Exodromy for stratified topological spaces

(after MacPherson, Treumann, Curry–Patel)

1 Stratified spaces

Def Let P be a finite poset. The Alexandroff topology on P is the topology whose opens are the upwards closed sets, e.g.

$$P_{\geq p} = \{ q \in P \mid q \geq p \}.$$

Ex In $P = \{0 < 1\} = [1]$, the opens are $\emptyset, \{1\}, [1]$.

(Sierpiński space)

[illegible]

Def A <u>P-stratified space</u> is a pair (X,f) where X is a topological space and $f: X \longrightarrow P$ a continuous function. (Often: abuse of notation (X,P).)

Ex (1) I

 $\longrightarrow \{0<1\}$.

(2) S^1

 $\longrightarrow \{0<1\}$.

(3) S^2

 $\longrightarrow \{0<1<2\}$.

(4) Products: if (X,P), (X',P') are stratified

spaces, then $(X \times X', P \times P')$ is stratified over $P \times P'$.

 $\longrightarrow$

$$\left\{ \begin{tikzcd} (1,0) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] & (0,1) \arrow[u] \end{tikzcd} \right\}.$$

(Older references: usually only $P = [n]$.)

<u>Notation</u> $X_{\geq p} := f^{-1}(P_{\geq p})$, $X_p = f^{-1}(p)$, etc ...

open in X locally closed

<u>Def</u> Let $\mathcal{C}$ be a category (e.g. poset). Then $\mathcal{C}^{\triangleright} = \mathcal{C} \amalg \{\infty\}$ with

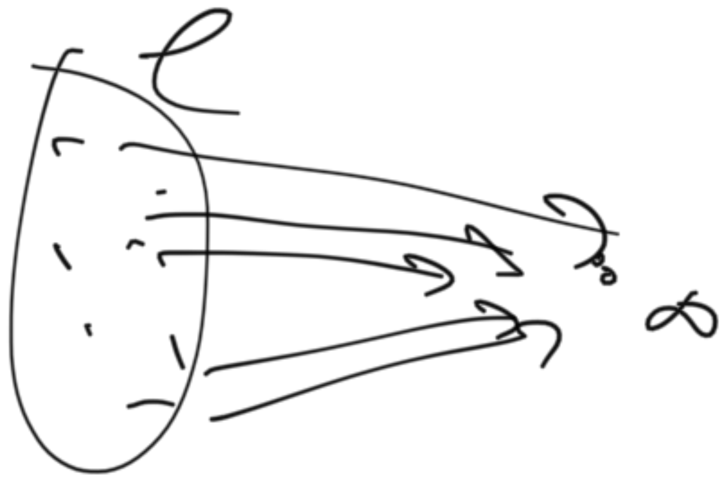
terminal

$$\mathcal{C}^{\triangleright}(x,y) = \begin{cases} \mathcal{C}(x,y) & x, y \in \mathcal{C} \\ * & y = \infty \\ \end{cases}$$

(right cone)

| $\emptyset$ $x = \infty$, $y \in \mathcal{C}$.

<u>Picture</u>



Similarly, $\mathcal{C}^{\triangleleft}$: (left cone)

<u>Rmk</u> If P is a poset, then so are $P^{\triangleleft}, P^{\triangleright}$.

<u>Ex</u> $[n]^{\triangleleft} \cong [n+1] \cong [n]^{\triangleright}$.

<u>Def</u> The <u>cone</u> CX of a topological space X is $(X \times \mathbb{R}_{>0}) \cup \{*\}$
where $U \subseteq CX$ is open iff $U \cap (X \times \mathbb{R}_{>0})$ is open

and $* \in U \Rightarrow X \times (0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$.

<u>Rmk</u> There is a natural map $(X \times \mathbb{R}_{\geq 0}) \coprod_{X \times 0} * \longrightarrow CX$,

which is a homeomorphism if X is compact Hausdorff.

