Overview of 'Exodromy'

Goal. Give an overview of the motivating ideas behind joint work with Barwick and Glasman and our main results.

> Also explain some of the reasons for the tools we use (e.g., ∞-categories).

Outline.

(1) Monodromy in topology and algebraic geometry } π_1

(2) MacPherson's classification of constructible sheaves in terms of exit-paths

motivation

(3) What we want and why ∞-categories

(4) Monodromy for the derived category of local systems

(5) Monodromy for schemes: étale homotopy theory

(6) Main Results of 'Exodromy'

monodromy for exit-paths

1 Classical Monodromy

Topology. T (nice) locally connected topological space, $t \in T$

> Most familiar formulation: if T is connected, then there's an equivalence

$$\begin{aligned} LC(T; \mathrm{Vect}(k)) &\xrightarrow{\sim} \mathrm{Rep}_k(\pi_1(T,t)) \\ L &\longmapsto L_t \circlearrowleft \pi_1(T,t) \end{aligned}$$

Since locally const. sheaves on $[0,1]$ are const.

> Basepoint free formulation:

fundamental groupoid

Put your favorite coeffs here.

$$\begin{aligned} LC(T; \mathrm{Set}) &\xrightarrow{\sim} \mathrm{Fun}(\Pi_1(T), \mathrm{Set}) \\ F &\longmapsto [x \longmapsto F_x] \end{aligned}$$

given a path $\gamma : [0,1] \longrightarrow T$

$$F_{\gamma(0)} \xleftarrow{\sim} \Gamma([0,1]; \gamma^* F) \xrightarrow{\sim} F_{\gamma(1)}$$

Algebraic geometry. X scheme, $\bar{x} \longrightarrow X$ geometric point

> Étale fundamental group: $\pi_1^{\text{ét}}(X, \bar{x}) \in \mathrm{Pro}(\mathrm{Grp}^{\mathrm{fin}})$

Example. Given a field k with separable closure $k^{\mathrm{sep}} \supset k$:

$$\pi_1^{\text{ét}}(\mathrm{Spec}(k), \mathrm{Spec}(k^{\mathrm{sep}})) \cong \mathrm{Gal}(k^{\mathrm{sep}}/k)$$

Example. $X/\mathbb{C}$ finite type:

$$\pi_1(X(\mathbb{C})^{\mathrm{an}})^{\wedge} \xrightarrow{\sim} \pi_1^{\text{ét}}(X)$$

Monodromy. X connected

$$\left(\begin{array}{l}\overbrace{\text{locally const constructible}}^{\text{can be trivialized on a finite cover w/ finite values}}\\ \text{étale sheaves of sets on } X\end{array}\right) \simeq \left(\begin{array}{l}\text{finite } \overbrace{\text{sets}}^{\text{modules / a finite ring, ...}} \text{ with a}\\ \text{continuous } \pi_1^{\text{ét}}(X)\text{-action}\end{array}\right)$$

Remark. The finiteness conditions can be removed if X is locally connected and we replace the profinite étale fundamental group with the SGA 3 étale fundamental pro-group.

Very Rough Goal of Exodromy. Remove the local constancy condition

> This already has an answer in topology!

2 Exit-paths & Constructible Sheaves
after MacPherson-Treumann

Stratified Spaces. A stratification of a topological space S by $\{0 < \cdots < n\}$ is a filtration by closed subsets

$$F_0(S) \subset \cdots \subset F_{n-1}(S) \subset F_n(S) = S. \quad \Big\} \text{ Strata } F_i(S) \setminus F_{i-1}(S)$$

More generally:

Def. Let P be a poset. The **Alexandroff topology** on P is the topology where

U is 'upwards closed'

$$[U \subset P \text{ open}] \iff [p \in U \text{ and } q \geq p \Rightarrow q \in U]$$

A **P-stratification** of a top space S is a continuous map $s: S \to P$.

> $p \in P$, **p-th stratum** $S_p := s^{-1}(p) \subset S$.

> A sheaf F on S is **constructible** if $\forall p \in P$

$$F|_{S_p} \text{ is locally constant.}$$

Thesis or 'Exit Paths and Constructible stacks'

Theorem ($\underbrace{\text{MacPherson, Treumann}}$). Let $S \to P$ be a nice stratified topological space. There is an equivalence of categories

E.g 'topologically stratified'

$$\mathrm{Cons}_P(S; \mathrm{Set}) \xrightarrow{\sim} \mathrm{Fun}(\underbrace{\mathrm{Exit}_P(S)}, \mathrm{Set})$$

exit-path category
- objects: points of S
- morphisms: homotopy classes of paths flowing from lower to higher strata
  - ↳ called 'exit-paths'

$$F \longmapsto [x \longmapsto F_x]$$

on morphisms acts by specialization

3 What we want & Why ∞-categories?

Want.

