SURFACES MINIMALES : THÉORIE VARIATIONNELLE ET APPLICATIONS

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1. Introduction

2. First variation formula

Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ be a submanifold. We use the notation $|\Sigma|$ for the volume of Σ. Consider $(x_1, \ldots, x_k)$ local coordinates on Σ and let

$$g_{ij}(x) = g\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big), \text{ for } 1 \leq i, j \leq k,$$

be the components of $g|_\Sigma$. The volume element of Σ is defined to be the differential k-form $d\Sigma = \sqrt{\det\, g_{ij}(x)}$. The volume of Σ is given by

$$\mathrm{Vol}(\Sigma) = \int_\Sigma d\Sigma.$$

Consider the variation of Σ given by a smooth map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$. Use $F_t(x) = F(x,t)$ and $\Sigma_t = F_t(\Sigma)$. In this section we are interested in the first derivative of $\mathrm{Vol}(\Sigma_t)$.

Definition 2.1 (Divergence)**.** Let X be a arbitrary vector field on $\Sigma^k \subset M$. We define its divergence as

$$\mathrm{div}_\Sigma X(p) = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle, \tag{1}$$

where $\{e_1, \dots, e_k\} \subset T_p\Sigma$ is an orthonormal basis and ∇ is the Levi-Civita connection with respect to the Riemannian metric g.

Lemma 2.2.

$$\frac{\partial}{\partial t} d\Sigma_t = div_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t. \tag{2}$$

Proof. Note that

$$\frac{\partial}{\partial t}\det g = \mathrm{tr}(g^{-1}\partial_t g)\det g,$$

where $g^{-1} = (g^{ij}) = (g_{ij})^{-1}$. Then

$$\frac{\partial}{\partial t}\det g = \sum_{i,j}(g^{ij}\partial_t g_{ij})\det g.$$

We can calculate the first derivative of the metric using the compatibility of ∇ with respect to g

$$\partial_t g_{ij} = g(\nabla_{\partial F/\partial t}\partial_i F, \partial_j F) + g(\partial_i F, \nabla_{\partial F/\partial t}\partial_j F),$$

where $\partial_i F = \partial F/\partial x_i$. Use the symmetry of ∇ to commute $\nabla_{\partial F/\partial t}\partial_i F = \nabla_{\partial_i F}\partial F/\partial t$. Put everything together to obtain

$$\begin{aligned} \frac{\partial}{\partial t}\det g &= 2\sum_{i,j} g^{ij} g(\nabla_{\partial_i F}\partial F/\partial t, \partial_j F)\det g \\ &= 2 div_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right)\det g. \end{aligned}$$

□

We have then:

Theorem 2.3 (First Variation Formula I)**.**

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} div_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t.$$

In order to characterize the submanifolds that are critical values for area, we need to introduce a important geometric object and some calculation.

2nd fundamental form and mean curvature: Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ be a submanifold. For each $p \in \Sigma$, define the second fundamental form of $\Sigma \subset M$ as

$$B(V, W) = \nabla_V W - (\nabla_V W)^T = (\nabla_V W)^N,$$

where V, W are tangent to Σ. B is a symmetric tensor. If $\{e_1, \ldots, e_k\} \subset T_p\Sigma$ is an orthonormal basis, we define the mean curvature vector to be

$$\overrightarrow{H} = \text{tr } B = \sum_{i=1}^{k} \left(\nabla_{e_i} e_i\right)^N.$$

Lemma 2.4.

$$div_\Sigma X = div_\Sigma X^T - \langle X^N, \overrightarrow{H} \rangle.$$

Proof. Decompose X in tangent and normal parts $X = X^T + X^N$ and write

$$div_\Sigma X = div_\Sigma X^T + \sum_{i=1}^{k} \langle \nabla_{e_i} X^N, e_i \rangle.$$

Since X^N is normal to Σ and e_i is tangent, we have

$$0 = e_i \langle X^N, e_i \rangle = \langle \nabla_{e_i} X^N, e_i \rangle + \langle X^N, \nabla_{e_i} e_i \rangle. \tag{3}$$

Then

$$\begin{aligned} div_\Sigma X &= div_\Sigma X^T - \sum_{i=1}^{k} \langle X^N, \nabla_{e_i} e_i \rangle \\ &= div_\Sigma X^T - \langle X^N, \overrightarrow{H} \rangle. \end{aligned}$$

□

Theorem 2.5 (Divergence Theorem)**.** *Let Σ be a compact submanifold of M and X be a tangential vector field on Σ. Then*

$$\int_\Sigma div_\Sigma X d\Sigma = \int_{\partial \Sigma} g(X, \nu) d\sigma,$$

where $\nu \in T\Sigma$ is the unique unit outward normal to $\partial\Sigma$ and $d\sigma$ is the volume element of $\partial\Sigma$.

Theorem 2.6 (First variation formula II)**.**

$$\frac{d}{dt} |\Sigma_t| = -\int_{\Sigma_t} \langle \partial F / \partial t, \overrightarrow{H} \rangle d\Sigma_t + \int_{\partial \Sigma_t} \langle \partial F / \partial t, \nu \rangle d\sigma_t.$$

Moreover, if $X = \partial F / \partial t$ vanishes on $\partial\Sigma$ at $t = 0$, then

$$\left. \frac{d}{dt} \right|_{t=0} |\Sigma_t| = -\int_\Sigma \langle X, \overrightarrow{H} \rangle d\Sigma.$$

Corollary 2.7.
$$\frac{d}{dt}\bigg|_{t=0} |\Sigma_t| = 0, \text{ for any } X, \text{ with } X = 0 \text{ on } \partial\Sigma, \text{ iff } \overrightarrow{H} = 0.$$

Definition 2.8. $\Sigma^k \subset M$ is a minimal submanifold of M if $\overrightarrow{H} = 0$.

3. Examples

(Geodesic). For $k = 1$, Σ is a smooth curve $\gamma : I \to M$ satisfying
$$0 = \overrightarrow{H} = \left(\nabla_{\frac{\gamma'}{|\gamma'|}} \frac{\gamma'}{|\gamma'|}\right)^N = \nabla_{\frac{\gamma'}{|\gamma'|}} \frac{\gamma'}{|\gamma'|}.$$
Geodesics are minimal surfaces of dimension 1.

(Minimal surfaces of euclidean space). Start with the particular case of a minimal surfaces given as the graph of a differentiable function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^k$. Let $F : \Omega \to \text{graph}(u)$ be given by $F(x) = (x, u(x))$. Use also the notation
$$\frac{\partial F}{\partial x_i} = \left(\frac{\partial}{\partial x_i}, \frac{\partial u}{\partial x_i}\right).$$
The euclidean metric of $\mathbb{R}^{k+1}$ restricted to graph(u) can be written as
$$g_{ij} = g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right) = \delta_{ij} + \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$
where $\delta_{ij} = 1$, if $i = j$, and equal zero otherwise.

Exercise: Prove that $\det g = 1 + |\nabla u|^2 = 1 + \sum_{i=1}^k (\partial u/\partial x_j)^2$ and conclude
$$|\text{graph}(u)| = \int_\Omega \sqrt{1 + |\nabla u|^2}dx.$$

Consider the variations of graph(u) given by the graphs of functions $u_t = u + tv$, for any fixed v so that $v = 0$ on $\partial\Omega$. Then

$$\frac{d}{dt}\bigg|_{t=0} |\text{graph}(u_t)| = \frac{d}{dt}\bigg|_{t=0} \int_\Omega \sqrt{1 + |\nabla(u + tv)|^2}dx$$
$$= \int_\Omega \langle \nabla v, \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\rangle dx = -\int_\Omega v \, \text{div}_{\mathbb{R}^k}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)dx.$$

The graph of a function u is a minimal surface iff u satisfies the minimal surface equation
$$\text{div}_{\mathbb{R}^k}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{4}$$
Equivalently, if u is a solution of the second order elliptic quasilinear P.D.E. given by:
$$\sum_{i,j}\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2}\right)\partial_i\partial_j u = 0. \tag{5}$$

Basic examples of minimal surfaces in $\mathbb{R}^3$.

- The plane: $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$.
- The catenoid: $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = \cosh z\}$. A bigraph, it is obtained by the revolution of the curve $y = \cosh z$ over the z-axis.
- The helicoid: $(u, v) \mapsto (u \cos v, u \sin v, v)$. This is a multigraph over $\mathbb{R}^2 \setminus \{0\}$ for the function $u(x, y) = \arctan(y/x)$.
- The Scherk's surface: graph(u), where $u(x, y) = \log \frac{\cos x}{\cos y}$, for $x, y \in (-\pi/2, \pi/2)$. Scherk's surface is a doubly periodic minimal surface.

It is important to classify and understand these minimal surfaces in $\mathbb{R}^3$. In this direction there is the Colding-Minicozzi Theory, that studies the structure of embedded minimal disks. One important application is

Theorem 3.1 (Meeks and Rosenberg, 2005)**.** *The only complete embedded simply connected minimal surfaces in $\mathbb{R}^3$ are the plane and the helicoid.*

4. Maximum principle

In this section, we apply the strong maximum principle for the minimal surface equation.

Theorem 4.1 (Strong maximum principle)**.** *Let v be a solution to*

$$\sum_{i,j} a_{ij} \partial_i \partial_j v + \sum_i b_i \partial_i v = 0,$$

where $(a_{ij}(x))_{ij}$ is positive definite for each $x \in \Omega$. If $v \leq 0$ and $v = 0$ at some point, then $v \equiv 0$.

The main result of this section is

Theorem 4.2 (Maximum principle)**.** *Let $u_1, u_2 : \Omega \to \mathbb{R}$ be solutions to minimal surface equation. Suppose Ω is connected, $u_1 \leq u_2$ and $u_1(p) = u_2(p)$ for some $p \in \Omega$. Then $u_1 \equiv u_2$.*

Proof. Let $v = u_1 - u_2$. We have

$$0 = \sum_{i,j} \left(\delta_{ij} - \frac{\partial_i u_1 \partial_j u_1}{1 + |\nabla u_1|^2} \right) \partial_i \partial_j v + \sum_{i,j} \left(\frac{\partial_i u_2 \partial_j u_2}{1 + |\nabla u_2|^2} - \frac{\partial_i u_1 \partial_j u_1}{1 + |\nabla u_1|^2} \right) \partial_i \partial_j u_2.$$

Since $v = u_1 - u_2 \leq 0$ and $v(p) = 0$, the strong maximum principle gives $v \equiv 0$. Then we conclude $u_1 = u_2$. □

Corollary 4.3. *Let $\Sigma_1^{n-1}, \Sigma_2^{n-1} \subset \mathbb{R}^n$ be connected minimal hypersurfaces, such that Σ_2 is on one side of Σ_1. If there exists $p \in \Sigma_1 \cap \Sigma_2 \neq \emptyset$, then $\Sigma_1 = \Sigma_2$.*

Remark. This result also holds for minimal hypersufaces $\Sigma_1, \Sigma_2 \subset M$ of a Riemannian manifold.