But for $X = \mathbb{R}_{>0}$, CX has fewer opens: e.g.

$$\{(x, y) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid y < x\} \cup \{*\}$$

is not open in CX:

<u>Rmk</u> If X is P-stratified, then CX is $P^{\triangleleft}$-stratified:

$$\bar{f} : CX \longrightarrow P^{\triangleleft}$$

$$* \longmapsto -\infty$$
$$(x,t) \longmapsto f(x).$$

<u>Picture</u>

$-\infty$ ↳ whatever f was (forgetting the $\mathbb{R}_{>0}$-coordinate)

<u>Def</u> (HA, <u>Def</u> A.5.5, with simplifying assumptions)
A P-stratified space X is <u>conically stratified</u> if every $x \in X_p$ (for all $p \in P$) has a neighbourhood U_x of the form $\underbrace{\mathbb{R}^q \times CY}$ for some $P_{>p}$-stratified space Y.

Lawrie: any ?

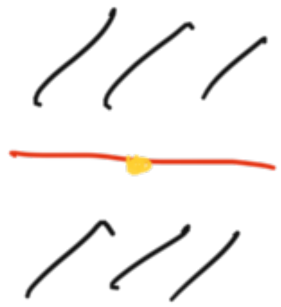
↳ $P^{\triangleleft}_{>p} = P_{\geq p}$-stratified.

Such an open U_x is called a <u>basic open</u>.

Ex Anything nice:

near equator:  $= \mathbb{R} \times$ 

$= \mathbb{R} \times C(\{x, y\})$.

near point:  $=$ 

$= C($$)$.

$LS^1 \cong S^0$.

Def A <u>morphism</u> $(X, P) \to (Y, Q)$ of stratified spaces is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ {} & {} \end{tikzcd}$$

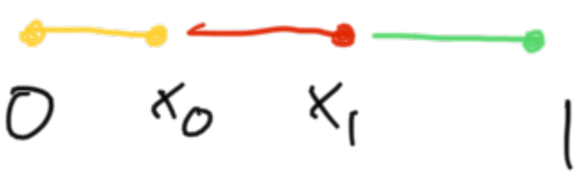
$$P \longrightarrow Q.$$

2 Exit paths

Def An $[n]$-stratification $I \xrightarrow{f} [n]$ is <u>monotone</u> if f is monotone.

Ex

$\{0 < 1\}$ monotone



$\{0 < 1 < 2\}$ monotone

└ denoted $(I, \{0, x_0, x_1, \dots, x_n, 1\})$.

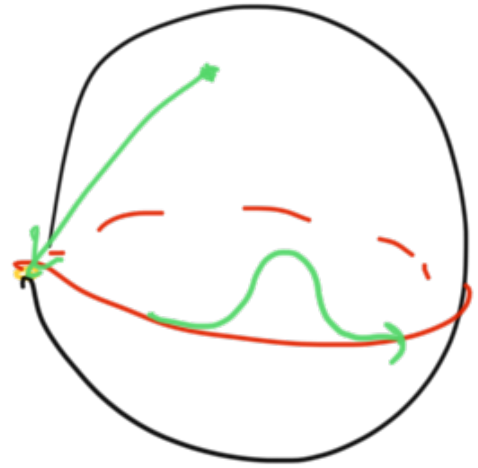
$\{0 < 1\}$ not monotone.

Def (Curry-Patel version)

An <u>exit path</u> in a stratified space (X, P) is a morphism $\alpha : (I, [n]) \to (X, P)$ from a monotone stratification to (X, P).

<u>Ex</u>

✓



✗

<u>Def</u> The <u>concatenation</u> of exit paths $\alpha : (I, [n]) \to (X, P)$, $\beta : (I, [m]) \to (X, P)$ with $\alpha(1) = \beta(0)$ is the map

$$\beta * \alpha : (I, [m+n]) \longrightarrow (X, P)$$

$\beta * \alpha : (I, \ldots) \to (X, P)$

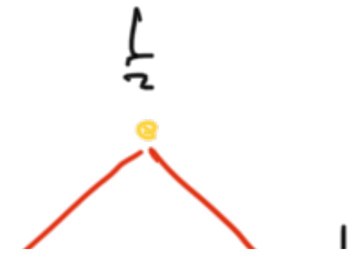
$$t \longmapsto \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

<u>Rmk</u> An isomorphism $(I,[n]) \to (I,[n])$ of monotone stratifications induces a reparametrisation on exit paths $(I,[n]) \to X$.

