Instituto de Matemática Pura e Aplicada

Doctoral Thesis

MIN-MAX THEORY FOR NONCOMPACT MANIFOLDS AND THREE-SPHERES WITH UNBOUNDED WIDTHS

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À minha família

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“The methods we will take up here are all variations on a basic result known to everyone who has done any walking in the hills: the mountain pass lemma.”

L. Nirenberg, Variational and topological methods in nonlinear problems, Bull. Am. Math. Soc., 4, 1981.

“Many of the world’s mountain ranges have presented formidable barriers to travel.”

(Adapted from the Wikipedia article: Mountain Pass)

Abstract

In this thesis, we deal with new min-max constructions of minimal surfaces and with an analysis of the min-max invariant, the width, on a class of Riemannian metrics of positive scalar curvature on the three-sphere. The work is divided in two independent parts.

We prove the existence of closed embedded minimal hypersurfaces in complete non-compact Riemannian manifolds containing a bounded open subset with smooth and strictly mean-concave boundary and a natural thickness assumption on the behavior of the geometry at infinity. With this intent, we develop a min-max theory for the area functional following Almgren and Pitts' setting, to produce minimal hypersurfaces with intersecting properties.

In the second part, we build a sequence of Riemannian metrics on the three-sphere with scalar curvature greater than or equal to 6 and arbitrarily large widths. The search for metrics with such properties is motivated by the rigidity result of min-max minimal spheres in three-manifolds obtained by F. Codá Marques and A. Neves. The construction of these examples is based on the connected sum procedure of positive scalar curvature metrics introduced by Gromov and Lawson. We develop analogies between the area of the boundaries of some special open subsets in our Riemannian three-spheres and 2-colorings of associated full binary trees. Via the relative isoperimetric inequality and elementary combinatorial arguments, we argue that the widths converge to infinity.

Keywords: Min-max theory, mean-curvature concave regions, maximum principle for varifolds, minimal surfaces in non-compact manifolds, relative isoperimetric inequality, scalar curvature.

Resumo

Nessa tese, lidamos com uma nova construção min-max de hipersuperfícies mínimas e com uma análise do invariante min-max, a width, em uma classe de métricas Riemannianas de curvatura escalar positiva na esfera tridimensional. Este trabalho é dividido em duas partes independentes.

Em primeiro lugar, provamos a existência de hipersuperfícies mínimas fechadas e mergulhadas em variedades Riemannianas completas e não compactas, contendo um subconjunto aberto e limitado com fronteira suave e estritamente côncava com relação à curvatura média e com uma hipótese de espessura na geometria dos fins. Com esta finalidade, é desenvolvida uma nova teoria min-max, seguindo a abordagem de Almgren e Pitts, que produz hipersuperfícies mínimas que intersectam um domínio previamente fixado.

Na segunda parte do trabalho, construímos uma sequência de métricas Riemannianas na esfera, com curvatura escalar maior do que ou igual a 6 e widths arbitrariamente grandes. A pergunta sobre a existência de métricas com tais propriedades é motivada pelo teorema de rigidez de esferas min-max em variedades tridimensionais obtido por F. Codá Marques e A. Neves. A construção desses exemplos é baseada no procedimento de soma conexa de métricas de curvatura escalar positiva introduzido por Gromov e Lawson. Criamos analogias entre a área do bordo de alguns abertos especialmente escolhidos nas nossas esferas Riemannianas e 2-colorações de árvores binárias associadas. Via a desigualdade isoperimétrica relativa e argumentos combinatórios, somos capazes de concluir que as widths convergem para infinito.

Palavras-chave: Teoria min-max, superfícies mínimas em variedades não compactas, princípio do máximo para varifolds, regiões côncavas com respeito à curvatura média, curvatura escalar, desigualdade isoperimétrica relativa.

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Introduction

Minimal surfaces, the extremizers of the area functional, are among the most important topics in differential geometry. Euler and Lagrange were the first to consider minimal surfaces, proving that if the graph of a C^2 function u is minimal in the Euclidean space, then u satisfies a second order elliptic quasi-linear partial differential equation. Later, Meusnier discovered a geometric characterization for these surfaces, by the vanishing of the mean-curvature. These two points of view explain why minimal surfaces are natural objects of study in geometric analysis.

Henceforth, many mathematicians contributed to the development of the theory and applied its ideas to settle deep problems and establish beautiful results in geometry. For instance, minimal surfaces have a strong link with problems involving the scalar curvature of three-manifolds, the proof of the positive mass conjecture in general relativity by Schoen and Yau [57] is one of the most important examples of this connection. Some other important recent works using minimal surfaces are the proof of the finite time extinction of the Ricci flow with surgeries starting at a homotopy 3-sphere by Colding and Minicozzi [18] and [19], the proof of the Willmore Conjecture and the existence of infinitely many closed minimal hypersurfaces in closed manifolds with positive Ricci curvature by Marques and Neves [36] and [37].

The existence of minimal submanifolds with some specific properties plays a fundamental role in the development of the theory. The most natural way to produce minimal surfaces is by minimizing the area functional in a fixed class. This idea was applied in many contexts: in a class of surfaces with same boundary, known as the Plateau's Problem; or in a homology class; or even in the class of surfaces $\Sigma \subset \Omega$, with $\partial\Sigma \subset \partial\Omega$, the free boundary problem. It was proved, by Schoen and Yau [56], Sacks and Uhlenbeck [50] and Freedman, Hass and Scott [28], that if Σ is a closed incompressible surface in a 3-manifold M, then there is a least area immersion with the same action in the π_1 level.

In the topology of 3-manifolds, there are two important types of surfaces, namely the Heegaard splittings and the incompressible surfaces. The results above show that this second type occur as minimal surfaces produced by minimization processes, and for this reason, they are stable, i.e., the Morse index is equal to zero. It is also possible to apply variational methods to construct higher index minimal surfaces. There are two basic approaches: applying Morse theory to the energy functional on the space of maps from a fixed surface, such as in the works of Sacks and Uhlenbeck [49], Micallef and Moore [39] and Fraser [27], or via a min-max argument for the area functional over classes of sweepouts. In some cases, these methods can be applied to realize Heegaard splitings as embedded minimal surfaces, see [48] and [17].

This min-max technique is the central object of study of this thesis, it was inspired by the work of Birkhoff [8] on the existence of simple closed geodesics in Riemannian 2-spheres, a question posed by Poincaré. Looking for closed geodesics, he interpreted the geometric point of view suggested by the Principle of Least Action as the iteration of a specific curve shortening process. This process is continuous with respect to variations of the beginning curve, so it can be applied to whole families of closed curves sweeping out a given Riemannian 2-sphere at the same time. Considering the longest curve in the family after each step of the shortening process, a subsequence of these converges to a closed geodesic.

In this thesis, we deal with new min-max constructions of minimal surfaces and with an analysis of the min-max invariant, the width, on a class of Riemannian metrics of positive scalar curvature on the three-sphere. The work is divided in two independent parts, which we detail now.

1. Min-max minimal surfaces in non-compact manifolds

There is no immersed closed minimal surface in the Euclidean space $\mathbb{R}^3$. This fact illustrates the existence of simple geometric conditions creating obstructions for a Riemannian manifold to admit closed minimal surfaces. In the Euclidean space, we can see the obstruction coming in the following way: by the Jordan-Brouwer separation theorem every connected smooth closed surface $\Sigma^2 \subset \mathbb{R}^3$ divides $\mathbb{R}^3$ in two components, one of them bounded, which we denote Ω. Start contracting a large Euclidean ball containing Ω until it touches Σ the first time. Let $p \in \Sigma$ be a first contact point, then the maximum principle says that the mean curvature vector of Σ at p is non-zero and points inside Ω. In particular, $\Sigma^2 \subset \mathbb{R}^3$ is not minimal.

In first part of this work we consider two natural and purely geometric properties that imply that a complete non-compact Riemannian manifold N admits a smooth closed embedded minimal hypersurface. In order to state the result, we introduce the following notation: we say that N has the $\star_k$-*condition* if there exists $p \in N$ and $R_0 > 0$, such that

$$(\star_k-\text{curvature}) \qquad \sup_{q \in B(p,R)} |\mathrm{Sec}_N|(q) \leq R^k$$

and

$$(\star_k-\text{injectivity radius}) \qquad \inf_{q \in B(p,R)} inj_N(q) \geq R^{-\frac{k}{2}},$$

for every $R \geq R_0$, where $|\mathrm{Sec}_N|(q)$ and $inj_N(q)$ denote, respectively, the maximum sectional curvature for 2-planes contained in the tangent space T_qN and the injectivity radius of N at q. For instance, if N has bounded geometry then the $\star_k$-condition holds for every positive k.

Our main result is:

Theorem 1. *Let (N^n, g) be a complete non-compact Riemannian manifold of dimension $n \leq 7$. Suppose:*

- *N contains a bounded open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary;*
- *N satisfies the $\star_k$-condition, for some $k \leq \frac{2}{n-2}$.*

Then, there exists a closed embedded minimal hypersurface $\Sigma^{n-1} \subset N$ that intersects Ω.

In the recent paper [21], Collin, Hauswirth, Mazet and Rosenberg prove that any complete non-compact hyperbolic three-dimensional manifold of finite volume admits a closed embedded minimal surface. These manifolds have a different behavior at infinity from those considered in Theorem 1, their ends are all thin hyperbolic cusps. On the other hand, our $\star_k$-condition on N can be seen as a thickness assumption on its ends. However, the two arguments can be applied together and we obtain the following:

Corollary 1. *Let (N^3, g) be a complete non-compact Riemannian manifold. Suppose that N contains a bounded open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary. Suppose also that each end of N is either a hyperbolic cusp or thick, satisfying the $\star_k$-condition for some $k \leq 2$. Then, there exists a closed embedded minimal surface $\Sigma^2 \subset N$ that intersects the mean-concave region Ω.*

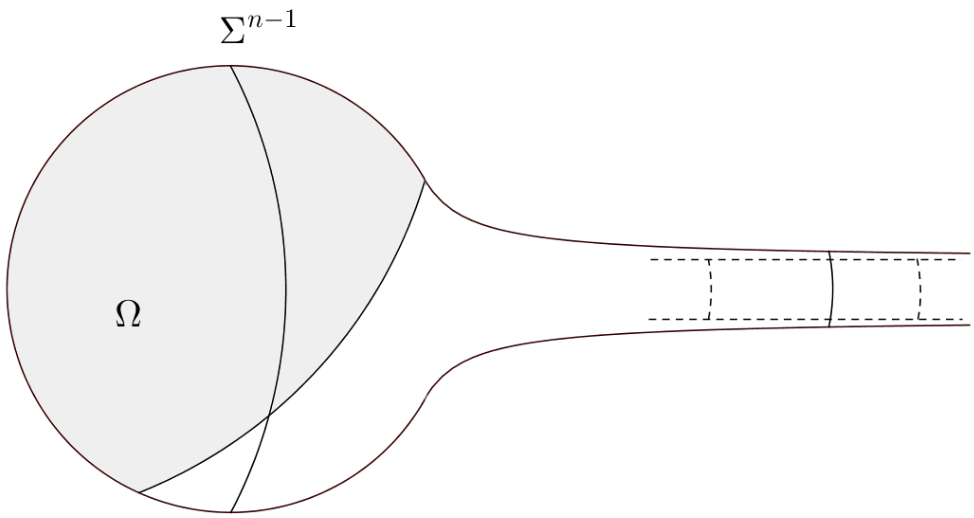


Figure 1: A complete non-compact Riemannian manifold, asymptotic to a cylinder and containing a mean-concave open set Ω. In this case, Theorem 1 could be applied.

In our arguments, the geometric behavior of the ends of M involving the $\star_k$-condition is used together with the monotonicity formula to provide a lower bound for the $(n-1)$-dimensional volume of minimal hypersurfaces in M that, simultaneously, intersect Ω and contain points very far from it.

The hypothesis involving the mean-concave bounded domain Ω comes from the theory of closed geodesics in non-compact surfaces. In 1980, Bangert proved the existence of infinitely many closed geodesics in a complete Riemannian surface M of finite area and homeomorphic to either the plane, or the cylinder or the Möbius band, see [7]. The first step in his argument is to prove that the finite area assumption implies the existence of locally convex neighborhoods of the ends of M.

As a motivation for our approach, we briefly discuss how Bangert uses the Lusternik-Schnirelmann's theory in the case that M is homeomorphic to the plane. If $C \subset M$ is a locally convex neighborhood of the infinity of M whose boundary $\partial C \neq \varnothing$ is not totally geodesic, he proves that M contains infinitely many closed geodesics intersecting $M - C$. To obtain one such curve, the idea is to apply the Lusternik-Schnirelmann's technique for a class Π of paths β defined on $[0, 1]$ and taking values in a finite-dimensional subspace of the space of piecewise C^1 closed curves, with the properties that the curves β_0 and β_1 have image in the interior of C, being β_0 non-contractible in C and

β_1 contractible in C. The min-max invariant in this case is the number

$$L(\Pi) = \inf_{\beta \in \Pi} \sup\{E(\beta_t) : \beta_t(S^1) \cap (M - \mathring{C}) \neq \varnothing\},$$

where $E(\gamma)$ denotes the total energy of a map $\gamma : S^1 \to M$, which is defined as the integral of the energy density $|\gamma'(s)|^2 ds$. Then, achieve $L(\Pi)$ as the energy of a closed geodesic intersecting $M - C$.

To prove the Theorem 1, we develop a min-max method that is adequate to produce minimal hypersurfaces with intersecting properties. Let us briefly describe our technique. Let (M^n, g) be a closed Riemannian manifold and Ω be an open subset of M. Consider a homotopy class Π of one-parameter sweepouts of M by codimension-one submanifolds. For each given sweepout $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we consider the number

$$L(S, \Omega) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \Sigma_t \cap \overline{\Omega} \neq \varnothing\},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated with the Riemannian metric. Define the width of Π with respect to Ω to be

$$L(\Pi, \Omega) = \inf\{L(S, \Omega) : S \in \Pi\}.$$

More precisely, our min-max technique is inspired by the discrete setting of Almgren and Pitts. The original method was introduced in [3] and [47] between the 1960's and 1980's, and has been used recently by Marques and Neves to answer deep questions in geometry, see [36] and [37]. The method consists of applications of variational techniques for the area functional. It is a powerful tool in the production of unstable minimal surfaces in closed manifolds. For instance, Marques and Neves, in the proof of the Willmore conjecture, proved that the Clifford Torus in the three-sphere is a min-max minimal surface. The min-max technique for the area functional appear also in a different setting, as introduced by Simon and Smith, in the unpublished work [52], or Colding and De Lellis, in the survey paper [17]. Other recent developments on this theory can be found in [22], [23], [32], [33], [63].

In the Almgren and Pitts' discrete setting, Π is a homotopy class in $\pi_1^{\#}(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$. We define the width of Π with respect to Ω, $\mathbf{L}(\Pi, \Omega)$, following the same principle as in the above discussion. Then, we prove:

Theorem 2. *Let (M^n, g) be a closed Riemannian manifold, $n \leq 7$, and $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ be a non-trivial homotopy class. Suppose that M contains an open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary. There exists a stationary integral varifold Σ whose support is a smooth embedded closed minimal hypersurface intersecting Ω and with $||\Sigma||(M) = \boldsymbol{L}(\Pi, \Omega)$.*

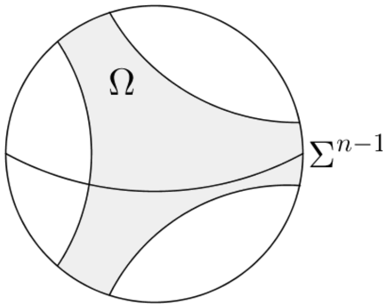


Figure 2: Example of mean-concave domain $\Omega \subset S^3$ for which Σ^{n-1} given by Theorem 2 is not entirely inside Ω.

Consider the unit three-sphere $S^3 \subset \mathbb{R}^4$ and let Ω be a mean-concave subset of S^3. Assume that points in $\mathbb{R}^4$ have normal coordinates (x, y, z, w). The min-max minimal surface Σ produced from our method, starting with the homotopy class Π of the standard sweepout $\{\Sigma_t\}$ of S^3, $\Sigma_t = \{w = t\}$ for $t \in [-1, 1]$, is a great sphere. Indeed, it is obvious that

$$\mathbf{L}(\Pi, \Omega) \leq \max\{\mathcal{H}^2(\Sigma_t) : t \in [-1, 1]\} = 4\pi.$$

This allows us to conclude that Σ must be a great sphere, because these are the only minimal surfaces in S^3 with area less than or equal to 4π.

The intersecting condition in Theorem 2 is optimal in the sense that it is possible that the support of the minimal surface Σ is not entirely in $\overline{\Omega}$. We illustrate this with two examples of mean-concave subsets of the unit three-sphere $S^3 \subset \mathbb{R}^4$ containing no great sphere. The first example is the complement of three spherical geodesic balls, which can be seen in Figure 2.

In order to introduce the second example, for each $0 < t < 1$, consider the subset of S^3 given by

$$\Omega(t) = \{(x, y, z, w) \in S^3 : x^2 + y^2 > t^2\}.$$

It is not hard to see that no $\Omega(t)$ contain great spheres. Moreover, the boundary of $\Omega(t)$ is a constant mean-curvature torus in S^3. If $0 < t < 1/\sqrt{2}$, the mean-curvature vector of $\partial\Omega(t)$ points outside $\Omega(t)$. In this case, $\Omega = \Omega(t)$ is a mean-concave subset of S^3 that contains no great sphere.

2. Positive scalar curvature metrics of large widths

Since the proof of the positive mass conjecture in general relativity by Schoen and Yau [57], and Witten [62], the rigidity phenomena involving the scalar curvature has been fascinating the geometers. These results play an important role in modern differential geometry and there is a vast literature about it, see ([5, 9, 10, 11, 12, 13, 14, 24, 35, 38, 40, 42, 45, 56]). Many of these works concern rigidity phenomena involving the scalar curvature and the area of minimal surfaces of some kind in three-manifolds.

The width of a Riemannian three-manifold (M^3, g), as we have seen in the first part of this introduction, is a very interesting geometrical invariant which is closely related to the production of unstable, closed, embedded minimal surfaces. It can be defined in different ways depending on the setting, but has always the intent to be the lowest value W for which it is possible to sweep M out using surfaces of g area at most W. In this part of the thesis, we use the smooth setting of Simon-Smith and Colding-De Lellis.

Let g be a Riemannian metric on the three-sphere. A *sweepout* of (S^3, g) is a one-parameter family $\{\Sigma_t\}$, $t \in [0, 1]$, of smooth 2-spheres of finite area which are boundaries of open subsets $\Sigma_t = \partial\Omega_t$, vary smoothly and become degenerate at times zero, $\Omega_0 = \varnothing$, and one, $\Omega_1 = S^3$.

The simplest way to sweep out the three-sphere is using the level sets of any coordinate function $x_i : S^3 \subset \mathbb{R}^4 \to \mathbb{R}$.

Let Λ be a set of sweepouts of (S^3, g). It is said to be saturated if given a map $\phi \in C^\infty([0, 1] \times S^3, S^3)$ such that $\phi(t, \cdot)$ are diffeomorphism of S^3, all of which isotopic to the identity, and a sweepout $\{\Sigma_t\} \in \Lambda$, we have $\{\phi(t, \Sigma_t)\} \in \Lambda$.

The *width* of the Riemannian metric g on S^3 with respect to the saturated set of sweepouts Λ is defined as the following min-max invariant:

$$W(S^3, g) = \inf_{\{\Sigma_t\}\in\Lambda} \sup_{t\in[0,1]} Area_g(\Sigma_t),$$

where $Area_g(\Sigma_t)$ denotes the surface area of the slice Σ_t with respect to g.

Marques and Neves [35] proved that the width of a metric g of positive Ricci curvature in S^3, with scalar curvature $R \geq 6$, satisfies the upper bound $W(S^3, g) \leq 4\pi$ and there exists an embedded minimal sphere Σ, of index one and surface area $Area_g(\Sigma) = W(S^3, g)$. They proved also that in case of equality $W(S^3, g) = 4\pi$, the metric g has constant sectional curvature one. The main purpose of this part of the work is to prove that this is no longer true without the assumption on the Ricci curvature. More precisely, we have:

Theorem 3. *For any $m > 0$ there exists a Riemannian metric g on S^3, with scalar curvature $R \geq 6$ and width $W(S^3, g) \geq m$.*

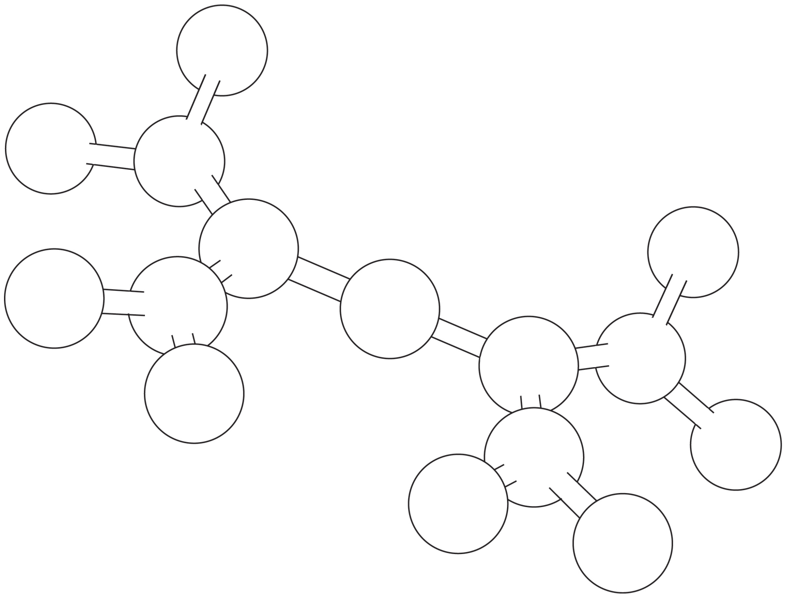


Figure 3: The metric on S^3 associated with the full binary tree with 8 leaves.

Remark 1. It is interesting to stress that the Riemann curvature tensors of the examples that we construct are uniformly bounded.

In order to prove Theorem 3, we construct a special sequence of metrics on the three-sphere using the connected sum procedure of Gromov and Lawson [30]. More precisely, given a full binary tree, we associate a spherical region for each node. These regions are subsets of the round three-sphere obtained by removing either one, two or three identical geodesic balls, depending on the vertex degree of the node only. Regions corresponding to neighboring nodes are glued together using a copy of a fixed tube, which is obtained by Gromov-Lawson's method. In Figure 3, it is provided a rough depiction of the metric associated with the full binary tree with 8 leaves.

The lower bounds that we obtain for the widths rely on a combinatorial argument and on the relative isoperimetric inequality. The first key step of our argument is the choice of a special slice $\Sigma_{t_0} = \partial\Omega_{t_0}$, for any fixed sweepout $\{\Sigma_t\}$ of S^3 with the metrics that we consider. Then, we induce a 2-coloring on the nodes of the associated binary tree, for which the color of each node depends on the volume of Ω_{t_0} on the corresponding spherical region. A 2-coloring of a tree is an assignment of one color, black or white, for each node. Finally, a joint application of a combinatorial tool about 2-colorings of full binary trees and the relative isoperimetric inequality on some compact three-manifolds with boundary give us lower bounds on the widths of the considered Riemannian metrics.

Liokumovich also used combinatorial arguments to construct Riemannian metrics of large widths on surfaces of small diameter, see [34].

Recalling our discussion in the first part, the foundational idea of the Almgren-Pitts min-max theory for the area functional is to achieve the width as the area of a closed minimal surface, possibly disconnected and with multiplicities. For our examples of metrics on S^3, it is expected that this min-max minimal surface will have multiple components, some of them stable with area strictly less than 4π. These surfaces correspond to the spherical slices of minimum area in the tubes of our construction.

Brendle, Marques and Neves [13] constructed non-spherical metrics with scalar curvature $R \geq 6$ on the hemisphere, which coincide with the standard round metric in a neighborhood of the boundary sphere. These metrics are counterexamples to the Min-Oo conjecture. Our result gives also a setting in which a scalar curvature rigidity result does not hold.

The standard argument to provide a positive lower bound for the width of a Riemannian metric is to consider the supremum of the isoperimetric profile. Given a Riemannian metric g on the three-sphere, its isoperimetric profile is the function $\mathcal{I} : [0, vol_g(S^3)] \to \mathbb{R}$ defined by

$$\mathcal{I}(v) = \inf\{Area_g(\partial\Omega) : \Omega \subset S^3 \text{ and } vol_g(\Omega) = v\}.$$

In particular, if $\{\Sigma_t\}$ is a sweepout of (S^3, g) and the associated open subsets are $\Omega_t \subset S^3$, then the volumes $vol_g(\Omega_t)$ assume all the values between zero and $vol_g(S^3)$. Then, we have

$$\sup\{\mathcal{I}(v) : v \in [0, vol_g(S^3)]\} \leq W(S^3, g).$$

If g has positive Ricci curvature and scalar curvature $R \geq 6$, the 4π upper bound on the supremum of the left-hand-side of the above expression was previously obtained by Eichmair [24]. We observed that the supremum of the isoperimetric profiles of the examples that we use also form an unbounded sequence. This claim is also proved via a combinatorial result and provides us a second proof of the content of Theorem 3.

Organization

The chapters are organized as follows. In Chapter 1, we explain the objects and some tool that we use to develop the new min-max theory. In Chapters 2, 3 and 4, we take account of the formal description of the method, the development of some further tools and its proofs. Chapter 5 is devoted to the proof our main result, Theorem 1. Finally, in Chapter 6, we discuss the result about the widths of metrics of positive scalar curvature.

CHAPTER 1

Preliminaries

In this chapter, we introduce and develop some preliminary notions and statements that are applied along this thesis. In the first section 1.1, we clarify which objects from geometric measure theory we are going to use and the related constructions and results. The main goal of this part is to explain what almost minimizing varifolds are and its regularity property. Then, in 1.2, we state a maximum principle type result of White, that plays a key role in our first main result. We discuss also one other ingredient that is related to this part of our work, the monotonicity formula 1.3. Finally, we describe the standard notation for dealing with discrete maps and generalized homotopies, this is the content of 1.4.

1.1 Geometric measure theory

In this section we introduce some ideas from Geometric Measure Theory and the corresponding notation that use along this thesis. Our main reference is the book of Simon [51]. Sometimes we will follow the notation in Pitts [47].

First, we give a brief exposition about varifolds and currents supported in Riemannian manifolds, which are objects that generalize the notion of smooth submanifolds in some sense. They have associated notions of dimension and volume. Currents are also oriented and have an associated definition of boundary, which give this theory a homological flavor. In the end of the section, we discuss two important situations in which special currents or varifolds happen to be smooth.

1.1.1 Varifolds

A k-dimensional varifold in $\mathbb{R}^L$ is a Radon measure (Borel regular and finite on compact subsets) in the Grassmannian of the unoriented k-dimensional planes of $\mathbb{R}^L$. Let $G_k(\mathbb{R}^L)$ denote this Grassmannian manifold, which can be interpreted as the trivial bundle $G_k(\mathbb{R}^L) = \mathbb{R}^L \times G(L,k)$, where $G(L,k)$ is the collection of all k-subspaces of $\mathbb{R}^L$. In order to make the above definition precise, in the next paragraph we describe the topology of the space $G_k(\mathbb{R}^L)$.

The distance between two subspaces $S, T \in G(L,k)$ is defined as the norm of the linear transformation $(P_S - P_T)$, where P_S and P_T are the orthogonal projections of $\mathbb{R}^L$ onto S and T, respectively. Then, $G_k(\mathbb{R}^L)$ is equipped with the product metric. We use $C_c(G_k(\mathbb{R}^L))$ to denote the set of the compactly supported continuous functions on $G_k(\mathbb{R}^L)$.

Since varifolds are Radon measures, it is natural to invoke the Riesz representation theorem, [51] Theorem 4.1, to construct or express examples as linear functionals on $C_c(G_k(\mathbb{R}^L))$. For instance, we can use it to explain how those measures generalize the notion of a k-dimensional submanifold: given a k-submanifold $\Sigma^k \subset \mathbb{R}^L$ (or, more generally, a $\mathcal{H}^k$-measurable and countably k-rectifiable set) and a positive locally $\mathcal{H}^k$-integrable function θ on Σ^k, we associate the following varifold

$$\mathbf{V}(\Sigma,\theta)(\phi) = \int_\Sigma \phi(x, T_x\Sigma)\cdot\theta(x)d\mathcal{H}^k(x), \tag{1.1}$$

for all $\phi \in C_c(G_k(\mathbb{R}^L))$. These are the rectifiable varifolds in $\mathbb{R}^L$.

Remark 2. The theory of k-rectifiable sets of $\mathbb{R}^L$, or countably k-rectifiable sets, is discussed in [51] Chapter 3. One important property of these sets is that they have a notion of tangent space for $\mathcal{H}^k$-almost all of its points.

Associated to a general k-dimensional varifold V in $\mathbb{R}^L$, there is a Radon measure on $\mathbb{R}^L$ that generalizes the notion of k-dimensional volume (with multiplicity). The mass of V, denoted by $||V||$, is defined by the formula:

$$||V||(A) = V(\{(x,v) \in G_k(\mathbb{R}^L) : x \in A\}), \tag{1.2}$$

for all Borel subsets $A \subset \mathbb{R}^L$ with compact closure. Note that, in the case of a rectifiable $V = \mathbf{V}(\Sigma,\theta)$, the value $V(\mathcal{A})$ for $\mathcal{A} \subset G_k(R^L)$ is the supremum of $\mathbf{V}(\Sigma,\theta)(\phi)$, where $\phi \in C_c(G_k(R^L))$, $|\phi| \leq 1$ and is supported in $\mathcal{A}$. This implies that $||\mathbf{V}(\Sigma,\theta)||(A)$ is simply the integration of θ, over $\Sigma \cap A$ and with respect to the k-dimensional Hausdorff measure $\mathcal{H}^k$.

The support of a varifold V in $\mathbb{R}^L$ is the support of its mass $||V||$. The space of varifolds, with its usual weak topology (convergence as varifolds is equivalent to convergence as measures), is metrizable. Which means that there exists a metric, the **F**-*metric*, that induces the varifold weak topology. It is defined in Pitts book [47].

One of the main properties of the varifolds is that the mass is well behaved when we take limits. More precisely, if V_i is a sequence of varifolds converging to V, then

$$||V||(G) \leq \liminf_{i\to\infty} ||V_i||(G), \tag{1.3}$$

for all open subsets $G \subset \mathbb{R}^L$, and

$$\limsup_{i\to\infty} ||V_i||(K) \leq ||V||(K), \tag{1.4}$$

for all $K \subset R^L$ compact. Moreover, if $A \subset \mathbb{R}^L$ has compact closure and $||V||(\partial A) = 0$, then $||V_i||(A)$ tends to $||V||(A)$.

For a Riemannian n-manifold M, we consider it is isometrically embedded in some Euclidean space $\mathbb{R}^L$. Then, we use $\mathcal{V}_k(M)$ to denote the closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M, in the weak topology. In this case, given $V \in \mathcal{V}_k(M)$, the mass $||V||$ is a Radon measure in M.

Given a C^1-map $F : M \to M$, the push-forward of $V \in \mathcal{V}_k(M)$ is denoted by $F_\#(V)$. It can be viewed as the image of V via F and, for $V = \mathbf{V}(\Sigma, \theta)$, is defined as the rectifiable varifold with support $F(\Sigma)$ and multiplicity function

$$\tilde{\theta}(y) = \sum_{x \in F^{-1}(y)\cap\Sigma} \theta(x). \tag{1.5}$$

The definition of $F_\#(V)$ for general V can be found in [51] page 233.

Let $\mathcal{X}(M)$ denote the space of smooth vector fields of M with the C^1-topology. The first variation $\delta : \mathcal{V}_k(M) \times \mathcal{X}(M) \to \mathbb{R}$ is defined as

$$\delta V(X) = \left.\frac{d}{dt}\right|_{t=0} ||F_{t\#}(V)||(M), \tag{1.6}$$

where $\{F_t\}_t$ is the flow of X. The first variation is continuous with respect to the product topology of $\mathcal{V}_k(M) \times \mathcal{X}(M)$. A varifold V is said to be *stationary in* M if $\delta V(X) = 0$ for every $X \in \mathcal{X}(M)$. In general, the first variation obeys the following formula (see [51] page 234):

$$\delta V(X) = \int \mathrm{div}_S X(x) dV(x, S). \tag{1.7}$$

1.1.2 Integral currents

The notion of k-dimensional currents in $\mathbb{R}^L$ is the analogue for differential k-forms of the Schwartz distributions for functions. For our purposes, it is more interesting to think of a k-dimensional current in $\mathbb{R}^L$, or simply a k-current, as an orientable Lipschitz k-submanifold of $\mathbb{R}^L$.

The general definition of a k-current is of a continuous linear functional on the space $\Omega^k_c(\mathbb{R}^L)$ of the compactly supported differential k-forms, which is equipped with its usual locally convex topology, see [51] page 131. For instance, if $\Sigma^k \subset \mathbb{R}^L$ is an orientable k-dimensional submanifold we can consider the linear functional induced by integration of k-forms on Σ:

$$[|\Sigma|](\omega) = \int_\Sigma \omega, \quad \text{for all } \omega \in \Omega^k_c(\mathbb{R}^L). \tag{1.8}$$

More generally, we can consider formal finite sums

$$T = \sum_{i=1}^{m} \theta_i \cdot [|\Sigma^k_i|], \tag{1.9}$$

where each Σ^k_i is an orientable k-submanifold and the coefficients are integers. These are the prototypical integral k-currents.