(1) An exit-path category for the étale topology

(2) Want this description to work for the derived category of constructible sheaves with perfect stalks: given a finite ring R and scheme X, we want an equivalence

$$\underbrace{D_{cons}}_{\text{with perfect stalks}}(X_{\acute{e}t}; R) \simeq \mathrm{Fun}^{cts}(\text{ÉtaleExit}(X), \overbrace{D_{perf}}^{\text{qis to a bounded complex of finite projective } R\text{-modules}}(R)).$$

with perfect stalks. Sometimes written 'ctf' for 'constructible locally of finite Tor-dim'

qis to a bounded complex of finite projective R-modules ⇕ compact object of D(R)

- Similarly with R replaced by $\mathbb{Z}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell, \mathbb{F}_q[\![t]\!], \ldots$

Why ∞-categories. Knowing an 'underived' statement

$$\left\{ \begin{array}{c} \text{constructible étale} \\ \text{sheaves of } R\text{-mods on } X \end{array} \right\} \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), \mathrm{Mod}(R)^{fin})$$

does **not** imply the derived result!

> This problem already appears in topology (and with monodromy).

Problematic Example. Consider the 2-Sphere S^2 (more generally, any reasonable connected, simply connected top. space).

- $\pi_1(S^2) \simeq *$, so monodromy tells us that

$$LC(S^2; Ab) \xrightarrow{\sim} Ab.$$

- So if 'deriving' passed through taking functors, we'd believe that the derived category of local systems on S^2 is $D(\mathbb{Z})$.
- This is the wrong answer!

$$H^*_{sheaf}(S^2; \mathbb{Z}) \cong H^*_{sing}(S^2; \mathbb{Z}) \cong \begin{cases} 0, & * \neq 0,2 \\ \mathbb{Z}, & * = 0,2 \end{cases}$$

The derived category of local systems on S^2 knows that $H^2(S^2; \mathbb{Z}) \neq 0$, but $D(\mathbb{Z})$ does not!

More precisely, this shows there is no equivalence

$$\begin{pmatrix} \text{derived local} \\ \text{systems on } S^2 \end{pmatrix} \simeq D(\mathbb{Z})$$

that sends $\mathbb{Z}_{S^2}$ to $\mathbb{Z}$. In particular, no sym. monoidal equivalence. Moreover, these categories are actually not equivalent

Real Problem. If C is a 1-category and D is an ∞-category, then generally $ho(Fun(C,D)) \not\simeq Fun(C, ho(D))$.

homotopy category

Note. This problem will always occur if we work with n-categories for any finite n: just replace S^2 by S^{n+1} in the example.

Upshot. At very least we need to work with dg-categories to get the results we're after.

Notes on this point.

(1) The theory of dg-categories embeds into the theory of ∞-categories.

(2) It is not actually harder to work with ∞-categories than dg-categories.

- Thanks to the work of Lurie, the tools for working with ∞-cats are better.

(3) It is actually useful to have 'nonabelian' results for sheaves of ∞-groupoids, and these immediately imply the results for derived categories.

4 Monodromy for topological spaces, redux

Notation. Write Spc for the ∞-category of **spaces** (aka ∞-groupoids).

> Spc can be obtained from the category Top of topological spaces by formally inverting the weak homotopy equivalences (in the ∞-categorical sense).
  - We write $\Pi_\infty : \mathrm{Top} \longrightarrow \mathrm{Spc}$ for the localization functor.

> If we think about $\Pi_\infty(T)$ as an $\underbrace{\infty\text{-groupoid}}_{\text{singular simplicial set } \mathrm{Sing}(T)}$, it has:

(0) Objects: points of T

(1) 1-morphisms: paths in T (paths are invertible!)

(2) 2-morphisms: homotopies between paths

(3) 3-morphisms: homotopies between homotopies

$\vdots$

Thm (Lurie [HA, Theorem A.4.19]). Let T be a nice locally contractible topological space (e.g., a CW-complex or manifold). There is an equivalence of ∞-categories

$$
\begin{tikzcd}
LC(T;\mathrm{Spc}) \arrow[r, "\simeq"] & \mathrm{Fun}(\Pi_\infty(T), \mathrm{Spc}) \arrow[d, "\wr"] \\
 & \mathrm{Spc}_{/\Pi_\infty(T)} \arrow[ul, dotted]
\end{tikzcd}
$$

'Grothendieck construction'
AKA
'straightening / unstraightening'

$$
\left[\begin{array}{l} U \longmapsto \mathrm{Map}_{/\Pi_\infty(T)}(\Pi_\infty(U), X) \\ \mathrm{Open}(T) \ni U \end{array}\right] \longleftarrow\!\shortmid \begin{array}{c} X \\ \downarrow \\ \Pi_\infty(T) \end{array}
$$

(to see this is a sheaf, need Lurie's van Kampen Thm [HA, Prop. A.3.2]: given a covering sieve S of $V \subset T$, $\operatorname{colim}_{U \in S} \Pi_\infty(U) \xrightarrow{\sim} \Pi_\infty(V)$ in Spc.