<u>Def</u> An <u>elementary homotopy</u> $\alpha \Rightarrow \beta$ from $\alpha : (I, \{0, \frac{1}{2}, 1\}) \to (X,P)$ to $\beta : (I, \{0,1\}) \to (X,P)$ is a map

$\xrightarrow{h} (X,P)$

whose restrictions to the edges are



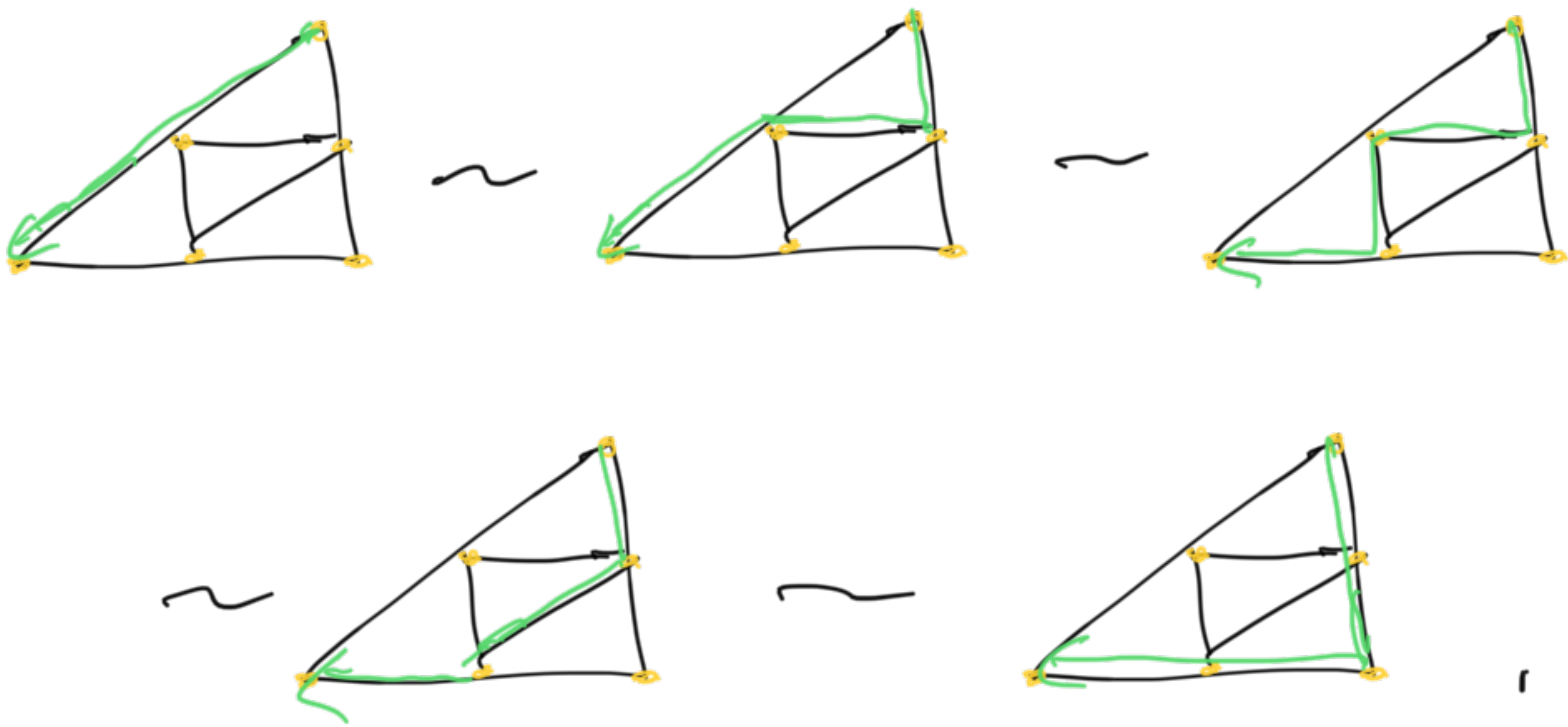
and such that the image of h is contained in a basic open neighbourhood U_x of $x = \alpha(0) = \beta(0)$.

Def Homotopy equivalence is the relation generated by

$$\gamma_1 * \alpha * \gamma_2 \sim \gamma_1 * \beta * \gamma_2$$

when $\alpha \Rightarrow \beta$, up to reparametrisation.

Picture



Def The <u>exit path category</u> $\underline{\mathrm{Exit}}_P(X)$ has

- objects: $|X|$ (points of X),
- morphisms: equivalence classes of exit paths.