Motivated by those simple examples, we associate to any k-current a support, a boundary and a notion of k-dimensional volume. Let T be a k-current. The *support of* T, denoted by $spt(T)$, is the smallest closed subset of $\mathbb{R}^L$ for which $T(\omega) = 0$, for all $\omega \in \Omega^k_c(\mathbb{R}^L)$ supported outside $spt(T)$. The boundary ∂T of T is the $(k-1)$-current defined by

$$\partial T(\omega) = T(d\omega), \tag{1.10}$$

where d denotes the usual exterior derivative of differential forms. By Stokes' Theorem, we have that $\partial[|\Sigma|] = [|\partial\Sigma|]$, for any $\Sigma \subset \mathbb{R}^L$ submanifold with boundary. The quantity that generalizes the k-dimensional volume is called *the mass of* T. For each $W \subset \mathbb{R}^L$ open, we define the mass of T in W by:

$$||T||(W) = \sup\{T(\omega) : \omega \in \Omega^k_c(\mathbb{R}^L), \quad spt(\omega) \subset W \text{ and } |\omega| \leq 1\}, \tag{1.11}$$

where $|\omega|$ is the maximum value of $\langle \omega(x), \omega(x)\rangle^{1/2}$, for $x \in \mathbb{R}^L$. We use $\langle \cdot, \cdot \rangle$ to denote both the extension of the canonical inner product of $\mathbb{R}^L$ to the spaces of higher dimensional vectors and covectors, for which an orthonormal basis $\{e_1, \ldots, e_L\}$ of $\mathbb{R}^L$ and its dual basis $\{e^1, \ldots, e^L\}$ induce orthonormal bases

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < i_2 < \ldots < i_k \leq L}$$

for the space of k-vectors, and

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}_{1 \leq i_1 < i_2 < \ldots < i_k \leq L}$$

of k-dimensional covectors. We use also $\langle \omega, v \rangle$ to denote the natural pairing of a k-vector v and a k-covector ω. The total mass of a current T, $||T||(\mathbb{R}^L)$, is usually denoted by $\mathbf{M}(T)$.

A consequence of the Riesz Representation Theorem is that if $||T||(W) < +\infty$, for all bounded open subsets $W \subset \mathbb{R}^L$, then $||T||$ is a Radon measure and there exists a $||T||$-measurable function $\vec{T}$ with values in the space of the unit k-dimensional vectors such that

$$T(\omega) = \int \langle \omega, \vec{T} \rangle d||T||, \quad \text{for all } \omega \in \Omega_c^k(\mathbb{R}^L). \tag{1.12}$$

From this formula, we see that many currents can be represented as an integration with respect to a measure. One important class of currents with this property is the class of the integral currents, which we introduce now.

Definition 1. A k-current T is an *integral current* if it can be expressed as in (1.12) and

(i) *spt*(T) is $\mathcal{H}^k$-measurable (Hausdorff measurable) and k-rectifiable (is covered by countably many Lipschitz k-dimensional charts);

(ii) there exists a locally $\mathcal{H}^k$-integrable, positive and integer valued function θ such that

$$||T||(W) = \int_W \theta \ \ d\mathcal{H}^k \llcorner spt(T), \tag{1.13}$$

for all open subsets $W \subset \mathbb{R}^L$. (Notation: $\mathcal{H}^k \llcorner A(W) = \mathcal{H}^k(A \cap W)$)

(iii) $\vec{T}$ is $\mathcal{H}^k$-measurable and, for $\mathcal{H}^k$-almost all $x \in spt(T)$, the unit k-vector $\vec{T}(x)$ can be expressed by

$$\vec{T}(x) = \tau_1 \wedge \cdots \wedge \tau_k, \tag{1.14}$$

for some $\{\tau_1, \ldots, \tau_k\} \subset T_x(spt(T))$ orthonormal basis.

Cut and paste via slicing theory

Let $U \subset \mathbb{R}^{n+k}$ be an open subset, $S, T \in \mathbf{I}_n(U)$ be integral currents with the same boundary $\partial T = \partial S$ and $Q \in \mathbf{I}_{n+1}(U)$ be so that $\partial Q = T - S$. Let f be a Lipschitz function defined on U. Let us describe the construction of cutting S and T along a slice of f on Q and gluing the three together.

We use $\langle Q, f, t_+\rangle$ to denote the slice of the current Q by f at t. It must be thought as the intersection of the support of Q with $f^{-1}(t)$, multiplicity equal to the restriction of that of Q and oriented in such a way that if the basis $\{\nabla f, v_1, \ldots, v_n\} \subset T_xQ$ is positively oriented, then $\{v_1, \ldots, v_n\}$ is also positive in the tangent space of the slice. The precise definition and the main properties of the slicing theory can be find in [51], Chapter 6.

Lemma 1. *Let $t \in \mathbb{R}$ be so that $||S||(\{f = t\}) = 0$ and $\langle Q, f, t_+\rangle \in \boldsymbol{I}_n(U)$. Then*

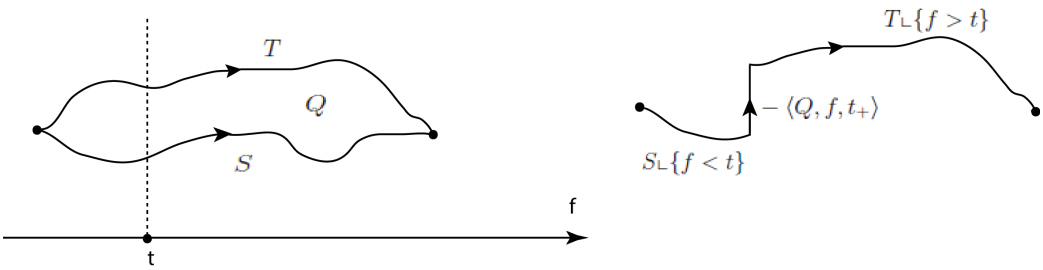
$$R = T \llcorner \{f > t\} - \langle Q, f, t_+\rangle + S \llcorner \{f < t\}, \tag{1.15}$$

has the following properties:

(i) $\partial R = \partial S = \partial T$;

(ii) $\boldsymbol{M}(R - S) \leq \boldsymbol{M}(T - S) + \boldsymbol{M}(\langle Q, f, t_+\rangle)$;

Moreover, if $||T||(\{f = t\}) = 0$, the analogue of (ii) with the roles of S and T interchanged also holds true.



Cut S and T along a slice of f on Q.

Proof. Apply the boundary operator to the expression

$$\langle Q, f, t_+\rangle = -\partial[Q \llcorner \{f > t\}] + (\partial Q) \llcorner \{f > t\}, \tag{1.16}$$

to obtain, for each $\omega \in \Omega^n_c(U)$,

$$\begin{aligned} \partial\langle Q, f, t_+\rangle(\omega) &= \partial[(\partial Q) \llcorner \{f > t\}](\omega) \\ &= [(T - S) \llcorner \{f > t\}](d\omega) \\ &= [T \llcorner \{f > t\} - S \llcorner \{f > t\}](d\omega) \\ &= T \llcorner \{f > t\}(d\omega) - S \llcorner \{f > t\}(d\omega) \\ &= \partial[T \llcorner \{f > t\}](\omega) - \partial[S \llcorner \{f > t\}](\omega). \end{aligned}$$

This calculation gives us

$$\partial[T \llcorner \{f > t\}] - \partial\langle Q, f, t_+\rangle(\omega) = \partial[S \llcorner \{f > t\}].$$

Now, it is possible to conclude that

$$\partial R = \partial[S \llcorner \{f > t\}] + \partial[S \llcorner \{f < t\}] = \partial S,$$

where the last equality holds because we chose $||S||(\{f = t\}) = 0$. The mass estimate (ii) follows directly from the expression defining the current R in the statement of the Lemma. □

Integral currents on manifolds

Let (M^n, g) be an orientable compact Riemannian manifold. We assume that M is isometrically embedded in $\mathbb{R}^L$. We denote by $B(p, r)$ the open geodesic ball in M of radius r and centered at $p \in M$.

We denote by $\mathbf{I}_k(M)$ the space of k-dimensional integral currents in $\mathbb{R}^L$ supported in M and $\mathcal{Z}_k(M)$ the space of integral currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$. Given $T \in \mathbf{I}_k(M)$, we denote by $|T|$ the integral varifold in M associated with T, which is obtained by forgetting the orientation of T.

The above spaces come with several relevant metrics. We use the standard notations $\mathcal{F}$ and $\mathbf{M}$ for the flat norm and mass norm on $\mathbf{I}_k(M)$, respectively. The flat metric is defined by the formula

$$\mathcal{F}(S, T) = \inf\{\mathbf{M}(P) + \mathbf{M}(Q) : P \in \mathbf{I}_k(M), Q \in \mathbf{I}_{k+1}(M), S - T + P = \partial Q\},$$

for $S, T \in \mathbf{I}_k(M)$. Convergence in the flat metric is sometimes stated as convergence in sense of volumes. And the mass norm is simply $\mathbf{M}(S, T) = \mathbf{M}(S - T)$.

Finally, the $\mathbf{F}$-*metric* on $\mathbf{I}_k(M)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$

We use also the k-dimensional Hausdorff measure $\mathcal{H}^k$ of subsets of M.

Dealing with min-max constructions, it frequently happens that we give an argument by contradiction as follows: we start with a path of surfaces (or integral k-currents, when working in the setting of Almgren-Pitts) that is continuous with respect to one of the above topologies, called sweepout, and with some extra property that we want to prove that does not occur. Then, we somehow deform the path to have the maximum area of slices decreased and find a contradiction. In general, the deformed path is continuous in a weaker sense as the original one. The continuity of the new path depends on the deformation that we need to perform. A key difference between flat metric and $\mathbf{F}$-metric convergences is the behavior of the mass of the limit current. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of integral k-currents. Mass is lower semi-continuous with respect to the flat metric (or weak convergence), i.e., if we have convergence of T_i in the flat metric,

$$\lim_{i\to\infty} \mathcal{F}(T_i, T) = 0,$$

then, for all open subsets $W \subset M$,

$$||T||(W) \leq \liminf_{i\to\infty} ||T_i||(W).$$

But, since the orientation is being considered, mass cancellation is allowed and the strict inequality may occur. On the other hand, convergence in the $\mathbf{F}$-metric implies convergence of the induced varifolds, and then the masses $\mathbf{M}(T_i)$ converge to the mass of the limit current. In particular, if we have cancellation of mass, the weak limits in the senses of currents and varifolds are not related by forgetting orientation only. Let us see some concrete examples.

Example 1. Let I be the rectifiable 1-dimensional varifold in $\mathbb{R}^2$ induced by a unit length line segment and $\{T_i\} \subset \mathcal{Z}_1(\mathbb{R}^2)$ be a sequence of boundaries $T_i = \partial S_i$ of integral 2-currents so that

$$spt(I) \subset int(spt(S_i)) \text{ and } \mathbf{M}(S_i) = area(S_i) \to 0, \text{ as } i \to \infty,$$

see Figure 1.1. In particular, we have that $\mathcal{F}(T_i, 0) \leq \mathbf{M}(S_i)$, which implies that T_i converges to the zero current with respect to the flat metric. In this case, all the mass is lost. On the other hand, for the the induced varifold $|T_i|$, we have

$$|T_i|(f) = \int_{T_i} f(x, T_x(T_i)) d\mathcal{H}^1(x) \to 2\int_I f(x, T_x I) d\mathcal{H}^1(x),$$

as i tends to infinity, for every $f \in C_c(G_1(\mathbb{R}^2))$. Hence, the $|T_i|$ converge to the varifold $2I = \mathbf{V}(I, 2)$. Since the limit varifold and the varifold induced by the weak limit current do not coincide, the sequence T_i does not admit any subsequence which converges in the $\mathbf{F}$-metric sense.

A similar construction can be considered in the case of higher dimensions, for instance, we can consider the integral 2-currents obtained by connecting two concentric spheres of radii converging to one by a small neck.

Remark 3. Actually, cancellation of mass occur if and only if the limit varifold and the varifold induced by the limit current do coincide. See paragraph 18f of the Subsection 2.1 of Pitts' book [47].

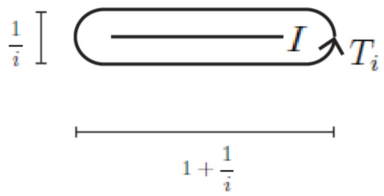


Figure 1.1: The phenomenon of cancellation of mass.

Example 2. Let Σ be an orientable closed surface with trivial normal bundle in a three-manifold. Suppose N is a unit normal vector field defined on Σ. Consider a sequence of smooth real functions u_i on Σ, which converges to the zero function in the C^1-topology. Then, the integral 2-currents $[|\Sigma_i|]$ induced by the graph surfaces

$$\Sigma_i = \{exp_x(u_i(x)N(x)) : x \in \Sigma\}$$

tends to $[|\Sigma|]$ with respect to the $\mathbf{F}$-metric. Indeed, the C^0 convergence implies the flat metric approximation and the C^1 convergence assures that no mass lost.

However, both of these convergence senses ($\mathcal{F}$ and $\mathbf{F}$) are far from implying convergence in the mass norm. These three metrics are related by the following general formula

$$\mathcal{F}(S,T) \leq \mathbf{F}(S,T) \leq 2\mathbf{M}(S-T).$$

We assume that $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ both have the topology induced by the flat metric. When endowed with a different topology, these spaces will be denoted either by $\mathbf{I}_k(M;\mathbf{M})$ and $\mathcal{Z}_k(M;\mathbf{M})$, in case of the mass norm, or $\mathbf{I}_k(M;\mathbf{F})$ and $\mathcal{Z}_k(M;\mathbf{F})$, if we use the $\mathbf{F}$-metric. If $U \subset M$ is an open set of finite perimeter, the associated current in $\mathbf{I}_n(M)$ is denoted by $[|U|]$.

Similarly to the case of varifolds, it is also possible to make a precise sense for the push-forward $F_\#(T)$ of a current T by a map F. For a detailed discussion see page 137 of [51]. This notion makes possible an important formula, the homotopy formula, that will be applied in Section 3.2. We finish this subsection by presenting this tool.

Let $U \subset M$ be a open subset and $h : [0,1] \times U \to \mathbb{R}^Q$ be a smooth homotopy between $f(x) = h(0,x)$ and $g(x) = h(1,x)$. If $T \in \mathbf{I}_k(U)$ and h is proper in $[0,1] \times U$, then $h_\#([|(0,1)|] \times T)$ is a well-defined integral $(k+1)$-dimensional current in $\mathbb{R}^Q$ and its boundary can expressed according to the following formula

$$\partial h_{\#}([|(0,1)|] \times T) = g_{\#}(T) - f_{\#}(T) - h_{\#}([|(0,1)|] \times \partial T). \quad (1.17)$$

1.1.3 Regularity Theorems

In this subsection we describe two results concerning regularity of codimension one varifolds and currents with special minimization properties.

The first result that we state is a classical one related to the solution of the codimension one Plateau problem in higher dimensions.

Theorem 4. *Let M^n be a Riemannian manifold properly embedded in $\mathbb{R}^L$ and T be a codimension 1 homologically mass-minimizing current in M, i.e.,*

$$\boldsymbol{M}(T) \leq \boldsymbol{M}(S), \quad \textit{for every current } S \textit{ with } \partial S = \partial T.$$

Then, $spt(T) - spt(\partial T)$ is a smooth minimal hypersurface of M except at a singular subset $Sing(T)$, which is empty if $n \leq 7$, is locally finite if $n = 8$, and has Hausdorff dimension at most $n - 8$, in case $n > 8$.

In the case of the Euclidean space $M^n = \mathbb{R}^n$, this was proved by Fleming [26] for $n = 3$, Almgren [4] for $n = 4$, Simons [59] for $n \leq 7$, and Federer [25] for $n \geq 8$. Its extension to general Riemannian manifolds is due to a maximum principle type result of Schoen and Simon [54].

Next, we state the regularity theorem for stationary varifolds which are almost minimizing. This is a variational property that enables us to approximate the varifold by arbitrarily close integral cycles which are themselves almost locally area minimizing. The key characteristic of varifolds that are almost minimizing in small annuli is regularity, which is not necessarily the case for general integral stationary varifolds.

In order to formally explain the almost minimizing property, consider the following notation. Let M^n be a compact Riemannian manifold and U be an open subset of M. For $p \in M$ and $0 < s < r$, use $B(p,r)$ and $A(p,s,r) = B(p,r) - \overline{B(p,s)}$ to denote the open geodesic ball of radius r and centered at p and the open annulus in M, respectively.

Given $\varepsilon, \delta > 0$, consider the set $\mathcal{A}(U; \varepsilon, \delta)$ of integer cycles $T \in \mathcal{Z}_{n-1}(M)$ for which the following happens: for every finite sequence

$$T = T_0, T_1, \ldots, T_m \in \mathcal{Z}_{n-1}(M),$$

such that

$$\operatorname{spt}(T_i - T) \subset U, \quad \mathbf{M}(T_i, T_{i-1}) \leq \delta \quad \text{and} \quad \mathbf{M}(T_i) \leq \mathbf{M}(T) + \delta,$$

it must be true that $\mathbf{M}(T_m) \geq \mathbf{M}(T) - \varepsilon$.

Definition 2. Let $V \in \mathcal{V}_{n-1}(M)$ be a rectifiable varifold in M. We say that V *is almost minimizing in* U if for every $\varepsilon > 0$, there exist $\delta > 0$ and $T \in \mathcal{A}(U; \varepsilon, \delta)$ satisfying $\mathbf{F}(V, |T|) < \varepsilon$.

Remark 4. This definition is basically the same as in Pitts' book, the difference is that we ask $|T|$ to be ε-close to V in the $\mathbf{F}$-metric on the whole M, not only in U. This creates no problem because our definition implies Pitts' and in the next step we prove the existence of almost minimizing varifolds in the sense of Definition 2. This is also observed in Remark 6.4 of [63].

Theorem 5. *Let M^n be a Riemannian manifold properly embedded in $\mathbb{R}^L$, $p \in M$ and $r > 0$. If $V \in \mathcal{V}_{n-1}(M)$, a limit of rectifiable varifolds, satisfies*

$$V \text{ is stationary in the metric ball } B(p,r),$$

and

$$V \text{ is almost minimizing in the annuli } A(p,s,r),$$

for all $0 < s < r$, then spt$(||V||) \cap B(p,r)$ is a smooth minimal hypersurface of $B(p,r)$ except at a singular subset of Hausdorff codimension at least 7.

Pitts [47] proved this theorem for $n \leq 6$. Later, Schoen and Simon [53] extended this result to any dimension.

1.2 The maximum principle

In [61], White proved a maximum principle type theorem for general varifolds. In this thesis, we use an application of this result which we describe now.

Let (M^n, g) be a closed Riemannian manifold and $\Omega \subset M$ be an open domain with smooth and strictly mean-concave boundary $\partial\Omega$. Consider the foliation $\{\partial\Omega_t\}$ of a small tubular neighborhood of $\partial\Omega$, where

$$\Omega_t = \{x \in M : d_{\partial\Omega}(x) < t\} \tag{1.18}$$

and $d_{\partial\Omega}$ is the signed distance function to $\partial\Omega$, which is negative in Ω. Let ν be the unit vector field normal to all $\partial\Omega_t$ and pointing outside Ω_t. Use $\overrightarrow{H}(p)$ to denote the mean-curvature vector of $\partial\Omega$ at $p \in \partial\Omega$. White's result and a covering argument on $\partial\Omega$ imply the following statement:

Corollary 2. *Let $\overrightarrow{H}(p) = H(p)\nu(p)$ and suppose that $H > \eta > 0$ over $\partial\Omega$. There exist $a < 0 < b$ and a smooth vector field X on $M - \Omega_a$ with*

$$X \cdot \nu > 0 \text{ on } \overline{\Omega_b} - \Omega_a$$

and

$$\delta V(X) \leq -\eta \int |X| d\mu_V, \tag{1.19}$$

for every $(n-1)$*-varifold* V *in* $M - \Omega_a$*.* *(*$\mu_V = ||V||$ *is the mass of* V*)*

We refer to X as the maximum principle vector field. Up to multiplying X by a constant, we can suppose that its flow $\{\Phi(s,\cdot)\}_{s\geq 0}$ is such that

$$\Phi(1, M - \Omega_a) \subset M - \overline{\Omega_b}. \tag{1.20}$$

The key property of this flow is that it is mass-decreasing for $(n-1)$-varifolds. Actually, $\{\Phi(s,\cdot)\}_{s\geq 0}$ is also mass-decreasing for n-varifolds.

For sake of completeness, we state White's result and indicate how it implies Corollary 2. The maximum principle proved in [61] is precisely:

Theorem 6. *Let* N *be a Riemannian manifold with boundary. Let* $p \in \partial N$ *be such that* $0 < \eta < k_1 + \ldots + k_{n-1}$*, where* $k_1 \leq \ldots \leq k_{n-1}$ *are the principal curvatures of* ∂N *at* p *with respect to the unit normal* $\nu(N)$ *that points into* N*. Then, there exists a compactly supported smooth vector field* X *on* N *such that* $X(p)$ *is a nonzero vector normal to* ∂N*,* $X \cdot \nu(N) \geq 0$ *on* ∂N *and satisfying (1.19) for every* $(n-1)$*-varifold in* N*.*

In his proof, White considers an extension of N near $p \in \partial N$. In the case that we are interested in, M is closed and $\Omega \subset M$ is a domain with smooth and strictly mean-concave boundary $\partial\Omega$. Let us suppose that $0 < \eta < k_1 + \ldots + k_{n-1}$, for each $p \in \partial\Omega$. Now, the manifold with boundary $N = M - \Omega$ has a natural extension near each $p \in \partial\Omega$ already. We apply White's construction to this setting to obtain:

Theorem 7. *For every* $p \in \partial\Omega$*, there exists* $t(p) < 0$ *and a smooth vector field* X *on* $N_{t(p)} = M - \Omega_{t(p)}$ *so that* $X(p)$ *is a nonzero vector normal to* ∂N*,* $X \cdot \nu \geq 0$ *on* $\partial N_t = \partial(M - \Omega_t)$*, for every* $t \geq t(p)$*, and satisfying (1.19) for every* $(n-1)$*-varifold in* $N_{t(p)}$*.*

Proof. Let V be a $(n-1)$-varifold in M. By first variation formula, expression (1.7), it is possible to conclude

$$\delta V(X) \leq \int \Psi_X(x) d\mu_V,$$

where $\Psi_X(x) = \max\{\mathrm{div}_P X(x) : P \text{ is a hyperplane in } T_xM\}$. Thus, we need a vector field X such that $\Psi_X(x) \leq -\eta|X(x)|$, for every x. Let

$$\Sigma = \{x \in M : d_g(x, N) = (d_g(x,p))^4\},$$

where $N = M - \Omega$. Consider a small open geodesic ball $B(p,r)$, in which Σ is a smooth hypersurface and there is a smooth, well defined nearest point retraction from $B(p,r)$ to Σ. Define $u : B(p,r) \to [-r,r]$ to be the signed distance function to Σ, positive in $N \cap B(p,r)$. Note that Σ is outside N and tangent to $\partial\Omega$ at p with second order contact. Therefore, we can take $r > 0$ sufficiently small so that the sum of the principal curvatures of the level sets of u is greater than η at each point $q \in B(p,r)$.

We can suppose also that $\nabla u(q) \cdot \nu > 0$, for every $q \in B(p,r)$. Let $K > 0$ be so that $|\tilde{k}_i(q)| \leq K$, for all $q \in B(p,r)$ and $1 \leq i \leq n-1$, where $\tilde{k}_i(q)$ is the i-th principal curvature of Σ_q at q, and Σ_q is the level set of u in which q lives. Choose $\varepsilon \leq (4K)^{-1/2}$ and $t(p) < 0$, such that $\{q \in N_{t(p)} \cap B(p,r) : u(q) \leq \varepsilon\}$ is a compact subset of $\{x \in B(p,r) : -\varepsilon < u(x)\}$. Finally, define a vector field on $N_{t(p)}$ by $X(q) = \phi(u(q))\nabla u(q)$, where

$$\phi(t) = \begin{cases} \exp\left(\frac{1}{t-\varepsilon}\right), & \text{if } -\varepsilon \leq t < \varepsilon \\ 0, & \text{if } t \geq \varepsilon. \end{cases}$$

As in White's proof, the choice of ε gives us that $\phi'(t) \leq -K\phi(t)$, for every $-\varepsilon \leq t < \varepsilon$. Following the same calculation done by White, we have that the matrix of the bilinear form $Q(u,v) = \langle u, \nabla_v X\rangle, u, v \in T_q(N_t)$, $q \in \mathrm{spt}(X)$, with respect to the orthonormal basis given by $\nabla u(q)$ and $(n-1)$ vectors that diagonalize the second fundamental form of Σ_q at q, is a diagonal matrix with diagonal elements $\phi'(u)$ and $-\phi(u)\tilde{k}_i(q)$, for $1 \leq i \leq n-1$. This concludes the proof. Indeed, let $e_1 = \nabla u(q)$ and $\{e_2, \ldots, e_n\} \subset T_q\Sigma_q$ be a basis that diagonalizes the second fundamental form of Σ_q. Given $P \subset T_qM$ hyperplane,

$$\mathrm{div}_P X(q) = Tr_P Q(q) = \sum_{i=1}^{n} |e_j^T|^2 \cdot Q(e_j, e_j),$$

where e_j^T is the orthogonal projection of e_j in P. Using $\sum_{j=1}^n |e_j^T|^2 = n-1$, the values of $Q(e_j,e_j)$ and that $\phi' \leq -K\phi \leq -\tilde{k}_j(q)\phi$, we infer that

$$\mathrm{div}_P X(q) \leq -\phi(u(q)) \sum_{j=2}^{n} (|e_j^N|^2 K + |e_j^T|^2 \tilde{k}_j(q)) < -\eta\phi(u(q)) = -\eta|X(q)|.$$

□

Proof of Corollary 2. Since $\partial\Omega$ is compact, we can choose $p_1, \ldots, p_m \in \partial\Omega$, negative numbers $t(p_1), \ldots, t(p_m)$, and vector fields X_i defined on $N_{t(p_i)}$ given by theorem 7, and such that $\{\mathrm{int}(\mathrm{spt}(X_i)\}_{i=1}^m$ is an open cover of $\partial\Omega$. Choose $a < 0$ and $b > 0$ so that we still have

$$(\partial\Omega_a) \cup (\partial\Omega_b) \subset \bigcup_{i=1}^{m} \mathrm{int}(\mathrm{spt}(X_i)),$$

and define the vector field $X = \sum_{i=1}^{m} X_i$ on N_a. Recall that $X_i \cdot \nu > 0$, when $X_i \neq 0$. Therefore $X \cdot \nu > 0$ on $\overline{\Omega_b} - \Omega_a$. Finally, given a $(n-1)$-varifold in N_a, we have

$$\delta V(X) = \sum_{i=1}^{m} \delta V(X_i) \leq -\eta \sum_{i=1}^{m} \int |X_i| d\mu_V \leq -\eta \int |X| d\mu_V.$$

□

1.3 The monotonicity formula

The intent of the present subsection is to use the monotonicity formula for minimal surfaces to obtain some consequences that we need in the arguments of Subsection 3.3.1 and Proposition 2 of Chapter 5. First of all, we sketch the proof of the monotonicity formula for k-dimensional submanifolds in general Riemannian manifolds. Then, we indicate the formulas that we use in the applications.

Let (M^n, g) be a closed Riemannian manifold, $p \in M$ and $r_0 = \text{inj}_M(p)$ its injectivity radius. Let $\Sigma^k \subset M$ be a submanifold with bounded mean-curvature, $||\overrightarrow{H}_\Sigma|| \leq H$, for some $H > 0$. We use $d : \Sigma \to \mathbb{R}$ to denote the distance function to p in M. Consider the vector field $X = d\nabla d$.

In exponential coordinates $\{x^1, \ldots, x^n\}$ around p, the coordinate vectors and the Riemannian metric are given by

$$\frac{\partial}{\partial x^j} = D(exp_p)_{x^i e_i}(e_j) \quad \text{and} \quad g_{ij}(x) = \delta_{ij} + O(|x|^2),$$

where $\{e_1, \ldots, e_n\} \subset T_pM$ is an orthonormal basis. If a vector w is given in local coordinates as $w = \sum_i w^i \frac{\partial}{\partial x_i}$, a simple calculation yields

$$g(w, w) = \sum_{i=1}^{n} (w^i)^2 + O(|x|^2).$$

In particular, if $g(w, w) = 1$, all the coordinates w^i are uniformly bounded. Also, in these coordinates, $X = d\nabla d$ can be expressed as $X = \sum_i x^i \frac{\partial}{\partial x_i}$. Thus, for all $g(w, w) = 1$, we have

$$g(\nabla_w X, w) = 1 + O(|x|^2). \tag{1.21}$$

Finally, this allows us to conclude that, near p,

$$\text{div}_\Sigma X^T = \text{div}_\Sigma X + \langle X, \overrightarrow{H}_\Sigma \rangle \geq k - (Cd^2 + Hd), \tag{1.22}$$

where $C = C(p) > 0$ is a constant depending on $p \in M$ only.

Let $l(r) = \int_{\Sigma \cap \partial B_r} d\sigma_r$ and $v(r) = \int_{\Sigma \cap B_r} d\Sigma$ be the $(k-1)$-dimensional volume of $\Sigma \cap \partial B_r$ and the k-dimensional volume of $\Sigma \cap B_r$, respectively, where $B_r = B(p, r)$ denotes the geodesic ball in M with radius r and centered at p. Then, using 1.22 and Stokes' theorem, we have

$$rl(r) \geq \int_{\Sigma \cap \partial B_r} \langle X^T, \nu \rangle d\sigma_r = \int_{\Sigma \cap B_r} div_\Sigma X^T d\Sigma \geq \int_{\Sigma \cap B_r} (k - Cd^2 - Hd) d\Sigma.$$

In conclusion, $rl(r) \geq (k - Cr^2 - Hr)v(r)$. Now, we use the co-area formula to obtain

$$v'(r) = \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla_\Sigma d|} d\sigma_r \geq l(r).$$

Putting everything together, $v'(r) \geq \frac{1}{r}(k - Cr^2 - Hr)v(r)$. This can be rewritten as

$$\frac{d}{dr}\left(\frac{e^{Ar^2 + Hr} v(r)}{r^k} \right) \geq 0, \tag{1.23}$$

where $A = C/2$, depends on p. From this we have:

Theorem 8. *Let (M^n, g) be a Riemannian manifold, $\Sigma^k \subset M$ be a minimal submanifold and $p \in M$. Then, there exists r_0 and $A > 0$, depending on p only, such that*

$$r \mapsto e^{Ar^2} \frac{v(r)}{r^k} \text{ is non-decreasing in } r, \text{ for } r < r_0. \tag{1.24}$$

Remark 5. If M is compact, we can choose r_0 and A uniform.

Remark 6. A similar monotonicity formula can be proved for stationary integral varifolds (or integral varifolds with bounded mean-curvature or first variation) in Riemannian manifolds. This was first observed by Allard [1]. Section 17 of [51] is also a good reference.

For the first application, observe that if we start with $p \in \Sigma$ and let r go to zero on (1.24), we conclude that

$$\mathcal{H}^k(\Sigma \cap B(p, r)) \geq \frac{\omega_k r^k}{e^{Ar^2}}, \tag{1.25}$$

where ω_k is the volume of the standard unit k-dimensional ball. This expression is applied in the proof of the small mass lemma in Chapter 3.

For the second application, in Chapter 5, we use a similar monotonicity formula, which involves bounds on the sectional curvatures. Let (M^n, g) be a closed Riemannian manifold, $p \in M$ and $r_0 = \mathrm{inj}_M(p)$ its injectivity radius. Suppose the following bound on the sectional curvatures near p:

$$\sup_{q \in B(p,r_0)} |\mathrm{Sec}_N|(q) \leq K.$$

Then, for $0 < r < \min\{r_0, K^{-1/2}\}$, minimality implies

$$\mathcal{H}^k(\Sigma \cap B(p,r)) \geq \frac{\omega_k}{e^{k\sqrt{K}r}} \cdot r^k. \tag{1.26}$$

We omit the proof of this second monotonicity type formula, but it can be proved following the same steps as in the book of Colding and Minicozzi [20].

1.4 Discrete maps in the space of codimension one integral cycles

We begin this section by introducing the domains of our discrete maps. More details can be found in [36] or [47].

- $I^n = [0,1]^n \subset \mathbb{R}^n$ and $I_0^n = \partial I^n = I^n - (0,1)^n$;
- for each $j \in \mathbb{N}$, $I(1,j)$ denote the cell complex of I^1 whose 0-cells and 1-cells are, respectively, $[0], [3^{-j}], \ldots, [1-3^{-j}]$,$[1]$ and $[0, 3^{-j}]$, $[3^{-j}, 2 \cdot 3^{-j}], \ldots, [1-3^{-j}, 1]$;
- $I(n,j) = I(1,j) \otimes \cdots \otimes I(1,j)$, n times;
- $I(n,j)_p = \{\alpha_1 \otimes \cdots \otimes \alpha_n : \alpha_i \in I(1,j) \text{ and } \sum_{i=1}^n \dim(\alpha_i) = p\}$;
- $I_0(n,j)_p = I(n,j)_p \cap I_0^n$, are the p-cells in the boundary;
- $\partial : I(n,j) \to I(n,j)$, the boundary homomorphism is defined by

  $$\partial(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{i=1}^n (-1)^{\sigma(i)} \alpha_1 \otimes \cdots \otimes \partial\alpha_i \otimes \cdots \otimes \alpha_n,$$

  where $\sigma(i) = \sum_{j<i} \dim(\alpha_i)$, $\partial[a,b] = [b] - [a]$ and $\partial[a] = 0$;
- $\mathbf{d} : I(n,j)_0 \times I(n,j)_0 \to \mathbb{N}$, is the grid distance, it is given by

  $$\mathbf{d}(x,y) = 3^j \sum_{i=1}^n |x_i - y_i|;$$

- $\mathbf{n}(i,j) : I(n,i)_0 \to I(n,j)_0$, the nearest vertex map satisfies

$$\mathbf{d}(x, \mathbf{n}(i,j)(x)) = \min\{\mathbf{d}(x,y) : y \in I(n,j)_0\}.$$

Let (M^n, g) denote an orientable compact Riemannian manifold.

