If ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \wedge \psi$, ...

View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

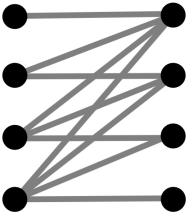
1) $xy \in E$ and $x = y$ are formulas.
2) If ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \wedge \psi$, ...
3) If ϕ is a formula, then so is $\exists x \phi$ and $\forall x \phi$.

View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

1) $xy \in E$ and $x = y$ are formulas.
2) If ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \wedge \psi$, $\dots$.
3) If ϕ is a formula, then so is $\exists x \phi$ and $\forall x \phi$.

A class $\mathcal{C}$ is **stable** if there is no formula $\psi(\bar{x}, \bar{y})$ st for all n there is $G \in \mathcal{C}$ with tuples $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n$ of vertices st

$$G \models \phi(\bar{a}_i, \bar{b}_j) \quad \leftrightarrow \quad i \leq j.$$



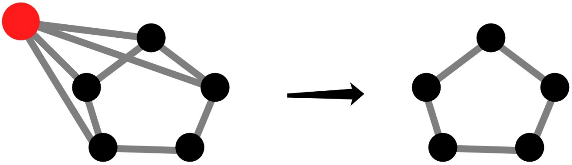
View G as a model over the signature with a binary relation E. Thus **formulas** in first-order logic are obtained as follows.

1) $xy \in E$ and $x = y$ are formulas.
2) If ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \wedge \psi$, ...
3) If ϕ is a formula, then so is $\exists x \phi$ and $\forall x \phi$.

A class $\mathcal{C}$ is **stable** if there is no formula $\psi(\bar{x}, \bar{y})$ st for all n there is $G \in \mathcal{C}$ with tuples $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n$ of vertices st

$$G \models \phi(\bar{a}_i, \bar{b}_j) \quad \leftrightarrow \quad i \leq j.$$

We also assume all classes are closed under deleting vertices.



Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the radius-r flipper game for each $r \in \mathbb{N}$.*

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the radius-r flipper game for each $r \in \mathbb{N}$.*

Model theory: Baldwin and Shelah.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the radius-r flipper game for each $r \in \mathbb{N}$.*

Model theory: Baldwin and Shelah.
Combinatorics: Dreier, Mählmann, Siebertz, and Toruńczyk.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

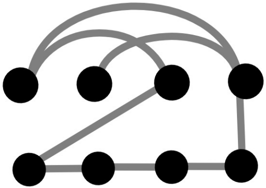
A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector.

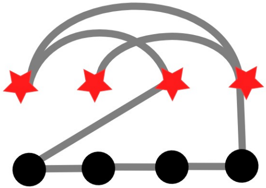


To **flip,** select $X \subseteq V(G)$ and replace $G[X]$ by its complement.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector.

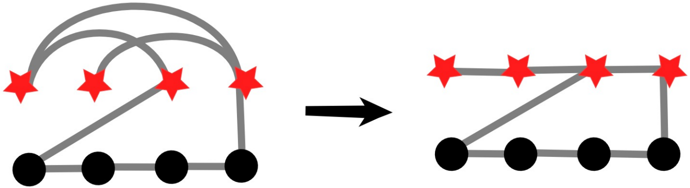


To **flip,** select $X \subseteq V(G)$ and replace $G[X]$ by its complement.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector.



To **flip,** select $X \subseteq V(G)$ and replace $G[X]$ by its complement.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

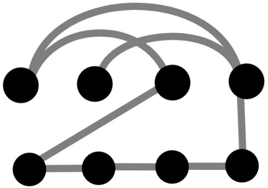
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



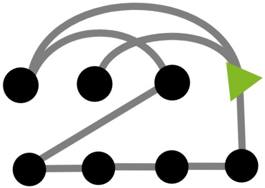
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



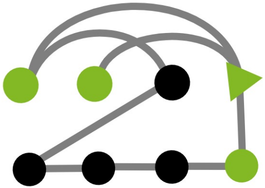
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



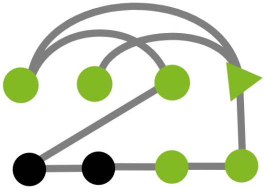
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:

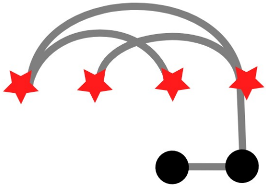
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



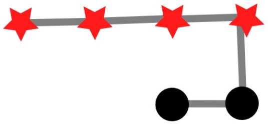
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



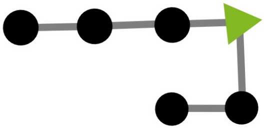
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



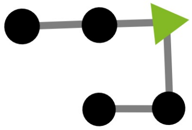
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



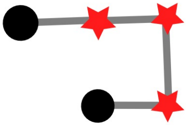
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



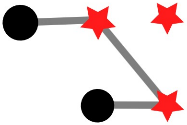
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



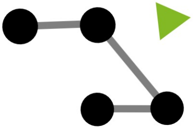
Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:



Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Example with $r = 2$:

Theorem (With Gajarský, Mählmann, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, & Toruńczyk.)

A class of graphs is **stable** *if and only if Flipper wins the* **radius-r flipper game** *for each $r \in \mathbb{N}$.*

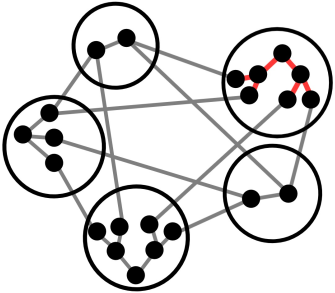
Two player game: Flipper and Connector. In each round:

1) If $|V(G)| = 1$ then Flipper wins in that round.
2) Else, Connector picks a vertex v and we restrict to $B_r(v)$.
3) Then Flipper performs a flip.

Flipper **wins the game** on a class $\mathcal{C}$ if there exists $t \in \mathbb{N}$ so that Flipper wins in $\leq t$ rounds on each $G \in \mathcal{C}$.

Theorem (Grohe, Kreutzer, & Siebertz 2019)

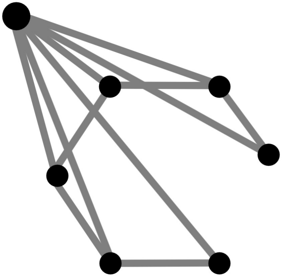
First-order **model-checking** *is fixed-parameter tractable on any class which is* **stable** *& excludes* $K_{t,t}$**-subgraph***.*



Theorem (Grohe, Kreutzer, & Siebertz 2019)

First-order **model-checking** *is fixed-parameter tractable on any class which is* **stable** *& excludes* $K_{t,t}$**-subgraph**.

Naive algorithm for determining if $G \models \phi$: $\mathcal{O}(n^{|\phi|})$.

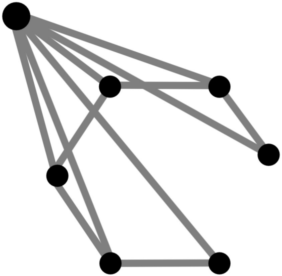


Theorem (Grohe, Kreutzer, & Siebertz 2019)

First-order **model-checking** *is fixed-parameter tractable on any class which is* **stable** *& excludes* $K_{t,t}$**-subgraph**.

Naive algorithm for determining if $G \models \phi$: $\mathcal{O}(n^{|\phi|})$.

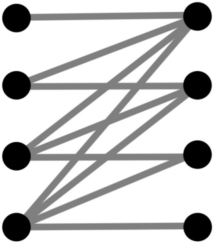
FO model-checking is FPT on $\mathcal{C}$ if there exists $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{R}$ so that the problem can be solved in time $f(|\phi|)n^c$.



Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

First-order model-checking is fixed-parameter tractable on any class which is **stable**.

Recall that we use first-order logic to “exclude”:

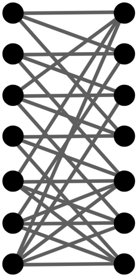


half-graph

Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

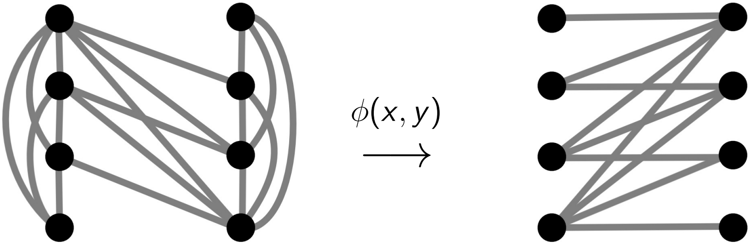
First-order model-checking is fixed-parameter tractable iff the class is **NIP**.