5. Calibration: area-minimizing surfaces

Let $u : \Omega \to \mathbb{R}$ be a solution to the minimal surface equation, where $\Omega \subset \mathbb{R}^n$. Then $u_t(x) = u(x) + t$ is also a solution and $\Sigma_t = \mathrm{graph}(u_t)$ form a foliation of $\Omega \times \mathbb{R}$ by minimal hypersurfaces. This means for every $p \in \Omega \times \mathbb{R}$ we can find t so that $p \in \Sigma_t$. For each vector $V \in T_p\mathbb{R}^{n+1}$, let V^T be the component of V in $T_p\Sigma_t$. Consider the following differential n-form

$$\omega(p)(X_1, \dots, X_n) = \mathrm{Vol}_{\Sigma_t}(X_1^T, \dots, X_n^T). \tag{6}$$

Observe we can also write ω as

$$\begin{aligned} \omega(p)(X_1, \dots, X_n) &= \mathrm{Vol}_{\mathbb{R}^{n+1}}(X_1^T, \dots, X_n^T, N) \\ &= (-1)^n \mathrm{Vol}_{\mathbb{R}^{n+1}}(X_1^T, \dots, X_n^T, N) \\ &= (-1)^n (\iota_N \mathrm{Vol}_{\mathbb{R}^{n+1}})(X_1, \dots, X_n), \end{aligned}$$

where N is the unit normal vector to Σ_t and $\iota_N \mathrm{Vol}_{\mathbb{R}^{n+1}}$ is the contraction of $\mathrm{Vol}_{\mathbb{R}^{n+1}}$ with the vector field N.

Cartan's formula: If $\mathcal{L}_X\omega$ is the Lie derivative of ω with respect to X, then

$$d(\iota_X\omega) + \iota_X(d\omega) = \mathcal{L}_X\omega.$$

Remark: If X is a vector field on (M^n, g), then

$$\mathcal{L}_X dV_g = (\mathrm{div}_g X) dV_g.$$

Using Cartan's formula and the previous remark we can calculate $d\omega$ as

$$\begin{aligned} (-1)^n d\omega &= d(\iota_N \mathrm{Vol}_{\mathbb{R}^{n+1}}) \\ &= \mathcal{L}_N \mathrm{Vol}_{\mathbb{R}^{n+1}} - \iota_N d(\mathrm{Vol}_{\mathbb{R}^{n+1}}) \\ &= \big(\mathrm{div}_{\mathbb{R}^{n+1}} N\big) \mathrm{Vol}_{\mathbb{R}^{n+1}}. \end{aligned}$$

Can use that N is explicitly given by $N = (1 + |\nabla u|^2)^{-1/2}(-\nabla u, 1)$ to conclude that

$$(-1)^n d\omega = \left(\sum_{i=1}^n \frac{\partial N}{\partial x_i}\right) \mathrm{Vol}_{\mathbb{R}^{n+1}} = -\mathrm{div}_{\mathbb{R}^n}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \mathrm{Vol}_{\mathbb{R}^{n+1}}.$$

Corollary 5.1. *$d\omega = 0$ iff graph(u) is minimal.*

This differential form ω has also the property

$$|\omega(X_1, \dots, X_n)| \le |X_1| \cdot \ldots \cdot |X_n|,$$

which we abbreviate for $|\omega| \le 1$.

Definition 5.2. (Harvey and Lawson, 1982) Let (M^n, g) be a Riemannian manifold. A differential k-form on $\Omega \subset M^n$ is a calibration if it satisfies

(1) $d\omega = 0$;
(2) $|\omega| \le 1$.

Definition 5.3. $\Sigma^k \subset M$ is calibrated by ω if $\omega|_\Sigma = \mathrm{Vol}_\Sigma$.

Theorem 5.4. *Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ be a calibrated submanifold. Then Σ minimizes volume in its homology class.*

Proof. Let Σ' be in the same homology class of Σ, then there exists a $(k+1)$-dimensional U so that $\Sigma' - \Sigma = \partial U$. Suppose Σ is calibrated by ω, then

$$0 = \int_U d\omega = \int_{\Sigma'} \omega - \int_\Sigma \omega.$$

Since $\omega|_\Sigma = \mathrm{Vol}_\Sigma$, we have

$$\mathrm{Vol}(\Sigma) = \int_\Sigma \omega = \int_{\Sigma'} \omega \leq \mathrm{Vol}(\Sigma').$$

□

Corollary 5.5. *Let $u : \overline{\Omega} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, be a solution to minimal surface equation. Then $\Sigma = graph(u)$ is calibrated. Therefore, for any $\Sigma' \subset \overline{\Omega} \times \mathbb{R}$, $\Sigma' = \Sigma$, we have $Vol(\Sigma) \leq Vol(\Sigma')$.*

Example: Complex submanifolds of $\mathbb{C}^n$.

Definition 5.6. $\Sigma \subset \mathbb{C}^n$ is complex if $T_p\Sigma$ is a complex vector subspace of $\mathbb{C}^n$, for every $p \in \Sigma$.

Consider the convention $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ and in coordinates $(z_1, \dots, z_n) = (x_1, \dots, x_n, y_1, \dots, y_n)$, with $z_k = x_k + iy_k$. The complex multiplication by i map is denoted by $J : \mathbb{C}^n \to \mathbb{C}^n$ and defined by

$$J(x_1, \dots, x_n, y_1, \dots, y_n) = (-y_1, \dots, -y_n, x_1, \dots, x_n).$$

Note that $J^2 = -I$. Consider also the following differential form on $\mathbb{C}^n$

$$\omega = \sum_{k=1}^n dx_k \wedge dy_k. \tag{7}$$

It can be seen that $\omega(X, Y) = \langle J(X), Y \rangle$ and that $d\omega = 0$. The differential form $\omega^k = \omega \wedge \dots \wedge \omega$, k-times, has complex degree k and satisfies $d\omega^k = 0$. To conclude that $\omega^k/k!$ calibrates complex k-dimensional submanifolds, use:

Exercise: (Wirtinger's Inequality) Let $V \subset \mathbb{C}^n$ be a vector subspace of real dimension $2k$ and let $\{X_1, \dots, X_{2k}\}$ be a basis of V. Then

$$\frac{\omega^k}{k!}(X_1, \dots, X_{2k}) \leq \mathrm{Vol}(X_1, \dots, X_{2k}).$$

Moreover, equality holds iff V is a complex subspace of $\mathbb{C}^n$.

Corollary 5.7. *Complex submanifolds of $\mathbb{C}^n$ are volume minimizers.*

Example: (Special Lagrangian submanifolds of $\mathbb{C}^n$) Let $\alpha = \mathrm{Re}(\Omega)$, where $\Omega = dz_1 \wedge \ldots \wedge dz_n$ is a complex valued n-differential form. Moreover, $d\alpha = \mathrm{Re}(d\Omega) = 0$ and $|\alpha| \leq 1$.

Definition 5.8. $\Sigma^n \subset \mathbb{C}^n$ is a special Lagrangian if Σ is calibrated by α.

Can consider also $\alpha_\theta = \mathrm{Re}(e^{i\theta}\Omega)$. It satisfies $d\alpha_\theta = 0$ and $|\alpha_\theta| \leq 1$.

Definition 5.9. $\Sigma^n \subset \mathbb{C}^n$ is Lagrangian if $J(T_p\Sigma) = T_p^\perp\Sigma$, for every $p \in \Sigma$.

For example, in $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, the real $\mathbb{R}^n$ is a Lagrangian.

Remark: If $\Sigma \subset \mathbb{C}^n$ is Lagrangian and $\overrightarrow{H} = 0$ (minimal), then Σ is calibrated by α_θ for some θ.

Consider the inner product on $\mathbb{C}^n$ given by $\langle\langle z, w\rangle\rangle = \sum_{k=1}^n z_k\overline{w_k}$. Let $\{e_1, \ldots, e_n\} \subset T_p\Sigma$ be an orthonormal basis such that

$$\{e_1, \ldots, e_n, J(e_1), \ldots, J(e_n)\}$$

is an orthonormal basis of $\mathbb{C}^n$. Define a $\mathbb{C}$-linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ by

$$A(\partial/\partial x_k) = e_k, \text{ for } k = 1, \ldots, n.$$

Then A is unitary and

$$\begin{aligned} \Omega(e_1, \ldots, e_n) &= \Omega(A(\partial/\partial x_1), \ldots, A(\partial/\partial x_n)) \\ &= (\det\nolimits_{\mathbb{C}} A)\Omega(\partial/\partial x_1, \ldots, \partial/\partial x_n) \\ &= \det\nolimits_{\mathbb{C}} A. \end{aligned}$$

Can write $\Omega_p(e_1, \ldots, e_n) = e^{i\theta(p)}$, for each $p \in \Sigma$. Here, $\theta(p)$ is called the Lagrangian angle.

Remark: Observe $d\theta(v) = \langle v, J(\overrightarrow{H})\rangle$, for each $v \in T\Sigma$. If Σ is Lagrangian and $\overrightarrow{H} = 0$, then θ is constant. In this case,

$$\mathrm{Re}(e^{-i\theta}\Omega)(e_1, \ldots, e_n) = \mathrm{Re}(\mathrm{Vol}(e_1, \ldots, e_n)) = \mathrm{Vol}(e_1, \ldots, e_n).$$

Therefore, Σ is volume minimizing.