Definition 3. Given $\phi : I(n,j)_0 \to \mathcal{Z}_{n-1}(M)$, we define its *fineness* as

$$\mathbf{f}(\phi) = \sup\left\{\frac{\mathbf{M}(\phi(x) - \phi(y))}{\mathbf{d}(x,y)} : x, y \in I(n,j)_0, x \neq y\right\}.$$

Remark 7. If we check that $\mathbf{M}(\phi(x) - \phi(y)) < \delta$, for every $\mathbf{d}(x,y) = 1$, then we can conclude directly that $\mathbf{f}(\phi) < \delta$.

Definition 4. Let $\phi_i : I(1,k_i)_0 \to \mathcal{Z}_{n-1}(M)$, $i = 1, 2$, be given discrete maps. We say that ϕ_1 *is* 1*-homotopic to* ϕ_2 *in* $(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ *with fineness* δ if we can find $k \in \mathbb{N}$ and a map

$$\psi : I(1,k)_0 \times I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(i) $\mathbf{f}(\psi) < \delta$;

(ii) $\psi([i-1], x) = \phi_i(\mathbf{n}(k, k_i)(x))$, $i = 1, 2$ and $x \in I(1,k)_0$;

(iii) $\psi(\tau, [0]) = \psi(\tau, [1]) = 0$, for $\tau \in I(1,k)_0$.

CHAPTER 2

Min-max theory for intersecting slices

In this chapter we describe the min-max theory that we developed to prove some existence of minimal hypersurfaces. The set up that we follow is similar to the original one introduced by Almgren and Pitts. The crucial difference is that we see the slices intersecting a closed subset $\overline{\Omega} \subset M$ only. The aim is to produce an embedded closed minimal hypersurface intersecting the given domain.

The chapter is divided in two section. Before going through the rigorous definitions and statements, we present the aims and main ideas of the new method in the form of a nontechnical outline. Later in the chapter, we introduce the formal description. Further tools and proofs are postponed to subsequent chapters.

2.1 Outline and some intuition

In this section, we outline the proof of our min-max result, Theorem 2. We present the main ideas, omitting the technical issues and reducing the use of the language of geometric measure theory.

The min-max minimal hypersurface of Theorem 2 arises as a limit in the weak sense of varifolds. We consider sweepouts of a closed manifold by codimension-one integral currents without boundary. For this reason, a minimum of this language is unavoidable to introduce our main ideas.

Let us give a brief idea about these structures. Currents and varifolds are measure-theoretic generalized submanifolds. They come with notions of dimension, support and mass. Currents come also with orientation and a notion of boundary, which is a current of one dimension lower. We use ∂T to denote the boundary of a current T and $\mathcal{Z}_{n-1}(M)$ to denote the space of integral $(n-1)$-currents without boundary and supported in M. The mass of $T \in \mathcal{Z}_{n-1}(M)$, $\mathbf{M}(T)$, is a generalized $(n-1)$-dimensional Hausdorff measure with multiplicities. On the other hand, the mass of a varifold V is a Radon measure $||V||$ on M. The space of varifolds with uniformly bounded masses has the remarkable property of being compact with respect to the weak varifold convergence. A class of varifolds that plays a key role in this work is the class of stationary varifolds, those are the varifolds that are critical with respect to the mass in M, $||V||(M)$. This generalizes the idea of minimal submanifolds of M. We can obtain a varifold $|T|$ out of a current $T \in \mathcal{Z}_{n-1}(M)$, simply by forgetting the orientation.

2.1.1 A new width and the intersecting property

Let M^n be an orientable closed Riemannian manifold of dimension n and Ω be a connected open subset of M with smooth and strictly mean-concave boundary, i.e., $\partial\Omega$ is smooth and its mean-curvature vector is everywhere non-zero and points outside Ω.

We consider continuous sweepouts $S = \{\Sigma_t\}$, $t \in [0,1]$, of M by integral $(n-1)$-currents with no boundary, $\Sigma_t \in \mathcal{Z}_{n-1}(M)$, and such that the slices Σ_t degenerate to the zero current at $t = 0$ and 1. For example, the level sets of the function $2^{-1}(1 + x_{n+1})$ in the standard n-sphere $S^n \subset \mathbb{R}^{n+1}$ compose a sweepout, or, more generally, given any Morse function $f : M \to [0,1]$, we can consider the sweepout of the level sets $\Sigma_t = \partial\left(\{x \in M : f(x) < t\}\right)$.

Two such sweepouts S^1 and S^2 are homotopic to each other if there is a continuous map ψ, defined on $[0,1]^2$, for which $\{\psi(s,t)\}_{t\in[0,1]}$ is a sweepout of M, for each $s \in [0,1]$, being $S^1 = \{\psi(0,t)\}_{t\in[0,1]}$ and $S^2 = \{\psi(1,t)\}_{t\in[0,1]}$. Once a homotopy class Π of sweepouts is fixed, we can run the min-max.

The key difference between our method and the original one by Almgren and Pitts is that we see only the slices Σ_t that intersect $\overline{\Omega}$. Given $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we use the following notation:

$$\text{dmn}_\Omega(S) = \{t \in [0,1] : \text{spt}(\Sigma_t) \cap \overline{\Omega} \neq \varnothing\},$$

where $\text{spt}(\Sigma_t)$ denotes the support of $\Sigma_t \in \mathcal{Z}_{n-1}(M)$, and

$$\mathbf{L}(S,\Omega) = \sup\{\mathbf{M}(\Sigma_t) : t \in \text{dmn}_\Omega(S)\}.$$

We define the width of Π with respect to Ω to be

$$\mathbf{L}(\Pi,\Omega) = \inf\{\mathbf{L}(S,\Omega) : S \in \Pi\}.$$

The standard setting of Almgren and Pitts coincides with ours when Ω is M. In this case, the min-max philosophy is to obtain sequences $S^k = \{\Sigma_t^k\}_{t\in[0,1]} \in \Pi$, $k = 1, 2, \ldots$, and $\{t_k\}_{k\in\mathbb{N}} \subset [0, 1]$, for which

$$\mathbf{L}(\Pi, M) = \lim_{k\to\infty} \mathbf{L}(S^k, M) = \lim_{k\to\infty} \mathbf{M}(\Sigma_{t_k}^k)$$

and such that $|\Sigma_{t_k}^k|$ converges, as varifolds, to a disjoint union of closed embedded smooth minimal hypersurfaces, denoted by V. In our approach, we show that it is possible to realize the width $\mathbf{L}(\Pi, \Omega)$ via a min-max sequence of slices $\Sigma_{t_k}^k$, as before, and with the extra properties that $t_k \in \mathrm{dmn}_\Omega(S^k)$, for every $k \in \mathbb{N}$, and the support of the min-max limit V intersects Ω.

The organization of our ideas is as follows:

1. First we prove that the width with respect to Ω is strictly positive for non-trivial homotopy classes, i.e., $\mathbf{L}(\Pi, \Omega) > 0$.

2. We can always find $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ such that

   $$\lim_{k\to\infty} \mathbf{L}(S^k, \Omega) = \mathbf{L}(\Pi, \Omega)$$

   and

   $$\sup\{\mathbf{M}(\Sigma) : \Sigma \text{ is a slice of } S^k, \text{ for some } k \in \mathbb{N}\} < \infty.$$

   This means that there exists a minimizing sequence of sweepouts of M for which the masses of all slices are uniformly controlled. Observe that, by the definition, there is no a priori control on the masses of non-intersecting slices. Such sequences of sweepouts are said to be *critical with respect to* Ω. In this case, a sequence of intersecting slices $\Sigma^k \in S^k$ satisfying

   $$\lim_{k\to\infty} \mathbf{M}(\Sigma^k) = \mathbf{L}(\Pi, \Omega),$$

   is said to be a min-max sequence in $\mathscr{S}$. We use $\mathcal{C}(\mathscr{S}, \Omega)$ to denote the set of limits (as varifolds) of min-max sequences in $\mathscr{S}$.

3. There exists $\mathscr{S} \subset \Pi$ critical with respect to Ω for which $V \in \mathcal{C}(\mathscr{S}, \Omega)$ imply either that V is stationary in M or that $\mathrm{spt}(V) \cap \Omega = \varnothing$.

4. Given $\mathscr{S} \subset \Pi$ as above, we can find a non-trivial intersecting min-max limit $V \in \mathcal{C}(\mathscr{S}, \Omega)$ that is almost minimizing in small annuli (as introduced by Pitts, see the formal Definition 2 in Chapter 1).

5. In conclusion, the obtained $V \in \mathcal{C}(\mathscr{S}, \Omega)$ intersects Ω, and by item 3 it is stationary in M. Since V is almost minimizing in small annuli and stationary, Theorem 5 of Chapter 1, implies that V is smooth.

2.1.2 The role of the mean-concavity

Let $U \subset \Omega$ be a subset with the property that the difference $\Omega - U$ is contained in small tubular neighborhood of $\partial\Omega$ in M. Since $U \subset \Omega$, we easily see that $\mathbf{L}(\Pi, U) \leq \mathbf{L}(\Pi, \Omega)$, for all homotopy classes of sweepouts Π. We use the mean-concavity assumption to prove that the inequality above is indeed an equality: $\mathbf{L}(\Pi, U) = \mathbf{L}(\Pi, \Omega)$. The main tool to prove this claim is the maximum principle for general varifolds of Brian White, [61].

More precisely, if we had the strict inequality $\mathbf{L}(\Pi, U) < \mathbf{L}(\Pi, \Omega)$ for some Π, we would be able to find

$$0 < \varepsilon < \mathbf{L}(\Pi, \Omega) - \mathbf{L}(\Pi, U)$$

and a sweepout $S = \{\Sigma_t\}_{t \in [0,1]} \in \Pi$, such that

$$\mathbf{L}(S, U) < \mathbf{L}(\Pi, \Omega) - \varepsilon.$$

In particular, if $t \in \mathrm{dmn}_\Omega(S)$ and $\mathbf{M}(\Sigma_t) \geq \mathbf{L}(\Pi, \Omega) - \varepsilon$, then $\Sigma_t \subset M - \overline{U}$. We use the maximum principle to deform those Σ_t, without increasing the mass, to a new slice Σ'_t, supported outside Ω. Then, we obtain $S' \in \Pi$ with $\mathbf{L}(S', \Omega) < \mathbf{L}(\Pi, \Omega) - \varepsilon$, and this is a contradiction.

2.1.3 Sketch of proofs

The existence of non-trivial homotopy classes goes back to Almgren. He proved, in [2], that the set of homotopy classes of sweepouts is isomorphic to the top dimensional homology group of M, $H_n(M^n, \mathbb{Z})$. Moreover, it is possible to prove the existence of positive constants $\alpha_0 = \alpha_0(M)$ and $r_0 = r_0(M)$ with the property that: given $p \in M$, $0 < r \leq r_0$ and a sweepout $S = \{\Sigma_t\}$, $0 \leq t \leq 1$, in a non-trivial homotopy class, we have

$$\sup_{t \in [0,1]} \mathbf{M}((\Sigma_t) \llcorner B(p, r)) \geq \alpha_0 r^{n-1},$$

where $(\Sigma_t) \llcorner B(p, r)$ is the restriction of Σ_t to the geodesic ball $B(p, r)$ of M, of radius r and centered at p. If we take $B(p, r) \subset\subset \Omega$, then

$$\mathbf{L}(S, \Omega) = \sup_{t \in \mathrm{dmn}_\Omega(S)} \mathbf{M}(\Sigma_t) \geq \sup_{t \in [0,1]} \mathbf{M}((\Sigma_t) \llcorner B(p, r)) \geq \alpha_0 r^{n-1}.$$

Therefore, $\mathbf{L}(\Pi, \Omega) > 0$, for non-trivial homotopy classes Π. The existence of such numbers α_0 and r_0 is due to Gromov, see Section 4.2.B in [29]. These lower bounds for the width also appear in [31] and Section 8 of [37].

We can construct a critical sequence of sweepouts with respect to Ω as follows. Take a minimizing sequence $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$, i.e., a sequence for which $\mathbf{L}(S^k, \Omega)$ tends to $\mathbf{L}(\Pi, \Omega)$ as k tend infinity. In order to obtain a critical sequence out of $\mathscr{S}$, we have to uniformly control the masses of the non-intersecting slices of the S^k's. Let $k \in \mathbb{N}$ and $[t_1, t_2] \subset [0, 1]$ be such that $[t_1, t_2] \cap \mathrm{dmn}_\Omega(S^k) = \{t_1, t_2\}$. We homotopically deform the path $\{\Sigma_t^k\}$, $t \in [t_1, t_2]$, keeping the ends fixed and without entering Ω to a new continuous path $\{\tilde{\Sigma}_t^k\}$, $t \in [t_1, t_2]$, in such a way that

$$\sup_{t\in[t_1,t_2]} \mathbf{M}(\tilde{\Sigma}_t^k) \leq C\left(\mathbf{M}(\Sigma_{t_1}^k) + \mathbf{M}(\Sigma_{t_2}^k)\right),$$

where $C > 0$ is a universal constant depending only on M. In this deformation, it is important that $(\Sigma_{t_2}^k - \Sigma_{t_1}^k)$ is the boundary of an integral n-current supported in $M - \Omega$. Since the masses of the extremal slices, $\Sigma_{t_1}^k$ and $\Sigma_{t_2}^k$, are already accounted in $\mathbf{L}(S^k, \Omega)$, we have the desired uniform control.

Let $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ be a critical sequence with respect to Ω. Intuitively, we can invoke the compactness theorem for varifolds with uniformly bounded mass to obtain a limit optimal sweepout $\mathscr{S}^* = \{\overline{\Sigma}_t\}_{t\in[0,1]} \in \Pi$, i.e.,

$$\overline{\Sigma}_t = \lim_{k\to\infty} |\Sigma_t^k|, \text{ for every } t \in [0,1], \quad \text{and} \quad \mathbf{L}(\mathscr{S}^*, \Omega) = \mathbf{L}(\Pi, \Omega).$$

Moreover, we can identify the critical set $\mathcal{C}(\mathscr{S}, \Omega)$ with the set of the maximal intersecting slices of $\mathscr{S}^*$, which is denoted by $\mathcal{C}(\mathscr{S}^*, \Omega)$. Observe that $\mathscr{S}^*$ does not belong necessarily to the same category of sweepouts as the S^k's, because the limit slices $\overline{\Sigma}_t$ are only varifolds, while Σ_t^k are integral currents. For the purpose of this outline, let us suppose for a while that there exists such a $\mathscr{S}^* = \{\overline{\Sigma}_t\}_{t\in[0,1]} \in \Pi$, optimal with respect to Ω.

The third step of our list is called the pull-tight argument and it is inspired by Theorem 4.3 in [47] and Proposition 8.5 in [36]. We deform $\mathscr{S}^*$ to obtain a better optimal sweepout $\mathscr{S}' = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$ with the extra property that:

$$\Sigma_t \in C(\mathscr{S}', \Omega) \Rightarrow \text{ either } \Sigma_t \text{ is stationary or } \mathrm{spt}(\Sigma_t) \cap \overline{\Omega} = \varnothing.$$

More precisely, we construct a continuous deformation $\{H(s,t)\}_{s,t\in[0,1]}$, that starts with $H(0,t) = \overline{\Sigma}_t$ and ends with $\mathscr{S}'$, whose key property is that

$$\mathbf{M}(H(1,t)) < \mathbf{M}(\overline{\Sigma}_t),$$

unless $\overline{\Sigma}_t$ is either stationary or $\mathrm{spt}(\overline{\Sigma}_t) \cap \overline{\Omega} = \varnothing$, for which it must be true that $H(s,t) = \overline{\Sigma}_t$, for every $s \in [0,1]$. The main improvement on our pull-tight deformation H is that it keeps unmoved the non-intersecting slices, allowing us to conclude that $\mathscr{S}'$ is also optimal with respect to Ω.

The final part is to prove that some $V \in \mathcal{C}(\mathscr{S}, \Omega)$, for the sequence $\mathscr{S} \subset \Pi$ with the properties stated in Step 3, is at the same time intersecting and almost minimizing in small annuli. For the precise notation, see Definition 2 and Theorem 10 in Section 2, or Theorem 4.10 in Pitts book [47]. The almost minimizing varifolds are natural objects in the min-max theory, which can be arbitrarily approximated by locally mass-minimizing currents. The proof of the regularity of almost minimizing varifolds uses the curvature estimates for stable minimal hypersurfaces of Schoen-Simon-Yau [55] and Schoen-Simon [53]. The dimension assumption, $n \leq 7$, is important in the application of these regularity results. In general, our argument produces a closed, embedded minimal hypersurface with a singular set of Hausdorff codimension at least 7, see Theorem 5.

Assuming that no $V \in \mathcal{C}(\mathscr{S}, \Omega)$ satisfies our assumption, we deform the critical sequence $\mathscr{S} \subset \Pi$ to obtain strictly better competitors, i.e., we obtain a sequence of sweepouts $\tilde{S}^k \in \Pi$ out of S^k such that

$$\mathbf{L}(\tilde{S}^k, \Omega) < \mathbf{L}(S^k, \Omega) - \rho,$$

for some uniform $\rho > 0$. Since $\mathscr{S}$ is critical with respect to Ω, we have $\tilde{S}^k \in \Pi$ and $\mathbf{L}(\tilde{S}^k, \Omega) < \mathbf{L}(\Pi, \Omega)$, for large $k \in \mathbb{N}$. This will give us a contradiction.

Recall that we have two types of varifolds $V \in \mathcal{C}(\mathscr{S}, \Omega)$, either

(1) $\operatorname{spt}(V) \cap \Omega \neq \varnothing$ and V is stationary in M

(2) or $\operatorname{spt}(V) \cap \Omega = \varnothing$.

Then, big intersecting slices of $\mathscr{S}$ are close either to intersecting non-almost minimizing varifolds or to non-intersecting varifolds. Let $U \subset U_1 \subset \Omega$ be open sets, with $\overline{U} \subset U_1$, $\overline{U_1} \subset \Omega$ and such that $\Omega - U$ is inside a small tubular neighborhood of $\partial\Omega$.

The deformation from S^k to $\tilde{S}^k$ is done in two steps. The first is based in the fact that if V is type (1), then it is not almost minimizing in small annuli centered at some $p = p(V) \in \operatorname{spt}(V)$. Then, there exists $\varepsilon(V) > 0$ such that given a slice $\Sigma_t^k \in S^k$ close to V and $\eta > 0$, we can find a continuous path $\{\Sigma_t^k(s)\}_{s\in[0,1]}$ with the following properties:

- $\Sigma_t^k(0) = \Sigma_t^k$
- $\operatorname{spt}(\Sigma_t^k(s) - \Sigma_t^k) \subset a(V)$
- $\mathbf{M}(\Sigma_t^k(s)) \leq \mathbf{M}(\Sigma_t^k) + \eta$
- $\mathbf{M}(\Sigma_t^k(1)) < \mathbf{M}(\Sigma_t^k) - \varepsilon(V)$,

where $a(V)$ is a small annulus centered at $p(V)$. Observe that if Σ_t^k is also close to a type (2) varifold, then it has small mass in U_1, $||\Sigma_t^k||(U_1)$. If we take $a(V)$ small enough, we can suppose that we always have either $a(V) \subset U_1$ or $a(V) \subset M - \overline{U}$. Anyway, we have

$$||\Sigma_t^k(s)||(\overline{U}) \leq ||\Sigma_t^k||(U_1) + \eta, \quad \text{for every } s \in [0,1].$$

Let $[t_1, t_2] \subset \mathrm{dmn}_\Omega(S^k)$ be a maximal interval of big intersecting slices Σ_t^k close to type (1) varifolds. Replace $\{\Sigma_t^k\}$, $t \in [t_1, t_2]$, by the continuous path $\{\tilde{\Sigma}_t^k\}$, $t \in [t_1, t_2]$, obtained as follows:

(i) first, use $\{\Sigma_{t_1}^k(s)\}_{s\in[0,1]}$ to go from $\Sigma_{t_1}^k$ to $\Sigma_{t_1}^k(1)$;

(ii) then use $\{\Sigma_t^k(1)\}$, $t \in [t_1, t_2]$, to reach $\Sigma_{t_2}^k(1)$;

(iii) finally use the inverse way of $\{\Sigma_{t_2}^k(s)\}_{s\in[0,1]}$ to come back to $\Sigma_{t_2}^k$.

Repeating the process on each maximal interval, we obtain $\tilde{\mathscr{S}} = \{\tilde{S}^k\}_{k\in\mathbb{N}}$, critical with respect to Ω and such that the big intersecting slices $\tilde{\Sigma}_t^k$ have small mass inside $\overline{U}$.

Each such slice is replaced by one outside $\overline{\Omega}$ via a two-steps deformation. First we deform $\tilde{\Sigma}_t^k$ inside Ω to a current not intersecting $\overline{U}$ and then we use the maximum principle to take it out of $\overline{\Omega}$. And we have a contradiction.

2.2 Formal description

Let (M^n, g) be an orientable closed Riemannian manifold and $\Omega \subset M$ be a connected open subset. We begin with the basic definitions following the works of Almgren, Pitts and Marques-Neves, see [3], [36], [37] and [47].

Definition 5. An

$$(1, \boldsymbol{M}) - \textit{homotopy sequence of mappings into } (\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$$

is a sequence of maps $\{\phi_i\}_{i\in\mathbb{N}}$

$$\phi_i : I(1, k_i)_0 \to \mathcal{Z}_{n-1}(M),$$

such that ϕ_i is 1-homotopic to ϕ_{i+1} in $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ with fineness δ_i and

(i) $\lim_{i\to\infty} \delta_i = 0$;

(ii) $\sup\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}(\phi_i) \text{ and } i \in \mathbb{N}\} < \infty$.

The notion of homotopy between two $(1, \mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$, is the following:

Definition 6. We say that $S^1 = \{\phi_i^1\}_{i\in\mathbb{N}}$ *is homotopic with* $S^2 = \{\phi_i^2\}_{i\in\mathbb{N}}$ if ϕ_i^1 is 1-homotopic to ϕ_i^2 with fineness δ_i and $\lim_{i\to\infty} \delta_i = 0$.

One checks that to be "homotopic with" is an equivalence relation on the set of $(1, \mathbf{M})$-homopoty sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$. An equivalence class is called a $(1, \mathbf{M})$-homotopy class of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$. We follow the usual notation $\pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ for the set of homotopy classes. These definitions are the same as in Pitts, [47].

The key difference is that our width considers only slices intersecting $\overline{\Omega}$. Given a map $\phi : I(1, k)_0 \to \mathcal{Z}_{n-1}(M)$ we defined its reduced domain by

$$\mathrm{dmn}_\Omega(\phi) = \{x \in I(1, k)_0 : \mathrm{spt}(||\phi(x)||) \cap \overline{\Omega} \neq \varnothing\}. \tag{2.1}$$

Definition 7. Let $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ be a homotopy class and $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$. We define

$$\mathbf{L}(S, \Omega) = \limsup_{i\to\infty} \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\}. \tag{2.2}$$

The *width of* Π *with respect to* Ω is the minimum $\mathbf{L}(S, \Omega)$ among all $S \in \Pi$,

$$\mathbf{L}(\Pi, \Omega) = \inf\{\mathbf{L}(S, \Omega) : S \in \Pi\}. \tag{2.3}$$

Keeping the notation in the previous definition, we write $V \in \mathbf{K}(S, \Omega)$ if $V = \lim_j |\phi_{i_j}(x_j)|$, for some increasing sequences $\{i_j\}_{j\in\mathbb{N}}$ and $x_j \in \mathrm{dmn}_\Omega(\phi_{i_j})$. Moreover, if $\mathbf{L}(S, \Omega) = \mathbf{L}(\Pi, \Omega)$, we say that S *is critical with respect to* Ω. In this case we consider the *critical set of* S *with respect to* Ω, defined by

$$\mathcal{C}(S, \Omega) = \{V \in \mathbf{K}(S, \Omega) : ||V||(M) = \mathbf{L}(S, \Omega)\}. \tag{2.4}$$

As in the classical theory, $\mathcal{C}(S, \Omega) \subset \mathcal{V}_{n-1}(M)$ is compact and non-empty, but here it is not clear whether there exists $V \in \mathcal{C}(S, \Omega)$ with $||V||(\overline{\Omega}) > 0$.

In the direction of proving that there are critical varifolds intersecting $\overline{\Omega}$, in Subsection 3.3.2, we construct a deformation process to deal with discrete maps whose big slices enter Ω with very small mass. For doing this, and from now on, we introduce our main geometric assumption:

$$\Omega \textit{ has smooth and strictly mean-concave boundary } \partial\Omega. \tag{2.5}$$

This deformation process is inspired by Pitts' deformation arguments for constructing replacements, Section 3.10 in [47], and Corollary 2.

We introduce some notation to explain that result, which is precisely stated and proved in Subsection 3.3.2. Following the notation in the Subsection 1.2, consider an open subset $U \subset \Omega$ such that

$$U \supset \overline{\Omega_a} = \{x \in \Omega : d(x, \partial\Omega) \leq -a\},$$

where $a < 0$ is given by Lemma 2. Lemma 9 guarantees the existence of positive constants η_0 and ε_2 depending on M, Ω and U, and C_1 depending only on M, with the following properties: given a discrete map $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ such that $\mathbf{f}(\phi) \leq \eta_0$ and, for some $L > 0$,

$$\mathbf{M}(\phi(x)) \geq L \Rightarrow ||\phi(x)||(U) < \varepsilon_2,$$

then, up to a discrete homotopy of fineness $C_1\mathbf{f}(\phi)$, we can suppose that

$$\max\{\mathbf{M}(\phi(x)) : x \in \mathrm{dmn}_\Omega(\phi)\} < L + C_1\mathbf{f}(\phi).$$

The proof is quite technical and lengthy, because it involves interpolation results, see Section 3.1 of Chapter 3. Despite the technical objects in that proof, the lemma has several applications in key arguments of this work.

It is important to generate $(1, \mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ out of a continuous map $\Gamma : [0, 1] \to \mathcal{Z}_{n-1}(M; \mathbf{F})$, with $\Gamma(0) = \Gamma(1) = 0$. Similarly to the discrete set up, use the notations

$$\mathrm{dmn}_\Omega(\Gamma) = \{t \in [0, 1] : \mathrm{spt}(||\Gamma(t)||) \cap \overline{\Omega} \neq \varnothing\} \tag{2.6}$$

and

$$L(\Gamma, \Omega) = \sup\{\mathbf{M}(\Gamma(t)) : t \in \mathrm{dmn}_\Omega(\Gamma)\}. \tag{2.7}$$

Theorem 9. *Let Γ be as above and suppose that it defines a non-trivial class in $\pi_1(\mathcal{Z}_{n-1}(M; \mathcal{F}), 0)$. Then, there exists a non-trivial homotopy class $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$, such that $\boldsymbol{L}(\Pi, \Omega) \leq L(\Gamma, \Omega)$.*

Remark 8. Proving this is a nice application of Lemma 9, combined with interpolation results. It is done in Subsection 3.3.3 of Chapter 3.

Observe that item (ii) in the definition of $(1, \mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ requires a uniform control on the masses of all slices. When we try to minimize the width in a given homotopy class, in order to construct a critical sequence with respect to Ω, there is no restriction about the non-intersecting slices. Then, it is possible that item (ii) fails in the limit. Precisely, if $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ is a homotopy class, via a diagonal sequence argument through a minimizing sequence $\{S^j\}_{j\in\mathbb{N}} \subset \Pi$, we can produce $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$, so that ϕ_i^* is 1-homotopic to ϕ_{i+1}^* with fineness tending to zero and

$$\lim_{i\to\infty} \max\{\mathbf{M}(\phi_i^*(x)) : x \in \mathrm{dmn}_\Omega(\phi_i^*)\} = \mathbf{L}(\Pi, \Omega).$$

But, perhaps, in doing this, we do not guarantee that the masses of the non-intersecting slices do not diverge. To overcome this difficulty, we prove the following statement:

Lemma 2. *There exists $C = C(M, \Omega) > 0$ with the following property: given a discrete map $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ of small fineness, we can find*

$$\tilde{\phi} : I(1, \tilde{k})_0 \to \mathcal{Z}_{n-1}(M)$$

such that:

(a) $\tilde{\phi}$ is 1-homotopic to ϕ with fineness $C \cdot \boldsymbol{f}(\phi)$;

(b) $\phi(dmn_\Omega(\phi)) = \tilde{\phi}(dmn_\Omega(\tilde{\phi}))$;

(c)

$$\max_{x\in dmn(\tilde{\phi})} \boldsymbol{M}(\tilde{\phi}(x)) \leq C \cdot (\max\{\boldsymbol{M}(\phi(x)) : x \in dmn_\Omega(\phi)\} + \boldsymbol{f}(\phi)) \,.$$

To produce a true competitor out of S^*, for each i large enough we replace ϕ_i^* by another discrete map, also denoted by ϕ_i^*, using Lemma 2. The new S^* has the same intersecting slices and, as consequence of items (a) and (c), the additional property of being an element of Π. This concludes the existence of critical S^* for $\mathbf{L}(\Pi, \Omega)$. The proof of Lemma 2 is based on a natural deformation of boundaries of n-currents in n-dimensional manifolds with boundary. Briefly, the deformation is the image of the given $(n-1)$-boundary via the gradient flow of a Morse function with no interior local maximum. We postpone the details to Section 3.2.

Actually, we prove a Pull-tight type Theorem, as Theorem 4.3 in [47] and Proposition 8.5 in [36]. Precisely, given $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), 0)$, we obtain:

Proposition 1. *There exists a critical sequence $S^* \in \Pi$. For each critical sequence S^*, there exists a critical sequence $S \in \Pi$ such that*

- *$\mathcal{C}(S, \Omega) \subset \mathcal{C}(S^*, \Omega)$, up to critical varifolds Σ with $||\Sigma||(\Omega) = 0$;*
- *every $\Sigma \in \mathcal{C}(S, \Omega)$ is either a stationary varifold or $||\Sigma||(\Omega) = 0$.*

In this statement, critical means critical with respect to Ω. The proof is postponed to Chapter 4.1. In the classical set up, the pull-tight gives a critical sequence for which all critical varifolds are stationary in M. In our case, it is enough to know that the intersecting critical varifolds are stationary.

The existence of non-trivial classes was proved by Almgren, in [2]. In fact, his result provides an isomorphism

$$F : \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\}) \to H_n(M). \tag{2.8}$$

Then, we apply the Proposition 8.2 of [37] to guarantee that the non-trivial classes have positive width, in the sense introduced in Definition 7.

Lemma 3. *If* $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ *and* $\Pi \neq 0$*, then* $\boldsymbol{L}(\Pi, \Omega) > 0$*.*

Given $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$, the ϕ_i's with sufficient large i can be extended to maps Φ_i continuous in the mass norm, respecting the non-intersecting property, see the interpolation result Theorem 11. The Proposition 8.2 in [37] provides a lower bound on the value of $\sup\{\mathbf{M}(\Phi(\theta)\llcorner B(p,r)) : \theta \in S^1\}$, for continuous maps $\Phi : S^1 \to \mathcal{Z}_{n-1}(M)$ in the flat topology. To obtain Lemma 3, apply Proposition 8.2 for those Φ_i and a small geodesic ball $\overline{B(p,r)} \subset \Omega$.

The next step in Almgren and Pitts' program is to prove the existence of a critical varifold $V \in \mathcal{C}(S, \Omega)$ with the property of being almost minimizing in small annuli. Recall the definition in subsection 1.1.3 of Chapter 1. The goal of our construction is to produce min-max minimal hypersurfaces with intersecting properties. In order to obtain this, we prove the following version of the existence theorem of almost minimizing varifolds:

Theorem 10. *Let* (M^n, g) *be a closed Riemannian manifold,* $n \leq 7$*, and* $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ *be a non-trivial homotopy class. Suppose that* M *contains an open subset* Ω *with smooth and strictly mean-concave boundary. There exists an integral varifold* V *such that*

(i) $||V||(M) = \boldsymbol{L}(\Pi, \Omega)$*;*

(ii) V *is stationary in* M*;*

(iii) $||V||(\Omega) > 0$*;*

(iv) for each $p \in M$*, there exists a positive number* r *such that* V *is almost minimizing in* $A(p, s, r)$ *for all* $0 < s < r$*.*

This is similar to Pitts' Theorem 4.10 in [47], the difference being that we prove that the almost minimizing and stationary varifold also intersects Ω. We postpone its proof to Subsection 4.2 of Chapter 4, it is a combination of Pitts' argument and Lemma 9. This is the last preliminary result to prove our main result, Theorem 2 as stated in the introduction, which says:

Main Theorem

Let (M^n, g) be a closed Riemannian manifold, $n \leq 7$, and

$$\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$$

be a non-trivial homotopy class. Suppose that M contains an open subset Ω, such that $\overline{\Omega}$ is a manifold whose boundary is smooth and strictly mean-concave. There exists a stationary integral varifold Σ whose support is a smooth embedded closed minimal hypersurface intersecting Ω and with

$$||\Sigma||(M) = \boldsymbol{L}(\Pi, \Omega).$$

The argument to prove Theorem 2 is simple now. Lemma 3 provides a non-trivial homotopy class, for which we can apply Theorem 10 and obtain an integral critical varifold V that is almost minimizing in small annuli and intersects Ω. The Pitts' regularity theory, Theorem 5, developed in Chapters 5 to 7 in [47], guarantees that $\mathrm{spt}(||V||)$ is an embedded smooth hypersurface.