Instead "exclude" an arbitrary bipartite graph:



bipartite graph

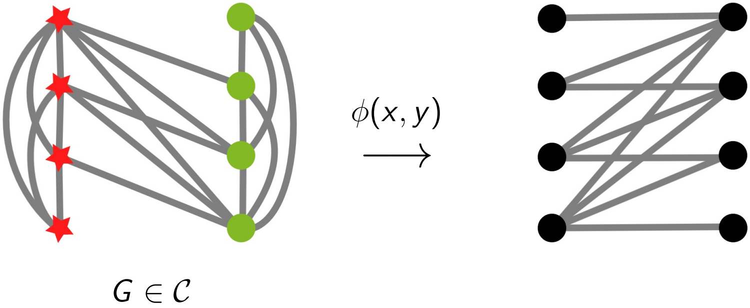
Use transductions to exclude something from $\mathcal{C}$:



$G \in \mathcal{C}$

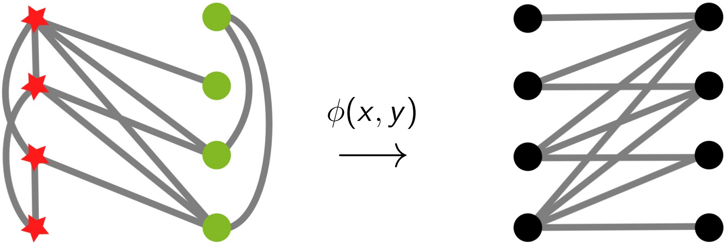
$$\phi(x, y) := \neg xy \in E$$

Use transductions to exclude something from $\mathcal{C}$:



$$\phi(x, y) := (\neg xy \in E) \wedge (c(x) \neq c(y))$$

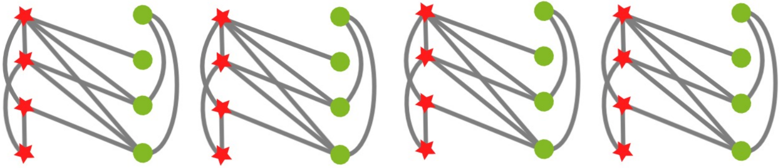
Use transductions to exclude something from $\mathcal{C}$:



$G \in \mathcal{C}$

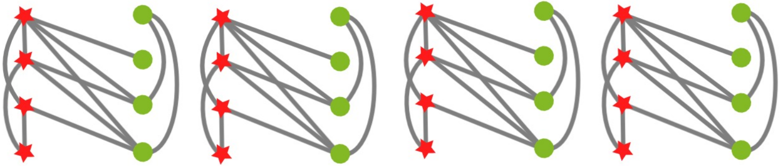
$$\phi(x, y) := (\neg xy \in E) \wedge (c(x) \neq c(y))$$

Use transductions to exclude something from $\mathcal{C}$:



$$\phi(x, y) := (\neg xy \in E) \wedge (c(x) \neq c(y)) \wedge (\neg x \text{ copy } y)$$

Use transductions to exclude something from $\mathcal{C}$:



For fixed $\phi(x, y)$, the resulting **transduction** of $\mathcal{C}$ is the class of all graphs which can be obtained this way.

$$\phi(x, y) := (\neg xy \in E) \wedge (c(x) \neq c(y)) \wedge (\neg x \text{ copy } y)$$

Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

First-order model-checking is fixed-parameter tractable iff the class is **NIP**.

Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

First-order model-checking is fixed-parameter tractable iff the class is **NIP**.

Conjecture (Gajarský, Pilipczuk, Toruńczyk)

A class has unbounded clique-width if and only if it has an FO-transduction which contains a subdivision of each wall.

Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

First-order model-checking is fixed-parameter tractable iff the class is **NIP***.*

Conjecture (Gajarský, Pilipczuk, Toruńczyk)

A class has unbounded clique-width if and only if it has an FO-transduction which contains a subdivision of each wall.

Thank you!