6. Second Variation Formula

Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ be a submanifold. Consider the variation $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$. Recall

$$\begin{aligned} \frac{\partial}{\partial t} d\Sigma_t &= \mathrm{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t \\ &= \big(\sum_{i,j} g^{ij}\langle \nabla_{\partial_i F}\partial F/\partial t, \partial_j F\rangle\big) d\Sigma_t, \end{aligned}$$

in local coordinates $(x_1, \dots, x_k)$ on Σ. Observe $\frac{\partial}{\partial t} g^{-1} = -g^{-1}(\partial_t g) g^{-1}$, i.e.,

$$\begin{aligned}\frac{\partial}{\partial t} g^{ij} &= -\sum_{k,l} g^{ik} \partial_t g_{kl} g^{lj} \\ &= -\sum_{k,l} g^{ik} \big(\langle \nabla_{\partial_k F}(\partial F/\partial t), \partial_l F\rangle + \langle \partial_k F, \nabla_{\partial_l F}(\partial F/\partial t)\rangle \big) g^{lj}.\end{aligned}$$

Using the Riemann Curvature Tensor, we have

$$\begin{aligned}\partial_t \langle \nabla_{\partial_i F} \partial_t F, \partial_j F\rangle &= \langle \nabla_{\partial_t F} \nabla_{\partial_i F} \partial_t F, \partial_j F\rangle + \langle \nabla_{\partial_i F} \partial_t F, \nabla_{\partial_t F} \partial_j F\rangle \\ &= \langle (\nabla_{\partial_t F} \nabla_{\partial_i F} - \nabla_{\partial_i F} \nabla_{\partial_t F}) \partial_t F, \partial_j F\rangle \\ &\quad + \langle \nabla_{\partial_i F} \nabla_{\partial_t F} \partial_t F, \partial_j F\rangle + \langle \nabla_{\partial_i F} \partial_t F, \nabla_{\partial_t F} \partial_j F\rangle \\ &= \langle R^M(\partial_t F, \partial_i F) \partial_t F, \partial_j F\rangle \\ &\quad + \langle \nabla_{\partial_i F}(\nabla_{\partial_t F} \partial_t F), \partial_j F\rangle + \langle \nabla_{\partial_i F} \partial_t F, \nabla_{\partial_j F} \partial_t F\rangle,\end{aligned}$$

then

$$\begin{aligned}\sum_{i,j} g^{ij} \partial_t \langle \nabla_{\partial_i F} \partial_t F, \partial_j F\rangle &= \sum_{i,j} g^{ij} \langle R^M(\partial_t F, \partial_i F) \partial_t F, \partial_j F\rangle \\ &\quad + \mathrm{div}_\Sigma(\nabla_{\partial_t F} \partial_t F) \\ &\quad + \sum_{i,j} g^{ij} \langle \nabla_{\partial_i F} \partial_t F, \nabla_{\partial_j F} \partial_t F\rangle.\end{aligned}$$

Use $X = \partial_t F = \partial F/\partial_t$, when $t = 0$, as simplified notation for the variational vector field. If $\{e_1, \dots, e_k\} \subset T\Sigma$ is an orthonormal basis, we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathrm{div}_{\Sigma_t}\big(\partial F/\partial t\big) &= \sum_{i=1}^k \langle R^M(X, e_i) X, e_i\rangle + \mathrm{div}_\Sigma(\nabla_X X) \\ &\quad + \sum_{i=1}^k \langle \nabla_{e_i} X, \nabla_{e_i} X\rangle \\ &\quad - \sum_{i,j=1}^k (\langle \nabla_{e_i} X, e_j\rangle + \langle \nabla_{e_j} X, e_i\rangle) \langle \nabla_{e_i} X, e_j\rangle.\end{aligned}$$

The third term on the above expression can be rewritten as

$$\begin{aligned}\sum_{i=1}^k \langle \nabla_{e_i} X, \nabla_{e_i} X\rangle &= \sum_{i=1}^k \langle \nabla^\perp_{e_i} X, \nabla^\perp_{e_i} X\rangle + \sum_{i,j,l} \langle \langle \nabla_{e_i} X, e_j\rangle e_j, \langle \nabla_{e_i} X, e_l\rangle e_l\rangle \\ &= |\nabla^\perp X|^2 + \sum_{i,j} \langle \nabla_{e_i} X, e_j\rangle^2.\end{aligned}$$

This simplifies the derivative $\mathrm{div}_\Sigma(\partial F/\partial t)$ to

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathrm{div}_{\Sigma_t}\big(\partial F/\partial t\big) &= \sum_{i=1}^k \langle R^M(X,e_i)X,e_i\rangle + \mathrm{div}_\Sigma(\nabla_X X) \\ &\quad + |\nabla^\perp X|^2 - \sum_{i,j=1}^k \langle \nabla_{e_j} X, e_i\rangle \langle \nabla_{e_i} X, e_j\rangle. \end{aligned}$$

Finally, we get the second variation formula for area

Theorem 6.1 (Second Variation Formula)**.**

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t| &= \frac{d}{dt}\Big|_{t=0} \left(\int_{\Sigma_t} div_{\Sigma_t}(\partial F/\partial t) d\Sigma_t \right) \\ &= \int_\Sigma \left(\sum_{i=1}^k |\nabla^\perp_{e_i} X|^2 + div_\Sigma(\nabla_X X) \right. \\ &\quad - \sum_{i=1}^k R^M(X,e_i,X,e_i) + \big(\sum_{i=1}^k \langle \nabla_{e_i} X, e_i\rangle\big)^2 \\ &\quad \left. - \sum_{i,j=1}^k \langle \nabla_{e_j} X, e_i\rangle\langle \nabla_{e_i} X, e_j\rangle \right) d\Sigma. \end{aligned}$$

Moreover, if X is normal,

$$\langle \nabla_{e_i} X, e_j\rangle = -\langle X, \nabla_{e_i} e_j\rangle = -\langle X, B(e_i,e_j)\rangle. \tag{8}$$

Theorem 6.2. *If X is normal and $\overrightarrow{H}=0$, then*

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t| &= \int_\Sigma \left(|\nabla^\perp X|^2 - \sum_{i=1}^k R^M(X,e_i,X,e_i) \right. \\ &\quad \left. - \sum_{i,j=1}^k \langle B(e_i,e_j), X\rangle^2 \right) d\Sigma + \int_{\partial\Sigma} \langle \nabla_X X, \nu\rangle d\sigma. \end{aligned}$$

Minimization Problems.

Plateau Problem: Given $\Gamma^{k-1} \subset M$ an oriented boundary. Find Σ^k with $\partial\Sigma = \Gamma$ and $|\Sigma| = \inf_{\partial\Sigma'=\Gamma} |\Sigma'|$. The natural variations here are: $X=0$ on Γ. Then Σ has zero mean curvature, $\overrightarrow{H}=0$, and $\partial\Sigma = \Gamma$.

Homology: Given $\alpha \in H_k(M,\mathbb{Z})$ an homology class. Find Σ^k with $[\Sigma]=\alpha$ and $|\Sigma| = \inf_{[\Sigma']=\alpha} |\Sigma'|$. In this case, $\overrightarrow{H}=0$ and Σ is closed.

Free Boundary Problem: Given $N^p \subset M^n$. Find Σ^k with $\partial\Sigma \subset N$ and $|\Sigma| = \inf_{\partial\Sigma' \subset N} |\Sigma'|$. The first variation formula gives

$$\frac{d}{dt}\bigg|_{t=0} |\Sigma_t| = -\int_\Sigma \langle \overrightarrow{H}, X\rangle d\Sigma + \int_{\partial\Sigma} \langle X, \nu\rangle d\sigma,$$

for any variation X such that $X|_{\partial\Sigma} \in TN$. Then $\overrightarrow{H} = 0$ and $\nu \perp TN$.

Jacobi Operator. Assume now Σ smooth, compact and $\overrightarrow{H} = 0$. For a given variation $X = \partial F/\partial t$, use the following notation for the first and second variations

$$\delta\Sigma(X) = \frac{d}{dt}\bigg|_{t=0} |\Sigma_t| \quad \text{and} \quad \delta^2\Sigma(X, X) = \frac{d^2}{dt^2}\bigg|_{t=0} |\Sigma_t|.$$

If X is normal to Σ and $X = 0$ on $\partial\Sigma$, then

$$\delta^2\Sigma(X, X) = -\int_\Sigma \langle L_\Sigma X, X\rangle d\Sigma,$$

where L_Σ is the Jacobi Operator

$$L_\Sigma X = \Delta^\perp X + \big(\sum_{i=1}^k R^M(X, e_i)e_i\big)^\perp + \sum_{i,j=1}^k \langle B(e_i, e_j), X\rangle B(e_i, e_j).$$

Remaks:

(1) The last term on Jacobi Operator is known as Simons' Operator

$$A(X) = \sum_{i,j=1}^k \langle B(e_i, e_j), X\rangle B(e_i, e_j).$$

(2) If X is normal and U is tangential, the normal connection $\nabla^\perp$ means

$$\nabla^\perp_U X = (\nabla_U X)^\perp.$$

(3) The normal Laplacian is defined as

$$\Delta^\perp X = \sum_{i=1}^k \left(\nabla^\perp_{e_i}\nabla^\perp_{e_i} X - \nabla^\perp_{(\nabla_{e_i} e_i)^T} X\right),$$

and satisfies the integration by parts formula

$$\int_\Sigma |\nabla^\perp X|^2 d\Sigma = -\int_\Sigma \langle X, \Delta^\perp X\rangle d\Sigma.$$

(4) L_Σ is an elliptic operator. Its spectrum is given by

$$\lambda_1 \le \lambda_2 \le \ldots \to +\infty.$$

There is a L^2-basis of eigenfunctions $\{X_i\}_i$, i.e., $L_\Sigma X_i + \lambda_i X_i = 0$. Moreover, for each i,

$$\delta^2\Sigma(X_i, X_i) = -\int_\Sigma \langle L_\Sigma X_i, X_i\rangle d\Sigma = \lambda_i \int_\Sigma |X_i|^2 d\Sigma.$$

Definition 6.3. The Morse index of Σ is the number of negative eigenvalues of L_Σ counted with multiplicities.

Definition 6.4. $\Sigma^k \subset M$ is stable if index of Σ is zero. In this case $\delta^2\Sigma(X,X) \geq 0$, for every admissible X.

Hypersurface Case: Suppose $\Sigma^{n-1} \subset M^n$ is two-sided (trivial normal bundle). In this case the admissible vector fields can be written as $X = fN$, for some $f \in C^\infty(\Sigma)$, where N is a globally defined unit normal to Σ. Then, the first variation formula simplify to

$$\delta\Sigma(X) = -\int_\Sigma f\langle \overrightarrow{H}, N\rangle d\Sigma = -\int_\Sigma fH d\Sigma, \tag{9}$$

where $H = \langle \overrightarrow{H}, N\rangle$ is the mean curvature function of Σ. About the second variation formula, we have

$$\begin{aligned} \delta^2\Sigma(X,X) &= \int_\Sigma \left(|\nabla_\Sigma f|^2 - (Ric(N,N) + |B|^2)f^2\right) d\Sigma \\ &= -\int_\Sigma fL_\Sigma f d\Sigma, \end{aligned} \tag{10}$$

where $L_\Sigma f = \Delta_\Sigma f + (Ric(N,N) + |B|^2)f$ and B is the second fundamental form. In this case, if Σ is stable, then

$$\int_\Sigma (Ric(N,N) + |B|^2) f^2 d\Sigma \leq \int_\Sigma |\nabla_\Sigma f|^2 d\Sigma, \text{ for every } f \in C_c^\infty(\Sigma). \tag{11}$$

This is known as the stability inequality.