CHAPTER 3

Further tools

In this chapter we present several tools that play a role in the proofs of our min-max results. We begin with some well known interpolations statements used since Almgren and Pitts introduced the min-max theory of minimal surfaces. This is the content of Section 3.1. In this section, we present also a simple enhancement of an interpolation result concerning the supports of the interpolated objects. In Section 3.2, we describe a natural deformation process of boundaries of n-currents in n-manifolds with boundary to the zero current. This deformation is similar to the flow of the minus gradient vector field of a Morse function without local maxima, and has the fundamental property of controlling the masses along the process by a universal constant times the mass of the initial boundary. Then, in Section 3.3, we develop one of the key ingredients of our arguments, namely, the discrete deformations of single currents (and of discrete sweepouts) which enter a mean-concave region Ω with small mass (for which big intersecting slices enter Ω with small mass) to a current supported outside Ω (to a sweepout of strictly smaller width with respect to Ω). These tools are applied to prove two results stated in Chapter 2: Lemma 2 and Theorem 9.

3.1 Interpolation results

Interpolation is an important tool for passing from discrete maps of small fineness to continuous maps in the space of integral cycles and vice-versa, the fineness and continuity being with respect to two different topologies.

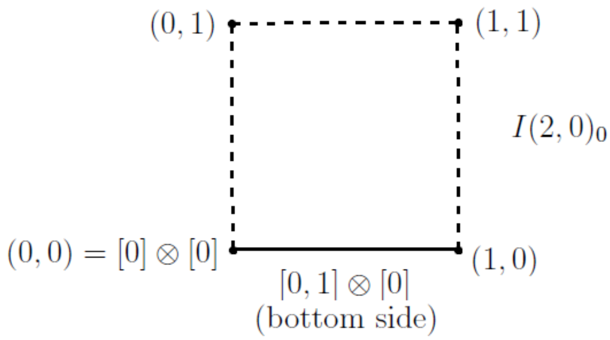


Figure 3.1: Domain of the discrete map ψ and associated cells.

This type of technique appeared already in the works of Almgren, Pitts, Marques and Neves [2, 3, 47, 36, 37]. In our approach, we mostly follow the Sections 13 and 14 of [36]. We make also a remark that is important in the arguments of this thesis concerning the supports of interpolating sequences.

We start by presenting conditions under which a discrete map can be approximated by a continuous map in the mass norm. The main result is important to prove Lemmas 9 and 3, and Proposition 1, which are key ingredients of the program that we presented in the previous chapter.

Let (M^n, g) be a closed Riemannian manifold. We observe from Corollary 1.14 in [2] that there exists $\delta_0 > 0$, depending only on M, such that for every

$$\psi : I(2,0)_0 \to \mathcal{Z}_{n-1}(M)$$

with $\mathbf{f}(\psi) < \delta_0$ and $\alpha \in I(2,0)_1$ with $\partial\alpha = [b] - [a]$, see Figure 3.1, we can find $Q(\alpha) \in \mathbf{I}_n(M)$ with

$$\partial Q(\alpha) = \psi([b]) - \psi([a]) \text{ and } \mathbf{M}(Q(\alpha)) = \mathcal{F}(\partial Q(\alpha)).$$

Let Ω_1 be a connected open subset of M, such that $\overline{\Omega_1}$ is a manifold with boundary. The first important result in this section is:

Theorem 11. *There exists $C_0 > 0$, depending only on M, such that for every map*

$$\psi : I(2,0)_0 \to \mathcal{Z}_{n-1}(M)$$

with $\boldsymbol{f}(\psi) < \delta_0$, we can find a continuous map in the mass norm

$$\Psi : I^2 \to \mathcal{Z}_{n-1}(M; \boldsymbol{M})$$

such that

(i) $\Psi(x) = \psi(x)$, *for all* $x \in I(2,0)_0$;

(ii) for every $\alpha \in I(2,0)_p$, $\Psi|_\alpha$ depends only on the values assumed by ψ on the vertices of α;

(iii)
$$\sup\{\boldsymbol{M}(\Psi(x)-\Psi(y)) : x,y \in I^2\} \leq C_0 \sup_{\alpha\in I(2,0)_1}\{\boldsymbol{M}(\partial Q(\alpha))\}.$$

Moreover, if $\boldsymbol{f}(\psi) < \min\{\delta_0, \mathcal{H}^n(\Omega_1)\}$ and

$$spt(||\psi(0,0)||) \cup spt(||\psi(1,0)||) \subset M - \overline{\Omega_1},$$

we can choose Ψ with $spt(||\Psi(t,0)||) \subset M - \overline{\Omega_1}$, for all $t \in [0,1]$.

Proof. The first part of this result is Theorem 14.1 in Marques and Neves, [36]. There the authors sketch the proof following the work of Almgren, Section 6 of [2], and ideas of Pitts, Theorem 4.6 of [47]. To prove our second claim, we follow that sketch. They start with Δ, a differentiable triangulation of M. Hence, if $s \in \Delta$ then the faces of s also belong to Δ. Given $s, s' \in \Delta$, use the notation $s' \subset s$ if s' is a face of s. Let $U(s) = \cup_{s\subset s'} s'$. Let Ω_2 be a small connected neighborhood of $\overline{\Omega_1}$, so that

$$\text{spt}(||\psi(0,0)||) \cup \text{spt}(||\psi(1,0)||) \cap \overline{\Omega_2} = \varnothing.$$

Up to a refinement of Δ, we can suppose that

$$s \in \Delta \text{ and } U(s) \cap (M - \Omega_2) \neq \varnothing \Rightarrow U(s) \subset M - \overline{\Omega_1}.$$

For $Q = Q([0,1]\otimes[0])$, we have $Q \in \mathbf{I}_n(M)$ and $\partial Q = \psi(1,0) - \psi(0,0)$ does not intersect Ω_2. Then, there are two possibilities: either $\text{spt}(||Q||) \cap \Omega_2 = \varnothing$ or $\Omega_2 \subset \text{spt}(||Q||)$. Note that in the second case we would have

$$\mathcal{H}^n(\Omega_1) < \mathcal{H}^n(\Omega_2) \leq ||Q||(\Omega_2) \leq \mathbf{M}(Q) \leq \mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega_1).$$

This is a contradiction and we have $\text{spt}(||Q||) \cap \Omega_2 = \varnothing$. By the construction of Ψ we know that

$$\text{spt}(||\Psi(t,0)||) \subset \bigcup\{U(s) : U(s) \cap \text{spt}(||Q||) \neq \varnothing\}.$$

But $U(s) \cap \text{spt}(Q) \neq \varnothing$ implies that $U(s) \cap (M - \Omega_2) \neq \varnothing$. Then, the choice of the triangulation gives us $U(s) \subset M - \overline{\Omega_1}$. This concludes the proof. □

As in [36], we also use the following discrete approximation result for continuous maps. Assume we have a continuous map in the flat topology $\Phi : I^m \to \mathcal{Z}_{n-1}(M)$, with the following properties:

- $\Phi|_{I_0^m}$ is continuous in the $\mathbf{F}$-metric
- $L(\Phi) = \sup\{\mathbf{M}(\Phi(x)) : x \in I^m\} < \infty$
- $\limsup_{r\to 0} \mathbf{m}(\Phi, r) = 0$,

where $\mathbf{m}(\Phi, r)$ is the concentration of mass of Φ in balls of radius r, i.e.:

$$\mathbf{m}(\Phi, r) = \sup\{||\Phi(x)||(B(p, r)) : x \in I^m, p \in M\}.$$

Theorem 12. *There exist sequences of mappings*

$$\phi_i : I(m, k_i)_0 \to \mathcal{Z}_{n-1}(M) \text{ and } \psi_i : I(1, k_i)_0 \times I(m, k_i)_0 \to \mathcal{Z}_{n-1}(M),$$

with $k_i < k_{i+1}$, $\psi_i([0], \cdot) = \phi_i(\cdot)$, $\psi_i([1], \cdot) = \phi_{i+1}(\cdot)|_{I(m,k_i)_0}$, *and sequences* $\{\delta_i\}_{i\in\mathbb{N}}$ *tending to zero and* $\{l_i\}_{i\in\mathbb{N}}$ *tending to infinity, such that*

(i) for every $y \in I(m, k_i)_0$

$$\boldsymbol{M}(\phi_i(y)) \leq \sup\{\boldsymbol{M}(\Phi(x)) : \alpha \in I(m, l_i)_m, x, y \in \alpha\} + \delta_i.$$

In particular,

$$\max\{\boldsymbol{M}(\phi_i(x)) : x \in I(m, k_i)_0\} \leq L(\Phi) + \delta_i;$$

(ii) $\boldsymbol{f}(\psi_i) < \delta_i$;

(iii)

$$\sup\{\mathcal{F}(\psi_i(y, x) - \Phi(x)) : (y, x) \in dmn(\psi_i)\} < \delta_i;$$

(iv) if $x \in I_0(m, k_i)_0$ *and* $y \in I(1, k_i)_0$, *we have*

$$\boldsymbol{M}(\psi_i(y, x)) \leq \boldsymbol{M}(\Phi(x)) + \delta_i.$$

Moreover, if $\Phi|_{\{0\}\times I^{m-1}}$ *is continuous in the mass topology then we can choose* ϕ_i *so that*

$$\phi_i(x) = \Phi(x), \text{ for all } x \in B(m, k_i)_0.$$

Remark 9. In case Φ is continuous in the F-metric in I^m, there is no concentration of mass. This is the content of lemma 15.2 in [36].

In the proof of our main lemma, in Subsection 3.3.2, we apply the following consequence of Theorem 12. Let $U, \Omega_1 \subset M$ be open subsets, being $\overline{\Omega_1}$ a manifold with boundary, and $\rho > 0$.

Assume we have a continuous map in the F-metric $\Psi : I^2 \to \mathcal{Z}_{n-1}(M)$. Suppose also that

- $t \in [0,1] \mapsto \Psi(0,t)$ is continuous in the mass norm;
- $\sup\{||\Psi(s,t)||(\overline{U}) : s \in [0,1] \text{ and } t \in \{0,1\}\} < \rho$;
- $\text{spt}(||\Psi(1,t)||) \subset M - \overline{\Omega_1}$, for every $t \in [0,1]$.

Corollary 3. *Given $\delta > 0$, there exists $k \in \mathbb{N}$ and a map*

$$\Psi_1 : I(2,k)_0 \to \mathcal{Z}_{n-1}(M),$$

with the following properties:

(i) $\boldsymbol{f}(\Psi_1) < \delta$*;*

(ii) $\sup\{||\Psi_1(\sigma,\tau)||(\overline{U}) : \sigma \in I(1,k)_0$ *and* $\tau \in \{0,1\}\} < \rho + \delta$*;*

(iii) $\sup\{||\Psi_1(1,\tau))||(\overline{\Omega_1}) : \tau \in I(1,k)_0\} < \delta$*;*

(iv) $\boldsymbol{M}(\Psi_1(x)) \leq \boldsymbol{M}(\Psi(x)) + \delta$*, for every* $x \in I(2,k)_0$*;*

(v) $\Psi_1(0,\tau) = \Psi(0,\tau)$*, if* $\tau \in I(1,k)_0$*.*

Items (i) and (v) are easy consequences of Theorem 12. The other items hold if we choose a sufficiently close discrete approximation. The proof of Corollary 3 involves a simple combination of the uniform continuity of $\mathbf{M} \circ \Psi$, compactness arguments and the Lemma 4.1 in [36]. For the convenience of the reader, we state this lemma here.

Lemma 4. *Let $\mathcal{S} \subset \mathcal{Z}_k(M;\boldsymbol{F})$ be a compact set. For every $\rho > 0$, there exists δ so that for every $S \in \mathcal{S}$ and $T \in \mathcal{Z}_k(M)$*

$$\boldsymbol{M}(T) < \boldsymbol{M}(S) + \delta \text{ and } \mathcal{F}(T - S) \leq \delta \Rightarrow \boldsymbol{F}(S,T) \leq \rho.$$

The last lemma that we discuss in this section is similar to Theorem 12 restricted to the case $m = 1$. It says that the hypothesis about the no concentration of mass is not required in this case. See Lemma 3.8 in [47].

Lemma 5. *Suppose $L, \eta > 0$, K compact subset of U and $T \in \mathcal{Z}_k(M)$. There exists $\varepsilon_0 = \varepsilon_0(L,\eta,K,U,T) > 0$, such that whenever*

- $S_1, S_2 \in \mathcal{Z}_k(M)$*;*
- $\mathcal{F}(S_1 - S_2) \leq \varepsilon_0$*;*
- *spt*$(S_1 - T) \cup$ *spt*$(S_2 - T) \subset K$*;*
- $\boldsymbol{M}(S_1) \leq L$ *and* $\boldsymbol{M}(S_2) \leq L$*,*

there exists a finite sequence $S_1 = T_0, T_1, \ldots, T_m = S_2 \in \mathcal{Z}_k(M)$ *with*

$$spt(T_l - T) \subset U, \quad \boldsymbol{M}(T_l - T_{l-1}) \leq \eta \text{ and } \boldsymbol{M}(T_l) \leq L + \eta.$$

This lemma is useful to approximate continuous maps in flat topology by discrete ones with small fineness in mass norm, but first we have to restrict the continuous map to finer and finer grids $I(1, k)_0$.

3.2 Deformations of boundaries on manifolds with boundary

In this section, we develop a deformation tool and apply it to give the details of the proof of Lemma 2, which concerns deforming non-intersecting slices to uniformly control their masses. This section is organized as follows: first, in Subsection 3.2.1, we state the deformation result. Then, in Subsection 3.2.2 we prove a Morse-type theorem that is applied in the proof of the deformation lemma. This lemma is proved in Subsection 3.2.3. Finally, in Subsection 3.2.4, it is applied in the proof of Lemma 2.

3.2.1 Deforming with controlled masses

In this subsection, we simply state the natural deformation tool that we develop to control the masses of non-intersecting slices.

Lemma 6. *Let* $(\tilde{M}^n, g)$ *be a compact Riemannian manifold with smooth boundary* $\partial\tilde{M}$. *There exists* $C = C(\tilde{M}) > 0$ *with the following property: given* $A \in \boldsymbol{I}_n(\tilde{M})$ *such that* $\boldsymbol{M}(A) + \boldsymbol{M}(\partial A) < \infty$, *we can find a map*

$$\phi : [0, 1] \to Z_{n-1}(\tilde{M})$$

continuous in the flat topology such that

(i) $\phi(0) = \partial A$ *and* $\phi(1) = 0$;

(ii) $\boldsymbol{M}(\phi(t)) \leq C \cdot \boldsymbol{M}(\partial A)$, *for every* $t \in [0, 1]$.

Moreover, if $spt(||A||) \subset int(\tilde{M})$, *there exists a compact subset* $K \subset int(\tilde{M})$, *such that* $spt(||\phi(t)||) \subset K$, *for every* $t \in [0, 1]$.

Remark 10. The construction makes clear that ϕ has no concentration of mass, i.e., $\limsup_{r\to 0} \sup\{||\Phi(x)||(B(p, r)) : x \in I^m, p \in \tilde{M}\} = 0$.

3.2.2 A Morse type theorem

Let (M^n, g) be a compact Riemannian manifold with smooth non-empty boundary ∂M and $f : M \to [0,1]$ be a Morse function on M, such that $f^{-1}(1) = \partial M$ and with no interior local maximum. The fundamental theorems in Morse theory describe how the homotopy type of the sublevel sets $M^a = \{x \in M : f(x) \leq a\}$ change as $a \in [0,1]$ varies, see Theorems 3.1 and 3.2 in [41]. In this section we develop a slightly different approach to those results. It is important in the proof of Lemma 6, which is a key ingredient for producing sweepouts with uniformly controlled non-intersecting slice, recall the discussion in Chapter 2.

Theorem 13. *Let $0 \leq c < d \leq 1$, $\lambda \in \{0, 1, \ldots, n-1\}$ and $p \in f^{-1}(c)$ be an index λ critical point of f. Suppose $\varepsilon > 0$ is such that $f^{-1}([c-\varepsilon, d])$ contains no critical points other than p. Then, for sufficiently small $\varepsilon > 0$, there exists a smooth homotopy*

$$h : [0,1] \times M^d \to M^d,$$

with the following properties:

1. *$h(1, \cdot)$ is the identity map of M^d;*
2. *$h(0, M^d)$ is contained in $M^{c-\varepsilon}$ with a λ-cell attached.*

Remark 11. The difference of the above statement and the classical ones, is that here we are able to guarantee that the homotopy is smooth by relaxing the condition of $h(1, \cdot)$ being a retraction onto the whole $M^{c-\varepsilon}$ with the λ-cell attached.

Proof. By the classical statements 3.1 and 3.2 in [41], we know that there exists a smooth homotopy between the identity map of M^d and a retraction of M^d onto $M^{c-\varepsilon} \cup H$, i.e., the sublevel set with a handle H attached. It is also observed that this set has smooth boundary. The final argument in the proof of Theorem 3.2 in Milnor's book is a vertical projection of the handle onto $M^{c-\varepsilon} \cup e^\lambda$, where e^λ is a λ-cell contained in H. See diagram 7, on page 19 of [41]. This projection is not adequate for us because it is not smooth. We adapt this step in that proof by defining a smooth projection.

Following the notation in [41], let $u_1, \ldots, u_n$ be a coordinate system in a neighborhood U of p so that the identity

$$f = c - (u_1^2 + \ldots + u_\lambda^2) + (u_{\lambda+1}^2 + \ldots + u_n^2), \tag{3.1}$$

holds throughout U. We use $\xi = u_1^2 + \ldots + u_\lambda^2$ and $\eta = u_{\lambda+1}^2 + \ldots + u_n^2$. For sufficiently small $\varepsilon > 0$, the λ-cell e^λ can be explicitly given by the points in U with $\xi \leq \varepsilon$ and $\eta = 0$. Consider $\delta > 0$ so that the image of U by the coordinate system contains the set of points with $\xi \leq \varepsilon + \delta$ and $\eta \leq \delta$. Let $\phi : (0, +\infty) \to (0, +\infty)$ be a function such that $\phi \in C^\infty(0, +\infty)$ and:

(a) $\phi(\xi) = 0$, if $\xi \leq \varepsilon$;

(b) $\phi(\xi) = \xi - \varepsilon$, if $\varepsilon + \delta \leq \xi$;

(c) $\phi(\xi) \leq \xi - \varepsilon$, for all $\xi \in (0, +\infty)$.

With these choices, we are able to redefine the projection for points classified as case 2 on page 19 of [41], i.e., $\varepsilon \leq \xi \leq \eta + \varepsilon$. Consider

$$(t, u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_\lambda, s_t u_{\lambda+1}, \ldots, s_t u_n),$$

where the number $s_t \in [0, 1]$ is defined by

$$s_t = t + (1 - t)\sqrt{\frac{\phi(\xi)}{\eta}}.$$

This map is smooth for points in case 2 because now the set $\phi(\xi) = \eta$ meets the boundary of e^λ smoothly. By (c), we see that the image of each point at time $t = 0$ is inside $M^{c-\varepsilon} \cup e^\lambda$. Using this new projection, the statement follows via the same program as in the proof of Theorem 3.2 in [41]. □

3.2.3 Constructing the deformation

Lemma 6 is about the construction of natural deformations starting with the boundary of a fixed n-dimensional integral current A in a compact manifold with boundary, and contracting it continuously and with controlled masses to the zero current. This generalizes the notion of cones in $\mathbb{R}^n$. Aiming the proof of that lemma, we developed a modified Morse-type Theorem that we discuss now.

Proof of Lemma 6. To make the notation simpler, in this proof we use M instead of $\tilde{M}$ to denote the manifold with boundary.

Step 1**:**

Consider a Morse function $f : M \to [0, 1]$ with $f^{-1}(1) = \partial M$ and no interior local maximum. Let $C(f) = \{p_1, \ldots, p_k\} \subset M$ be the critical set of f, with $c_i = f(p_i)$ and $\text{index}(f, p_i) = \lambda_i \in \{0, 1, \ldots, n-1\}$. Suppose, without loss of generality, $0 = c_k < \cdots < c_1 < 1$.

We adapt Morse's fundamental theorems to construct a list of homotopies

$$h_i : [0,1] \times M_i \to M_i, \quad \text{for } i = 1, \dots, k,$$

defined on sublevel sets $M_1 = M$ and $M_i = M^{c_{i-1}-\varepsilon_{i-1}} = \{f \leq c_{i-1} - \varepsilon_{i-1}\}$, for $i = 2, \dots, k$, with sufficiently small $\varepsilon_i > 0$. Those maps are constructed in such a way we have the following properties:

- h_i is smooth for all $i = 1, \dots, k$;
- $h_i(1, \cdot)$ is the identity map of M_i;
- $h_i(0, M_i)$ is contained in M_{i+1} with a λ_i-cell attached, if $i \leq k-1$;
- $h_k(0, M_k) = f^{-1}(0)$.

This is achieved by Theorem 13, in Subsection 3.2.2.

Step 2:

Recall $A \in \mathbf{I}_n(M)$ and $h_0(0, M)$ is contained in the union of M_1 with a λ_1-cell. Since $\lambda_1 < n$, the support of $A^1 = h_1(0, \cdot)_\# A$ is a subset of M_2. Indeed, it is an integral n-dimensional current and the λ_1-cell has zero n-dimensional measure. Inductively, we observe:

$$A^i := h_i(0, \cdot)_\# A^{i-1} \text{ has support in } M_{i+1}, \text{ for } i = 2, \dots, k,$$

with the additional notation $M_{k+1} = f^{-1}(0)$. In particular, this allows us to concatenate the images of A by the sequence of homotopies. Consider

$$\Phi : [0,1] \to \mathbf{I}_n(M)$$

starting with $\Phi(0) = A$ and inductively defined by

$$\Phi(t) = h_i(i - kt, \cdot)_\# \Phi((i-1)/k),$$

for $t \in [(i-1)/k, i/k]$ and $i = 1, \dots, k$.

Step 3:

The homotopy formula, equation (1.17) of Chapter 1, tells us

$$\Phi(t) - \Phi(s) = (h_i)_\# \left([|(i - ks, i - kt)|] \times \partial\Phi((i-1)/k) \right), \qquad (3.2)$$

for $(i-1)/k \leq t \leq s \leq i/k$. In fact, the boundary term vanishes,

$$\partial (h_i)_\# \left([|(i - ks, i - kt)|] \times \Phi((i-1)/k) \right) = 0,$$

because $[|(i-ks, i-kt)|] \times \Phi((i-1)/k)$ is $(n+1)$-dimensional, while h_i takes values in a n-dimensional space.

Step 4:

The mass of the product current $[|(i-ks,i-kt)|]\times\partial\Phi((i-1)/k)$ is the product measure of Lebesgue's measure on $(i-ks,i-kt)$ and the mass of $\partial\Phi((i-1)/k)$. Since all homotopies $h_1,\ldots,h_k$ are smooth, we conclude that Φ is continuous in the mass norm and that there exists a positive constant $C=C(M)>0$ such that

$$\mathbf{M}(\partial\Phi(t))\leq C\cdot\mathbf{M}(A),\quad \text{for all } t\in[0,1].$$

Step 5:

Consider ϕ being the boundary map applied to Φ, i.e., $\phi(t)=\partial\Phi(t)$. Since Φ is continuous in the mass topology, it follows directly ϕ is continuous in the flat metric. Moreover, all maps $h_i(t,\cdot)$ with $t>0$ are diffeomorphisms, this implies ϕ is continuous in the F-metric, up to finite points.

The last claim in the statement of Lemma 6 follows from construction, because $\mathrm{spt}(||A||)\subset\mathrm{int}(\tilde{M})$ implies there exists a small $\varepsilon>0$ such that $\mathrm{spt}(||A||)$ is contained in the sublevel set $K:=\{f\leq 1-\varepsilon\}$. □

3.2.4 Controlling the non-intersecting slices

Finally, we present the proof of Lemma 2. As a matter of reading convenience, let us precisely restate the result that we are going to prove. In our main setting, Lemma 2 says that there exists $C=C(M,\Omega)>0$ with the following property: given a discrete map

$$\phi: I(1,k)_0\to\mathcal{Z}_{n-1}(M)$$

of small fineness, we can find

$$\tilde{\phi}: I(1,\tilde{k})_0\to\mathcal{Z}_{n-1}(M)$$

such that:

(a) $\tilde{\phi}$ is 1-homotopic to ϕ with fineness $C\cdot\mathbf{f}(\phi)$;

(b) $\phi(\mathrm{dmn}_\Omega(\phi))=\tilde{\phi}(\mathrm{dmn}_\Omega(\tilde{\phi}))$;

(c)

$$\max_{x\in\mathrm{dmn}(\tilde{\phi})}\mathbf{M}(\tilde{\phi}(x))\leq C\cdot\left(\max\{\mathbf{M}(\phi(x)):x\in\mathrm{dmn}_\Omega(\phi)\}+\mathbf{f}(\phi)\right).$$

Proof of Lemma 2. Consider a discrete map $\psi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ such that $\mathbf{f}(\psi) \leq \min\{\delta_0, \mathcal{H}^n(\Omega)\}$. Recall the choice of δ_0 in Section 3.1. We replace ψ with another discrete map of small fineness. Observe, for each $v \in I(1,k)_0 - \{1\}$, we can find the isoperimetric choice $A(x) \in \mathbf{I}_n(M)$ of $\psi(v+3^{-k})-\psi(v)$, i.e., $\partial A(v) = \psi(v+3^{-k})-\psi(v)$ and $\mathbf{M}(A(v)) = \mathcal{F}(\partial A(v))$.

Consider $x, y \in I(1,k)_0$ with the following properties:

- $[x,y] \cap \mathrm{dmn}_\Omega(\psi) = \varnothing$;
- $x - 3^{-k}$ and $y + 3^{-k}$ belong to $\mathrm{dmn}_\Omega(\psi)$.

Observe that, for every $z \in [x, y-3^{-k}] \cap I(1,k)_0$, we have

$$\mathrm{spt}(||A(z)||) \subset M - \Omega. \tag{3.3}$$

In fact, observe $\psi(z)$ and $\psi(z+3^{-k})$ have zero mass in $\overline{\Omega}$ and $\psi(z+3^{-k}) = \psi(z)+\partial A(z)$, so, the possibilities are either $\mathrm{spt}(||A(z)||)$ contains Ω or do not intersect it. The first case is not possible because $A(z)$ is an integral current with mass smaller than $\mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega)$. Next, we explain how ψ is modified on each such $[x,y]$, which is called a maximal interval of non-intersecting slices. Observe that $\mathrm{spt}(||\psi(x)||) \cup \mathrm{spt}(||\psi(y)||) \subset M - \overline{\Omega}$. Let $\tilde{M} = M - \Omega$ and consider the integral current

$$A = \sum A(z) \in \mathbf{I}_n(\tilde{M}),$$

where we sum over all $z \in [x, y-3^{-k}] \cap I(1,k)_0$. Note that $\partial A = \psi(y) - \psi(x)$ and that $\mathrm{spt}(||A||)$ do not intersect $\partial\tilde{M}$. Let $\phi : [0,1] \to \mathcal{Z}_{n-1}(\tilde{M})$ be the map obtained via Lemma 6 applied to the chosen A.

To produce a discrete map, we have to discretize ϕ. Since we are dealing with a one-parameter map, we can directly apply Lemma 5. Consider a compact set $K \subset M - \overline{\Omega} =: U$, such that

$$\mathrm{spt}(||\phi(t)||) \subset K, \quad \text{for all } t \in [0,1]. \tag{3.4}$$

This is related to the extra property of ϕ that we have in Lemma 6, because $\mathrm{spt}(||A||) \subset M - \overline{\Omega}$. Following the notation of Lemma 5, choose $\varepsilon_0 > 0$ making that statement work with $L = C \cdot \mathbf{M}(\partial A)$, $\eta \leq \mathbf{f}(\psi)$, $T = 0$, and the fixed $K \subset U$. The constant $C = C(M, \Omega) > 0$ is the one in Lemma 6. Let $k_1 \in \mathbb{N}$ be large enough, so that $\phi_1 := \phi|_{I(1,k_1)_0}$ is ε_0-fine in the flat topology. In this case, for all $\theta \in I(1,k_1)_0$, we have:

- $\mathcal{F}(\phi_1(\theta + 3^{-k_1}) - \phi(\theta)) \leq \varepsilon_0$;
- $\mathrm{spt}(\phi(\theta)) \subset K$;

- $\mathbf{M}(\phi(\theta)) \leq L = C \cdot \mathbf{M}(\partial A)$.

Lemma 5 says that we can take $\tilde{k} \geq k_1$, so that ϕ_1 admits extension $\tilde{\phi}$ to $I(1, \tilde{k})_0$ with fineness $\mathbf{f}(\tilde{\phi}) \leq \eta \leq \mathbf{f}(\psi)$, $\text{spt}(\tilde{\phi}(\theta)) \subset M - \overline{\Omega}$ and with uniformly controlled masses

$$\mathbf{M}(\tilde{\phi}(\theta)) \leq L + \eta \leq C \cdot \mathbf{M}(\partial A) + \mathbf{f}(\psi).$$

We replace $\psi|_{[x,y]\cap I(1,k)_0}$ with a discrete map defined in a finer grid

$$\tilde{\psi} : I(1, k + \tilde{k})_0 \cap [x, y] \to \mathcal{Z}_{n-1}(M),$$

such that

$$\tilde{\psi}(w) = \psi(y) - \tilde{\phi}((w - x) \cdot 3^k),$$

if $w \in I(1, k + \tilde{k})_0 \cap [x, x + 3^{-k}]$, and $\tilde{\psi}(w) = \psi(y)$, otherwise. This map has the same fineness as $\tilde{\phi}$, and so $\mathbf{f}(\tilde{\psi}) \leq \mathbf{f}(\psi)$. Also, $\tilde{\psi}$ and ψ agree in x and y,

$$\tilde{\psi}(x) = \psi(y) - \tilde{\phi}(0) = \psi(y) - \partial A = \psi(x) \text{ and } \tilde{\psi}(y) = \psi(y).$$

Moreover, no slice $\tilde{\psi}(w)$ intersects $\overline{\Omega}$ and

$$\begin{aligned} \mathbf{M}(\tilde{\psi}(w)) &\leq \mathbf{M}(\psi(y)) + C \cdot \mathbf{M}(\partial A) + \mathbf{f}(\psi) \\ &\leq C \cdot (\max\{\mathbf{M}(\psi(z)) : z \in \text{dmn}_\Omega(\psi)\} + \mathbf{f}(\psi)) . \end{aligned} \tag{3.5}$$

The second line is possible, because $x - 3^{-k}$ and $y + 3^{-k}$ belong to $\text{dmn}_\Omega(\psi)$. Also, the constant appearing in the last inequality is bigger than the original, but still depending only on M and Ω.

Observe one can write the original ψ restricted to $[x, y] \cap I(1, k)_0$, similarly to the expression that defines $\tilde{\psi}$, as $\psi(w) = \psi(y) - \hat{\phi}(w/3)$, where

$$\hat{\phi} : I(1, k + 1)_0 \to \mathcal{Z}_{n-1}(M)$$

is given by

- $\hat{\phi}(\theta) = \partial A$, if $\theta \in [0, x/3] \cap I(1, k + 1)_0$;
- $\hat{\phi}(\theta) = \psi(y) - \psi(3\theta)$, if $\theta \in [x/3, y/3] \cap I(1, k + 1)_0$;
- $\hat{\phi}(\theta) = 0$, if $\theta \in [y/3, 1] \cap I(1, k + 1)_0$.

We concatenate the inverse direction of $\tilde{\phi}$ with $\hat{\phi}$ to construct

$$\overline{\phi} : I(1, \overline{k} + 1)_0 \to \mathcal{Z}_{n-1}(M), \tag{3.6}$$

where $\overline{k} = \max\{\tilde{k}, k + 1\}$, and

$$\overline{\phi}(\theta) = \begin{cases} \tilde{\phi} \circ \mathbf{n}(\overline{k}, \tilde{k})(1 - 3\theta) & \text{if } 0 \leq \theta \leq 3^{-1} \\ \hat{\phi} \circ \mathbf{n}(\overline{k}, k + 1)(3\theta - 1) & \text{if } 3^{-1} \leq \theta \leq 2 \cdot 3^{-1} \\ 0 & \text{if } 2 \cdot 3^{-1} \leq \theta \leq 1. \end{cases}$$

Claim 1. *$\boldsymbol{f}(\overline{\phi}) \leq \boldsymbol{f}(\psi)$ and $\overline{\phi}$ is 1-homotopic to zero in $(\mathcal{Z}_{n-1}(M;\boldsymbol{M}), \{0\})$ with fineness $C_1 \cdot \boldsymbol{f}(\psi)$.*

The constant $C_1 > 0$ is uniform, in the sense that it does not depend on ψ. The proof of this claim finishes the argument, because the discrete homotopy between $\overline{\phi}$ and the zero map tells us that the initial map ψ and the $\tilde{\psi}$ we built are 1-homotopic with the same fineness. Claim 1 is a consequence of Almgren's Isomorphism. □

3.3 Ruling out small intersecting mass

In this section we develop deformations of objects that enter a region $\Omega \subset M$ with small mass. This tool plays a key role in the present work. We divide the section in three parts, first we discuss how to deform single currents with that small area property and what properties this deformation must satisfy. This is done in Subsection 3.3.1. Then, in Subsection 3.3.2, we apply the method to deform sweepouts whose big intersecting slices have the small-area property. Lemma 9 is the main result that is proven in this part. We end the section with Subsection 3.3.3, in which we make the first application of Lemma 9. Namely, we discretize continuous sweepouts minding the intersecting property to prove Theorem 9 of Chapter 2.

3.3.1 Deforming single currents

Let (M^n, g) be a compact Riemannian manifold isometrically embedded in the Euclidean space $\mathbb{R}^L$. Consider open subsets $W \subset\subset U$ in M. The goal of this section is to prove that it is possible to deform an integral current T with small mass in U to a current T^* outside W. The deformation is discrete, with support in U, arbitrarily small fineness and the masses along the deformation sequence can not increase much. In this part, the mean-concavity is not used and the statements do not involve Ω.

Lemma 7. *There exists $\varepsilon_1 > 0$ with the following property: given*

$$T \in \mathcal{Z}_{n-1}(M^n) \text{ with } ||T||(U) < \varepsilon_1 \text{ and } \eta > 0,$$

it is possible to find a sequence $T = T_1, \ldots, T_q \in \mathcal{Z}_{n-1}(M^n)$ such that

(1) $spt(T_l - T) \subset U$;

(2) $\boldsymbol{M}(T_l - T_{l-1}) \leq \eta$;

(3) $\boldsymbol{M}(T_l) \leq \boldsymbol{M}(T) + \eta$;

(4) $spt(T_q) \subset M - \overline{W}$.