Immediate consequence. For any ring R,

$$\left\{\begin{matrix}\text{derived } \infty\text{-cat. of local} \\ \text{of } R\text{-modules on } T\end{matrix}\right\} \simeq \mathrm{Fun}(\Pi_\infty(T), \underbrace{D(R)}_{\substack{\text{derived } \infty\text{-cat} \\ \text{of } R}})$$

- More generally, Spc or $D(R)$ can be replaced by any reasonable 'coefficients': precisely, any 'presentable ∞-category'.

Key takeaway. Nonabelian coefficients are the universal example and we should prove theorems for sheaves of spaces when possible.

Exodromy, redux

Theorem (Lurie [HA, Theorem A.9.3]). Let P be a poset satisfying the ascending chain condition and S a nice P-stratified space. There is an equivalence of ∞-categories

$$\mathrm{Cons}_P(S;\mathrm{Spc}) \simeq \mathrm{Fun}(\underbrace{\mathrm{Exit}_P(S)}_{\substack{\text{exit-path}\\ \infty\text{-category of } S}}, \mathrm{Spc})$$

5 Etale homotopy Theory
after Artin-Mazur-Friedlander

Let X be a scheme and $\bar{x} \to X$ a geometric point.

'π-finite spaces':
- $\pi_0(S)$ finite
- $\pi_i(S,s)$ finite $\forall\, i>0,\ s \in S$
- $\pi_i(S)=0$ for $i>>0$

Étale π_1

$$\pi_1^{\acute{e}t}(X,\bar{x}) \in \mathrm{Pro}(\mathrm{Grp}^{fin})$$

$$\pi_1^{\acute{e}t}(\mathrm{Spec}(k), \mathrm{Spec}(k^{sep})) \cong \mathrm{Gal}(k^{sep}/k)$$

$X/\mathbb{C}$ finite type

$$\pi_1(X(\mathbb{C})^{an})^{\wedge} \xrightarrow{\sim} \pi_1^{\acute{e}t}(S)$$

connected

$$\left(\begin{array}{c}\text{locally const constructible}\\ \text{étale sheaves of sets on } X\end{array}\right)$$

$$\text{\rotatebox{90}{$\simeq$}}$$

$$\left(\begin{array}{c}\text{finite sets with a}\\ \text{continuous } \pi_1^{\acute{e}t}(X)\text{-action}\end{array}\right)$$

Étale homotopy

$$\hat{\Pi}_\infty^{\acute{e}t}(X) \in \mathrm{Pro}(\mathrm{Spc}_\pi)$$

$$\pi_1(\hat{\Pi}_\infty^{\acute{e}t}(X), \bar{x}) \simeq \pi_1^{\acute{e}t}(X,\bar{x})$$

$X/\mathbb{C}$ finite type

$$\Pi_\infty(X(\mathbb{C})^{an})^{\wedge}_{\pi} \xrightarrow{\sim} \hat{\Pi}_\infty^{\acute{e}t}(X)$$

can actually be defined so this $\simeq$ holds

$$\mathrm{LCC}_{\acute{e}t}(X;\mathrm{Spc}) \simeq \mathrm{Fun}^{cts}(\hat{\Pi}_\infty^{\acute{e}t}(X), \mathrm{Spc}_\pi)$$

$$\begin{tikzcd}
\mathrm{Spc}_\pi^{op} \arrow[r, "{\mathrm{Fun}(\Pi_\infty(-),\mathrm{Spc}_\pi)}"] \arrow[d] & \mathrm{Cat}_\infty \\
\mathrm{Pro}(\mathrm{Spc}_\pi)^{op} \arrow[ur, dashed, "{\mathrm{Fun}^{cts}(-,\mathrm{Spc}_\pi)}"'] &
\end{tikzcd}$$

extension by univ. property

6 Main Results of 'Exodromy'

First. Need to define 'ÉtaleExit(X)'

Suggestive Observation. Let F be a sheaf on a scheme X, and $\bar{x} \to X$ and $\bar{\eta} \to X$ geometric points such that $x \in \overline{\{\eta\}}$. Every étale specialization $\alpha : \bar{\eta} \to \operatorname{Spec}(\mathcal{O}^{sh}_{X,x})$ gives rise to a specialization map

$$sp_\alpha : F_{\bar{x}} \longrightarrow F_{\bar{\eta}}.$$

If F is constructible, then F is lcc if and only if all specialization maps are equivalences.