Corollary 6.5. *No two-sided closed minimal hypersurface $\Sigma^{n-1} \subset M$ can be stable if $Ric_M > 0$.*

Proof. Otherwise, plug $f \equiv 1$ in the stability inequality to obtain

$$0 < \int_\Sigma (Ric(N,N) + |B|^2) d\Sigma \leq \int_\Sigma |\nabla_\Sigma f|^2 d\Sigma = 0.$$

This is a contradiction. $\square$

7. Monotonicity Formula

Theorem 7.1 (Monotonicity formula for minimal submanifolds in $\mathbb{R}^n$)**.** *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal surface (compact and with boundary $\partial\Sigma$), so that $\partial\Sigma \cap \overline{B}_t = \emptyset$, where $p \in \mathbb{R}^n$, $0 < s < t$ and $B_r = B_r^{\mathbb{R}^n}(p)$. Then*

$$\frac{Vol(\Sigma \cap B_t)}{t^k} - \frac{Vol(\Sigma \cap B_s)}{s^k} = \int_{(B_t \setminus B_s)\cap\Sigma} \frac{|(x-p)^N|^2}{|x-p|^{k+2}} d\Sigma. \tag{12}$$

Remark: In particular, the density function

$$\theta_p(r) = \frac{\mathrm{Vol}(\Sigma \cap B_r)}{\omega_k r^k},$$

is non-decreasing in r, where ω_k is the k-th dimensional volume of the k-th dimensional unit disk. Note that if $r \mapsto \theta_p(r)$ is constant, then Σ is a cone with vertex p, because $(x-p)^N \equiv 0$.

Definition 7.2. The density of Σ at p is defined to be $\theta_p = \lim_{r\to 0} \theta_p(r)$. In particular, if $p \in \Sigma$ and Σ is smooth at p, then $\theta_p \geq 1$.

Corollary 7.3. *If $p \in \Sigma$ and Σ is smooth at p, then*

$$\frac{Vol(\Sigma \cap B_r)}{\omega_k r^k} \geq \theta_p \geq 1.$$

Moreover, if $Vol(\Sigma \cap B_r) = Vol(D_r^k) = \omega_k r^k$, for some $r > 0$, then Σ has to be a cross sectional disk of radius r.

Proof. The first claim is an immediate consequence of the monotonicity formula. In case of equality $\mathrm{Vol}(\Sigma \cap B_r) = \mathrm{Vol}(D_r^k)$, the monotonicity formula also gives that Σ is a cone with vertex p. Since it is smooth at p, the Σ has to be a cross sectional disk. □

Co-area Formula: Let Σ^k be a manifold, $h : \Sigma \to \mathbb{R}$ be a proper Lipschitz function and $f \in L^1_{loc}(\mathbb{R})$. Then

$$\int_{\{s\leq h\leq t\}} f|\nabla h| d\Sigma = \int_s^t \left(\int_{\{h=r\}} f d\sigma_r \right) dr. \tag{13}$$

Remark: A good reference for the co-area formula is the book "Lecture on Geometric Measure Theory", by Leon Simon.

Natural ambient vector fields. As usual, let $e_1, \dots, e_n$ be the constant vector fields given by the canonical direction on $\mathbb{R}^n$ and consider also the vector field $x \mapsto (x-p) = \sum_{i=1}^n (x_i - p_i) e_i$.

Proposition 7.4. *If $\Sigma^k \subset \mathbb{R}^n$, then*

(1) $div_\Sigma e_i = 0$
(2) $div_\Sigma (x-p) = k$.

This follows immediately from $\nabla_v e_i = 0$ and $\nabla_v x = v$, for every $v \in \mathbb{R}^n$.

Proposition 7.5. *If $\Sigma^k \subset \mathbb{R}^n$ is minimal ($\overrightarrow{H} = 0$), then*

(1) $div_\Sigma (e_i)^T = 0$
(2) $div_\Sigma (x-p)^T = k$.

Remarks:

- $\Delta_\Sigma f = \mathrm{div}(\nabla_\Sigma f)$
- $e_i = \nabla x_i$, then $(e_i)^T = \nabla_\Sigma x_i$
- $(x-p) = \frac{1}{2}\nabla|x-p|^2$, then $(x-p)^T = \frac{1}{2}\nabla_\Sigma|x-p|^2$.

Corollary 7.6. *If $\Sigma^k \subset \mathbb{R}^n$ is minimal ($\overrightarrow{H} = 0$), then*

(1) $\Delta_\Sigma x_i = 0$
(2) $\Delta_\Sigma |x-p|^2 = 2k$.

Proposition 7.7 (Convex hull property)**.** *Let $\Sigma^k \subset \mathbb{R}^n$ be a compact minimal surface with boundary $\partial\Sigma$. Then Σ is contained in the convex hull Conv($\partial\Sigma$) of the boundary.*

Definition 7.8. Let S be a subset of $\mathbb{R}^n$. Its convex hull is defined as being the intersection of all half-spaces containing S.

Proof. A half-space H of $\mathbb{R}^n$ can be written as $H = \{h \leq 0\}$, for some function $h(x) = \sum_{i=1}^n a_i x_i + b$. Suppose $\partial\Sigma \subset H$. Since $h|_{\partial\Sigma} \leq 0$ and $\Delta_\Sigma h = 0$, then, by maximum principle, $h \leq 0$ in Σ. This is true for each half-space containing $\partial\Sigma$, so $\Sigma \subset \mathrm{Conv}(\partial\Sigma)$. ☐

Corollary 7.9. *Let $\Sigma^2 \subset \mathbb{R}^n$ be a minimal disk and B be a ball so that $\partial\Sigma \cap B = \emptyset$. Then, each component of $\Sigma \cap B$ is simply connected.*

Proof. Suppose, by contradiction, there is a connect component of $\Sigma\cap B$ that is not simply connected. Take a closed curve $\gamma \subset \Sigma \cap B$ that does not bound a disk in $\Sigma \cap B$. Because Σ is a disk, can find $\Omega\Sigma$ so that $\partial\Omega = \gamma$. Since Ω is minimal, the convex hull property property gives $\Omega \subset \mathrm{Conv}(\gamma) \subset B$. This is a contradiction. ☐

In general, the same proof gives

Corollary 7.10. *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold and B be a ball so that $\partial\Sigma \cap B = \emptyset$. Then, the induced homomorphism $H_{k-1}(\Sigma\cap B) \to H_{k-1}(\Sigma)$ is injective.*

Proof of monotonicity formula: Let $d : \Sigma \to \mathbb{R}$ be the distance function to p, $d(x) = |x-p|$ and $v_k(r) = \mathrm{Vol}(\Sigma \cap B_r)$. By co-area formula, can write

$$\begin{aligned} v_k(r) &= \int_{\Sigma\cap B_r} 1 d\Sigma \\ &= \int_{d \leq r} \frac{1}{|\nabla_\Sigma d|} |\nabla_\Sigma d| d\Sigma \\ &= \int_0^r \left(\int_{\Sigma\cap\partial B_\tau} \frac{1}{|\nabla_\Sigma d|} d\sigma_\tau \right) d\tau \end{aligned}$$

and, then

$$v_k'(r) = \int_{\Sigma\cap\partial B_r} \frac{1}{|\nabla_\Sigma d|} d\sigma_r.$$

Since Σ is minimal we have $\mathrm{div}_\Sigma (x-p)^T = k$, so

$$\int_{\Sigma\cap\partial B_r} \langle (x-p)^T, \nu\rangle d\sigma_r = \int_{\Sigma\cap B_r} \mathrm{div}_\Sigma (x-p)^T d\Sigma = k v_k(r).$$

Now we can use the above expressions to calculate

$$\begin{aligned}\frac{d}{dr}\left(\frac{v_k(r)}{r^k}\right) &= \frac{v_k'(r)}{r^k} - \frac{k v_k(r)}{r^{k+1}} \\ &= \frac{1}{r^k}\int_{\Sigma\cap\partial B_r} \frac{1}{|\nabla_\Sigma d|} d\sigma_r - \frac{1}{r^{k+1}}\int_{\Sigma\cap\partial B_r} \langle (x-p)^T, \nu\rangle d\sigma_r.\end{aligned}$$

But $\langle (x-p)^T, \nu\rangle = |(x-p)^T|$ and

$$1 - |\nabla_\Sigma d|^2 = |(\nabla d)^N|^2 = \frac{|(x-p)^N|^2}{|x-p|^2}.$$

Then

$$\begin{aligned}\frac{d}{dr}\left(\frac{v_k(r)}{r^k}\right) &= \frac{1}{r^k}\int_{\Sigma\cap\partial B_r} \left(\frac{1}{|\nabla_\Sigma d|} - |\nabla_\Sigma d|\right) d\sigma_r \\ &= \int_{\Sigma\cap\partial B_r} \frac{1}{r^k}\cdot\frac{1}{|\nabla_\Sigma d|}\cdot\frac{|(x-p)^N|^2}{|x-p|^2} d\sigma_r \\ &= \int_{\Sigma\cap\partial B_r} \frac{1}{|\nabla_\Sigma d|}\cdot\frac{|(x-p)^N|^2}{|x-p|^{k+2}} d\sigma_r.\end{aligned}$$

Integrating from s to t and applying the co-area formula again, we obtain

$$\begin{aligned}\frac{v_k(t)}{t^k} - \frac{v_k(s)}{s^k} &= \int_s^t \left(\int_{\Sigma\cap\partial B_r} \frac{1}{|\nabla_\Sigma d|}\cdot\frac{|(x-p)^N|^2}{|x-p|^{k+2}} d\sigma_r\right) dr \\ &= \int_{(B_t\setminus B_s)\cap\Sigma} \frac{|(x-p)^N|^2}{|x-p|^{k+2}} d\Sigma.\end{aligned}$$

Monotonicity formula on Riemannian manifolds: Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ be a minimal submanifold ($\overrightarrow{H} = 0$). Take $p \in M$, $r_0 <$ injectivity radius of M at p, $0 < s < t < r_0$. Let $d : \Sigma \to \mathbb{R}$ be the distance function to p, $d(x) = d_g(x, p)$. Consider the vector field $X = d\nabla d$.

In exponential coordinates around p, the Riemannian metric is given by

$$g_{ij}(x) = \delta_{ij} + O(|x|^2).$$

Divergence in coordinates: If a vector field X is given in local coordinates as $X = \sum_i X^i \frac{\partial}{\partial x_i}$, then its divergence has the following expression

$$\mathrm{div}_g X = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(\sqrt{\det g} X^i\big).$$

Exercise: If $X = d\nabla d$, then $\mathrm{div}_\Sigma X = k + O(|x|^2)$.