Remark 12. The constant ε_1 depends only on M and $d_g(W, M-U)$.

The main ingredients to prove this result are: the key element is a discrete deformation process used by Pitts in the construction of replacements, see section 3.10 in [47], and the monotonicity formula, Section 1.3.

Lemma 8 (Pitts' deformation argument)**.** *Let $K \subset U$ be subsets of M, with K compact and U open. Given $T \in \mathcal{Z}_k(M)$ and $\eta > 0$, there exists a finite deformation $T = T_1, \dots, T_q \in \mathcal{Z}_k(M)$ satisfying items (1), (2), (3) of the statement of Lemma 7 and with the additional properties that T_q is locally area-minimizing in int(K) and $\boldsymbol{M}(T_q) \leq \boldsymbol{M}(T)$.*

The idea of this lemma is to consider all finite fine deformations of T and minimize the mass of the last slice T_q. This can not be done directly because the space of integral currents of bounded masses is compact in the weak topology, but not in the mass norm. Then, the proof is divided in two steps, first we consider the minimization process among lists which are fine with respect to the flat metric. Secondly, we apply interpolation arguments to achieve the desired fineness.

Proof of Lemma 8. Fixed $\varepsilon > 0$, consider all lists $T = S_1, \dots, S_q \in \mathcal{Z}_k(M)$, with the following properties:

(a) $\text{spt}(S_l - T) \subset K$

(b) $\mathcal{F}(S_l - S_{l-1}) \leq \varepsilon$

(c) $\mathbf{M}(S_l) \leq \mathbf{M}(T) + \varepsilon$.

Consider $M(T, K, \varepsilon) := \inf \mathbf{M}(S_q)$, where the infimum is taken among all finite sequences as above (q is allowed to vary). The interesting remark is that the infimum is attained by a finite sequence and the last integral current S_q of an optimal list is locally area-minimizing in the interior points of K.

Indeed, consider a minimizing sequence of lists:

$$T = S_1^j, S_2^j, \dots, S_{q_j}^j \in \mathcal{Z}_k(M), \quad j = 1, 2, \dots$$

such that $\lim_j \mathbf{M}(S_{q_j}^j) = M(T, K, \varepsilon)$. Observe $\mathbf{M}(S_{q_j}^j) \leq \mathbf{M}(T) + \varepsilon$, for all j. Then, by the compactness theorem for integral currents, Theorem 27.3 in [51], we can suppose that $S_{q_j}^j$ weakly converge, as currents, to an integral current T'. Since we are dealing with a sequence of integer multiplicity cycles of uniformly bounded masses, we have that $\mathcal{F}(S_{q_j}^j, T')$ tends to zero, see Theorem 31.2 in [51]. For sufficiently large j, consider the sequence $T =$

$S_1^j, S_2^j, \ldots, S_{q_j}^j, S_{q_j+1}^j = T'$. It satisfy (a), (b), (c) and $\mathbf{M}(S_{q_j+1}^j) = M(T, K, \varepsilon)$. In other words, the infimum is attained.

Since we can take the constant list $S_l = T$, we easily see that $\mathbf{M}(S_q) \leq \mathbf{M}(T)$, where S_q is the last current of any optimal list.

Let $p \in \text{int}(K)$ and choose $r > 0$ such that $B(p, r) \subset \text{int}(K)$. Since S_q is an integral current, we can make r small enough in such a way that $||S_q||(B(p, r)) < \varepsilon/2$. Then, if $\mathbf{M}(S) < \mathbf{M}(S_q)$ for some $S \in \mathcal{Z}_k(M)$ with $\text{spt}(S - S_q) \subset B(p, r)$, the sequence $T = S_1, S_2, \ldots, S_q, S$, would still have the properties (a) to (c) and $\mathbf{M}(S) < \mathbf{M}(S_q) = M(T, K, \varepsilon)$. But this is a contradiction.

For small ε, we modify the obtained $\{S_l\}_{l=1}^q$ using Lemma 5, to construct a new sequence with small fineness in the mass norm. Following Lemma 5, choose $\varepsilon_0 = \varepsilon_0(\mathbf{M}(T) + \eta/2, \eta/2, K, U, T)$. Consider $0 < \varepsilon \leq \min\{\varepsilon_0, \eta/2\}$ and a sequence $\{S_l\}_{l=0}^q$, satisfying (a), (b), (c) and $\mathbf{M}(S_q) = M(T, K, \varepsilon)$. For each $l = 2, \ldots, q$, we have

- $S_{l-1}, S_l \in \mathcal{Z}_k(M)$
- $\mathcal{F}(S_l - S_{l-1}) \leq \varepsilon \leq \varepsilon_0$
- $\text{spt}(S_{l-1} - T) \cup \text{spt}(S_l - T) \subset K$
- $\mathbf{M}(S_{l-1})$ and $\mathbf{M}(S_l) \leq \mathbf{M}(T) + \varepsilon \leq \mathbf{M}(T) + \eta/2$.

Interpolate to obtain $S_{l-1} = T_0^l, T_1^l, \ldots, T_{m(l)}^l = S_l \in \mathcal{Z}_k(M)$, such that

$$\text{spt}(T_j^l - T) \subset U, \quad \mathbf{M}(T_j^l - T_{j-1}^l) \leq \eta/2 \leq \eta \text{ and } \mathbf{M}(T_j^l) \leq \mathbf{M}(T) + \eta.$$

Putting all the T_j^l lists together, we conclude the argument. □

The second tool that we use in the proof of Lemma 7, the monotonicity formula, says that there are $C, r_0 > 0$, such that for any minimal submanifold $\Sigma^k \subset M$ and $p \in \Sigma$, we have

$$\mathcal{H}^k(\Sigma \cap B(p, r)) \geq Cr^k, \quad \text{for all } 0 < r < r_0.$$

This follows from expression (1.25) of Section 1.3. Now, we proceed to the proof of the small mass deformation lemma.

Proof of 7: Consider a compact subset K with $W \subset\subset K \subset U$, and let

$$T \in \mathcal{Z}_{n-1}(M^n) \text{ with } ||T||(U) < \varepsilon_1 \text{ and } \eta > 0,$$

be given, $\varepsilon_1 > 0$ to be chosen. Apply Corollary 8 to these $K \subset U$, T and η, to obtain a special finite sequence $T = T_1, \ldots, T_q \in \mathcal{Z}_{n-1}(M)$. The items

(1) to (3) of Lemma 7 follow automatically. Since $\operatorname{spt}(T_q - T) \subset U$ and $\mathbf{M}(T_q) \leq \mathbf{M}(T)$, we have $||T_q||(U) \leq ||T||(U) \leq \varepsilon_1$.

Note that the induced varifold $|T_q|$ is stationary in $\operatorname{int}(K)$, being locally area-minimizing in this open set. Suppose T_q is not outside W, take $p \in W \cap \operatorname{spt}(T_q)$ and $0 < r = 2^{-1}\min\{r_0, d_g(W, M - K)\}$. Then, applying the monotonicity formula for stationary integral varifolds, Remark 6, we have

$$Cr^{n-1} \leq ||T_q||(B(p,r)) \leq ||T_q||(U) \leq \varepsilon_1.$$

In order to conclude the proof, we choose $\varepsilon_1 < Cr^{n-1}$. □

3.3.2 Discrete sweepouts

Let (M^n, g) be a closed embedded submanifold of $\mathbb{R}^L$ and $\Omega \subset M$ be an open subset with smooth and strictly mean-concave boundary $\partial\Omega$. Recall the domains $\Omega_a \subset \Omega \subset \Omega_b$ and the maximum principle vector field X that we considered in Subsection 1.2. Let $U \subset \Omega$ be an open subset such that $\overline{\Omega_a} \subset U$. In this section we consider discrete maps $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ with small fineness for which slices $\phi(x)$ with mass greater than a given $L > 0$ have small mass in U. The goal here is to extend the construction of Subsection 3.3.1 and deform the map ϕ via a 1-homotopy with small fineness.

In order to state the precise result, consider the constants: $C_0 = C_0(M)$ and $\delta_0 = \delta_0(M)$ as introduced in Section 3.1, and $\varepsilon_1(U, \Omega_a)$ as given by Lemma 7 for $W = \Omega_a$ and the fixed U. Consider also the following combinations

- $(3 + C_0)\eta_0 = \varepsilon_2 = \min\{\varepsilon_1(U, \Omega_a), 5^{-1}\delta_0, 5^{-1}\mathcal{H}^n(\Omega_a)\}$;
- $C_1 = 3C_0 + 7$.

Observe that $C_1 = C_1(M)$, but ε_2 and η_0 depend also on U and Ω_a.

Assume we have a discrete map $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ with fineness $\mathbf{f}(\phi) \leq \eta_0$ and satisfying the property that, for some $L > 0$,

$$\mathbf{M}(\phi(x)) \geq L \Rightarrow ||\phi(x)||(\overline{U}) < \varepsilon_2. \tag{3.7}$$

Lemma 9. *There exists $\tilde{\phi} : I(1,N)_0 \to \mathcal{Z}_{n-1}(M)$ 1-homotopic to ϕ with fineness $C_1 \cdot \mathbf{f}(\phi)$, with the following properties:*

$$L(\tilde{\phi}) = \max\{\mathbf{M}(\tilde{\phi}(x)) : x \in dmn_\Omega(\tilde{\phi})\} < L + C_1 \cdot \mathbf{f}(\phi)$$

and such that the image of $\tilde{\phi}$ is equal to the union of $\{\phi(x) : \mathbf{M}(\phi(x)) < L\}$ with a subset of $\{T \in \mathcal{Z}_{n-1}(M) : ||T||(\overline{U}) \leq 2\varepsilon_2\}$.

Remark 13. If $0 < \varepsilon \leq \varepsilon_2$ and we assume that $||\phi(x)||(\overline{U}) < \varepsilon$, in (3.7), we can still apply the lemma. Moreover, the image of $\tilde{\phi}$ will coincide with $\{\phi(x) : \mathbf{M}(\phi(x)) < L\}$ up to slices with the property $||\tilde{\phi}(x)||(\overline{U}) \leq \varepsilon + C_1 \cdot \eta$.

Overwiew of the proof of Lemma 9

Deforming each big slice using the small mass procedure we obtain a map ψ defined in a 2-dimensional grid. The first difficulty that arises is that the obtained map has fineness of order ε_2 instead of $\mathbf{f}(\phi)$. We correct this using the interpolation results, see Section 3.1. The fine homotopy that we are able to construct ends with a discrete map $\tilde{\phi}$ whose intersecting slices $\tilde{\phi}(x)$ with mass exceeding L by much do not get very deep in Ω, i.e., $\text{spt}(\tilde{\phi}(x) \llcorner \Omega)$ is contained in a small tubular neighborhood of $\partial\Omega$ in M.

The second part is the application of the maximum principle. The idea is to continuously deform the slices supported in $M - \Omega_a$ of the map $\tilde{\phi}$, produced in the first step, via the flow $\{\Phi(s, \cdot)\}_{s\geq 0}$ of the maximum principle vector field, see Corollary 2. Since $\Phi(1, M - \Omega_a) \subset M - \overline{\Omega_b}$, the bad slices $\tilde{\phi}(x)$ end outside $\overline{\Omega_b}$. But this deformation is continuous only with respect to the F-metric and we need a map with small fineness in the mass norm. This problem is similar to the difficulty that arises in the classical pull-tight argument. We produce then a discrete version of the maximum principle deformation that is arbitrarily close, in the F-metric, to the original one. This correction creates one more complication, because the approximation can create very big intersecting slices with small mass in Ω_b. To overcome this we apply the small mass procedure again.

Proof of Lemma 9. Consider the set

$$\mathcal{K} = \{x \in I(1, k)_0 : \mathbf{M}(\phi(x)) \geq L\}.$$

Let $\alpha, \beta \in \mathcal{K}$. A subset $[\alpha, \beta] \cap I(1, k)_0$ is called a maximal interval on $\mathcal{K}$ if $[\alpha, \beta] \cap I(1, k)_0 \subset \mathcal{K}$ and $\alpha - 3^{-k}, \beta + 3^{-k} \notin \mathcal{K}$.

We describe the construction of the homotopy on each maximal interval. Observe that for every $x \in [\alpha, \beta] \cap I(1, k)_0$, we have $||\phi(x)||(\overline{U}) < \varepsilon_2 \leq \varepsilon_1$.

Let $N_1 = N_1(\phi)$ be a positive integer so that, for each $x \in [\alpha, \beta] \cap I(1, k)_0$, we can apply Lemma 7 to $W = \Omega_a \subset\subset U$ and find sequences

$$\phi(x) = T(0, x), T(1, x), \dots, T(3^{N_1}, x) \in \mathcal{Z}_{n-1}(M),$$

with fineness at most $\eta = \mathbf{f}(\phi)$, controlled supports and masses

$$\text{spt}(T(l, x) - \phi(x)) \subset U \quad \text{and} \quad \mathbf{M}(T(l, x)) \leq \mathbf{M}(\phi(x)) + \eta, \tag{3.8}$$

and ending with an integral cycle $T(3^{N_1}, x))$ whose support is contained in $M - \overline{\Omega_a}$. Observe that $||\phi(x)||(\Omega) = 0$ imply that $T(l, x)$ is constant $\phi(x)$. Then, we perform the first step of the deformation. Consider

$$\psi : I(1, N_1)_0 \times ([\alpha, \beta] \cap I(1, k)_0) \to \mathcal{Z}_{n-1}(M)$$

defined by $\psi(l,x) = T(l \cdot 3^{N_1}, x)$, for every $x \in [\alpha,\beta] \cap I(1,k)_0$ and $l \in I(1,N_1)_0$. It follows directly from the construction that the map ψ satisfies:

$$\sup\{\mathbf{M}(\psi(l,x)) : l \in I(1,N_1)_0 \text{ and } x \in \{\alpha,\beta\}\} \leq L + 2\eta, \tag{3.9}$$

and

$$\operatorname{spt}(||\psi(1,x)||) \subset M - \overline{\Omega_a}, \quad \text{for every } x \in [\alpha,\beta] \cap I(1,k)_0. \tag{3.10}$$

It is also an easy fact that

$$\mathbf{M}(\psi(3^{-N_1},x) - \psi(3^{-N_1}, x + 3^{-k})) \leq 3\eta, \tag{3.11}$$

for $x \in [\alpha, \beta - 3^{-k}] \cap I(1,k)_0$. Observe that (3.8) implies that

$$\sup\{||\psi(l,x)||(\overline{U}) : l \in I(1,N_1)_0\} \leq ||\phi(x)||(\overline{U}) + \eta \leq \varepsilon_2 + \eta, \tag{3.12}$$

for every x, and

$$||\psi(1,x)||(\overline{U}) \leq ||\phi(x)||(\overline{U}) \leq \varepsilon_2. \tag{3.13}$$

Moreover, we check that

$$\mathbf{f}(\psi) \leq 3\eta + 4\varepsilon_2 \leq 5\varepsilon_2 \leq \min\{\delta_0, \mathcal{H}^n(\Omega_a)\}. \tag{3.14}$$

We have to prove that $\mathbf{M}(\psi(l,x) - \psi(l,x')) \leq 5\varepsilon_2$, always that $|x - x'| \leq 3^{-k}$. Indeed, rewrite that difference as

$$(\psi(l,x) - \phi(x)) + (\phi(x) - \phi(x')) + (\phi(x') - \psi(l,x')). \tag{3.15}$$

The first and third terms are similar and, if they are non-zero, their analysis follow the steps: $\mathbf{M}(\psi(l,x) - \phi(x)) = ||\psi(l,x) - \phi(x)||(U)$, because $\psi(l,x)$ is constructed in such a way that $\operatorname{spt}(\psi(l,x) - \phi(x)) \subset U$ and

$$\begin{aligned} ||\psi(l,x) - \phi(x)||(U) &\leq ||\psi(l,x)||(U) + ||\phi(x)||(U) \\ &\leq \eta + 2||\phi(x)||(U) \\ &< \eta + 2\varepsilon_2. \end{aligned}$$

In the above estimate, we have used only (3.12) and that $||\phi(x)||(U) < \varepsilon_2$. The mass of the second term of (3.15) is at most η, then

$$\mathbf{M}(\psi(l,x) - \psi(l,x')) \leq 3\eta + 4\varepsilon_2 \leq 5\varepsilon_2.$$

The last inequality follows from the special choices of $\eta \leq \eta_0$ and $3\eta_0 \leq \varepsilon_2$.

In order to get the desired fineness on the final 1-homotopy, we have to interpolate. For each $l \in I(1,N_1)_0 - \{0,1\}$ and $x \in [\alpha, \beta - 3^{-k}] \cap I(1,k)_0$, we apply Theorem 11 to the restriction of ψ to the four corner vertices of $[l, l+3^{-N_1}] \times [x, x+3^{-k}]$. This is allowed because of the expression (3.14). The result of this step is a continuous map in the mass norm

$$\Psi : [3^{-N_1}, 1] \times [\alpha, \beta] \to \mathcal{Z}_{n-1}(M), \tag{3.16}$$

that extends ψ. Moreover, in the 1-cells of the form $[l, l+3^{-N_1}] \times \{x\}$, the interpolating elements $\Psi(s, x)$ differ from $\psi(l, x)$ or $\psi(l+3^{-N_1}, x)$ in the mass norm at most by a factor of $C_0\mathbf{M}(\psi(l, x) - \psi(l+3^{-N_1}, x)) \leq C_0\eta$. This remark implies that, for $s \in [3^{-N_1}, 1]$ and $x \in \{\alpha, \beta\}$, we have

$$\mathbf{M}(\Psi(s, x)) \leq L + (2 + C_0)\eta, \tag{3.17}$$

and, together with expression (3.12), for the same s and x, we arrive at

$$||\Psi(s, x)||(\overline{U}) \leq (\varepsilon_2 + \eta) + C_0\eta. \tag{3.18}$$

Since $\Psi(1, \alpha) = \psi(1, \alpha)$, we have better estimates $\mathbf{M}(\Psi(1, \alpha)) \leq L + 2\eta$ and $||\Psi(1, \alpha)||(\overline{U}) \leq \varepsilon_2$, at time $s = 1$. The same holds for β. A similar assertion holds for the 1-cells $\{l\} \times [x, x+3^{-k}]$ and gives us

$$\mathbf{M}(\Psi(3^{-N_1}, t) - \psi(3^{-N_1}, x)) \leq 3C_0\eta, \tag{3.19}$$

for $x \in [\alpha, \beta] \cap I(1, k)_0$ and $t \in [\alpha, \beta]$ with $|t - x| \leq 3^{-k}$. Property (3.19) implies that restrictions of $\Psi(3^{-N_1}, \cdot)$ to any $[\alpha, \beta] \cap I(1, N)_0$ are 1-homotopic to the original ϕ with fineness at most $(3C_0 + 1)\mathbf{f}(\phi)$. Finally, since $\mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega_a)$ and $\text{spt}(||\psi(1, x)||) \subset M - \overline{\Omega_a}$, we can suppose that

$$\text{spt}(||\Psi(1, t)||) \subset M - \overline{\Omega_a}, \quad \text{for every } t \in [\alpha, \beta]. \tag{3.20}$$

Recall that $\{\Phi(s, \cdot)\}_s$ is the flow of the maximum principle vector field X and consider the map

$$(s, t) \in [1, 2] \times [\alpha, \beta] \mapsto \Phi(s-1, \cdot)_\#(\Psi(1, t)) =: \Psi(s, t).$$

This map is continuous in the F-metric, because each map $\Phi(s-1, \cdot)$ is diffeomorphism. Since $\Phi(s, \cdot)$ is a mass-decreasing flow, we have

$$\mathbf{M}(\Psi(s, t)) \leq \mathbf{M}(\Psi(1, t)) \leq \max\{\mathbf{M}(\Psi(1, t)) : t \in [\alpha, \beta]\} < \infty, \tag{3.21}$$

for every $t \in [\alpha, \beta]$ and $s \in [1, 2]$. In particular, the estimates (3.17) and (3.18) hold in $[3^{-N_1}, 2]$. Moreover, because of (1.20), we have

$$\text{spt}(||\Psi(2, t)||) \subset M - \overline{\Omega_b}, \quad \text{for every } t \in [\alpha, \beta]. \tag{3.22}$$

The final homotopy must be a fine discrete map defined on a 2d grid. In order to attain this, we interpolate once more, but now via Corollary 3. Let $0 < \varepsilon_3 \leq \varepsilon_1(\Omega_b, \Omega)$ be chosen in such a way that we can apply Lemma 7 with the sets Ω and Ω_b, and with $\varepsilon_3 \leq \eta$. Apply Corollary 3 for a sufficiently small δ, to obtain a number $N_2 = N_2(\phi) \geq N_1 + k$ and a discrete map

$$\Psi_1 : I(1, N_2)_0 \times ([\alpha, \beta] \cap I(1, N_2)_0) \to \mathcal{Z}_{n-1}(M),$$

such that

(i) $\mathbf{f}(\Psi_1) < \eta$;

(ii)

$$\sup\{||\Psi_1(\sigma, \tau)||(\overline{U}) : \sigma \in I(1, N_2)_0 \text{ and } \tau \in \{\alpha, \beta\}\} < \varepsilon_2 + (2 + C_0)\eta;$$

(iii)

$$\sup\{||\Psi_1(1, \tau))||(\overline{\Omega_b}) : \tau \in [\alpha, \beta] \cap I(1, N_2)_0\} < \varepsilon_3;$$

(iv)

$$\sup\{\mathbf{M}(\Psi_1(\sigma, x)) : \sigma \in I(1, N_2)_0 \text{ and } x \in \{\alpha, \beta\}\} \leq L + (3 + C_0)\eta;$$

(v) $\Psi_1(0, \tau) = \Psi(3^{-N_1}, \tau)$, if $\tau \in [\alpha, \beta] \cap I(1, N_2)_0$.

The next step is a second application of the Lemma 7, now for the slices $\Psi_1(1, \tau)$. The choice of ε_3 guarantee that there exists $N_3 = N_3(\phi) \in \mathbb{N}$ and an extension of Ψ_1 to the discrete domain

$$\text{dmn}(\Psi_1) = (I(1, N_2)_0 \cup (I(1, N_3)_0 + \{1\})) \times ([\alpha, \beta] \cap I(1, N_2)_0),$$

where $I(1, N_3)_0 + \{1\} = \{\lambda + 1 : \lambda \in I(1, N_3)_0\}$. This map has the following properties: in the same spirit as (3.14), we have, respectively,

$$\mathbf{f}(\Psi_1) \leq 3\eta + 4\varepsilon_3 \leq 7\eta. \tag{3.23}$$

Moreover, similarly to (3.9), (3.10) and (3.12), we have, respectively,

$$\mathbf{M}(\Psi_1(\sigma, \tau)) \leq L + (4 + C_0)\eta, \tag{3.24}$$

for all $(\sigma, \tau) \in \text{dmn}(\Psi_1)$ and $\tau \in \{\alpha, \beta\}$,

$$\text{spt}(||\Psi_1(2, \tau)||) \subset M - \overline{\Omega}, \tag{3.25}$$

for every $\tau \in ([\alpha, \beta] \cap I(1, N_2)_0)$, and

$$||\Psi_1(\sigma, \tau)||(\overline{U}) < \varepsilon_2 + (2 + C_0)\eta, \tag{3.26}$$

for all $(\sigma, \tau) \in \text{dmn}(\Psi_1)$ and $\tau \in \{\alpha, \beta\}$.

The last part is the organization of the homotopy. Take $N = N(\phi)$ sufficiently large such that it is possible to define a map

$$\Psi' : I(1, N)_0 \times ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1, N)_0) \to \mathcal{Z}_{n-1}(M)$$

in the following way:

- if $j = 0, 1, \ldots, 3^{N_2}$ and $\tau \in ([\alpha, \beta] \cap I(1, N)_0)$,

$$\Psi'(j \cdot 3^{-N}, \tau) = \Psi_1(j \cdot 3^{-N_2}, \mathbf{n}(N, N_2)(\tau));$$

- if $j = 0, 1, \ldots, 3^{N_3}$ and $\tau \in ([\alpha, \beta] \cap I(1, N)_0)$,

$$\Psi'(3^{-N+N_2} + j \cdot 3^{-N}, \tau) = \Psi_1(1 + j \cdot 3^{-N_3}, \mathbf{n}(N, N_2)(\tau));$$

- if $0 < \lambda_2 \leq \lambda_1 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N}$,

$$\Psi'(\lambda_1, \alpha - \lambda_2) = \Psi'(\lambda_1 - \lambda_2, \alpha) \text{ and } \Psi'(\lambda_1, \beta + \lambda_2) = \Psi'(\lambda_1 - \lambda_2, \beta);$$

- if $0 < \lambda_1 < \lambda_2 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N}$,

$$\Psi'(\lambda_1, \alpha - \lambda_2) = \phi(\alpha) \text{ and } \Psi'(\lambda_1, \beta + \lambda_2) = \phi(\beta);$$

- if $0 \leq \lambda_1 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N} < \lambda_2 \leq 3^{-k}$

$$\Psi'(\lambda_1, \alpha - \lambda_2) = \phi(\mathbf{n}(N, k)(\alpha - \lambda_2))$$

  and

$$\Psi'(\lambda_1, \beta + \lambda_2) = \phi(\mathbf{n}(N, k)(b + \lambda_2));$$

- if $(3^{N_2} + 3^{N_3}) \cdot 3^{-N} \leq \lambda \leq 1$ and $\tau([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1, N)_0)$, put

$$\Psi'(\lambda, \tau) = \Psi'((3^{N_2} + 3^{N_3}) \cdot 3^{-N}, \tau).$$

In order to obtain a 1-homotopy, we need to take N such that

$$(3^{N_2} + 3^{N_3}) \cdot 3^{-N} < \frac{1}{2} \cdot 3^{-k}.$$

Extend Ψ' to $I(1,N)_0 \times I(1,N)_0$, using the above construction near each maximal interval on $\mathcal{K}$ and putting $\Psi'(\lambda,\tau) = \phi(n(N,k)(\tau))$ on the complement. This map is a homotopy and has fineness $\mathbf{f}(\Psi') \leq 7\eta$. Then, the obtained map $\tilde{\phi}(\cdot) = \Psi'(1,\cdot)$ is 1-homotopic to the original discrete map ϕ with fineness at most $\max\{7, 3C_0+1\}\mathbf{f}(\phi) \leq C_1\mathbf{f}(\phi)$. The $(3C_0+1)$ factor comes from the first deformation step, recall the expression (3.19).

Moreover, if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1,N)_0)$ we have three possibilities for $\tilde{\phi}(x)$: it coincides either with some $\Psi_1(\sigma,\alpha)$, or $\Psi_1(\sigma,\beta)$ or $\Psi_1(2,\tau)$. By (3.24) and (3.25), we conclude that if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1,N)_0)$, then either $\mathbf{M}(\tilde{\phi}(x)) \leq L + C_1\mathbf{f}(\phi)$ or $x \notin \mathrm{dmn}_\Omega(\tilde{\phi})$. In particular, if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap \mathrm{dmn}_\Omega(\tilde{\phi}))$, then

$$\mathbf{M}(\tilde{\phi}(x)) \leq L + (4 + C_0)\eta < L + C_1\mathbf{f}(\phi). \tag{3.27}$$

If $x \in \mathrm{dmn}_\Omega(\tilde{\phi})$ and $x \notin [\alpha - 3^{-k}, \beta + 3^{-k}]$, for any $[\alpha,\beta]$ maximal on $\mathcal{K}$, then $\tilde{\phi}(x)$ also appear in ϕ and $\mathbf{M}(\tilde{\phi}(x)) < L$. This concludes the proof. □

3.3.3 Application: constructing discrete sweepouts

Next, we apply Lemma 9 to prove Theorem 9 of Chapter 2.

Proof of Theorem 9. Let ϕ_i, ψ_i, δ_i be given by Theorem 12 applied to the map Γ. It follows from property (iv) of Theorem 12 and the fact that $\Gamma(0) = \Gamma(1) = 0$ that, for all $y \in I(1,k_i)_0$ and $x \in \{0,1\}$, we have

$$\mathbf{M}(\psi_i(y,x)) \leq \delta_i. \tag{3.28}$$

Define $\bar{\psi}_i : I(1,k_i)_0 \times I(1,k_i)_0 \to \mathcal{Z}_{n-1}(M)$ by $\bar{\psi}_i(y,x) = 0$ if $x \in \{0,1\}$ and $\bar{\psi}_i(y,x) = \psi_i(y,x)$ otherwise. Define also $\bar{\phi}_i(x) = \bar{\psi}_i([0],x)$ for $x \in I(1,k_i)_0$. Note that $\mathbf{f}(\bar{\psi}_i) < 2\delta_i$, by (3.28) and Theorem 12 part (ii).

Then, we obtain $\{\bar{\phi}_i\}_{i\in\mathbb{N}}$, that is an $(1,\mathbf{M})$-homotopy sequence of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}), 0)$. But, we can not control its width by $L(\Gamma,\Omega)$ yet. To simplify notation, let us keep using ϕ_i and ψ_i instead of $\bar{\phi}_i$ and $\bar{\psi}_i$.

Since Γ is continuous in the $\mathbf{F}$-metric, $\mathbf{M} \circ \Gamma$ is uniformly continuous in I. Combine this with item (i) of Theorem 12 to conclude that

$$\mathbf{M}(\phi_i(y)) \leq \mathbf{M}(\Gamma(y)) + \frac{1}{i} + \delta_i. \tag{3.29}$$

From Lemma 4, properties (iii) of Theorem 12 and (3.29), we get

$$\lim_{i\to\infty} \sup\{\mathbf{F}(\phi_i(x) - \Gamma(x)) : x \in \mathrm{dmn}(\phi_i)\} = 0.$$

Recall the domains Ω_t starting with $\Omega = \Omega_0$ defined in Subsection 1.2. Also Ω_a, $a < 0$, fixed by Corollary 2.

Let $U \subset\subset \Omega$ be an open subset with $\overline{\Omega_a} \subset U$ and $0 < \varepsilon = \varepsilon_2(U, \Omega_a)$ be given by Lemma 9 with respect to the subsets U and Ω_a. Observe that

$$\mathcal{K} = \{\Gamma(t) : t \in [0,1] \text{ and } ||\Gamma(t)||(\Omega) = 0\}$$

is compact with respect to the $\mathbf{F}$-metric. Since $\overline{U} \subset \Omega$, it is possible to find $\rho > 0$ so that $\mathbf{F}(\mathcal{K}, T) < \rho$ implies $||T||(\overline{U}) < \varepsilon_2$. This construction of ρ does not involve the interpolation step, so we can suppose that, for large $i \in \mathbb{N}$,

$$\sup\{\mathbf{F}(\phi_i(x) - \Gamma(x)) : x \in \operatorname{dmn}(\phi_i)\} < \rho. \tag{3.30}$$

If $x \in I(1, k_i)_0$ is such that $x \in \mathcal{T}(\Gamma, \Omega)$, expression (3.29) gives us

$$\mathbf{M}(\phi_i(x)) \leq L(\Gamma, \Omega) + \frac{1}{i} + \delta_i.$$

Otherwise, $\Gamma(x) \in \mathcal{K}$ and (3.30) imply $||\phi_i(x)||(\overline{U}) < \varepsilon_2$. If $i \in \mathbb{N}$ is sufficiently large we apply Lemma 9 to obtain $\tilde{\phi}_i$ 1-homotopic to ϕ_i with fineness tending to zero and such that

$$\max\{\mathbf{M}(\tilde{\phi}_i(x)) : x \in \operatorname{dmn}_\Omega(\tilde{\phi}_i)\} \leq L(\Gamma, \Omega) + \frac{1}{i} + \delta_i + C_1 \cdot \mathbf{f}(\phi_i).$$

□

CHAPTER 4

The pull-tight and combinatorial arguments

In this chapter, we apply our tools from Chapter 3 to prove that the min-max program developed by Almgren and Pitts, the classical case $\Omega = M$, is suitable also in the new setting that we present in this thesis. The chapter is divided in three sections. In Section 4.1, we explain our generalization of the pull-tight argument and prove Proposition 1 stated in Chapter 2. In Section 4.2, we see that the original deformations and combinatorial argument done by Almgren and Pitts to prove existence of stationary almost minimizing varifolds do not affect the property of intersecting Ω with small mass. Then, we obtain Theorem 10. In Section 4.3, we include a detailed proof of the original combinatorial argument.

4.1 Generalizing the pull-tight argument

The classical pull-tight argument is based on the construction of an area decreasing flow letting still the stationary varifolds. Flowing all slices of a critical sequence S^*, we produce a better competitor S, for which critical varifolds are stationary in M. In our setting, we use a slightly different flow, because we let unmoved also the non-intersecting varifolds.

Precisely, let $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$ be a fixed critical sequence with respect to Ω. Consider the set $A_0 \subset \mathcal{V}_{n-1}(M)$ of varifolds with $||V||(M) \leq 2C$ and with one of the following properties: either V is stationary in M or $||V||(\Omega) = 0$. Here, $C = \sup\{\mathbf{M}(\phi_i^*(x)) : i \in \mathbb{N} \text{ and } x \in \mathrm{dmn}(\phi_i^*)\}$.

Following the same steps as in Section 15 of [36], we get a map

$$H : [0,1] \times (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \leq 2C\}) \to (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \leq 2C\}),$$

the pull-tight map, whose key properties are

(i) H is continuous in the product topology;

(ii) $H(t,T) = T$ for all $0 \leq t \leq 1$ if $|T| \in A_0$;

(iii) $||H(1,T)||(M) < ||T||(M)$ unless $|T| \in A_0$.