It has a natural functor $\underline{\mathrm{Exit}}_P(X) \to P$.

$$\begin{aligned} x &\longmapsto f(x) \\ \alpha &\longmapsto \big(f(\alpha(0)) \leq f(\alpha(1))\big) \end{aligned}$$

Ex (0) If $P = \{*\}$, then $\underline{\mathrm{Exit}}_P(X) = \Pi_1(X)$ (fundamental groupoid) (assume X is "nice").

(1) Take the $\{0<1\}$-stratification on $\mathbb{R}$ given by

.

Then

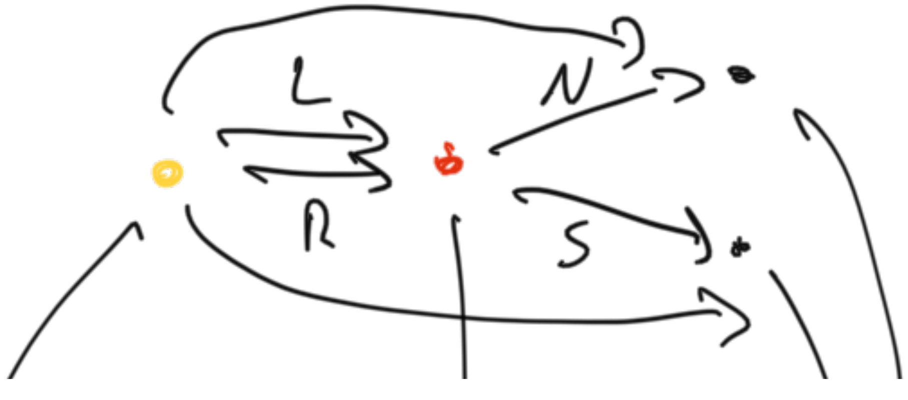
$$\underleftarrow{\mathrm{Exit}}_p(\mathbb{R}) \simeq$$

(no Automorphisms)

(2) Consider

$$\longrightarrow \{0 < 1 < 2\}.$$

Then $\underleftarrow{\mathrm{Exit}}_p(X)$ is equivalent to



$N \circ L = N \circ R$

$S \circ L = S \circ R$.

Aut=1 Aut=1 Aut=1

3 van Kampen theorem

Def A complete covering of X is a collection $\underline{U} = \{U_i\}$ of open subsets such that each $U_i \cap U_j$ (and X) is covered by $U_k \in \underline{U}$.

Prop (Stratified van Kampen theorem) (X, P) conically stratified topological space.
For any complete covering $\underline{U}$ of X, the map

$$\underset{\overrightarrow{U \in \underline{U}}}{\text{2-colim}}\ \underline{\mathrm{Exit}}_P(U) \longrightarrow \underline{\mathrm{Exit}}_P(X)$$

is an equivalence.

<u>Pf</u> (Sketch)

- On objects: $|X| = \varinjlim_{U \in \underline{\mathcal{U}}} |U|$.

- On morphisms:
  - * surjectivity: break down a path into parts in a single U
  - * injectivity: reduce to a single elementary homotopy in a basic open U_x. Do some subdivision to get a finite chain of elementary homotopies in U_x. $\square$

4. Constructible sheaves & exodromy equivalence

Recall: a sheaf of sets $\mathcal{F}$ on (X, P) is <u>constructible</u>

if each $\mathcal{F}|_{X_p}$ is locally constant. Write $\underline{\mathrm{Cons}}_P(X, \underline{\mathrm{Set}})$ for the category of constructible sheaves.

<u>Thm</u> (MacPherson, Treumann, ...) finite

Let (X, P) be a conically stratified space. Then the categories $\underline{\mathrm{Cons}}_P(X, \underline{\mathrm{Set}})$ and $[\underline{\mathrm{Exit}}_P(X), \underline{\mathrm{Set}}]$ are equivalent.

<u>Rmk</u> If (X, P) is conically stratified, and $x \in X_p$ then basis neighbourhoods of x are $U_x \cong \mathbb{R}^d \times CY$.

stratified $\quad$ $P_{>p}$-stratified.