Adapting the euclidean proof: Let $l(r) = \int_{\Sigma\cap\partial B_r} d\sigma_r$ and $v(r) = \int_{\Sigma\cap B_r} d\Sigma$. Take $C = C(p) > 0$ so that $\mathrm{div}_\Sigma X^T \geq k - Cr^2$, for sufficiently small r, this constant is given by previous exercise. Then

$$rl(r) \geq \int_{\Sigma\cap\partial B_r} \langle X^T, \nu\rangle d\sigma_r = \int_{\Sigma\cap B_r} div_\Sigma X^T d\Sigma \geq \int_{\Sigma\cap B_r} (k - Cr^2) d\Sigma.$$

In conclusion, $rl(r) \geq (k - Cr^2)v(r)$. It is important that Σ is minimal to obtain that. Now, as in euclidean case, we use co-area formula to obtain

$$v'(r) = \int_{\Sigma\cap\partial B_r} \frac{1}{|\nabla_\Sigma d|} d\sigma_r \geq l(r).$$

Putting everything together, $v'(r) \geq \frac{k}{r}(1 - Cr^2)v(r)$. This can be rewritten

$$\frac{d}{dr} \log\left(\frac{e^{Ar^2} v(r)}{r^k}\right) \geq 0,$$

where $A = C/2$, depends on p. From this we have

Theorem 7.11 (Monotonicity in Riemannian manifolds)**.** *Let (M^n, g) be a Riemannian manifold, $\Sigma^k \subset M$ be a minimal submanifold and $p \in M$. Then there exists r_0 and $A > 0$, depending on p, such that*

$$r \mapsto e^{Ar^2} \frac{v(r)}{r^k} \text{ is non-decreasing in } r, \text{ for } r < r_0.$$

Remark: If M is compact, we can choose r_0 and A uniform.

Corollary 7.12. *Let M^n be a closed Riemannian manifold. Then there exists $\delta > 0$ such that, for any closed minimal submanifold $\Sigma^k \subset M$ we have $Vol(\Sigma) \geq \delta$.*

Proof. Take r_0 and A as in the monotonicity formula. Suppose they are uniform, as in the remark. If $\Sigma^k \subset M$ is a closed minimal submanifold, then for each $p \in \Sigma$, we have

$$e^{A(r_0/2)^2} \frac{\mathrm{Vol}(\Sigma \cap B_{r_0/2})}{(r_0/2)^k} \geq \lim_{r\to 0^+} e^{Ar^2} \frac{\mathrm{Vol}(\Sigma \cap B_r)}{r^k} \geq \omega_k.$$

Therefore,

$$\mathrm{Vol}(\Sigma) \geq \frac{\omega_k (r_0/2)^k}{e^{A(r_0/2)^2}} =: \delta > 0.$$

□

8. Bernstein Theorem

Theorem 8.1 (Bernstein Theorem, 1916)**.** *Every entire minimal graph in $\mathbb{R}^3$ is a plane.*

This means if $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then u is a affine map: $u(x, y) = ax + by + c$.

The first remark is that minimal graphs are area-minimizing, we proved this using calibration. This implies they have polynomial volume growth.

Indeed, if $\Sigma^{n-1} \subset \mathbb{R}^n$ is area-minimizing and $B_r(p)$ is a euclidean ball intersecting Σ, such that Σ splits $B_r(p)$ in two components, then $\mathrm{Vol}(\Sigma \cap B_r(p))$ is at most the volume of the least component of $\partial B_r(p) \setminus \Sigma$:

$$\mathrm{Vol}(\Sigma \cap B_r(p)) \leq \frac{1}{2}\sigma_{n-1} r^{n-1},$$

where σ_{n-1} is the volume of the unit $(n-1)$-dimensional sphere. In particular, if $\Sigma^2 \subset \mathbb{R}^{\not\models}$ is a minimal graph, we have

$$\mathrm{area}(\Sigma \cap B_r) \leq Cr^2. \tag{14}$$

Logarithmic cut-off trick. Recall the stability inequality, equation (11). We can also choose the test functions on the stability inequality to be only Lipschitz with compact support. For each $R > 0$, consider

$$\eta_R(r) = \begin{cases} 1 & \text{if } r \leq R \\ 2 - \frac{\log r}{\log R} & \text{if } R \leq r \leq R^2 \\ 0 & \text{if } R^2 \leq r. \end{cases}$$

Let $p \in \Sigma$ and $r(x) = d_\Sigma(p, x)$ be the intrinsic distance to p is Σ. We have $|\nabla_\Sigma r| = 1$. Note that $r(x) \leq |x - p|$ implies $B_r^\Sigma(p) \subset \Sigma \cap B_r(p)$, and we see that Σ has quadratic area growth also with respect to the intrinsic balls:

$$\mathrm{area}(B_r^\Sigma(p)) \leq Cr^2. \tag{15}$$

Plugging $\eta_R \circ r = \eta_R(r)$ in the stability inequality,

$$\begin{aligned} \int_\Sigma |A|^2 \eta_R^2(r) d\Sigma &\leq \int_\Sigma |\nabla_\Sigma \eta_R(r)|^2 d\Sigma \\ &= \int_\Sigma [\eta_R'(r)]^2 |\nabla_\Sigma r| d\Sigma \\ &= \frac{1}{(\log R)^2} \int_{B_{R^2}^\Sigma \setminus B_R^\Sigma} \frac{1}{r^2} d\Sigma. \end{aligned} \tag{16}$$

Assume $\log R \in \mathbb{N}$, then

$$\begin{aligned} \sum_{k=\log R+1}^{2\log R} \int_{B_{e^k}^\Sigma \setminus B_{e^{k-1}}^\Sigma} \frac{1}{r^2} d\Sigma &\leq \sum_{k=\log R+1}^{2\log R} \int_{B_{e^k}^\Sigma \setminus B_{e^{k-1}}^\Sigma} \frac{1}{e^{2k-2}} d\Sigma \\ &\leq \sum_{k=\log R+1}^{2\log R} C \frac{1}{e^{2k-2}} (e^k)^2 = e^2 C \log R, \end{aligned} \tag{17}$$

where the last estimate is a consequence of the quadratic area growth. In conclusion,

$$\int_\Sigma |A|^2 \eta_R^2(r) d\Sigma \leq \frac{e^2 C}{\log R}. \tag{18}$$

Make $R \to \infty$ to conclude $\int_\Sigma |A^2| d\Sigma = 0$. This shows Σ is totally geodesic, then must be a plane. This proves Bernstein theorem.

9. The Stability Condition

A two-sided minimal hypersurface $\Sigma^{n-1} \subset M^n$ is called stable if

$$-\int_\Sigma \phi L\phi d\Sigma \geq 0, \text{ for all } \phi \in C_c^\infty(\Sigma),$$

where $L\phi = \nabla_\Sigma\phi + (|A|^2 + \mathrm{Ric}(N,N))\phi$, is the stability operator. Let $\Omega \subset\subset \Sigma$. The spectrum of L with respect to Ω is given by a sequence $\lambda_1 \leq \lambda_2 \leq \to \infty$, with respective eigenfunctions u_i, i.e.,

$$Lu_i + \lambda_i u_i = 0,$$

with $u_i = 0$ on $\partial\Omega$. The eigenspace of λ is defined as

$$E_\lambda = \{\phi \in C^\infty(\Sigma) \cap C_c^0(\Sigma) : L\phi + \lambda\phi = 0 \text{ in } \Omega \text{ and } \phi = 0 \text{ on } \partial\Omega\}.$$

Stability in $\Omega \subset \Sigma$ means that $\lambda_1(L,\Omega) \geq 0$, where

$$\lambda_1(L,\Omega) = \inf\left\{Q(\phi) := \frac{\int_\Omega \phi L\phi d\Sigma}{\int_\Omega \phi^2 d\Sigma} : \phi \neq 0 \text{ and } \phi = 0 \text{ on } \partial\Omega\right\}. \tag{19}$$

Lemma 9.1. *Suppose Ω is connected. If $u : \overline{\Omega} \to \mathbb{R}$ is continuous, smooth in Ω, $u = 0$ on $\partial\omega$, $u \neq 0$ and $Lu + \lambda_1 u = 0$, then $|u| > 0$ in Ω. Moreover,* $\dim E_{\lambda_1} = 1$.

Remark: This statement is actually true for any operator of type $L = \nabla + q$, where q is a function. This is an application of strong maximum principle:

Theorem 9.2 (Strong maximum principle)**.** *Suppose*

$$\sum_{i,j} a_{ij}(x)\partial_i\partial_j v + \sum_i b_i(x)\partial_i v + c(x)v = 0,$$

where $(a_{ij}(x))_{ij}$ is positive definite for every $x \in \Omega$. If $v \leq 0$ and $v(p) = 0$ for some $p \in \Omega$, then $v \equiv 0$.

Proof. Observe $|\nabla|u|| = |\nabla u|$ almost everywhere, this implies $Q(|u|) = Q(u) = \lambda_1$, and then $|u|$ is again a first eigenfunction of L:

$$\begin{cases} \Delta|u| + q|u| = 0 & \text{in } \Omega \\ |u| = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle we conclude $|u| > 0$. To proof the second assumption, suppose by contradiction we have two linearly independent eigenfunctions $u_1, u_2 \in E_{\lambda_1}$, then $\int_\Omega u_1 u_2 d\Sigma = 0$. But this is not possible because both u_1 and u_2 do not change sign inside Ω. This also proves that the first eigenfunction is the unique with the property of not changing the sign. □

Proposition 9.3. *Let $\Sigma^{n-1} \subset M^n$ be a two-sided minimal hypersurface and $\Omega \subset \Sigma$ be a bounded domain. If there exists $u > 0$ in Ω such that $Lu = 0$, then Ω is stable.*

Proof. The operator L here is given by $L = \Delta + q$, where $q = |A|^2 + \mathrm{Ric}(N, N)$. Consider $w := \log u$ and observe that

$$\Delta w = \Delta(\log u) = \sum_{i=1}^{n-1} \left(\frac{u_i}{u}\right)_i = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -q - |\nabla w|^2. \tag{20}$$

For any $f \in C_c^\infty(\Sigma)$, we have

$$-2\int_\Omega f\langle \nabla f, \nabla w\rangle d\Sigma = \int_\Omega f^2 \Delta w = -\int_\Omega q f^2 d\Sigma - \int_\Omega |\nabla w|^2 f^2 d\Sigma. \tag{21}$$

By Young inequality, we infer

$$\begin{aligned}\int_\Omega q f^2 d\Sigma + \int_\Omega |\nabla w|^2 f^2 d\Sigma &= 2\int_\Omega f\langle \nabla f, \nabla w\rangle d\Sigma \\ &\leq \int_\Omega |\nabla f|^2 d\Sigma + \int_\Omega f^2 |\nabla w|^2 d\Sigma,\end{aligned}$$

and then, we see that stability holds, $\int q f^2 \leq \int |\nabla f|^2$. □

Exercise: Let $\Sigma^{n-1} \subset M^n$ be a two-sided minimal surface, with unit normal vector N, and X be a Killing field on M. Then $L(\langle N, X\rangle) = 0$. (A vector field X on M is called a Killing field is its flow is by ambient isometries.)