Direct application of H on the slices of S^*, as in Section 15 of [36], does not necessarily provide a better competitor, because this involves discrete approximations. Indeed, it is possible that the approximation near very big non-intersecting slices of ϕ_i^* creates bad intersecting slices. To overcome this difficulty, we make the approximation very close and use Lemma 9.

Proof of Proposition 1. We follow closely the proof given in Proposition 8.5 of [36]. The first claim about existence of critical sequences was already proven in Chapter 2.

We concentrate now in the pull-tight deformation of a given a critical sequence $S^* \in \Pi$. Suppose $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$, and set

$$C = \sup\{\mathbf{M}(\phi_i^*(x)) : i \in \mathbb{N} \text{ and } x \in \mathrm{dmn}(\phi_i^*)\} < +\infty.$$

Consider the following compact subsets of $\mathcal{V}_{n-1}(M)$

$$A = \{V \in \mathcal{V}_{n-1}(M) : ||V||(M) \leq 2C\};$$
$$A_0 = \{V \in A : \text{either } V \text{ is stationary in } M \text{ or } ||V||(\Omega) = 0\}.$$

For sake of completeness and convenience, we provide the details of the construction of the pull-tight map H with the properties (i), (ii) and (iii), as stated above. Consider the following further compact subsets of $\mathcal{V}_{n-1}(M)$

$$A_1 = \{V \in A : \mathbf{F}(V, A_0) \geq 2^{-1}\},$$
$$A_i = \{V \in A : 2^{-i} \leq \mathbf{F}(V, A_0) \leq 2^{-i+1}\}, \text{ for } i \in \{2, 3, \ldots\}$$

Observe that $A = \cup_{i=0}^{\infty} A_i$. For every $V \in A_i$, $i \geq 1$, we choose a vector field $X_V \in \mathcal{X}(M)$ with $|X_V|_{C^1} \leq 1$ and such that

$$\delta V(X_V) \leq \frac{2}{3}\inf\{\delta V(Y) : Y \in \mathcal{X}(M) \text{ with } |Y|_{C^1} \leq 1\} < 0.$$

This number is strictly negative because all stationary elements of A belong to A_0. For fixed V, the map

$$S \in \mathcal{V}_{n-1}(M) \mapsto \delta S(X_V)$$

is continuous. Hence we can find, for every $V \in A_i$ with $i \geq 1$, a radius $0 < r_V < 2^{-i}$ so that we have

$$\delta S(X_V) \leq \frac{1}{2} \inf\{\delta S(Y) : Y \in \mathcal{X}(M) \text{ with } |Y|_{C^1} \leq 1\} < 0,$$

for every $S \in \mathbf{B}^{\mathbf{F}}_{r_V}(V)$. The compactness of the sets A_i in the $\mathbf{F}$-metric implies that the open cover $\mathbf{B}^{\mathbf{F}}_{r_V}(V)$ admits a finite subcover. Thus we can find positive integers $q_i \in \mathbb{N}$ and

- a set of radii $\{r_{ij}\}_{j=1}^{q_i}$, $r_{ij} < 2^{-i}$;
- a set of varifolds $\{V_{ij}\}_{j=1}^{q_i} \subset A_i$;
- a set of vector fields $\{X_{ij}\}_{j=1}^{q_i} \subset \mathcal{X}(M)$ with $|X_{ij}|_{C^1} \leq 1$;
- a set of balls $U_{ij} = \mathbf{B}^{\mathbf{F}}_{r_{ij}}(V_{ij}) \cap A$, $j = 1, \ldots, q_i$, with $A_i \subset \bigcup_{j=1}^{q_i} U_{ij}$;
- a set of positive real numbers $\{\varepsilon_{ij}\}_{j=1}^{q_i}$ such that

$$\delta S(X_{ij}) \leq -\varepsilon_{ij} < 0 \text{ for all } S \in U_{ij}, j = 1, \ldots, q_i.$$

The choice $r_{ij} < 2^{-i}$ implies that $\{U_{ij}\}$, $i \in \mathbb{N}$ and $1 \leq j \leq q_i$, is a locally finite covering of $A - A_0$. Therefore, we can choose a partition of unity $\{\phi_{ij}\}$ of $A - A_0$ with support$(\phi_{ij}) \subset U_{ij}$.

We define

$$X : A \to \mathcal{X}(M),$$

continuous in the $\mathbf{F}$-metric, by

$$X(V) = 0 \quad \text{if } V \in A_0,$$
$$X(V) = \mathbf{F}(V, A_0) \cdot \sum_{i \in \mathbb{N}, 1 \leq j \leq q_i} \phi_{ij}(V) X_{ij} \quad \text{if } V \in A - A_0.$$

It follows that

$$\delta V(X(V)) = 0 \text{ if } V \in A_0 \text{ and } \delta V(X(V)) < 0 \text{ if } V \in A - A_0.$$

This implies tha we can find a continuous function

$$h : A \to [0, 1]$$

such that

- $h = 0$ on A_0 and $h(V) > 0$ if $V \in A - A_0$,
- and $||f(s,V)_{\#}(V)||(M) < ||f(t,V)_{\#}(V)||(M)$ if $0 \leq t < s \leq h(V)$,

where $\{f(t,V)(\cdot)\}_{t\geq 0}$ denotes the one-parameter group of diffeomorphisms generated by $X(V)$.

Now let

$$H : [0,1] \times (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \leq 2C\}) \to (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \leq 2C\})$$

be given by

$$H(t,T) = f(t,|T|)_{\#}(T) \quad \text{if } 0 \leq t \leq h(|T|),$$
$$H(t,T) = f(h(|T|),|T|)_{\#}(T) \quad \text{if } h(|T|) \leq t \leq 1.$$

This pull-tight map has the desired properties (i)-(iii).

We now proceed to the construction of a sequence $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$ such that $\mathcal{C}(S,\Omega) \subset A_0 \cap \mathcal{C}(S^*,\Omega)$, up to critical varifolds V with $||V||(\Omega) = 0$. Denote the domain of ϕ_i^* by $I(1,k_i)_0$, and let $\delta_i = \mathbf{f}(\phi_i^*)$. Up to subsequence, we can suppose that, for $x \in I(1,k_i)_0$, either

$$\mathbf{M}(\phi_i^*(x)) < \mathbf{L}(\Pi,\Omega) + \frac{1}{i} \quad \text{or} \quad \operatorname{spt}(||\phi_i^*(x)||) \subset M - \overline{\Omega}.$$

For sufficiently large i, apply Theorem 11 to obtain continuous maps in the mass norm

$$\bar{\Omega}_i : [0,1] \to \mathcal{Z}_{n-1}(M;\mathbf{M}),$$

such that for all $x \in I(1,k_i)_0$ and $\alpha \in I(1,k_i)_1$ we have

$$\bar{\Omega}_i(x) = \phi_i^*(x) \quad \text{and} \quad \sup_{y,z\in\alpha}\{\mathbf{M}(\bar{\Omega}_i(z) - \bar{\Omega}_i(y))\} \leq C_0\delta_i. \tag{4.1}$$

Moreover, for every $x \in [0,1]$, either

$$\mathbf{M}(\overline{\Omega}_i(x)) < \mathbf{L}(\Pi,\Omega) + \frac{1}{i} + C_0\delta_i \quad \text{or} \quad \operatorname{spt}(||\overline{\Omega}_i(x)||) \subset M - \overline{\Omega}.$$

Consider the continuous map in the $\mathbf{F}$-metric

$$\Omega_i : [0,1] \times [0,1] \to \mathcal{Z}_{n-1}(M;\mathbf{F}), \quad \Omega_i(t,x) = H(t,\overline{\Omega}_i(x)).$$

Observe that, for every $(t,x) \in [0,1]^2$ and large $i \in \mathbb{N}$, either

$$\mathbf{M}(\Omega_i(t,x)) \leq \mathbf{M}(\overline{\Omega}_i(x)) < \mathbf{L}(\Pi,\Omega) + \frac{1}{i} + C_0\delta_i$$

or

$$\mathrm{spt}(||\Omega_i(t,x)||) \subset M - \overline{\Omega}.$$

For each i, the map Ω_i above has no concentration of mass, because it is continuous in the $\mathbf{F}$-metric, and has uniformly bounded masses because of property (iii) of H. Then, we can apply Theorem 12 for Ω_i to obtain

$$\bar{\phi}_{ij} : I(1, s_{ij})_0 \times I(1, s_{ij})_0 \to \mathcal{Z}_{n-1}(M)$$

such that

(a) $\mathbf{f}(\bar{\phi}_{ij}) < \frac{1}{j}$;

(b)
$$\sup\{\mathcal{F}(\bar{\phi}_{ij}(t,x) - \Omega_i(t,x)) \,:\, (t,x) \in I(2, s_{ij})_0\} \leq \frac{1}{j};$$

(c)
$$\mathbf{M}(\bar{\phi}_{ij}(t,x)) \leq \mathbf{M}(\Omega_i(t,x)) + \frac{1}{j} \quad \text{for all } (t,x) \in I_0(2, s_{ij})_0;$$

(d) $\bar{\phi}_{ij}([0],x) = \Omega_i(0,x) = \bar{\Omega}_i(x)$ for all $x \in I(1, s_{ij})_0$.

From Lemma 4, properties (b), and (c), we get

$$\lim_{j\to\infty} \sup\{\mathbf{F}(\bar{\phi}_{ij}(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij})_0\} = 0.$$

Hence, using a diagonal sequence argument, can find $\{\bar{\phi}_i = \bar{\phi}_{ij(i)}\}$ such that

$$\lim_{i\to\infty} \sup\{\mathbf{F}(\bar{\phi}_i(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij(i)})_0\} = 0. \tag{4.2}$$

We define $\hat{\phi}_i : I(1, s_{ij(i)})_0 \times I(1, s_{ij(i)})_0 \to \mathcal{Z}_{n-1}(M)$ to be equal to zero on $I(1, s_{ij(i)})_0 \times \{0, 1\}$, and equal to $\bar{\phi}_i$ otherwise.

Recall the domains Ω_t starting with $\Omega = \Omega_0$ defined in Subsection 1.2, as well as the fixed Ω_a, $a < 0$. We hope this notation do not cause confusion with the maps Ω_i. Here we use sets Ω_t with negative t close to zero only.

Let $U_i = \Omega_{-i^{-1}}$, $\overline{\Omega_a} \subset U_i$ be such that $U_i \subset U_{i+1}$ and $U_i \subset\subset \Omega$. Choose $0 < \varepsilon_i < \varepsilon_2(U_i, \Omega_a)$, where $\varepsilon_2(U_i, \Omega_a)$ is the required to apply Lemma 9 with respect to the sets U_i and Ω_a, and such that $\lim_{i\to\infty} \varepsilon_i = 0$. The sets

$$K_i = \{\Omega_i(1,x) : x \in [0,1] \text{ and } ||\Omega_i(1,x)||(\Omega) = 0\}$$

are compact with respect to the $\mathbf{F}$-metric. Then, for each $i \in \mathbb{N}$, it is possible to find $\rho_i > 0$, so that $\mathbf{F}(\mathcal{K}_i, T) < \rho_i$ implies $||T||(\overline{U_i}) < \varepsilon_i$.

Since this construction of ρ_i does not involve the last interpolation step we can suppose the choice of $j(i)$ has the additional properties $j(i) \geq i$ and

$$\sup\{\mathbf{F}(\hat{\phi}_i(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij(i)})_0\} < \rho_i. \tag{4.3}$$

If $x \in I(1, s_{ij})_0$ is such that $\mathbf{M}(\Omega_i(1,x)) < \mathbf{L}(\Pi, \Omega) + i^{-1} + C_0\delta_i$, then item (c) above and the choice of $j(i)$ give us

$$\mathbf{M}(\hat{\phi}_i(1,x)) \leq \mathbf{M}(\Omega_i(1,x)) + \frac{1}{j(i)} < \mathbf{L}(\Pi, \Omega) + \frac{2}{i} + C_0\delta_i.$$

Otherwise, we have $\text{spt}(||\Omega_i(1,x)||) \subset M - \overline{\Omega}$ and $\Omega_i(1,x) \in K_i$. By (4.3), $||\hat{\phi}_i(1,x)||(\overline{U_i}) < \varepsilon_i$. If $i \in \mathbb{N}$ is sufficiently large, we apply Lemma 9 to obtain ϕ_i 1-homotopic to $\hat{\phi}_i(1,\cdot)$ with fineness tending to zero and such that

$$\max\{\mathbf{M}(\phi_i(x)) : x \in \text{dmn}_\Omega(\phi_i)\} \leq \mathbf{L}(\Pi, \Omega) + \frac{2}{i} + C_0\delta_i + C_1\mathbf{f}(\hat{\phi}_i(1,\cdot)). \tag{4.4}$$

This works only for large i, because there exists a $\eta_0(U_i, \Omega_a)$, such that $\mathbf{f}(\hat{\phi}_i(1,\cdot)) \leq \eta_0(U_i, \Omega_a)$ is required to use Lemma 9. Indeed, by Remark 12, the choice of η_0 and $U_i \subset U_{i+1}$, we conclude that $\eta_0(U_i, \Omega_a) \leq \eta_0(U_{i+1}, \Omega_a)$, while $\mathbf{f}(\hat{\phi}_i(1,\cdot))$ decreases to zero as i tends to infinity. Then, if i is sufficiently large, we have $\mathbf{f}(\hat{\phi}_i(1,\cdot)) \leq \eta_0(U_1, \Omega_a) \leq \eta_0(U_i, \Omega_a)$.

Since $\mathbf{f}(\hat{\phi}_i)$ tends to zero, we obtain that ϕ_i is 1-homotopic to $\hat{\phi}_i([0],\cdot)$ in $(\mathcal{Z}_{n-1}(M;\mathbf{M}), 0)$ with fineness tending to zero. On the other hand, it follows from (4.1) and property (d) that $\hat{\phi}_i([0],\cdot)$ is 1-homotopic to ϕ_i^* in $(\mathcal{Z}_{n-1}(M;\mathbf{M}), 0)$ with fineness tending to zero. Hence, $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$. Furthermore, it follows from (4.4) that S is critical with respect to Ω.

We are left to show that $\mathcal{C}(S,\Omega) \subset A_0 \cap \mathcal{C}(S^*, \Omega)$, up to varifolds with zero mass in Ω. First we compare $\mathcal{C}(S,\Omega)$ with the critical set of $\{\hat{\phi}_i(1,\cdot)\}_i$. By Lemma 9 and Remark 13, each $\phi_i(x)$ either coincides with some slice of $\hat{\phi}_i(1,\cdot)$ or $||\phi_i(x)||(U_i) < \varepsilon_i + C_1\mathbf{f}(\hat{\phi}_i(1,\cdot))$. Then, $\mathcal{C}(S,\Omega) \subset \mathcal{C}(\{\hat{\phi}_i(1,\cdot)\}_i, \Omega)$, up to critical varifolds with zero mass in Ω. To conclude the proof, we claim that $\mathcal{C}(\{\hat{\phi}_i(1,\cdot)\}_i, \Omega) \subset A_0 \cap \mathcal{C}(S^*, \Omega)$. Here we omit the proof of this fact, because it is exactly the same as in the end of the Section 15 of [36]. □

4.2 Intersecting almost minimizing varifolds

In this section we prove Theorem 10. In the argument of proof we use ideas of the original proof of Pitts for the analogous min-max Theorem with $\Omega = M$. In this part, we explicitly state which result from [47] we use, see Claim 2 below. In the next section, we include the proof of this claim.

Proof of Theorem 10. **Part 1:**

Let $\Omega_a \subset \Omega \subset \Omega_b$ be fixed as in Subsection 1.2. Consider open subsets $\overline{\Omega_a} \subset U \subset U_1 \subset \Omega$ such that $\overline{U} \subset U_1$ and $\overline{U_1} \subset \Omega$. Let $S \in \Pi$ be given by Proposition 1. Write $S = \{\phi_i\}_{i\in\mathbb{N}}$ and let $I(1, k_i)_0 = \mathrm{dmn}(\phi_i)$. Consider also $\varepsilon_2 = \varepsilon_2(U, \Omega_a)$ be given by Lemma 9.

As in the original work of Pitts, our argument is by contradiction, we homotopically deform S to decrease its width with respect to Ω. This will create a contradiction with the fact that S is a critical sequence.

Part 2:

Observe that $\mathcal{C}_0(S, \Omega) := \{V \in \mathcal{C}(S, \Omega) : ||V||(\Omega) = 0\}$ is compact in the weak sense of varifolds. Then, it is possible to find $\varepsilon > 0$ such that

$$T \in \mathcal{Z}_{n-1}(M) \text{ and } \mathbf{F}(|T|, \mathcal{C}_0(S, \Omega)) < \varepsilon \Rightarrow ||T||(\overline{U_1}) < 2^{-1}\varepsilon_2. \tag{4.5}$$

Part 3:

Given $V \in \mathcal{C}(S, \Omega)$ with $||V||(\Omega) > 0$, by the contradiction assumption, there exists $p = p(V) \in \mathrm{spt}(||V||)$ such that V is not almost minimizing in small annuli centered at p. For $\mu = 1, 2$, choose sufficiently small

$$a_\mu(V) = A(p, s_\mu, r_\mu) \text{ and } A_\mu(V) = A(p, \tilde{s}_\mu, \tilde{r}_\mu), \tag{4.6}$$

with the following properties:

- $\tilde{r}_1 > r_1 > s_1 > \tilde{s}_1 > 3\tilde{r}_2 > 3r_2 > 3s_2 > 3\tilde{s}_2$;
- if $p \in U_1$, then $A_\mu(V) \subset U_1$;
- if $p \notin \overline{U}$, then $A_\mu(V) \cap \overline{U} = \varnothing$;
- V is not almost minimizing in $a_\mu(V)$.

Since V is not almost minimizing in $a_\mu(V)$, for $\mu = 1, 2$, there exists $\varepsilon(V) > 0$ with the following property: given

$$T \in \mathcal{Z}_{n-1}(M), \quad \mathbf{F}(|T|, V) < \varepsilon(V), \quad \mu \in \{1, 2\} \text{ and } \eta > 0, \tag{4.7}$$

we can find a finite sequence $T = T_0, T_1, \ldots, T_q \in \mathcal{Z}_{n-1}(M)$ such that

(a) $\mathrm{spt}(T_l - T) \subset a_\mu(V)$;

(b) $\mathbf{M}(T_l - T_{l-1}) \leq \eta$;

(c) $\mathbf{M}(T_l) \leq \mathbf{M}(T) + \eta$;

(d) $\mathbf{M}(T_q) < \mathbf{M}(T) - \varepsilon(V)$.

The properties of those annuli $a_\mu(V)$ concerning the sets U and U_1 make them slightly smaller than in Pitts' original choice. The aim with this is to obtain the property that $||T||(U_1)$ controls $||T_l||(\overline{U})$, for every l. Indeed, (a) implies that $\text{spt}(T_l - T)$ is always contained in either U_1 or $M - \overline{U}$. In the first case, we can use item (c) to prove that

$$||T_l||(\overline{U}) \leq |T_l||(U_1) \leq ||T||(U_1) + \eta.$$

In the second case, it follows that

$$||T_l||(\overline{U}) = ||T||(\overline{U}) \leq ||T||(U_1).$$

Part 4:

$\mathcal{C}(S,\Omega)$ is compact. Take $V_1, V_2, \ldots, V_\nu \in \mathcal{C}(S,\Omega)$ such that

$$\mathcal{C}(S,\Omega) \subset \bigcup_{j=1}^{\nu} \{V \in \mathcal{V}_{n-1}(M) : \mathbf{F}(V, V_j) < 4^{-1}\varepsilon(V_j)\}, \tag{4.8}$$

where $\varepsilon(V_j)$ is the one we chose in Part 3 in the case of intersecting varifolds, $||V_j||(\Omega) > 0$, or $\varepsilon(V_j) = \varepsilon$ as chosen in Part 2, otherwise.

Part 5:

Let $\delta > 0$ and $N \in \mathbb{N}$ be such that: given

$$i \geq N, \quad x \in \text{dmn}_\Omega(\phi_i) \quad \text{and} \quad \mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S,\Omega) - 2\delta, \tag{4.9}$$

there exists $f_1(x) \in \{1, \ldots, \nu\}$ with

$$\mathbf{F}(|\phi_i(x)|, V_{f_1(x)}) < 2^{-1}\varepsilon(V_{f_1(x)}). \tag{4.10}$$

The existence of such numbers can be seen via a contradiction argument. Moreover, choose δ and N satisfying the following two extra conditions

- $\delta \leq \min\{2^{-1}\varepsilon(V_j) : j = 1, \ldots \nu\}$;
- $i \geq N$ implies $\mathbf{f}(\phi_i) \leq \min\{\delta, 2^{-1}\varepsilon_2, \delta_0\}$.

Where $\delta_0 = \delta_0(M)$ is defined in Section 3.1 and $\varepsilon_2 = \varepsilon_2(U, \Omega_a)$ in Part 1.

Part 6:

In this step of our proof, we use the construction done in Theorem 4.10 of [47], Parts 7 to 18, to state a deformation result for discrete maps. Fix $i \geq N$. Let $\alpha, \beta \in \mathrm{dmn}(\phi_i) = I(1, k_i)_0$ be such that

(1) $[\alpha, \beta] \cap \mathrm{dmn}(\phi_i) \subset \mathrm{dmn}_\Omega(\phi_i)$;

(2) if $x \in (\alpha, \beta] \cap \mathrm{dmn}(\phi_i)$, then $\mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S, \Omega) - \delta$;

(3) if $x \in [\alpha, \beta] \cap \mathrm{dmn}(\phi_i)$, then $||V_{f_1(x)}||(\Omega) > 0$.

Claim 2. *There exist a sequence $\{\delta_i\}_{i \geq N}$ tending to zero, $N(i) \geq k_i$ and*

$$\psi_i : I(1, N(i))_0 \times ([\alpha, \beta] \cap I(1, N(i))_0) \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(i) $\lim_{i\to\infty} \boldsymbol{f}(\psi_i) = 0$*;*

(ii) $\psi_i([0], x) = \phi_i(\boldsymbol{n}(N(i), k_i)(x))$*;*

(iii) $\boldsymbol{M}(\psi_i(1, \zeta)) < \boldsymbol{M}(\psi_i(0, \zeta)) - \delta + \delta_i$*, for every* $\zeta \in [\alpha, \beta] \cap I(1, N(i))_0$*;*

(iv) $\psi_i(j, \alpha) = \phi_i(\alpha)$*, for every* $j \in I(1, N(i))_0$*;*

(v) $\{\psi_i(\lambda, x) : \lambda \in I(1, N(i))_0\}$ *describes the deformation obtained in Part 3, starting with* $T = \phi_i(x)$*, supported in some* $a_\mu(V_{f_1(x)})$ *and fineness* $\eta = \delta_i$*, for every* $x \in (\alpha, \beta] \cap dmn(\phi_i)$*.*

Remark 14. If we include $x = \alpha$ in the hypothesis (2), the construction yields a map ψ_i, for which (v) also holds for $x = \alpha$, instead of (iv).

Originally, Pitts wrote this argument using 27 annuli. It is suggested by [17], when dealing with one-parameter sweepouts it is enough to take only 2. This argument is presented in the next section.

Part 7:

Consider $\alpha, \beta \in \mathrm{dmn}(\phi_i) = I(1, k_i)_0$, such that $[\alpha, \beta]$ is maximal for the property: if $x \in [\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap \mathrm{dmn}(\phi_i)$, then

$$x \in \mathrm{dmn}_\Omega(\phi_i), \quad \mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S, \Omega) - \delta \text{ and } ||V_{f_1(x)}||(\Omega) > 0.$$

Let $\{\delta_i\}_{i\in\mathbb{N}}$ and $N(i) \in \mathbb{N}$ be as in Claim 2. Set $n_i = N(i) + k_i + 1$ and

$$L_i = \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\} - \delta + \delta_i. \tag{4.11}$$

Claim 3. *There exists a map*

$$\psi_i : I(1,n_i)_0 \times ([\alpha,\beta] \cap I(1,n_i)_0) \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(a) $\lim_{i\to\infty} \boldsymbol{f}(\psi_i) = 0$*;*

(b) $\psi_i([0],\cdot) = \phi_i \circ \boldsymbol{n}(n_i,k_i)$*;*

(c) $\psi_i(\lambda,\alpha) = \phi_i(\alpha)$ *and* $\psi_i(\lambda,\beta) = \phi_i(\beta)$ *for every* $\lambda \in I(1,n_i)_0$*;*

(d) $\max\{\boldsymbol{M}(\psi_i(1,\zeta)) : \zeta \in [\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,n_i)_0\} < L_i$*.*

Moreover, if $\zeta \in [\alpha,\beta] \cap I(1,n_i)_0$ *and* $\boldsymbol{M}(\psi_i(1,\zeta)) \geq L_i$*, then*

$$||\psi_i(1,\zeta)||(\overline{U}) \leq 2^{-1}\varepsilon_2 + \delta_i. \tag{4.12}$$

Proof of Claim 3. Apply Claim 2 on $[\alpha + 3^{-k_i}, \beta - 3^{-k_i}]$ to obtain the map

$$\psi_i : I(1,N(i))_0 \times ([\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,N(i))_0) \to \mathcal{Z}_{n-1}(M).$$

In order to perform the extension of this ψ_i to $I(1,n_i)_0 \times ([\alpha,\beta] \cap I(1,n_i)_0)$, we analyze the possibilities for $\phi_i(\alpha)$ and $\phi_i(\beta)$.

If $\alpha \in \mathrm{dmn}_\Omega(\phi_i)$ and $||V_{f_1(\alpha)}||(\Omega) > 0$, we can apply Claim 2 directly on $[\alpha, \beta - 3^{-k_i}]$. Then, we have that the map ψ_i is already defined on $I(1,N(i))_0 \times ([\alpha, \beta - 3^{-k_i}] \cap I(1,N(i))_0)$. Extend it to the desired domain simply by $\psi_i \circ \mathbf{n}(n_i, N(i))$. Observe that the choice of $[\alpha,\beta]$ implies that

$$\mathbf{L}(S,\Omega) - 2\delta \leq \mathbf{M}(\phi_i(\alpha)) < \mathbf{L}(S,\Omega) - \delta.$$

By item (iv) of Claim 2, $\psi_i(\lambda,\alpha) = \phi_i(\alpha)$, for every $\lambda \in I(1,n_i)_0$. Item (iii) of the same statement implies that

$$\max\{\mathbf{M}(\psi_i(1,\zeta)) : \zeta \in [\alpha, \beta - 3^{-k_i}] \cap I(1,n_i)_0\} < L_i.$$

Suppose now that α satisfies one of the following properties:

(I) either $\alpha \in \mathrm{dmn}_\Omega(\phi_i)$ and $||V_{f_1(\alpha)}||(\Omega) = 0$;

(II) or $\alpha \notin \mathrm{dmn}_\Omega(\phi_i)$.

In both cases, the extension of ψ_i on $I(1,n_i)_0 \times ([\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,n_i)_0)$ is given by $\psi_i \circ \mathbf{n}(n_i, N(i))$. We complete the extension in such a way that

$$\{\psi_i(\lambda,\zeta) : \zeta \in [\alpha, \alpha + 3^{-k_i}] \cap I(1,n_i)_0\} \subset \{\psi_i(j, \alpha + 3^{-k_i})\}_{j \in I(1,N(i))_0} \cup \{\phi_i(\alpha)\}.$$

The motivation for doing this is that $\phi_i(\alpha)$ and all $\psi_i(j, \alpha+3^{-k_i})$ have already small mass inside $\overline{U}$. Let us first prove this claim about the masses inside $\overline{U}$ and later we conclude the construction of the map ψ_i.

In case (I), we observe that $V_{f_1(\alpha)} \in \mathcal{C}_0(S, \Omega)$ and that

$$\mathbf{F}(|\phi_i(\alpha + 3^{-k_i})|, V_{f_1(\alpha)}) \leq \mathbf{f}(\phi_i) + \mathbf{F}(|\phi_i(\alpha)|, V_{f_1(\alpha)}) < \varepsilon,$$

where the last estimate is a consequence of the choice of N and f_1 in Part 5. In particular, $\mathbf{F}(|\phi_i(\alpha + 3^{-k_i})|, \mathcal{C}_0(S, \Omega)) < \varepsilon$. Item (v) of Claim 2 and the comments in the end of Part 3 imply that, for every $j \in I(1, N(i))_0$,

$$||\psi_i(j, \alpha + 3^{-k_i})||(\overline{U}) \leq 2^{-1}\varepsilon_2 + \delta_i. \tag{4.13}$$

Case (II) is simpler because $||\phi_i(\alpha)||(\overline{\Omega}) = 0$ directly implies that

$$||\phi_i(\alpha + 3^{-k_i})||(\overline{\Omega}) \leq \mathbf{f}(\phi_i) \leq 2^{-1}\varepsilon_2.$$

Then, a similar analysis tells us that expression (4.13) also holds in this case.

Finally, define ψ_i on $I(1, n_i)_0 \times ((\alpha, \alpha + 3^{-k_i}) \cap I(1, n_i)_0)$ by

$$\psi_i(\lambda, \alpha + 3^{-k_i} - \zeta \cdot 3^{-n_i}) = \psi_i(\max\{0, \mathbf{n}(n_i, N(i))(\lambda) - \zeta \cdot 3^{-N(i)}\}, \alpha + 3^{-k_i}).$$

Put $\psi_i(\lambda, \alpha) = \phi_i(\alpha)$, for every $\lambda \in I(1, n_i)_0$. Then, Claim 3 holds. □

Part 8:

Let $\psi_i : I(1, n_i)_0 \times I(1, n_i)_0 \to \mathcal{Z}_{n-1}(M)$ be the map obtained in such a way that for each interval $[\alpha, \beta]$ as in Part 7, its restriction to

$$I(1, n_i)_0 \times ([\alpha, \beta] \cap I(1, n_i)_0)$$

is the map of Claim 3, and $\psi_i(\lambda, \zeta) = (\phi_i \circ \mathbf{n}(n_i, k_i))(\zeta)$, otherwise.

Take $S^* = \{\phi_i^*\}_{i \in \mathbb{N}}$, where $\phi_i^* = \psi_i(1, \cdot)$ is defined on $I(1, n_i)_0$. Because the maps ψ_i have fineness tending to zero, $S^* \in \Pi$.

Part 9:

The sequence S is critical with respect to Ω, this implies that

$$\lim_{i\to\infty} \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\} = \mathbf{L}(S, \Omega).$$

In particular, $\lim_{i\to\infty} L_i = \mathbf{L}(S, \Omega) - \delta$. Let i be sufficiently large, so that

$$i \geq N, \quad \mathbf{f}(\phi_i^*) \leq \eta_0, \quad L_i < \mathbf{L}(S, \Omega) - 2^{-1}\delta \quad \text{and} \quad \delta_i < 2^{-1}\varepsilon_2,$$

where $\eta_0 = \eta_0(U, \Omega_a)$ is given by Lemma 9. This choice implies that the maps ϕ_i^* have the following property:

$$\mathbf{M}(\phi_i^*(\zeta)) \geq L_i \Rightarrow ||\phi_i^*(\zeta)||(\overline{U}) < \varepsilon_2. \tag{4.14}$$

Apply Lemma 9 to produce $\tilde{S} = \{\tilde{\phi}_i\}_{i\in\mathbb{N}}$ homotopic with S^*, such that

$$\max\{\mathbf{M}(\tilde{\phi}_i(x)) : x \in \mathrm{dmn}_\Omega(\tilde{\phi}_i)\} < L_i + C_1\mathbf{f}(\phi_i^*),$$

where $C_1 = C_1(M)$ is also given by Lemma 9. In particular, $\tilde{S} \in \Pi$ and

$$\mathbf{L}(\tilde{S}, \Omega) \leq \mathbf{L}(S, \Omega) - \delta = \mathbf{L}(\Pi, \Omega) - \delta.$$

This is a contradiction. □

4.3 Slices near intersecting critical varifolds

In this section we include Pitts' arguments in the proof of Claim 2. Follow the notation in the proof of Theorem 10. Assume also the additional notation

$$[\alpha, \beta] \cap dmn(\phi_i) = \{z_0, z_1, \ldots, z_s\} \subset I(1, k_i)_0.$$

For each $j \in \{1, \ldots, \nu\}$, consider

$$s(V_j) = \min\{\tilde{r}_\mu - r_\mu, s_\mu - \tilde{s}_\mu : \mu = 1, 2\},$$

where $\tilde{r}_\mu, \tilde{s}_\mu, r_\mu, s_\mu$ are number depending on V_j as chosen in Part 3 of the proof of Theorem 10. Let

$$s = \min\{s(V_j) : j = 1, \ldots, \nu\} \tag{4.15}$$

and define $\delta_i = 2\mathbf{f}(\phi_i)\,(1 + 8/s)$.

We start with the cut and paste argument:

Claim 4. *Fix $x_1, x_2 \in [z_0, z_s] \cap dmn(\phi_i)$ with $|x_1 - x_2| = 0$ or 3^{-k_i}, and $\mu \in \{1, 2\}$. There exists a positive integer $N_1 = N_1(i)$ and finite sequences*

$$T(m, 1), T(m, 2), \ldots, T(m, 3^{N_1}) \in \mathcal{Z}_{n-1}(M),$$

for $m = 1, 2$, with the following properties:

(a) $T(m, 1) = \phi_i(x_m)$;

(b) $spt(T(m, l) - T(m, 1)) \subset A_\mu(V_{f_1(x_1)})$;

(c) $\boldsymbol{M}(T(m,l) - T(m,l-1)) \leq \delta_i$;

(d) $\boldsymbol{M}(T(1,l) - T(2,l)) \leq \delta_i$;

(e) $\boldsymbol{M}(T(m,l)) \leq \boldsymbol{M}(\phi_i(x_m)) + \delta_i$;

(f) $\boldsymbol{M}(T(m,3^{N_1})) < \boldsymbol{M}(\phi_i(x_m)) - \delta + \delta_i/2$.