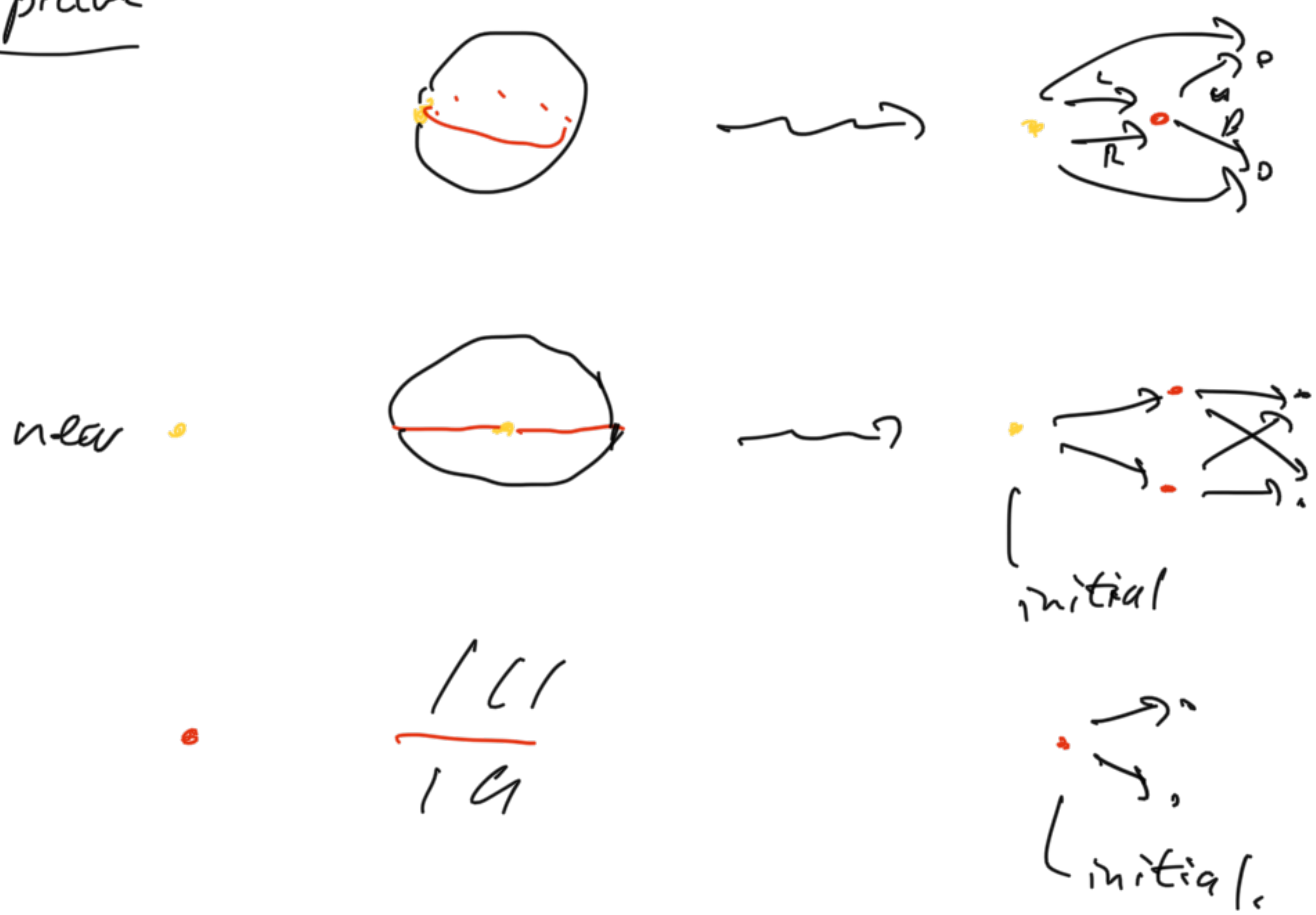
Then $x \in \underline{\mathrm{Exit}}_P(U_x)$ is <u>initial</u>:

$\mathbb{R}^d \times CY$

$\underline{\mathrm{Exit}}_P(x, y) = \{\alpha : x \to y \text{ exit}\}/\simeq$.

(..., cone pt).