Remark 1: This exercise gives a new proof of the fact that minimal graphs are stable. Suppose $\Sigma^{n-1} \subset \mathbb{R}^n$ is a minimal graph $\Sigma = \mathrm{graph}(u)$. Observe the function $\langle \partial/\partial x_n, N\rangle$ is positive along Σ. Moreover, by exercise it satisfies $L(\langle \partial/\partial x_n, N\rangle) = 0$. Use the previous proposition to conclude stability.

Remark 2: The solutions to $Lu = 0$ are called Jacobi fields.

Definition 9.4. A Riemannian manifold Σ is parabolic if the only positive superharmonic, $\Delta u \leq 0$, functions u on Σ are the constant functions.

9.1. **Proposition.** *If* $\dim \Sigma = 2$ *and area*$(\Sigma \cap B_r) \leq Cr^2$*, then* Σ *is parabolic.*

Proof. Consider $w := \log u$, where u is a function on Σ so that $u > 0$ and $\Delta u \leq 0$. We have then

$$\Delta w = \frac{\Delta w}{w} - \frac{|\nabla u|^2}{u^2} \leq -|\nabla w|^2.$$

If $X = \nabla w$, then $\mathrm{div}X \geq |X|^2$. For any cut-off function $\eta \in C_c^\infty(\Sigma)$, we see

$$\begin{aligned}\int_\Sigma \eta^2 |X|^2 d\Sigma &\leq \int_\Sigma \eta^2 \mathrm{div}X d\Sigma \\ &= -2\int_\Sigma \eta\langle X, \nabla\eta\rangle d\Sigma \\ &\leq \frac{1}{2}\int_\Sigma \eta^2|X|^2 + \int_\Sigma |\nabla\eta|^2.\end{aligned}$$

So there exists a positive constant $C > 0$ such that, for every smooth compactly supported test functions η, we have

$$\int_\Sigma \eta^2 |X|^2 d\Sigma \leq C \int_\Sigma |\nabla \eta|^2. \tag{22}$$

Apply the logarithmic cut-off trick to get $X \equiv 0$ and conclude u is constant. □

Corollary 9.5. *Any entire minimal graph is parabolic.*

In particular, if we define $u = \langle \partial_{x_3}, N \rangle > 0$, we get $\Delta u = -|A|^2 u \leq 0$. By parabolicity, we have u is constant and then $0 = \Delta u$. This implies $|A|^2 = 0$ and Σ must be a plane.

9.2. **Stability and positive curvature.** We have now some application of the stability condition in the presence of a positive curvature assumption, in order to get topological and geometric restrictions.

9.2.1. *Fundamental group and sec > 0:* Let $\gamma \subset M$ be a closed geodesic in M^n and X be a normal variation of γ. The second variation formula is this case is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |\gamma_t| = \int_\gamma [|\nabla^\perp X|^2 - R^M(X, \gamma', X, \gamma')] ds. \tag{23}$$

Remark: If there exists a parallel variation X and $\sec_M > 0$, then γ is unstable.

Theorem 9.6 (Synge)**.** *Let M^{2n} be a close orientable Riemannian manifold with positive sectional curvature. Then $\pi_1(M) = \{1\}$.*

Proof. Suppose, by contradiction, $\pi_1(M) = \{1\}$ and find γ, a closed geodesic minimizing length in a nontrivial homology class. Consider the parallel transport $P : \langle \gamma' \rangle_p^\perp \to \langle \gamma' \rangle_p^\perp$ along γ, for some fixed $p \in \gamma$. It is a linear orientation-preserving isometry. By linear algebra and dimension assumption, there exists $v \in \langle \gamma' \rangle_p^\perp$ such that $P(v) = v$. If V is the normal variation on γ given by parallel transport of v, the second variation formula (23) gives

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |\gamma_t| = -\int_\gamma R^M(X, \gamma', X, \gamma') ds < 0.$$

This contradicts the fact that γ minimizes length in its homology class. □

Corollary 9.7. *There is no metric with sec > 0 is $\mathbb{R}P^2 \times \mathbb{R}P^2$.*

Synge gives no restriction in the case of the product of two-spheres. In fact, this is a famous open problem:

Hopf Conjecture: Does $S^2 \times S^2$ has a metric with sec > 0?

9.2.2. *First Betti number and Ric > 0:* There is an analogous result to Synge theorem for the first Betti number, $b_1(M)$, of Riemannian manifolds with positive Ricci curvature.

Theorem 9.8. *Let M^n be an orientable closed Riemannian manifold with $Ric_M > 0$. Then, $b_1(M) = 0$.*

Proof. By Poincaré duality, we know that $H^1(M^n, \mathbb{Z}) \approx H_{n-1}(M^n, \mathbb{Z})$. Suppose that $H_{n-1}(M^n, \mathbb{Z}) \neq 0$ and choose a nontrivial $a \in H_{n-1}(M^n, \mathbb{Z})$. Minimize the volume in the class a to find a closed smooth minimal hypersurface $\Sigma^{n-1} \subset M^n$ such that

- $[\Sigma] = a$;
- $\mathrm{Vol}(\Sigma) = \min_{\Sigma' \in a} \mathrm{Vol}(\Sigma')$.

In particular, Σ is stable. Choose $f = 1$ in the stability inequality to obtain

$$0 = \int_\Sigma |\nabla f|^2 d\Sigma \geq \int_\Sigma (\mathrm{Ric}(N, N) + |A|^2) f^2 d\Sigma > 0,$$

and this is a contradiction. Then $b_1(M) = 0$. □

Remark: Although the result is true for any dimension, this particular proof applies only in case $n \leq 7$, to be sure the volume minimizer is smooth. One can use Bochner Technique give a proof that work without dimension assumption.

Theorem 9.9 (Bonnet-Myers)**.** *Let M^n be a complete Riemannian manifold with $Ric_M \geq (n-1)k > 0$. Then $diam(M) \leq \pi/\sqrt{k}$. In particular, M is compact and $\pi_1(M)$ is finite.*

The proof of this result is a contradiction argument. If $\mathrm{diam}(M) > \pi/\sqrt{k}$, then there exists a minimizing geodesic of length $l > \pi/\sqrt{k}$. Then we produce a variation of this geodesic with $L''(0) < 0$ and get a contradiction.

9.2.3. *Positive Scalar Curvature:* We describe now the Schoen-Yau ideas to deal with the case where the positive curvature assumption is on the scalar curvature instead of sectional or Ricci curvatures.

Let $\Sigma^{n-1} \subset M^n$ be a hypersurface and $p \in \Sigma$. Consider $\{e_1, \ldots, e_{n-1}\} \subset T_p\Sigma$ orthonormal basis and let $e_n = N$ be the unit vector normal to Σ. The second fundamental form is written as $h_{ij} = \langle B(e_i, e_j), N\rangle$. Recall the Gauss equation:

$$R^\Sigma_{ijkl} = R^M_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.$$

Sum in i and k to obtain

$$\mathrm{Ric}^\Sigma_{jl} = \mathrm{Ric}^M_{jl} - R^M_{njnl} + Hh_{jl} - \sum_i h_{il}h_{ij}.$$

Now, sum in j and l to get

$$R^\Sigma = R^M - \mathrm{Ric}^M_{nn} - \mathrm{Ric}^M_{nn} + H^2 - \sum_{i,j} h^2_{ij}.$$

This can be rewritten as

$$R^{\Sigma} = R^M - 2\mathrm{Ric}(N,N) + H^2 - |A|^2, \tag{24}$$

where R^{Σ} and R^M are the scalar curvatures of Σ and M, respectively. From this, in case $H = 0$, we can write the integrand of the stability inequality in terms of scalar curvature and the norm of second fundamental form:

$$\mathrm{Ric}(N,N) + |A|^2 = \frac{R^M}{2} - \frac{R^{\Sigma}}{2} + \frac{|A|^2}{2}. \tag{25}$$

If $\Sigma^2 \subset M^3$ is a closed two-sided stable minimal surface, then $R^{\Sigma} = 2K_{\Sigma}$ and the above expression together with stability equation gives

$$\int_{\Sigma} |\nabla_{\Sigma} f|^2 d\Sigma \geq \int_{\Sigma} \left(\frac{R^M}{2} - K_{\Sigma} + \frac{|A|^2}{2} \right) f^2 d\Sigma, \tag{26}$$

for every $f \in C_c^{\infty}(\Sigma)$. Take $f \equiv 1$ and apply Gauss-Bonnet to obtain:

$$\begin{aligned} 0 \ &\geq \ \int_{\Sigma} \left(\frac{R^M}{2} - K_{\Sigma} + \frac{|A|^2}{2} \right) d\Sigma \\ &= \ -2\pi\chi(\Sigma) + \frac{1}{2} \int_{\Sigma} \left(R^M + |A|^2 \right) d\Sigma. \end{aligned} \tag{27}$$

Theorem 9.10 (Schoen-Yau)**.** *Let (M^3, g) be a compact orientable Riemannian manifold and $\Sigma^2 \subset M$ be a stable closed oriented minimal surface, then*

(a) *If $R^M > 0$, then Σ is diffeomorphic to S^2;*
(b) *If $R^M \geq 0$, then either*
  (1) *Σ is diffeomorphic to S^2*
  (2) *Σ is a flat, totally geodesic torus.*

Proof. By stability condition and Gauss-Bonnet, we have (27). So, in case of $R^M \geq 0$, we conclude

$$2\pi\chi(\Sigma) \geq \frac{1}{2} \int_{\Sigma} \left(R^M + |A|^2 \right) d\Sigma \geq 0,$$

then Σ is diffeomorphic either to S^2 or to a torus T^2. If $\Sigma \approx T^2$, then $\chi(\Sigma) = 0$ and we conclude $R^M = 0$ along Σ and $|A|^2 = 0$. Since

$$\int_{\Sigma} \phi L \phi d\Sigma \geq 0, \text{ for all } \phi \in C^{\infty}(\Sigma),$$

and $\int 1L(1)d\Sigma = 0$, we see the constant function 1 is the first eigenfunction of the stability operator L. Finally, $L(1) = 0$ and we have $\mathrm{Ric}(N,N) = 0$ on Σ and also Σ is flat, $R^{\Sigma} = 0$. □

An important consequence of this theorem is:

Theorem 9.11 (Schoen-Yau and Gromov-Lawson)**.** *T^3 has no metric of positive scalar curvature.*

Indeed, any (T^3, g) contains a stable minimal torus if $R > 0$.

Remark: The previous result is also valid for any dimensions. If $n \leq 7$, there is also a proof with minimal hypersurfaces. In general, the argument is due to Gromov and Lawson.