To fix notation, given (x_1, μ) as in the previous claim, use

$$T_{(x_1,\mu)}(m,l), \text{ for } m = 1,2 \text{ and } l = 1,\ldots,3^{N_1}, \tag{4.16}$$

to denote the obtained two sequences. Note that $N_1 = N_1(i, x_1, \mu)$, but there are only finitely many values for each parameter, if $i \geq N$ is fixed. Take this number big enough, up to repetition of the last term on each sequence, so that $N_1 = N_1(i)$.

Proof. Observe that we always have that $x_1, x_2 \in \mathrm{dmn}_\Omega(\phi_i)$,

$$\mathbf{M}(\phi_i(x_1)) \text{ and } \mathbf{M}(\phi_i(x_2)) \geq L(S,\Omega) - 2\delta$$

and

$$||V_{f_1(x_1)}||(\Omega) \text{ and } ||V_{f_1(x_2)}||(\Omega) > 0.$$

Observe that $\mathbf{M}(\phi_i(x_2) - \phi_i(x_1)) \leq \mathbf{f}(\phi_i) \leq \delta_0$. Then, there exists an integral current $Q \in \mathbf{I}_n(M^n)$, so that

$$\partial Q = \phi_i(x_2) - \phi_i(x_1) \text{ and } \mathbf{M}(Q) = \mathcal{F}(\partial Q) \leq \delta.$$

Let $p = p(V_{f_1(x_1)}) \in \mathrm{spt}(||V_{f_1(x_1)}||)$ be the center of the annulus $a_\mu(V_{f_1(x_1)})$ and $u(x) = d_g(x,p)$ be the distance to p in the Riemannian manifold (M^n, g).

Using that ϕ_i takes values on the space of integral currents, the slicing theory (recall the notation we use in Subsection 1.1.2 or see Chapter 6 of Simon's book [51]) and the co-area formula, it is possible to choose numbers $r_\mu < t_1 < \tilde{r}_\mu$ and $\tilde{s}_\mu < t_2 < s_\mu$ so that

- $||\phi_i(x_m)||(\{u = t_1 \text{ or } t_2\}) = 0$, for $m = 1,2$;
- $\langle Q, u, t_1+\rangle, \langle Q, u, t_2+\rangle \in \mathbf{I}_{n-1}(M)$;
- $\mathbf{M}(\langle Q, u, t_1+\rangle) \leq 2\mathbf{f}(\phi_i)/(\tilde{r}_\mu - r_\mu)$;
- $\mathbf{M}(\langle Q, u, t_2+\rangle) \leq 2\mathbf{f}(\phi_i)/(s_\mu - \tilde{s}_\mu)$.

Define $T(m,1) = \phi_i(x_m)$, to guarantee that item (a) holds. The next step is to define

$$T(1,2) = T(1,1) = \phi_i(x_1)$$

and

$$\begin{aligned} T(2,2) &= \phi_i(x_2)\llcorner\{u > t_1\} - \langle Q, u, t_1+\rangle + \phi_i(x_1)\llcorner\{t_2 < u < t_1\} \\ &\quad +\langle Q, u, t_2+\rangle + \phi_i(x_2)\llcorner\{u < t_2\}. \end{aligned}$$

By Lemma 1 and the choices of the slices, $T(2,2) \in \mathcal{Z}_{n-1}(M)$ and, for $T = \phi_i(x_1)$ or $\phi_i(x_2)$,

$$\mathbf{M}(T(2,2) - T) \leq \mathbf{M}(\phi_i(x_2) - \phi_i(x_1)) + 4\mathbf{f}(\phi_i)/s < \delta_i,$$

where s is the one of expression (4.15). These $T(m,2)$ satisfy (b), (c), (d) and (e) for $l = 2$. In order to construct the terms $T(1,l)$ for $l > 2$, observe that

$$\mathbf{F}(|\phi_i(x_1)|, V_{f_1(x_1)}) < \varepsilon(V_{f_1(x_1)})/2. \tag{4.17}$$

Since $||V_{f_1(x_1)}||(\Omega) > 0$, we can apply the deformation obtained in Part 3 to $\phi_i(x_1)$, the given μ and $\eta = \delta_i/2$ to get a sequence

$$T(1,2) = \phi_i(x_1), T(1,3), \ldots, T(1,3^{N_1}) \in \mathcal{Z}_{n-1}(M)$$

that satisfy (b), (c), (e), for every l, and, for the last term,

$$\mathbf{M}(T(1,3^{N_1})) < \mathbf{M}(\phi_i(x_1)) - \varepsilon(V_{f_1(x_1)}) \leq \mathbf{M}(\phi_i(x_1)) - \delta. \tag{4.18}$$

Observe that N_1 depends on i, x_1 and μ only. Finally, the general definition for $m = 2$ and $l \geq 2$ is

$$\begin{aligned} T(2,l) &= T(1,l) + (\phi_i(x_2) - \phi_i(x_1))\llcorner(\{u > t_1\} \cup \{u < t_2\}) \quad (4.19) \\ &\quad -\langle Q, u, t_1+\rangle + \langle Q, u, t_2+\rangle. \end{aligned}$$

This gives (b) and (c) for every admissible m and l, immediately from the same properties for $m = 1$. Item (d) for $l > 2$ follows from the same assertion for $l = 2$, because

$$T(1,l) - T(2,l) = T(1,2) - T(2,2).$$

In order to compare the masses in each sequence, observe that the general formula (4.19) gives also a formula for the mass of the integral current $T(2,l)$:

$$\begin{aligned} \mathbf{M}(T(2,l)) &= ||\phi_i(x_2)||(\{u > t_1\} \cup \{u < t_2\}) + ||T(1,l)||(\{t_2 < u < t_1\}) \\ &\quad +\mathbf{M}(\langle Q, u, t_1+\rangle) + \mathbf{M}(\langle Q, u, t_2+\rangle) \\ &\leq \mathbf{M}(\phi_i(x_2)) + (||T(1,l)|| - ||\phi_i(x_1)||)(\{t_2 < u < t_1\}) \\ &\quad +(||\phi_i(x_1)|| - ||\phi_i(x_2)||)(\{t_2 < u < t_1\}) + 4\mathbf{f}(\phi_i)/s \\ &\leq \mathbf{M}(\phi_i(x_2)) + (||T(1,l)|| - ||\phi_i(x_1)||)(\{t_2 < u < t_1\}) + \delta_i/2. \end{aligned}$$

To conclude, use the known properties for $m = 1$, to obtain

$$(||T(1,l)|| - ||\phi_i(x_1)||)(\{t_2 < u < t_1\}) \leq \frac{\delta_i}{2}$$

and

$$(||T(1,3^{N_1})|| - ||\phi_i(x_1)||)(\{t_2 < u < t_1\}) < -\delta.$$

□

We intend to let $\phi_i(z_0)$ still and deform the other $\phi_i(z_k)$, $k \geq 1$, to decrease their masses. To perform this deformation with small fineness, we need more room in the domains of our discrete maps, then, we consider refinements. Choose $N_2 = k_i + N_1 + 1$ and define

$$f_2 : [z_0, z_s] \cap I(1,N_2)_0 \to \{0, 1, \ldots, 3^{N_1}\}$$

by $f_2(v) = \min\{\mathbf{d}(v, z_0), 3^{N_1}\}$, where $\mathbf{d} : I(1,N_2)_0 \times I(1,N_2)_0 \to \mathbb{N}$ is the grid distance $\mathbf{d}(v, z_0) = 3^{N_2}|v - z_0|$. This map satisfies:

Claim 5. *(a)* $\boldsymbol{M}(\phi_i \circ n(N_2, k_i)(v)) \geq L(S,\Omega) - \delta \Rightarrow f_2(v) = 3^{N_1}$*;*

(b) $\rho \in [z_1, z_s] \cap I(1,N_2)_1, \{v, v'\} \subset \rho^0 \Rightarrow |f_2(v) - f_2(v')| \leq 1$.

Proof. The choice of N_2 is relevant for the analysis of the vertices v of $I(1,N_2)_0$ inside $[z_0, z_1]$, because with that, there are $3^{N_1} + (3^{N_1} - 1)/2$ such vertices with $n(N_2, k_i)(v) = z_0$. Then it is possible to go from zero to 3^{N_1} satisfying (b), when you run from z_0 to z_1 on $I(1,N_2)_0$, before the middle of the interval $[z_0, z_1]$. This proves the claim. Item (a) is true because we are assuming that $\phi_i(z_0)$ is not very big, i.e., $\mathbf{M}(\phi_i(z_0)) < L(S,\Omega) - \delta$. □

The idea now is to start the process with the refinement $\phi_i \circ n(N_2, k_i)$ on $I(1,N_2)_0$. Considering those $I(1,N_2)_0$ refinements, we still have the problem of controlling the deformations of each pair of neighboring slices with different nearest projection in $I(1,k_i)_0$, i.e.,

$$(\phi_i \circ n(N_2, k_i))(u) \text{ and } (\phi_i \circ n(N_2, k_i))(v)$$

with $|u - v| = 3^{-N_2}$ and $|n(N_2, k_i)(u) - n(N_2, k_i)(v)| = 3^{-k_i}$. To deal with this difficulty, let $f_3 : I(1,N_2) \to I(1,N_2)_0$ be any vertex choice map, so that $f_3(\rho) \in \rho^0$, for $\rho \in I(1,N_2)$. Then, define the map

$$f_4 : [z_0, z_s] \cap I(1,N_2) - \{[z_0]\} \to \{1, \ldots, \nu\}$$

via the expression

$$f_4(\rho) = f_1 \circ n(N_2, k_i) \circ f_3(\rho). \tag{4.20}$$

Fix a cell $[z_k, z_{k+1}] \in I(1, k_i)_1$. Note that, for $\rho \in I(1, N_2) \cap [z_k, z_{k+1}]$, $f_4(\rho) = f_1(z_k)$, if z_k is closer to ρ than z_{k+1}, and $f_4(\rho) = f_1(z_{k+1})$, if z_{k+1} is closer to ρ than z_k. Moreover, $f_4(\rho)$ depends on the choice map f_3 for the middle cell $[v_k^+, v_{k+1}^-]$ of $[z_k, z_{k+1}]$, but still with $f_4([v_k^+, v_{k+1}^-]) = f_1(z_k)$ or $f_1(z_{k+1})$. The notation middle interval here means

$$v_k^+ = z_k + \frac{(3^{N_1+1} - 1)}{2 \cdot 3^{N_1+1}} \text{ and } v_{k+1}^- = v_k^+ + \frac{1}{3^{N_1+1}}, \tag{4.21}$$

so that $[v_k^+, v_{k+1}^-]$ is the unique 1-cell in $[z_k, z_{k+1}] \cap I(1, N_2)_1$ whose vertex has different nearest points $n(N_2, k_i)(v_k^+) = z_k$ and $n(N_2, k_i)(v_{k+1}^-) = z_{k+1}$.

The map f_3 is obviously not unique. In the next step, we introduce a map

$$f_5 : [z_0, z_s] \cap I(1, N_2) - \{[z_0]\} \to 2^M,$$

where 2^M denotes the set of subsets of M, so that $f_5(\rho) = A_\mu(V_{f_4(\rho)})$, for all $\rho \in (z_0, z_s] \cap I(1, N_2)$ with the additional property $f_5(\rho) \cap f_5(v) = \varnothing$, whenever $\rho \in [z_0, z_s] \cap I(1, N_2)_1$ and $v \in \rho^0$. This is slightly different to that f_5 that Pitts defined, because here we use only two annuli. There, Pitts was able to define f_5 for any map f_3 as above, but here we have to make choices on f_3 in order to produce the desired f_5:

- define $f_3([v_0^+, v_1^-]) = v_0^+$;
- choose disjoint $f_5([v_0^+, v_1^-])$ and $f_5(v_1^-)$;
- let $f_5(v_0^+)$ be the annulus $A_\mu(V_{f_4(v_0^+)})$ that is not $f_5([v_0^+, v_1^-])$;
- for $\rho \in [z_0, v_0^+] \cap I(1, N_2) - \{[z_0]\}$, define

$$f_5(\rho) = \begin{cases} f_5(v_0^+) & \text{if } \rho \in I(1, N_2)_0 \\ f_5([v_0^+, v_1^-]) & \text{if } \rho \in I(1, N_2)_1; \end{cases}$$

- let $f_5([v_1^-, v_1^- + 3^{-N_2}])$ be the $A_\mu(V_{f_4([v_1^-, v_1^- + 3^{-N_2}])})$ that is not $f_5(v_1^-)$;
- for $\rho \in [v_1^-, v_1^+] \cap I(1, N_2)$, define

$$f_5(\rho) = \begin{cases} f_5(v_1^-) & \text{if } \rho \in I(1, N_2)_0 \\ f_5([v_1^-, v_1^- + 3^{-N_2}]) & \text{if } \rho \in I(1, N_2)_1; \end{cases}$$

- up to relabeling, let $f_5(v_1^+) = A_1(V_{f_4(v_1^+)})$. Choose disjoint

$$A_{\mu(1)}(V_{f_4(v_1^+)}) \text{ and } A_{\mu(2)}(V_{f_4(v_2^-)}).$$

If $\mu(1) = 1$, choose $f_3([v_1^+, v_2^-]) = v_2^-$ and put

$$f_5([v_1^+, v_2^-]) = A_{\mu(2)}(V_{f_4(v_2^-)}).$$

If $\mu(1) = 2$, choose $f_3([v_1^+, v_2^-]) = v_1^+$ and let

$$f_5([v_1^+, v_2^-]) = A_2(V_{f_4(v_1^+)});$$

- inductively, define f_5 in the whole $[z_0, z_s] \cap I(1, N_2) - \{[z_0]\}$.

At this point, we are able to say what deformation sequences we actually use to produce the ψ_i, those appear in the map f_6 that we define now. Later, in order to get a map ψ_i with fineness tending to zero, we still have to consider a finer grid $[z_0, z_s] \cap I(1, N_3)_0$ in which we organize the cycles in f_6.

The $f_6(\rho, v, l)$ elements have three parameters, being $\rho \in [z_0, z_s] \cap I(1, N_2)$, $v \in \rho^0$ and $l \in \{0, 1, \ldots, 3^{-N_1}\}$. For each fixed (ρ, v), $\{f_6(\rho, v, l)\}_l$ is one of the already known deformation sequences starting with $\phi_i(\mathbf{n}(N_2, k_i)(v))$. The parameter ρ gives the annulus $f_5(\rho)$, where the deformation is supported. Precisely, recall the notaion (4.16), define:

- $f_6(z_0, z_0, l) = \phi_i(z_0)$, for $l \in \{0, 1, \ldots, 3^{N_1}\}$.
- if $\rho \in (z_0, z_s] \cap I(1, N_2)_0$ or $\rho \in [z_0, z_s] \cap I(1, N_2)_1$, consider

$$x_1 = \mathbf{n}(N_2, k_i) \circ f_3(\rho), \quad x_2 = \mathbf{n}(N_2, k_i)(v) \quad \text{and} \quad \mu \in \{1, 2\},$$

such that $A_\mu(V_{f_4(\rho)}) = f_5(\rho)$. Define

$$f_6(\rho, v, l) = \begin{cases} \phi_i(x_2) & \text{if } l = 0 \\ T_{(x_1, \mu)}(2, l) & \text{if } l = 1, \ldots, f_2(v) \\ T_{(x_1, \mu)}(2, f_2(v)) & \text{if } l = f_2(v), \ldots, 3^{N_1}. \end{cases}$$

Take $N_3 = N_1 + N_2 + 2$ and define $f_7 : I(1, N_3)_0 \to I(1, N_2)$ such that $f_7(w)$ is the unique cell of least dimension in $I(1, N_2)$ containing w. In order to simplify the notation, from now on, use $\mathbf{n} = \mathbf{n}(N_3, N_2)$.

Given $w \in I(1, N_3)_0$, the deformation sequence $\{\psi_i(\lambda, w)\}_\lambda$ starts always with $\psi_i(0, w) = \phi_i(\mathbf{n}(N_3, k_i)(w))$. Then we deform it using those $f_6(\rho, v, l)$ for the following choices:

- $v = \mathbf{n}(w)$;
- ρ being a face of $f_7(w)$;

- $\mathbf{n}(w) \in \rho^0$;
- $l \leq f_8(w, \rho)$,

where $f_8(w, \rho)$ is the last ingredient we need to introduce to write ψ_i. Note that v is determined by w. And there are at most two possibilities for ρ. In case we have two possibilities, $w \in I(1, N_3)_0$ is not a vertex of $I(1, N_2)_0$ and ρ is equal either to the 1-cell $f_7(w)$ or to the vertex $\mathbf{n}(w)$.

The maps $f_8(\cdot, \rho)$ are constructed as follows:

- if $\rho = [z_0]$, then $\mathbf{n}(w) = z_0$ and we put $f_8(w, z_0) = 0$;
- for $\rho \in (z_0, z_s] \cap I(1, N_2)_0$ and $w \in I(1, N_3)_0$ with $\mathbf{n}(w) = \rho$, we define

$$f_8(w, \rho) = \max\{0, f_2(\rho) - \gamma\},$$

  where the number γ is either equal to zero, if $|w - \rho| \leq 3^{-(N_2+1)}$, or

$$\gamma = \inf\{\mathbf{d}(w, \tilde{w}) : \tilde{w} \in I(1, N_3)_0, |\tilde{w} - \rho| \leq 3^{-(N_2+1)}\},$$

  otherwise, where $\mathbf{d} : I(1, N_3)_0 \times I(1, N_3)_0 \to \mathbb{N}$ is the grid distance;
- for $\rho \in [z_0, z_s] \cap I(1, N_2)_1$, write $\rho = [x, y]$ and let $S \subset [x, y]$ be the set of vertices in the middle third, i.e.,

$$S = [x + 3^{-(N_2+1)}, y - 3^{-(N_2+1)}] \cap I(1, N_3)_0.$$

  Given $w \in I(1, N_3)_0$ with $f_7(w) = \rho$, define

$$f_8(w, \rho) = \max\{0, f_2(\mathbf{n}(w)) - \mathbf{d}(w, S)\}.$$

Construction of ψ_i:

At this point, we are ready to write the map

$$\psi_i : I(1, N(i))_0 \times ([z_0, z_s] \cap I(1, N(i))_0) \to \mathcal{Z}_{n-1}(M),$$

and conclude the argument. Note that we simplified notation $N(i) = N_3(i)$.

If $w \in [z_0, z_s] \cap I(1, N_3)_0$ and $\mathbf{n}(w) = z_0$, put

$$\psi_i(j, w) = \phi_i(z_0), \tag{4.22}$$

for each $j \in I(1, N_3)_0$.

From now on, suppose $\mathbf{n}(w) \neq z_0$, then $f_2(\mathbf{n}(w)) > 0$. We start with

$$\psi_i(j, w) = \phi_i(\mathbf{n}(N_3, k_i)(w)), \tag{4.23}$$

for $j \in \{0, 3^{-N_3}\}$. Next, for each $\rho \in [z_0, z_s] \cap I(1, N_2)$ that is a face of $f_7(w)$ with $\mathbf{n}(w) \in \rho^0$ and $1 \leq j \cdot 3^{N_3} \leq 3^{N_1}$, define

$$\psi_i(j, w) \llcorner f_5(\rho) = f_6(\rho, \mathbf{n}(w), \min\{j \cdot 3^{N_3}, f_8(w, \rho)\}) \llcorner f_5(\rho). \tag{4.24}$$

For $Z = M - \cup\{f_5(\rho) : \rho \text{ is a face of } f_7(w) \text{ and } \mathbf{n}(w) \in \rho^0\}$ and $0 \leq j \cdot 3^{N_3} \leq 3^{N_1}$, put

$$\psi_i(j, w) \llcorner Z = \phi_i(\mathbf{n}(N_3, k_i)(w)) \llcorner Z. \tag{4.25}$$

Finally, for $3^{N_1} \leq j \cdot 3^{N_3} \leq 3^{N_3}$, we set

$$\psi_i(j, w) = \psi_i(3^{N_1 - N_3}, w). \tag{4.26}$$

Properties of ψ_i:

To verify $\lim_{i\to\infty} \mathbf{f}(\psi_i) = 0$, we claim:

Claim 6. *$\mathbf{f}(\psi_i) \leq 2\delta_i$.*

Proof. Each $\{\psi_i(j, w)\}_j$ is $(2\delta_i)$-fine in the mass norm. In fact, the support of this deformation sequence is at most two of the $f_5(\rho)$ annuli. On each such annuli, the deformation follows some sequence $f_6(\rho, v, l)$, which is δ_i-fine by Claim 4. So, we need only to check the fineness for terms of the form $\mathbf{M}(\psi_i(j, w) - \psi_i(j, \tilde{w}))$, where $w, \tilde{w} \in \text{dmn}(\psi_i)$ are so that $|w - \tilde{w}| = 3^{-N_3}$. In this case, there exists unique $\rho \in I(1, N_2)_1$ with $w, \tilde{w} \in \rho$. Set $\rho = [\rho_1, \rho_2]$.

We divide the analysis in three cases, depending on which third of ρ the vertices w and $\tilde{w}$ lie:

(1) $w, \tilde{w} \in [\rho_1, \rho_1 + 3^{-(N_2+1)}]$;

(2) $w, \tilde{w} \in [\rho_1 + 3^{-(N_2+1)}, \rho_2 - 3^{-(N_2+1)}]$;

(3) $w, \tilde{w} \in [\rho_2 - 3^{-(N_2+1)}, \rho_2]$.

In the first case, the deformations of $\psi_i(0, w)$ and $\psi_i(0, \tilde{w})$ coincide in $f_5(\rho_1)$, because $\mathbf{n}(N_3, k_i)(w) = \mathbf{n}(N_3, k_i)(\tilde{w})$ and $f_8(w, \rho_1) = f_8(\tilde{w}, \rho_1)$ are equal to $f_2(\rho_1)$. Furthermore, in the annulus $f_5(\rho)$ they are also deformed by the same $f_6(\rho, \mathbf{n}(w), l)$, but we may have $|f_8(w, \rho) - f_8(\tilde{w}, \rho)| = 1$. In this case, item (c) of Claim 4 again implies fineness δ_i between $\psi_i(j, w)$ and $\psi_i(j, \tilde{w})$. Case (3) is completely analogous to this one.

Consider now case (2). If w and $\tilde{w}$ are not two different middle vertices, their deformation inside $f_5(\rho)$ coincide, because $\mathbf{n}(N_3, k_i)(w) = \mathbf{n}(N_3, k_i)(\tilde{w})$ and $f_8(w, \rho) = f_2(\mathbf{n}(w)) = f_8(\tilde{w}, \rho)$. Then, $\psi_i(j, w)$ and $\psi_i(j, \tilde{w})$ differ at most in one of the annuli $f_5(\rho_1)$ and $f_5(\rho_2)$. Indeed, it is not possible $f_8(w,\rho_1) \neq 0$ and $f_8(w,\rho_2) \neq 0$, simultaneously. In case of difference, we argument using item (c) of Claim 4 once more to conclude the fineness is at most δ_i in this case.

Finally, if $[w,\tilde{w}]$ is the middle 1-cell of $I(1,N_3)_0$ inside ρ and the deformation sequences $\{\psi_i(j,w)\}_j$ and $\{\psi_i(j,\tilde{w})\}_j$ are not the same, then ρ is the middle 1-cell of $I(1,N_2)$ inside a 1-cell of $I(1,k_i)$. And we can use item (d) of Claim 4 to conclude $\mathbf{M}(\psi_i(j,w) - \psi_i(j,\tilde{w})) \leq \delta_i$.

□

Observe that, if $\mathbf{n}(N_3,k_i)(w) \in \{z_1,\dots,z_s\}$, there is $\rho \in [z_0,z_s] \cap I(1,N_2)$, face of $f_7(w)$ with $\mathbf{n}(w) \in \rho^0$ and $f_8(w,\rho) = f_2(\mathbf{n}(w)) = 3^{N_1}$. The mass of $\psi_i(1,w)$ on $f_5(\rho)$ is strictly lower than that of $\psi_i(0,w)$, indeed, depending on ρ, we can estimate this difference by:

- if ρ is a 0-cell,

$$\begin{aligned} ||\psi_i(1,w)||(f_5(\rho)) &< ||\psi_i(0,w)||(f_5(\rho)) - \varepsilon(V_{f_4(\rho)}) \\ &\leq ||\psi_i(0,w)||(f_5(\rho)) - \delta; \end{aligned}$$

- if ρ is a 1-cell,

$$||\psi_i(1,w)||(f_5(\rho)) < ||\psi_i(0,w)||(f_5(\rho)) - \delta + \frac{\delta_i}{2}.$$

Since mass of $\psi_i(0,w)$ increases at most δ_i on each annuli in which the deformation is not trivial, we conclude the following mass decay estimate:

$$\mathbf{M}(\psi_i(1,w)) \leq \mathbf{M}(\psi_i(0,w)) - \delta + 3\frac{\delta_i}{2}.$$

The extra information about the deformation of $\phi_i(z_k)$ are consequence of the fact that $f_8(z_k,\rho) = 0$, in case $\rho \in I(1,N_2)$ is a 1-cell. Then, they are deformed by $f_6(z_k,z_k,l)$ only. This concludes the argument.

CHAPTER 5

Min-max minimal hypersurfaces in non-compact manifolds

In this chapter, we use the min-max theory for intersecting slices developed in Chapter 2 to prove the main result of this thesis, Theorem 1 as stated in the Introduction. More precisely, we prove existence of closed embedded minimal hypersurfaces in a complete non-compact manifold N containing a bounded region Ω with smooth and strictly mean-concave boundary and with a thickness assumption on the geometry at infinity. This is a short chapter, in Section 5.1, we give an overview of the proof. Then, in the subsequent sections we detail two steps of our argument. In Section 5.2, we explain how do we use the $\star_k$-condition together with the monotonicity formula. Finally, in Section 5.3, we show how to restrict ourselves to the setting described in Chapter 2, in which the ambient manifold is compact and without boundary.

5.1 Overview of the proof

Proof of Theorem 1. We divide the proof in five steps.

Step 1:

Since the n-dimensional manifold N satisfies the $\star_k$-condition for some value $k \leq \frac{2}{n-2}$, there exist $p \in N$ and $R_0 > 0$ such that the estimates

$$\sup_{q \in B(p,R)} |\mathrm{Sec}_N|(q) \leq R^k$$

and

$$\inf_{q\in B(p,R)} inj_N(q) \geq R^{-\frac{k}{2}},$$

hold for geodesic balls $B(p,R)$ centered at p, of radius $R \geq R_0$, as defined in the Introduction. Choose a non-negative proper Morse function $f : N \to [0,+\infty)$ and let $\{\Sigma_t\}_{t\geq 0}$ be the one-parameter sweepout of the integral cycles induced by the level sets of f:

$$\Sigma_t := \partial\big(\{x \in N : f(x) < t\}\big) \in \mathcal{Z}_{n-1}(N). \tag{5.1}$$

This family has the special property that $\mathcal{H}^{n-1}(\Sigma_t)$ is a continuous function. Since $\overline{\Omega} \subset N$ is compact, we conclude

$$L := L\left(\{\Sigma_t\}_{t\geq 0}\right) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \text{spt}(||\Sigma_t||) \cap \overline{\Omega} \neq \varnothing\} < +\infty. \tag{5.2}$$

Step 2**:**

Consider $t_0 > 0$ such that Σ_{t_0} is a smooth regular level of f and large enough to satisfy the following property: any connected minimal hypersurface $\Sigma^{n-1} \subset N$, intersecting $\overline{\Omega}$, with non-empty boundary and $\inf_{\partial\Sigma} f \geq t_0$, must satisfy $\mathcal{H}^{n-1}(\Sigma) \geq 2L$.

The existence of such t_0 is a consequence of the monotonicity formula for minimal hypersurfaces and the fact that N satisfies the $\star_k$-condition for some $k \leq \frac{2}{n-2}$. We accomplish this argument in Section 5.2.

Step 3**:**

It is possible to obtain a compact Riemannian manifold without boundary (M^n, h), containing an isometric copy of $\{f \leq t_0\}$ and such that f extends to M as a Morse function, which is denoted by f_1. In Section 5.3 below, we perform a construction that gives one possible M. Since $\overline{\Omega} \subset \{f < t_0\}$, we have a copy of Ω inside M, which is also denoted Ω. Suppose $f_1(M) = [0,1]$.

Step 4**:**

Let $\Gamma = \{\Gamma_t\}_{t\in[0,1]}$ be the sweepout of M given by $\Gamma_t = f_1^{-1}(t)$. Consider the set of intersecting times

$$\text{dmn}_\Omega(\Gamma) = \{t \in [0,1] : \text{spt}(||\Gamma_t||) \cap \overline{\Omega} \neq \varnothing\}.$$

Observe that Γ_t coincides with the slice Σ_t, for every $0 \leq t \leq t_0$, and that $t \notin \text{dmn}_\Omega(\Gamma)$, for $t_0 < t \leq 1$. In particular,

$$L(\Gamma, \Omega) := \sup\{\mathcal{H}^{n-1}(\Gamma_t) : t \in \text{dmn}_\Omega(\Gamma)\}$$

coincides with the number L defined in Step 1. We apply now the Min-max Theory developed in Chapter 2. There exists a non-trivial closed embedded minimal hypersurface $\Sigma^{n-1} \subset M$ with $\mathcal{H}^{n-1}(\Sigma) \leq L$ and intersecting $\overline{\Omega}$.

Since our Min-max methods follow the discrete setting of Almgren and Pitts, we still have to construct out of Γ a non-trivial homotopy class $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$, such that $\mathbf{L}(\Pi, \Omega) \leq L(\Gamma, \Omega) = L$. This is the content of Theorem 9. We can apply this result because Γ is continuous in the $\mathbf{F}$-metric and non-trivial.

Step 5**:**

The choice of t_0 in Step 2 guarantees that any component of Σ^{n-1} that intersects $\overline{\Omega}$, can not go outside $\{f \leq t_0\}$. Otherwise, this would imply that

$$2L \leq \mathcal{H}^{n-1}(\Sigma) \leq L.$$

In conclusion, any intersecting component of Σ is a closed embedded minimal hypersuface in the open manifold N. $\square$

5.2 A slice far from the mean-concave region

Let us prove that the $\star_k$-condition implies that any minimal hypersurface that intersects Ω and, at the same time, goes far from Ω have large Hausdorff measure $\mathcal{H}^{n-1}$. Let $\Sigma^{n-1} \subset N$ be a minimal hypersurface as in Step 2 above. The main tool for this subsection is the following consequence of the monotonicity formula, see expression (1.26) of Subsection 1.3.

Proposition 2. *For every $q \in B(p, R)$ and $0 < s < R^{-\frac{k}{2}}$, we have*

$$\mathcal{H}^{n-1}(\Sigma \cap B(q, s)) \geq \frac{\omega_{n-1}}{e^{(n-1)\sqrt{R^k} s}} \cdot s^{n-1}, \tag{5.3}$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Consider $R_0 \leq R_1$ and $l \in \mathbb{N}$, for which $\overline{\Omega} \subset B(p, R_1)$ and $B(p, R_1 + l) \subset \{f < t_0\}$, t_0 to be chosen. Recall that Σ intersects $\overline{\Omega}$ and it is not contained in the sublevel set $\{f < t_0\}$. Then, for every $i \in \{1, 2 \ldots, l\}$, there are points $q_{ij} \in \Sigma$, $j \in \{1, 2, \ldots, \lfloor \sqrt{(R_1 + i)^k} \rfloor\}$, whose distance in N to p are given by

$$d(q_{ij}, p) = R_1 + i - 1 + \frac{2j - 1}{2\sqrt{(R_1 + i)^k}}.$$

Observe that $q_{ij} \in B(p, R_1+i)$ and that the balls $B_{ij} = B(q_{ij}, 2^{-1}(R_1+i)^{-\frac{k}{2}})$ are pairwise disjoint. Apply Proposition 2 to conclude that

$$\begin{aligned} \mathcal{H}^{n-1}(\Sigma) &\geq \sum_{i=1}^{l}\sum_{j} \mathcal{H}^{n-1}(\Sigma \cap B_{ij}) \\ &\geq \frac{\omega_{n-1}}{(2\sqrt{e})^{n-1}} \sum_{i=1}^{l} \left(\lfloor (R_1+i)^{\frac{k}{2}} \rfloor \cdot (R_1+i)^{-\frac{k(n-1)}{2}} \right). \end{aligned} \tag{5.4}$$

Since $k \leq \frac{2}{n-2}$, if we keep R_1 fixed and let $l \in \mathbb{N}$ go to infinity, then the right-hand side of expression (5.4) also tends to infinity. Choose $l \in \mathbb{N}$ large, for which that is greater than $2L$, where $L = L(\{\Sigma_t\}_{t\geq 0})$ is the number we considered in Step 1 above. This concludes the argument, because we chose t_0 such that $B(p, R_1+l) \subset \{f < t_0\}$.

5.3 A closed manifold containing the mean-concave region

Since t_0 is a regular value of f, $f^{-1}([t_0, t_0+3\varepsilon])$ has no critical points for sufficiently small $\varepsilon > 0$. Moreover, there exists a natural diffeomorphism

$$\xi : f^{-1}([t_0, t_0+3\varepsilon]) \to \Sigma_{t_0} \times [0, 3\varepsilon],$$

that identifies $f^{-1}(t_0)$ with $\Sigma_{t_0} \times \{0\}$. Suppose that $\Sigma_{t_0} \times [0, 3\varepsilon]$ has the product metric $g|_{\Sigma_{t_0}} \times \mathcal{L}$, where $\mathcal{L}$ denotes the Lebesgue measure. Consider the pullback metric $g_1 = \xi^*(g|_{\Sigma_{t_0}} \times \mathcal{L})$ and choose a smooth bump function $\varphi : [t_0, t_0+3\varepsilon] \to [0,1]$, such that

- $\varphi(t) = 1$, for every $t \in [t_0, t_0+\varepsilon]$, and
- $\varphi(t) = 0$, for every $t \in [t_0+2\varepsilon, t_0+3\varepsilon]$.