Def. Let X be a qcqs scheme. The pro-category $\mathrm{Gal}(S)$ has underlying category the category $\mathrm{Pt}(\mathrm{Sh}_{ét}(X))$ of points of the étale topos of X:

> Objects: geometric points $\bar{x} \to X$ $\left(\kappa(\bar{x}) = \kappa(x)^{sep}\right)$

> Morphisms:

$$\mathrm{Hom}_{\mathrm{Gal}(X)}(\bar{x}, \bar{x}') := \left\{ \begin{tikzcd} & \mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \arrow[d] \\ & X & \bar{x}' \arrow[l] \arrow[ul, dotted] \end{tikzcd} \right\}$$

Nonempty iff $x \in \overline{\{x'\}} \subset X$

See [SGA 4, Exposé VIII, §7]

$$\cong \mathrm{Hom}_{\mathrm{Sch}_X}(\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}), \mathrm{Spec}(\mathcal{O}^{sh}_{X,x'}))$$

Profinite Structure

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \cong \lim U \text{ “}= \bigcap_{\substack{\text{étale nbds} \\ U \text{ of } \bar{x}}} U\text{”}$$

$$\begin{tikzcd} & U \arrow[d, "\text{aff ét}"] \\ \bar{x} \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

Cofiltered

> So $\mathrm{Hom}_{\mathrm{Sch}_S}(\bar{x}', \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})) \cong \lim_U \mathrm{Hom}_{\mathrm{Sch}}(\bar{x}', U)$

$$\cong \lim_U \mathrm{Hom}_{\mathrm{Sch}}(\bar{x}', U)$$

$\coprod$ of spectra of fin sep. extensions

Results

Thm 1 (BGH). Let X be a qcqs scheme. There's an equivalence of ∞-categories

$$\mathrm{Cons}_{\acute{e}t}(X; \mathrm{Spc}) \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), \mathrm{Spc}_\pi).$$

For any finite ring R, there's an equivalence

$$\underbrace{D_{cons}(X; R)}_{\substack{\text{constructible derived} \\ \infty\text{-category}}} \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), \underbrace{D_{perf}(R)}_{D(R)^\omega = D(R)^{dual}})$$

$$F \longmapsto [\bar{x} \longmapsto F_{\bar{x}}]$$

- Similarly, $D_{cons}(X; \mathbb{Z}_\ell) \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), D_{perf}(\mathbb{Z}_\ell))$.
- If $|X|$ is noetherian, then

$$D_{cons}(X; \bar{\mathbb{Q}}_\ell) \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), D_{perf}(\bar{\mathbb{Q}}_\ell)).$$

Need condensed mathematics to do this correctly

Relation to Étale Homotopy Theory

Note. Can regard $\mathrm{Gal}(X)$ as a pro-category.

Notation. Write $B : \mathrm{Pro}(\mathrm{Cat}_\infty) \to \mathrm{Pro}(\mathrm{Spc})$ for the 'invert everything'/ 'classifying prospace' functor $\{C_i\}_{i \in I} \mapsto \{BC_i\}_{i \in I}$

- This is the left adjoint to the inclusion $\mathrm{Pro}(\mathrm{Spc}) \hookrightarrow \mathrm{Pro}(\mathrm{Cat}_\infty)$.

Thm 2 (H.) Let X be a qcqs scheme. There's a natural morphism

$$\underbrace{\Pi^{\text{ét}}_\infty(X)}_{\substack{\text{non-profinitely complete}\\ \text{étale homotopy type}}} \longrightarrow B\mathrm{Gal}(X)$$

that becomes an equivalence on homotopy pro-groups.

also called an 'equivalence after protruncation'

- This is the 'right' notion of an equivalence for pro-spaces

Reconstruction of schemes & Anabelian Geometry

Theorem (Neukirch–Uchida). Let K and L be numberfields. Then

$$K \cong L \iff \mathrm{Gal}(\bar{K}/K) \cong \mathrm{Gal}(\bar{L}/L)$$

as profinite groups

Theorem (Pop). The same is true with 'number fields' replaced by 'infinite fields finitely generated over their prime fields'.

Question. Is there a global version of Neukirch–Uchida?

Theorem 3 (BGH, after Voevodsky). Let k be a finitely generated field of characteristic 0. Then $X \mapsto \mathrm{Gal}(X)$ is a complete invariant of normal k-schemes.

The target of the functor is a little involved, but morally topological categories w/ a $\mathrm{Gal}(\bar{k}/k)$-action

doesn't just detect isomorphisms: detects <u>all</u> morphisms

$$\mathrm{Hom}_{\mathrm{Sch}_k}(X,Y) \cong \mathrm{Hom}(\mathrm{Gal}(X), \mathrm{Gal}(Y))$$