Theorem 9.12 (Fischer-Colbrie and Schoen)**.** *Let (M^3, g) be a oriented Riemannian manifold with $R^M \geq 0$ and $\Sigma^2 \subset M$ be a complete oriented noncompact stable minimal surface. Then the universal cover $\tilde{\Sigma}$ of Σ is conformally equivalent to $\mathbb{C}$.*

9.3. **Corollary.** *If $\Sigma^2 \subset \mathbb{R}^3$ is complete oriented stable minimal surface, then it is a totally geodesic plane.*

9.4. **Theorem.** [Fischer-Colbrie and Schoen] *Let M^n be a complete noncompact manifold. The following statements are equivalent:*

(1) $\lambda_1(L, \Omega) \geq 0$, *for every* $\Omega \subset\subset M$*;*
(2) $\lambda_1(L, \Omega) > 0$, *for every* $\Omega \subset\subset M$*;*
(3) *there exists* $u > 0$ *on* M *that solves* $Lu = 0$.

Proof: (3) $\Rightarrow$ (1)*:* this is lemma 9.3.

(1) $\Rightarrow$ (2)*:* Suppose $\Omega \subset\subset \Omega'$ and $\lambda_1(L, \Omega) \geq 0$. Choose a function u such that

$$Q(u) = \frac{\int_\Omega uLu}{\int_\Omega u^2} = \lambda_1(L, \Omega). \tag{28}$$

Suppose $\lambda_1(L, \Omega') = \lambda_1(L, \Omega)$ and define $\tilde{u}$ on Ω' by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise .} \end{cases}$$

Observe that $Q(\tilde{u}) = Q(u) = \lambda_1(L, \Omega)$, then $\tilde{u}$ is a first eigenfunction to L in Ω'. This is a contradiction, because $\tilde{u}$ vanishes in some open set. Since we always have $\lambda_1(L, \Omega') \leq \lambda_1(L, \Omega)$, in this case we conclude

$$0 \leq \lambda_1(L, \Omega') < \lambda_1(L, \Omega).$$

(2) $\Rightarrow$ (3)*:* The goal is to construct solutions u_R to

$$\begin{cases} Lu_R = 0 & \text{in } B_R(p) \\ u_R > 0 & \text{in } B_R(p) \\ u_R = 1 & \text{on } \partial B_R(p), \end{cases}$$

where $p \in M$ and $R > 0$ are fixed. Suppose we have achieved our goal. Consider $\tilde{u}_R(x) = u_R(x)/u_R(p)$ and observe $L\tilde{u}_R = 0$ and $\tilde{u}_R(p) = 1$.

9.5. **Harnack Inequality.** *Consider a second order elliptic operator*

$$Lu = \sum_{i,j} a_{ij}(x)\partial_i\partial_j u + \sum_i b_i(x)\partial_i u + c(x)u,$$

where $(a_{ij}(x))_{ij}$ is positive definite for every $x \in \Omega$ and let $K \subset \Omega$ be a compact subset. There exists $C > 0$, depending only on a_{ij}, b_i, c and K, such that any positive solution of $Lu = 0$ satisfies

$$\sup_K u \leq C \inf_K u.$$

As a consequence of this inequality, if we fix a compact subset K containing p, there is a positive $C = C(K)$ so that

$$\sup_K \tilde{u}_R \leq C \inf_K \tilde{u}_R \leq C,$$

for sufficiently large R. In particular, $||\tilde{u}_R||_{C^0(K)} \leq C$.

9.6. **Schoulder Estimates.** *Consider a second order elliptic operator*

$$Lu = \sum_{i,j} a_{ij}(x)\partial_i\partial_j u + \sum_i b_i(x)\partial_i u + c(x)u,$$

where $(a_{ij}(x))_{ij}$ is positive definite for every $x \in \Omega$ and $K \subset \Omega' \subset\subset \Omega$. There exists $C > 0$, depending only on a_{ij}, b_i, c and K, Ω', such that any solution of $Lu = f$ satisfies

$$||u||_{C^{2,\alpha}(K)} \leq C\left(||u||_{C^0(\Omega')} + ||f||_{C^\alpha(\Omega')}\right).$$

Use Shoulder estimates, to obtain

$$|\tilde{u}_R|||_{C^{2,\alpha}(K)} \leq C, \text{ for every } R.$$

Apply Arzela-Ascoli to extract a subsequence $\tilde{u}_{R_j}$ so that $\tilde{u}_{R_j} \to u$ in C^2_{loc}. Follow form the construction of maps $\tilde{u}_R$ that

$$\begin{cases} Lu = 0 & \text{in } M \\ u \geq 0 & \text{in } M \\ u = 1 & \text{at } p. \end{cases}$$

By Maximum principle, we conclude $u > 0$ in M. The existence of such a function imply stability, condition (1), by lemma 9.3. Need to prove the starting goal.

Write $u_R = v_R + 1$ and try to solve

$$\begin{cases} Lv_R = -L(1) & \text{in } B_R(p) \\ v_R = 0 & \text{on } \partial B_R(p). \end{cases}$$

9.7. **Fredholm Alternative.** *Either the problem*

$$\begin{cases} Lv = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution, or the problem

$$\begin{cases} Lv = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

can always be solved.

In our case, $\lambda_1(L, \Omega) > 0$ implies the homogeneous system in the Fredholm alternative has $v = 0$ as the only solution with $\Omega = B_R(p)$. Then it is possible to find v_R as we want. Define $u_R = v_R + 1$ and observe $Lu_R = 0$ and $u_R = 1$ on $\partial B_R(p)$. Suppose we have $u_R < 0$ somewhere. Take a connected component Ω of $\{u_R < 0\}$, we have

$$\begin{cases} Lu_R = 0 & \text{in } \Omega \\ u_R = 0 & \text{on } \partial\Omega. \end{cases}$$

This contradicts $\lambda_1(L, \Omega) > 0$. Therefore $u_R \geq 0$. Moreover, by maximum principle, $u_R > 0$ in $B_R(p)$, as we need. This finishes the proof of Fischer-Colbrie and Schoen theorem about the stability condition.

An application of this result is:

Corollary 9.13. *Let M^3 be an oriented Riemannian manifold and $\Sigma^2 \subset M$ be a stable oriented minimal surface, then any covering $f : \tilde{\Sigma} \to M$ of Σ is again stable.*

Remark: In this statement, we have a minimal immersion $f : \Sigma \to M$ and a isometric covering map $\pi : \tilde{\Sigma} \to \Sigma$. The claim is that the minimal immersion$f \circ \pi : \tilde{\Sigma} \to M$ is stable.

Proof. Since Σ is stable, there exists a $u > 0$ on Σ with $Lu = 0$. Consider the function $\tilde{u} = u \circ \pi$ and observe this is also a positive solution of $L_{\Sigma}\tilde{u} = 0$. This implies $\tilde{\Sigma}$ is stable. □

Example: Consider the constant curvature metric on $\mathbb{R}P^3$ and let $\Sigma^2 = \mathbb{R}P^2$ be the canonical projective plane inside $\mathbb{R}P^3$. Although Σ is area-minimizing surface inside $\mathbb{R}P^3$, and in particular stable, its double cover $f : S^2 \to \mathbb{R}P^3$ is not stable.

9.8. **Theorem.** *Let (M^3, g) be a complete, oriented 3-manifold with positive scalar curvature and $\Sigma^2 \subset M^3$ be a non-compact, oriented, stable minimal surface. Then the universal cover $\tilde{\Sigma}$ of Σ is conformally equivalent to $\mathbb{C}$.*

This result has the following important consequence

9.9. **Corollary.** [Do Carmo and Peng, Fischer-Colbrie and Schoen] *Let $\Sigma^2 \subset \mathbb{R}^3$ be a complete oriented stable minimal surface, then Σ is a flat plane.*

Proof of 9.9. Let $\tilde{f} : \tilde{\Sigma} \to \mathbb{R}^3$ be the immersion of the universal cover in $\mathbb{R}^3$. Since Σ is stable, Theorem 9.8 implies $\tilde{\Sigma}$ is conformally equivalent to $\mathbb{C}$. We have also $\tilde{f}$ is stable and the stability condition gives

$$\int_{\tilde{\Sigma}} |\nabla f|^2 - |A|^2 f^2 \geq 0, \quad \text{for all } f \in C_c^\infty(\tilde{\Sigma}). \tag{29}$$

By item (3) in Theorem 9.4, there exists a function $u > 0$ such that $\Delta u + |A|^2 u = 0$. In particular, $\Delta u = -|A|^2 u \leq 0$, i.e., u is superharmonic. Recall conformal changes of metric formula for the Laplace operator in this case is:

$$\text{if} \quad \tilde{g} = e^{2v} g, \quad \text{then} \quad \Delta_{\tilde{g}}(f) = e^{-2v} \Delta_g(f). \tag{30}$$

Then $\Delta_{\mathbb{C}} u \leq 0$, and we conclude u is constant. This implies $\Delta_g u = 0$ and, finally, $|A|^2 = 0$. Being totally geodesic and complete, Σ is a flat plane. $\square$

In these notes we give two proofs for Theorem 9.8, the first following the ideas in the original arguments and later we give a proof using Colding and Minicozzi curvature estimates.

In the first proof we use the following property of dimension $n = 2$:

$$\int |\nabla_g f|^2 dv_g \quad \text{is conformally invariant.} \tag{31}$$

Proof 1 *of 9.8.* In this case stability condition means

$$\int_{\tilde{\Sigma}} |\nabla f|^2 - \left(\frac{R^M}{2} - K_{\tilde{\Sigma}} + \frac{|A|^2}{2} \right) \cdot f^2 \geq 0, \tag{32}$$

for every $f \in C_c^\infty(\tilde{\Sigma})$. Because $R^M \geq 0$, we conclude

$$\int_{\tilde{\Sigma}} |\nabla f|^2 + K_{\tilde{\Sigma}} f^2 \geq 0, \quad \text{for all } f \in C_c^\infty(\tilde{\Sigma}). \tag{33}$$

Suppose, $\tilde{\Sigma}$ with the induced metric g is conformally equivalent to the unit disk $\mathbb{D}^2$ and let $g = e^{2u} ds^2$, where $ds^2 = dxdy$ is the standard metric on $\mathbb{D}^2$. In this case, we have

$$dv_g = e^{2u} dxdy \quad \text{and} \quad K_{\tilde{\Sigma}} = e^{-2u} \Delta_0 u. \tag{34}$$

Then, stability condition gives

$$\int_{\tilde{\Sigma}} \left(|\nabla_g f|^2 + e^{-2u} (\Delta_0 u) f^2 \right) dv_g \geq 0, \quad \text{for all } f \in C_c^\infty(\tilde{\Sigma}). \tag{35}$$

Apply expression (31) to conclude

$$\int_{\mathbb{D}^2} \left(|\nabla_0 f|^2 + (\Delta_0 u) f^2 \right) dxdy \geq 0, \quad \text{for all } f \in C_c^\infty(\mathbb{D}^2). \tag{36}$$

Choose $\zeta \in C_c^\infty(\mathbb{D}^2)$ and $f = \zeta \cdot e^{-u}$ to apply the stability inequality and obtain, via integration by parts,

$$\int_{\mathbb{D}^2} \zeta^2 e^{-2u} |\nabla_0 u|^2 dxdy - \int_{\mathbb{D}^2} e^{-2u} |\nabla_0 \zeta|^2 dxdy \leq 0, \tag{37}$$

and then

$$\int_{\mathbb{D}^2} \zeta^2 |\nabla_g e^{-u}|^2 dv_g - \int_{\mathbb{D}^2} |\nabla_g \zeta|^2 dxdy \leq 0. \tag{38}$$

Use r to denote the intrinsic distance in $\tilde{\Sigma}$ with respect to g. Let $\zeta = \zeta(r)$ be a radial function such that

- $\zeta(r) = 1$, if $0 \leq r \leq R$;
- $0 \leq \zeta(r) \leq 1$ and $|\zeta'| \leq c \cdot R^{-1}$, everywhere;
- $\zeta(r) = 0$, if $2R \leq r$.