On $f^{-1}([t_0, t_0+3\varepsilon])$, mix the original g and the product metric g_1 using the smooth function φ, to obtain

$$h_1(x) = (\varphi \circ f)(x) g(x) + (1 - (\varphi \circ f)(x)) g_1(x). \tag{5.5}$$

This metric admits the trivial smooth extension $h_1 = g$ over $f^{-1}([0, t_0])$. Summarizing, we produced a Riemannian manifold with boundary

$$M_1 = (\{f \leq t_0 + 3\varepsilon\}, h_1),$$

with the following properties:

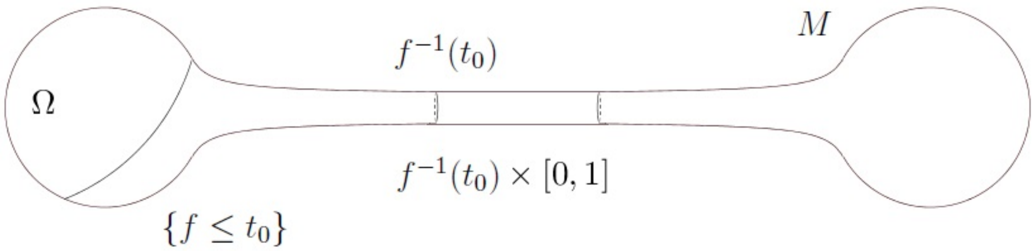


Figure 5.1: The closed manifold to which we apply our min-max technique

(i) M_1 contains an isometric copy of $\{f \leq t_0\}$ with the metric g;

(ii) near $\partial M_1 = f^{-1}(t_0 + 3\varepsilon)$, $h_1 = g_1$ is the product metric.

Then, it is possible to attach two copies of M_1 via the identity map of ∂M_1. Doing this we obtain a closed manifold M with a smooth Riemannian metric h, that coincides with h_1 on each half, see Figure 5.1. Precisely, $M = M_1 \cup_{\mathcal{I}} M_1$, where $\mathcal{I}$ denotes the identity map of ∂M_1.

The metric h is smooth because of item (ii). Moreover, item (i) says that M has an isometric copy of $\{f \leq t_0\}$. Finally, let us construct a Morse function f_1 on M that coincides with f in the first piece M_1. On the second half M_1, put $f_1 = 2(t_0 + 3\varepsilon) - f$.

CHAPTER 6

Metrics of positive scalar curvature and unbounded widths

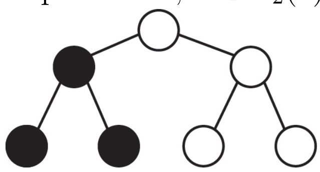
The content of this chapter is independent of the previous results of this thesis and is contained in [44]. As state in the Introduction of this work, we prove existence of metrics of scalar curvature greater than or equal to 6 on the three-sphere and arbitrarily large widths. The method involves combinatorial arguments, which are developed in Section 6.1. In Section 6.2, we explain the construction of the examples. Finally, in Sections 6.3 and 6.4 we give bounds for the geometric objects that we are interested in, the widths and the supremum of the isoperimetric profiles.

6.1 Combinatorial results

In this section, we state and prove our combinatorial results. For each positive integer $m \in \mathbb{N}$, we use T_m to denote the full binary tree, for which all the 2^m leaves, nodes of vertex degree one, have depth m. Recall that the vertex degree of a node is the number of edges incidents to it. Observe that T_m has $2^{m+1} - 1$ nodes, being 2^k nodes on the k-th level of depth.

We consider 2-colorings of the nodes of T_m. A 2-coloring of T_m is an assignment of one color, black or white, for each node. Fixed a 2-coloring, an edge is said to be a *dichromatic edge* if it connects nodes with different colors. We are interested in 2-colorings which minimize the quantity of dichromatic edges for a fixed number of black nodes or leaves. See Figure 6.1 below.

Figure 6.1: A 2-coloring of the binary tree T_2 with three black nodes and one dichromatic edge exactly. In particular, $3 \in B_2(1)$.



Definition 8. Let $m, d \in \mathbb{N}$ be positive integers. We define $B_m(d)$ to be the set of values b, with $1 \leq b \leq 2^{m+1} - 1$, for which there exists a 2-coloring of T_m with d dichromatic edges and b black nodes exactly.

The first statement that we prove in this section is the following upper bound on the size of the sets $B_m(d)$:

Lemma 10. *For every $m, d \in \mathbb{N}$, it follows that $\#B_m(d) \leq 2^d m^d$.*

Proof. Consider a 2-coloring $\mathcal{C}$ of the nodes of T_m which has d dichromatic edges exactly. Let k be the highest level of depth of T_m for which one of those d edges joins a node on the k-th level to a node on the $(k+1)$-th level of depth. Choose one of these deeper dichromatic edges and observe that it determines a monochromatic component of $\mathcal{C}$ which is a copy of $T_{m-(k+1)}$, for some $0 \leq k \leq m-1$. Changing the color of the nodes of this $T_{m-(k+1)}$ yields a 2-coloring $\mathcal{C}'$ with $d-1$ dichromatic edges and b black nodes, for some $b \in B_m(d-1)$. Since we changed the colors of the nodes in the copy of $T_{m-(k+1)}$ only and they all have the same color on $\mathcal{C}$, the number of black nodes of $\mathcal{C}$ is either $b + (2^{m-k} - 1)$ or $b - (2^{m-k} - 1)$. Therefore,

$$B_m(d) \subset \{b \pm (2^{m-k} - 1) : b \in B_m(d-1) \text{ and } 0 \leq k \leq m-1\},$$

and we conclude that

$$\#B_m(d) \leq 2m \cdot \#B_m(d-1)$$

and the statement follows inductively and from the fact that $2^{m+1} - 1$ is the only element of $B_m(0)$. $\square$

Definition 9. For $m \in \mathbb{N}$ and $1 \leq b \leq 2^{m+1} - 1$, let $d'_m(b)$ denote the minimum integer $d \in \mathbb{N}$ for which $b \in B_m(d)$. In other words, any 2-coloring of T_m with b black nodes exactly has at least $d'_m(b)$ dichromatic edges.

As a consequence of the above estimate we prove a qualitative result that guarantees that there is no uniform bound on the values $d'_m(b)$.

Proposition 3. *There exist integers $b(m)$, with $1 \leq b(m) \leq 2^{m+1} - 1$ and such that $\{d'_m(b(m))\}_{m\in\mathbb{N}}$ is an unbounded sequence.*

Proof. Suppose, by contradiction, there exists $D \in \mathbb{N}$ such that $d'_m(b) \leq D$, for all $m \in \mathbb{N}$ and $1 \leq b \leq 2^{m+1} - 1$. Then, each such b belongs to $B_m(d)$, for some $d \leq D$. By Lemma 10, we have

$$\#\Big(\bigcup_{d\leq D} B_m(d)\Big) \leq \sum_{d\leq D} 2^d m^d.$$

But, for fixed $m \in \mathbb{N}$, there are $2^{m+1} - 1$ possible values for b, which can not be controlled by the polynomial right hand side of the above expression. □

Since the vertex degrees of the nodes of the considered trees do not exceed 3, we can use Proposition 3 to prove the following:

Corollary 4. *Given $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $b(m) < 2^{m+1} - 1$, such that any 2-coloring of T_m with $b(m)$ black nodes exactly has at least k pairs of neighboring nodes with different colors. Moreover, for $1 \leq b \leq 2^{m+1} - 1$, any 2-coloring of T_m with b black nodes exactly has at least $(k - |b - b(m)|)/5$ pairwise disjoint pairs of neighboring nodes with different colors.*

Proof. Indeed, let m and $b(m)$ be such that $d'_m(b(m)) \geq k$. This choice is allowed by Proposition 3. Then, any 2-coloring of T_m with $b(m)$ black nodes exactly has at least k pairs of neighboring nodes with different colors.

For $1 \leq b \leq 2^{m+1} - 1$, we can estimate $d'_m(b)$ using the formula:

$$|d'_m(t) - d'_m(s)| \leq |t - s|. \tag{6.1}$$

To prove this relation, we need to verify $|d'_m(t) - d'_m(t+1)| \leq 1$ only. Consider a 2-coloring of T_m with $t + 1$ black nodes and $d'_m(t + 1)$ dichromatic edges exactly. Let N be a black node of this coloring with the property that no other black node lives in a level deeper than its level of depth. Changing the color of N to white we obtain a 2-coloring with exactly t black nodes and at most $d'_m(t+1)+1$ dichromatic edges. This implies that $d'_m(t) \leq d'_m(t+1)+1$. Analogously, we obtain $d'_m(t+1) \leq d'_m(t)+1$, and we are done with the proof of expression (6.1).

The choice of m and $b(m)$, together with equation (6.1) gives us that $d'_m(b) \geq k - |b - b(m)|$. Observe that each pair of neighboring nodes has a common node with four other pairs of neighboring nodes at most. This allows us to conclude that any 2-coloring of T_m with b black nodes exactly has at least $(k - |b - b(m)|)/5$ pairwise disjoint pairs of neighboring nodes with different colors. And this concludes the proof of the corollary. □

Corollary 4 is key in the proof of the lower bound that we provide for the supremum of the isoperimetric profiles of the Riemannian metrics that we construct in the next section. This is done in Section 6.4.

The rest of this section is devoted to the discussion of an interesting quantitative statement related to the previous results. It also can be applied to provide estimates for the widths of our examples.

Definition 10. We define the *dichromatic value of t leaves in T_m* as the least number of dichromatic edges of a 2-coloring of T_m with t black leaves exactly. We denote this number by $d_m(t)$.

The following statement about dichromatic values is the analogous of formula (6.1) for a fixed number of black leaves. We omit its proof here.

Lemma 11. $|d_m(t) - d_m(s)| \leq |s - t|$.

Theorem 14. *For each integer $m > 1$, let*

$$a(m) = \begin{cases} 1 + 2 + 2^3 + \ldots + 2^{m-2}, & \text{if } m \text{ is odd} \\ 1 + 2^2 + 2^4 + \ldots + 2^{m-2}, & \text{if } m \text{ is even.} \end{cases}$$

Then, $d_m(a(m)) \geq \lceil \frac{m}{2} \rceil$.

Proof. The proof is by induction. Define $a(1) = 1$. The initial cases, $m = 1$ and 2, are very simple. Suppose that the statement is true for $m - 1$ and $m - 2$. Let $\mathcal{C}$ be a 2-coloring of T_m with $a = a(m)$ black leaves exactly.

Let us count the number of different types of nodes in the $(m-1)$-th level of depth of T_m. Use α to denote the number of nodes which have two neighbor leaves of different colors on $\mathcal{C}$ and, similarly, let β be the number of nodes which have two black neighbor leaves. Then, we can write the number of black leaves as $a(m) = \alpha + 2 \cdot \beta$. In particular, this expression implies that α is an odd number and

$$\beta = \frac{1-\alpha}{2} + \frac{a(m)-1}{2} = \frac{1-\alpha}{2} + a(m-1) - m', \tag{6.2}$$

where $m' = 1$ if m is even and $m' = 0$ if m is odd. Indeed, it is easily seen that $a(m) - 1 = 2 \cdot (a(m-1) - m')$, for every $m \geq 3$.

Next, we induce a 2-coloring $\mathcal{C}'$ on the nodes of T_{m-1}. Let $\mathcal{T}_{m-2} \subset \mathcal{T}_{m-1} \subset T_m$ be such that T_m minus $\mathcal{T}_{m-1}$ is the set of leaves of T_m, and $\mathcal{T}_{m-1}$ minus $\mathcal{T}_{m-2}$ is the set of leaves of $\mathcal{T}_{m-1}$. We begin to define $\mathcal{C}'$ on T_{m-1}, identified with $\mathcal{T}_{m-1}$, asking $\mathcal{C}'$ to be equal to $\mathcal{C}$ on $\mathcal{T}_{m-2}$ and on the α leaves of $\mathcal{T}_{m-1}$ which have neighbor leaves of different colors on $\mathcal{C}$. The β leaves of $\mathcal{T}_{m-1}$ which have two black neighbor leaves of $\mathcal{C}$ on T_m are colored black on $\mathcal{C}'$. The remaining leaves of $\mathcal{T}_{m-1}$ have two white neighbor leaves of $\mathcal{C}$ on T_m and receive the color white on $\mathcal{C}'$. The important properties of $\mathcal{C}'$ are:

(i) The number of black leaves of $\mathcal{C}'$ on $\mathcal{T}_{m-1}$ is greater than or equal to β and at most $\alpha + \beta$;

(ii) The number of dichromatic edges of $\mathcal{C}$ is at least α plus the number of dichromatic edges of $\mathcal{C}'$.

The first of these properties follows directly from the construction. To prove the second, we begin by observing that the dichromatic edges of $\mathcal{C}'$ that are in $\mathcal{T}_{m-2}$ are, automatically, dichromatic edges of $\mathcal{C}$ on T_m. Then, we analyze cases to deal with the dichromatic edges of $\mathcal{C}'$ that use leaves of $\mathcal{T}_{m-1}$. We omit this simple analysis.

Let us use t to denote the number of black leaves of $\mathcal{C}'$ on $\mathcal{T}_{m-1}$. From (6.2) and property (i) above, we conclude that

$$\frac{1-\alpha}{2} - m' \leq t - a(m-1) \leq \frac{1+\alpha}{2} - m'. \tag{6.3}$$

By Lemma 11 and the induction hypothesis, we have

$$d_{m-1}(t) \geq d_{m-1}(a(m-1)) - \frac{1+\alpha}{2} \geq \left\lceil \frac{m-1}{2} \right\rceil - \frac{1+\alpha}{2}. \tag{6.4}$$

By definition, $\mathcal{C}'$ has at least $d_{m-1}(t)$ dichromatic edges. This implies, together with property (ii) and equation (6.4), that

$$\#\{\text{dichromatic edges of } \mathcal{C}\} \geq \left\lceil \frac{m-1}{2} \right\rceil + \frac{\alpha-1}{2}. \tag{6.5}$$

Since α is odd, if m is even we have that the number of dichromatic edges of $\mathcal{C}$ is at least $\lceil (m-1)/2 \rceil = \lceil m/2 \rceil$ and we are done. Otherwise, m is odd and equation (6.5) provides us

$$\#\{\text{dichromatic edges of } \mathcal{C}\} \geq \left\lceil \frac{m}{2} \right\rceil - 1 + \frac{\alpha-1}{2} = \left\lceil \frac{m}{2} \right\rceil + \frac{\alpha-3}{2}.$$

If $\alpha \geq 3$, the induction process ends and we are done.

From now on, we suppose that m is odd and $\alpha = 1$. The arguments are going to be similar to the previous one, but one level above on T_m. Recall that $\alpha = 1$ means that there is a unique node of T_m which is neighbor of leaves with different colors. Observe that each node on the $(m-2)$-th level of T_m, leaf of $\mathcal{T}_{m-2}$, is associated with four leaves of T_m, the leaves at edge-distance two. A unique node on this level is associated with an odd number (1 or 3) of black leaves of $\mathcal{C}$, because $\alpha = 1$. We denote this node and odd number by N and φ, respectively.

Let us use θ and γ to denote the number of nodes on the $(m-2)$-th level of T_m which are associated, respectively, with two and four black leaves of $\mathcal{C}$. Then, we can write the number of black leaves of $\mathcal{C}$ as $a(m) = \varphi + 2\theta + 4\gamma$.

In particular, this expression implies that $\varphi + 2\theta \equiv 3 \bmod 4$ and

$$\gamma = -\frac{1+\varphi+2\theta}{4} + a(m-2).$$

Next, we induce a 2-coloring $\mathcal{C}''$ on the nodes of T_{m-2}. We follow the same $\mathcal{T}_{m-2} \subset T_m$ notation that we introduced before. And now, we still need the analogous $\mathcal{T}_{m-3} \subset \mathcal{T}_{m-2}$. We choose $\mathcal{C}''$ to coincides with $\mathcal{C}$ on the nodes of $\mathcal{T}_{m-3}$ and on the θ leaves of $\mathcal{T}_{m-2}$ which are associated with two black leaves of $\mathcal{C}$. The γ leaves of $\mathcal{T}_{m-2}$ which are associated with four black leaves of $\mathcal{C}$ are colored black. The node N is colored white if $\varphi = 1$ and black if $\varphi = 3$. The leaves of $\mathcal{T}_{m-2}$ which remain uncolored receive the color white. As in the previous step, the important properties of $\mathcal{C}''$ are:

(I) The number of black leaves of $\mathcal{C}''$ is between the values $(\varphi-1)/2+\gamma$ and $(\varphi-1)/2+\gamma+\theta$;

(II) The number of dichromatic edges of $\mathcal{C}$ is at least $1+\theta$ plus the number of dichromatic edges of $\mathcal{C}'$.

By the same reasoning that we used to obtain equation (6.5), we have

$$\#\{\text{dichromatic edges of } \mathcal{C}\} \geq \left\lceil \frac{m-2}{2} \right\rceil + \frac{1+\varphi+2\theta}{4}.$$

Using that m is odd, $\varphi = 1$ or 3 and $\varphi + 2\theta \equiv 3 \bmod 4$, we conclude that the right hand side of the above expression is greater than or equal to $\lceil m/2 \rceil$. This finishes the induction step. □

An easy consequence of Theorem 14 is the following:

Corollary 5. *Given $m \in \mathbb{N}$ there exists $a(m) \in \mathbb{N}$, $1 \leq a(m) \leq 2^m$, such that any 2-coloring of T_m with $a(m)$ black leaves exactly has at least $(\lceil m/2 \rceil)/5$ pairwise disjoint pairs of neighboring nodes with different colors.*

6.2 Constructing the examples

In this section, we introduce our examples. We begin with a brief discussion about the Gromov-Lawson metrics of positive scalar curvature. Then, for each full binary tree we construct an associated metric of scalar curvature greater than or equal to 6 on the three-sphere.

6.2.1 Gromov-Lawson metrics

Gromov and Lawson developed a method that is adequate to perform connected sums of manifolds with positive scalar curvature, see [30]. They proved the following statement:

Let (M^n, g) be a Riemannian manifold of positive scalar curvature. Given $p \in M$, $\{e_1, \dots, e_n\} \subset T_pM$ an orthonormal basis, and $r_0 > 0$, it is possible to define a positive scalar curvature metric g' on the punctured geodesic ball $B(p, r_0) - \{p\}$ that coincides with g near the boundary $\partial B(p, r_0)$, and such that $(B(p, r_1) - \{p\}, g')$ is isometric to a half-cylinder, a product manifold $S^2(\rho) \times [0, +\infty)$ of a 2-sphere of radius ρ and a half-line, for some $r_1 > 0$.

Remark 15. Since $(B(p, r_1) - \{p\}, g')$ is isometric to a half-cylinder, there exists $0 < r < r_1$ so that the g' volume of $B(p, r_0) - B(p, r)$ is bigger than one half of the g volume of the removed geodesic ball $B(p, r_0)$.

6.2.2 Fundamental blocks

In order to construct our examples, we use the above metrics to build three types of fundamental blocks.

The first type is obtained from the above construction using the standard round metric on $M = S^3$, any $p \in S^3$ and orthonormal basis and $r_0 = 1$. We use $S_1 = (S^3 - B(p, r), g')$ to denote this block, where $0 < r < r_1 < r_0 = 1$ is chosen as in Remark 15. Observe that S_1 is a manifold with boundary, has positive scalar curvature and it has a product metric near the boundary two-sphere ∂S_1. To obtain the second fundamental block, S_2, we perform the same steps with $M = S_1$ and choosing the new removed geodesic ball to be antipodally symmetric to $B(p, 1)$. Finally, the third block, S_3, is obtained from S^3 after three application of Gromov-Lawson procedure by removing three disjoint geodesic balls $B(p_i, 1)$, $i = 1, 2$ and 3.

Summarizing, the fundamental blocks S_1, S_2 and S_3 are obtained from the standard three-sphere by removing one, two or three geodesic balls, respectively, and attaching a copy of a fixed piece for each removed ball. Each fundamental block has a product metric near its boundary spheres. Up to a re-scaling, admit that they have scalar curvature greater than or equal to 6. Also, by Remark 15, we can suppose that the attached piece has volume bigger than one half of the volume of the removed balls.

6.2.3 Metrics associated with binary trees

For each full binary tree T_m, we use the fundamental blocks to construct an associated metric on S^3.

In our examples, each node of T_m will be associated to a fundamental block. Following the notation of Subsection 6.2.2, for a node of degree k we associate a fundamental block of type S_k. We connect the blocks which correspond to neighboring nodes of T_m by identifying one boundary sphere of the first to a boundary sphere of the second with reverse orientations. After performing all identifications, we obtain a metric on S^3, which is denoted by g_m and has scalar curvature $R \geq 6$.

The metric g_m decomposes S^3 in $2^{m+1}-1$ disjoint closed regions, which are isometric to the standard three-sphere with either one, two or three identical disjoint geodesic balls removed, and $2^{m+1}-2$ connecting tubes. Moreover, by construction, the tubes are isometric to each other and their volume is greater than the volume of one of the removed geodesic balls.

6.3 Lower bounds on the widths

We begin this section by recalling what sweepouts and widths of Riemannian metrics on S^3 are. After reintroducing these interesting geometrical objects, we prove that the widths of the metrics g_m constructed in Section 6.2 converge to infinity.

We use $I = [0,1] \subset \mathbb{R}$ to denote the closed unit interval. The 2-dimensional Hausdorff measure on S^3 induced by a Riemannian metric g is denoted by $\mathcal{H}^2$. Let us remember the definition of sweepouts.

A *sweepout* of (S^3, g) is a family $\{\Sigma_t\}_{t\in I}$ of smooth 2-spheres, which are boundaries of open sets $\Sigma_t = \partial\Omega_t$ such that:

1. Σ_t varies smoothly in $(0,1)$;
2. $\Omega_0 = \varnothing$ and $\Omega_1 = S^3$;
3. Σ_t converges to Σ_τ, in the Haursdorff topology, as $t \to \tau$;
4. $\mathcal{H}^2(\Sigma_t)$ is a continuous function of $t \in I$.

Let Λ be a set of sweepouts of (S^3, g). It is said to be saturated if given a map $\phi \in C^\infty(I \times S^3, S^3)$ such that $\phi(t,\cdot)$ are diffeomorphism of S^3, all of which isotopic to the identity, and a sweepout $\{\Sigma_t\}_{t\in I} \in \Lambda$, we have $\{\phi(t,\Sigma_t)\}_{t\in I} \in \Lambda$. The *width of* (S^3, g) *associated with* Λ is the following min-max invariant:

$$W(S^3, g, \Lambda) = \inf_{\{\Sigma_t\}\in\Lambda} \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t).$$

From now on, we fix a saturated set of sweepouts Λ. The main result of this work is the following:

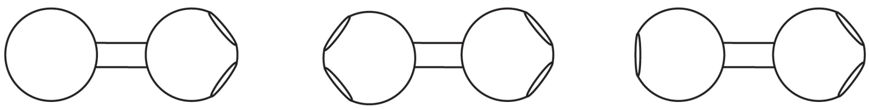


Figure 6.2: The possible regions that we obtain after gluing together two neighboring spherical regions.

Theorem 15. *The sequence* $\{g_m\}_{m\in\mathbb{N}}$ *of Riemannian metrics on* S^3 *that we constructed satisfies:*

$$\lim_{m\to\infty} W(S^3, g_m, \Lambda) = +\infty.$$

Proof. Recall that g_m is a metric on S^3 which is related to the full binary tree T_m. Also, there are $2^{m+1}-1$ disjoint closed subsets of (S^3, g_m) isometric to the standard round metric on the three-sphere with either one, two or three identical disjoint balls removed. These spherical regions are associated to nodes of T_m and two of them are glued to each other if, and only if, their corresponding nodes are neighbors in T_m. It is also important to recall that there is a tube connecting such neighboring regions, all of which isometric to each other. For convenience, let us denote these spherical regions by:

- L_i if it corresponds to a leaf of T_m;
- B if it corresponds to the node of degree 2;
- A_j if it corresponds to a node of degree 3.

After connecting any two neighboring spherical regions, we obtain a region $\mathcal{A}$ which is isometric to one of the domains depicted in Figure 6.2.

In the figure, the first region was obtained by gluing a spherical region of type L_i to its only A_j neighbor. The others are obtained by gluing either two A_j regions or the B region to one its two A_j neighbors.

Choose $\alpha > 0$ such that 2α is strictly less then the volume of one A_j. Then, $0 < \alpha < vol(\mathcal{A}) - \alpha < vol(\mathcal{A})$, for any of the possible $\mathcal{A}$'s. By the relative isoperimetric inequality, there exists $C > 0$ such that for any open subset $\Omega \subset \mathcal{A}$ of finite perimeter and $\alpha \leq vol(\Omega) \leq vol(\mathcal{A}) - \alpha$, we have

$$\mathcal{H}^2(\partial\Omega \cap int(\mathcal{A})) \geq C. \tag{6.6}$$

Moreover, since we have three possible isometric types of $\mathcal{A}$'s only, we can suppose that this constant does not depend on the type of $\mathcal{A}$.

Let $\{\Sigma_t\}_{t\in I}$ be a sweepout of (S^3, g_m). Consider the associated open sweepout $\{\Omega_t\}$, for which $\Sigma_t = \partial\Omega_t$. Let $a(m)$ be the integer provided by Corollary 5. Choose the least $t_0 \in I$ for which we have $vol(\Omega_{t_0} \cap L_i) \geq \alpha$, for at least $a(m)$ values of $i \in \{1, 2, 3, \ldots, 2^m\}$.

Observe that, at most $a(m) - 1$ of the L_i's can satisfy $vol(\Omega_{t_0} \cap L_i) > \alpha$. Up to a reordering of their indices, suppose that $vol(\Omega_{t_0} \cap L_i) \geq \alpha$, for $i = 1, 2, \ldots, a(m)$, and $vol(\Omega_{t_0} \cap L_i) \leq \alpha$, otherwise.

On T_m, consider the 2-coloring defined in the following way: the leaves associated to $L_1, \ldots, L_{a(m)}$ are colored black, the other leaves are colored white and the nodes which are not leaves are colored black if, and only if, the volume of Ω_{t_0} inside the corresponding spherical region is greater than or equal to α. This 2-coloring of T_m has exactly $a(m)$ black leaves.

By Corollary 5, the constructed coloring has at least $m/10$ pairwise disjoint pairs of neighboring nodes with different colors. Observe that each such pair gives one $\mathcal{A}$ type region for which we have

$$\alpha \leq vol(\Omega_{t_0} \cap \mathcal{A}) \leq vol(\mathcal{A}) - \alpha. \tag{6.7}$$

This follows because we chose α in such a way that the volume of our spherical regions are greater than 2α. By equations (6.6) and (6.7) we conclude that $\mathcal{H}^2(\partial\Omega_{t_0} \cap int(\mathcal{A})) \geq C$. Since this holds for $m/10$ pairwise disjoint $\mathcal{A}$ type regions, we have $\mathcal{H}^2(\Sigma_{t_0}) \geq C \cdot m/10$. This concludes our argument. □

6.4 On the isoperimetric profiles

In this section, we discuss the fact that isoperimetric profiles $\mathcal{I}_m$ of the metrics g_m are not uniformly bounded. This part also relies on a combinatorial argument, we use Corollary 4. The idea is to decompose S^3 into $2^{m+1} - 1$ pieces of identical g_m volumes, all of which being the union of one spherical region (L_i, B or A_j) with a portion of their neighboring tubes.

This decomposition of S^3 by balanced pieces is only possible because we chose the tube large enough to have g_m volume greater than the spherical volume μ of the removed geodesic balls. This allows us to decompose S^3 into $2^{m+1} - 2$ pieces with volume $vol_{g_0}(S^3) + \tau - 2\mu$ and one piece with volume $vol_{g_0}(S^3)$, and with the other desired properties, where g_0 is the standard round metric on S^3 and τ is the volume of the gluing tube. The piece with volume $vol_{g_0}(S^3)$ is the one related to the only node of degree 2 in T_m. For each pair of neighboring spherical regions, the boundary of the associated balanced regions has exactly one component in the connecting tube. This component is a spherical slice which splits the volume τ of the tube as $\tau - \mu$ plus μ, as depicted in Figure 6.3.

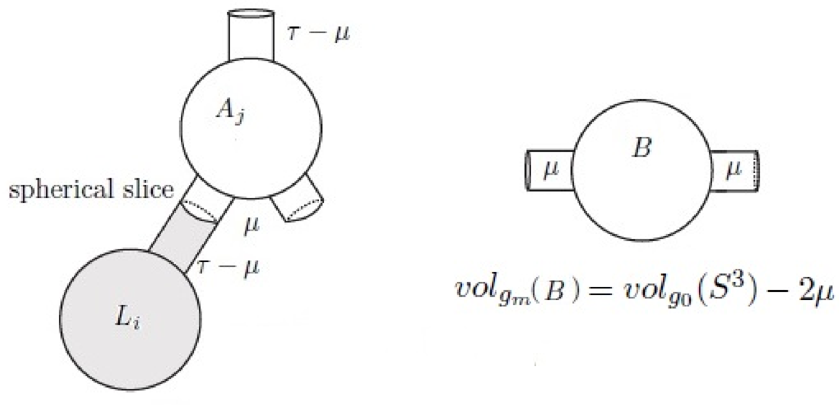


Figure 6.3: The balanced regions.

There are three types of balanced regions, depending on the number of boundary components, all of which we denote by $\mathcal{M}$. In this part, we use the isoperimetric inequality in its full generality: there exists $C > 0$ so that

$$\min\{\mathcal{H}^3(\Omega), \mathcal{H}^3(\mathcal{M} - \Omega)\}^{2/3} \le C \cdot \mathcal{H}^2(\partial\Omega \cap int(\mathcal{M})), \tag{6.8}$$

for every $\Omega \subset \mathcal{M}$. Since we have three types of $\mathcal{M}$ regions only, we suppose that $C > 0$ associated to $\mathcal{M}$ does not depend on its type.

Suppose, by contradiction, that there exists $L > 0$ such that $\mathcal{I}_m(v) \le L$, for every $m \in \mathbb{N}$ and $v \in [0, vol_{g_m}(S^3)]$.

For any $k \in \mathbb{N}$, let $m, b(m) \in \mathbb{N}$ be the integers provided by Corollary 4. Take $v(m) = b(m) \cdot (vol_{g_0}(S^3) + \tau - 2\mu)$ and let $\Omega \subset S^3$ be a subset with $vol_{g_m}(\Omega) = v(m)$ and such that $\mathcal{H}^2(\partial\Omega) \le L$. Observe that

$$\sum_{\mathcal{M}} \mathcal{H}^2(\partial\Omega \cap int(\mathcal{M})) \le \mathcal{H}^2(\partial\Omega) \le L.$$

Using the relative isoperimetric inequality, equation (6.8), we obtain:

$$\sum_{\mathcal{M}} \min\{\mathcal{H}^3(\Omega \cap \mathcal{M}), \mathcal{H}^3(\mathcal{M} - \Omega)\}^{2/3} \le C \cdot L.$$

Which implies that

$$\sum_{\mathcal{M}} \min\{\mathcal{H}^3(\Omega \cap \mathcal{M}), \mathcal{H}^3(\mathcal{M} - \Omega)\} \le C_1, \tag{6.9}$$

where $C_1 = (C \cdot L)^{3/2}$. Let $\mathcal{M}_1$ be the set of the $\mathcal{M}$ type regions for which $2 \cdot \mathcal{H}^3(\Omega \cap \mathcal{M}) < \mathcal{H}^3(\mathcal{M})$. Similarly, the $\mathcal{M}$ regions satisfying the opposite inequality, $2 \cdot \mathcal{H}^3(\Omega \cap \mathcal{M}) \geq \mathcal{H}^3(\mathcal{M})$, compose $\mathcal{M}_2$. Equation (6.9) implies

$$\sum_{\mathcal{M} \in \mathcal{M}_1} \mathcal{H}^3(\Omega \cap \mathcal{M}) + \sum_{\mathcal{M} \in \mathcal{M}_2} \mathcal{H}^3(\mathcal{M} - \Omega) \leq C_1. \tag{6.10}$$

Using that $\mathcal{H}^3(\mathcal{M} - \Omega) = \mathcal{H}^3(\mathcal{M}) - \mathcal{H}^3(\Omega \cap \mathcal{M})$ and the fact that

$$\sum_{\mathcal{M} \in \mathcal{M}_1} \mathcal{H}^3(\Omega \cap \mathcal{M}) + \sum_{\mathcal{M} \in \mathcal{M}_2} \mathcal{H}^3(\Omega \cap \mathcal{M}) = vol_{g_m}(\Omega) = v(m), \tag{6.11}$$

we easily conclude

$$\left| v(m) - \sum_{\mathcal{M} \in \mathcal{M}_2} \mathcal{H}^3(\mathcal{M}) \right| \leq C_1. \tag{6.12}$$

By the choice of $v(m)$, equation (6.12) implies that $|\#\mathcal{M}_2 - b(m)| \leq C_2$, where $C_2 = (C_1 + |\tau - 2\mu|)/(vol_{g_0}(S^3) + \tau - 2\mu)$ is a uniform constant.

Consider the 2-coloring $\mathcal{C}$ of T_m whose black nodes are those associated with the balanced regions in $\mathcal{M}_2$. Then, $\mathcal{C}$ has $\#\mathcal{M}_2$ black nodes exactly. By Corollary 4, there are at least $(k - |\#\mathcal{M}_2 - b(m)|)/5$ pairwise disjoint pairs of neighboring nodes with different colors. By a reasoning similar to the one that we used in the end of Section 6.3, we have that

$$\mathcal{H}^2(\partial\Omega) \geq C_3 \cdot (k - |\#\mathcal{M}_2 - b(m)|)/5, \tag{6.13}$$

for some $C_3 > 0$, which does not depend on m. Recalling that $\mathcal{H}^2(\partial\Omega) \leq L$ and $|\#\mathcal{M}_2 - b(m)| \leq C_2$, the above expression provides a uniform upper bound on k, which is arbitrary. This is a contradiction and we are done.

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