Local picture



Construction of functors

Define $\Phi : [\underline{\mathrm{Exit}_p}(X), \underline{\mathrm{Set}}] \rightleftarrows \underline{\mathrm{Cons}_p}(X) : \Psi$ as follows:

- Given $\mathcal{F} : \underline{\mathrm{Exit}_p}(X) \to \underline{\mathrm{Set}}$, define $\mathrm{Open}(X)^{op}$

$$\Phi(\mathcal{F}) : \mathcal{T}(X) \longrightarrow \underline{Set}$$

$$U \longmapsto \lim_{x \in \underleftarrow{Exit}_p(U)} \mathcal{F}(x)$$

$$(U \subseteq V) \longmapsto \lim_{x \in \underleftarrow{Exit}_p(V)} \mathcal{F}(x) \to \lim_{x \in \underleftarrow{Exit}_p(U)} \mathcal{F}(x).$$

Then $\Phi(\mathcal{F})$ is a constructible sheaf:

* If $\underline{U}$ is a complete covering, van Kampen gives

$$\underleftarrow{Exit}_p(X) = 2\text{-}\underrightarrow{\mathrm{colim}}_{U \in \underline{U}} \underleftarrow{Exit}_p(U).$$

Thus,

$$\Phi(\mathcal{F})(X) = \lim_{x \in \underleftarrow{Exit}_p(X)} \mathcal{F}(x) = \lim_{x \in \left(2\text{-}\underrightarrow{\mathrm{colim}}_{U \in \underline{U}} \underleftarrow{Exit}_p(U)\right)} \mathcal{F}(x)$$

$$= \lim_{U \in \underline{\mathcal{U}}} \ \lim_{x \in \underline{\mathrm{Exit}}_P(U)} \mathcal{F}(x) = \lim_{U \in \underline{\mathcal{U}}} \mathcal{F}(U),$$

which is the sheaf condition.

* If $x \in X_p$, and $U_x \subseteq V_x$ are basic opens, then x is initial in both $\underline{\mathrm{Exit}}_P(U_x)$ and $\underline{\mathrm{Exit}}_P(V_x)$, so

$$\Phi(\mathcal{F})(U_x) = \mathcal{F}(x) = \Phi(\mathcal{F})(V_x).$$

This shows that $\Phi(\mathcal{F})|_{V_x}(V_x) \to \Phi(\mathcal{F})|_{U_x}(U_x)$ is an isomorphism.

But the $U_x \cap X_p$ form a basis of neighbourhoods of $x \in X_p$, so $\Phi(\mathcal{F})|_{X_p}$ is locally constant.

• Given $\mathcal{G} \in \underline{\mathrm{Cons}}_P(X)$, define

$$\Psi(\mathcal{G}) : \mathrm{Exit}_P(X) \to \mathrm{Set}$$

$$\Psi(\mathcal{G}) : \underline{\mathrm{Exit}}_P(X) \to \underline{\mathrm{Set}}$$

$$x \longmapsto \mathcal{G}_x = \mathcal{G}(\mathcal{U}_x) \text{ for any basic open } \mathcal{U}_x \ni x,$$

If $\alpha : (I, [n]) \to (X, P)$ is an exit path from x to y contained in $\mathcal{U}_x$, define $\Psi(\mathcal{G})(\alpha)$ as the <u>specialisation map</u>

$$\mathcal{G}_x \xleftarrow{\sim} \mathcal{G}(\mathcal{U}_x) \longrightarrow \mathcal{G}_y .$$

Check:

* It does not depend on $\mathcal{U}_x$;
* Check that it factors through elementary homotopies.

<u>Rmk</u> Clearly Φ and Ψ are functorial.

To check that they are inverses:

$$\Phi(\mathcal{F}) = \operatorname{colim} \lim \mathcal{F}(y)$$

$$\cdots_{U \ni x} \quad \cdots_{y \in \mathrm{Exit}_p(U)}$$

$$= \underset{\substack{\longrightarrow \\ U_x \ni x}}{\mathrm{colim}} \ \mathcal{F}(x) = \mathcal{F}(x),$$

so $\Psi \circ \Phi = \mathrm{id}$. Similarly, using the Sheaf condition and the fact that $\mathcal{G}_x = \mathcal{G}(U_x)$, we recover

$$\mathcal{G}(U) = \lim_{\substack{V \in \underline{\mathcal{U}}_{std} \\ | \\ U_x}} \mathcal{G}(V) = \lim_{x \in \underleftarrow{\underline{\mathrm{Exit}}}_p(U)} \mathcal{G}_x . \quad \square$$