Plugging this function in the expression above allows us to conclude

$$\int_{B_R} |\nabla_g e^{-u}|^2 dv_g \leq \frac{c}{R^2} \int_{\mathbb{D}^2} dxdy \leq \frac{c}{R^2}. \tag{39}$$

Letting $R \to \infty$, we see $\int |\nabla_0 e^{-u}| = 0$, then u is constant. This is a contradiction, because ds^2 is not complete. By Uniformization Theorem, $\tilde{\Sigma}$ is conformally equivalent to $\mathbb{C}$. □

Colding and Minicozzi estimates. Let $\Sigma^2 \subset \mathbb{R}^3$. We use $B^\Sigma_{r_0}(x_0)$ to denote the intrinsic ball in Σ of radius r_0 and centered at x_0, not intersecting the cut locus of x_0. Apply Gauss-Bonnet Theorem to $\partial B_r(x_0)$, with $0 < r < r_0$, to obtain

$$\int_{B_r} K d\Sigma + \int_{\partial B_r} K_g(s) ds = 2\pi. \tag{40}$$

Use $L(r) = \int_{\partial B_r} ds$ to denote the length of geodesic circle of radius r. The first variation formula gives

$$L'(r) = \int_{\partial B_r} K_g(s) ds = 2\pi - \int_{B_r} K d\Sigma. \tag{41}$$

Let $A(r) = \int_{B_r} dv_g$ be the area of the geodesic disk of radius r. By co-area formula, $A(r)$ can be rewritten as

$$A(r) = \int_0^r \left(\int_{\partial B_t} ds \right) dt = \int_0^r L(t) dt. \tag{42}$$

Integrating, we obtain

(a)
$$L(r_0) = 2\pi r_0 - \int_0^{r_0} \left(\int_{B_r} K d\Sigma \right) dr;$$

(1)
$$A(r_0) = \pi r_0^2 - \int_0^{r_0} \int_0^t \left(\int_{B_r} K d\Sigma \right) dr dt;$$

(c)
$$A(r_0) = \pi r_0^2 - \int_{B_{r_0}} \frac{(r_0 - r)^2}{2} K d\Sigma;$$

Proof of (c). Consider $f(t) = \int_0^t \left(\int_{B_r} K d\Sigma \right) dr$ and observe $f(0) = 0$. Take $g(t) = 2^{-1} \cdot (r_0 - t)^2$ and rewrite the integral of f, using integral by parts:
$$\begin{aligned}
\int_0^{r_0} f(t)dt &= \int_0^{r_0} g(t) f''(t) dt \\
&= \int_0^{r_0} \frac{(r_0 - t)^2}{2} \left(\int_{\partial B_t} K d\sigma_t \right) dt \\
&= \int_0^{r_0} \left(\int_{\partial B_t} \frac{(r_0 - t)^2}{2} \int_{\partial B_t} K d\sigma_t \right) dt \\
&= \int_{B_{r_0}} \frac{(r_0 - r)^2}{2} K d\Sigma.
\end{aligned}$$
□

Suppose $\Sigma^2 \subset \mathbb{R}^3$ is a stable minimal surface. Gauss equation gives $2K = -|A|^2$. Plugging $f = \frac{r_0 - r}{2}$ in the stability inequality to obtain
$$\int_{B_{r_0}} |A|^2 \frac{(r_0 - r)^2}{4} \le \int_{B_{r_0}} \left| \nabla \left(\frac{r_0 - r}{2} \right) \right|^2. \tag{43}$$
Hence, we can conclude
$$A(r_0) - \pi r_0^2 = - \int_{B_{r_0}} \frac{(r_0 - r)^2}{2} K d\Sigma \le \int_{B_{r_0}} \left| \nabla \left(\frac{r_0 - r}{2} \right) \right|^2 \le \frac{A(r_0)}{4}.$$
As a corollary of last expression we have the following result:

9.10. **Theorem.** *If $\Sigma^2 \subset \mathbb{R}^3$ is a stable minimal surface and $B^\Sigma_{r_0}(x_0) \subset \Sigma$ does not intersect the cut locus of x_0, then area$(B^\Sigma_{r_0}(x_0)) \le \frac{4\pi}{3} r_0^2$.*

Back to the formula
$$A(r_0) = \pi r_0^2 + \int_{B^\Sigma_{r_0}} \frac{|A|^2}{4} (r_0 - r)^2,$$
observe, for each $0 < t < r_0$, we have
$$\frac{4\pi}{3} r_0^2 \ge A(r_0) \ge \pi r_0^2 + \int_{B^\Sigma_{r_0 - t}} \frac{|A|^2}{4} (r_0 - r)^2 \ge \pi r_0^2 + \frac{t^2}{4} \int_{B^\Sigma_{r_0 - t}} |A|^2.$$
Then we have the second consequence of this analysis: for $t = r_0/2$ we get
$$\int_{B^\Sigma_{\frac{r_0}{2}}} |A|^2 \le \frac{16\pi}{3}. \tag{44}$$

At this point we are able to present the second proof of Theorem 9.8.

Proof 2 *of 9.8.* Let $\Sigma^2 \subset \mathbb{R}^3$ be an oriented stable complete minimal surface and $\tilde{\Sigma}$ be the universal cover of Σ with the covering metric. The stability condition passes to $\tilde{\Sigma}$ and then, Colding-Minicozzi estimates give $\text{area}(B^{\tilde{\Sigma}}_{r_0}(x_0)) \leq \frac{4\pi}{3} r_0^2$. Use Proposition 9.1 to conclude $\tilde{\Sigma}$ is parabolic. Because of that $\tilde{\Sigma}$ is conformally equivalent to $\mathbb{C}$. $\square$

About higher dimension, this is still an open problem. Actually, there is a conjecture by Richard Schoen that says the result must be the same in euclidean 4-dimensional space:

Conjecture. *Let $\Sigma^3 \subset \mathbb{R}^4$ be an oriented stable complete minimal surface. Then Σ is a hyperplane.*

10. Simons' Equation

Let M^n be a Riemannian manifold and $\Sigma^{n-1} \subset M^n$. For a given $p \in \Sigma$, consider $\{e_i\}_{i=1}^{n-1}$ an orthonormal basis of $T_p\Sigma$ and take $e_n = N$ to be the unit normal to Σ in M.

Theorem 10.1 (Simons' Equation)**.**

$$\Delta|A|^2 = 2|\nabla A|^2 + 2\sum_{i,j} h_{ij}\nabla_i\nabla_j H - 2|A|^4 + 2H\,trA^3 + Rm * h * h + \nabla Rm * h.$$

Corollary 10.2. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a minimal hypersurface. Then*

$$\Delta|A|^2 = 2|\nabla A|^2 - 2|A|^4.$$

Definitions, notations and useful formulas.

Covariant derivative. If T is a tensor $T(X_1, \ldots, X_r)$ of order r, its covariant derivative is a tensor of order $r+1$, denoted by ∇T and defined as

$$\nabla T(X_1, \ldots, X_{r+1}) = X_{r+1}T(X_1, \ldots, X_r) - \sum_{i=1}^{r} T(X_1, \ldots, \nabla_{X_{r+1}}X_i, \ldots, X_r).$$

Index notation. Let T be a tensor of order r, the index notation for T and its derivative on the chosen orthonormal basis are

$$T_{i_1\ldots i_r} = T(e_{i_1}, \ldots, e_{i_r})$$

and

$$T_{i_1\ldots i_r;j} = \nabla_{e_j}T(e_{i_1}, \ldots, e_{i_r}) = \nabla_j T(e_{i_1}, \ldots, e_{i_r}) = \nabla T(e_{i_1}, \ldots, e_{i_r}, e_j).$$

Elements of Simons' equation. In the Simons' equation h is the symmetric tensor of order 2 defined by

$$h_{ij} = \langle B(e_i, e_j), N\rangle = \langle A(e_i), e_j\rangle = -\langle \nabla_{e_i} e_n, e_j\rangle,$$

where B is the second fundamental form of $\Sigma \subset M$ and A is the shape operator. The Riemann curvature tensor R^M_{ijkl} appears on the formula as Rm. The last two terms with the $*$ notation must be understood as sums of type $h_{ab}h_{cd}R^M_{ijkl}$, without derivatives, in the case of $Rm * h * h$. The norm squared of shape operator is

$$|\nabla A|^2 = \sum_{i,j,k=1}^{n-1} h^2_{ij;k}.$$

Finally, the Laplace operator applied on $|A|^2$ means

$$\Delta|A|^2 = \Delta_\Sigma\big(\sum_{i,j} h^2_{ij}\big) = \sum_k \big(\sum_{i,j} h^2_{ij}\big)_{;kk}$$

where the derivatives here are the intrinsic covariant derivatives in Σ.

Fundamental equations. Recall the fundamental Gauss and Codazzi equations, respectively given by:

$$R^\Sigma_{ijkl} = R^M_{ijkl} + h_{il}h_{jk} - h_{ik}h_{jl}$$

and

$$R^M_{nijk} = h_{ik;j} - h_{ij;k}.$$

Second covariant derivative. Let T be a tensor of order r on Σ^{n-1} and V, W be vector fields on Σ. The following formula shows that in order to commute covariant derivatives of T, one has to add a curvature correction term

$$(\nabla_V\nabla_W T - \nabla_W\nabla_V T)(X_1, \ldots, X_r) = \sum_{i=1}^r T(X_1, \ldots, R(W,V)X_i, \ldots, X_r),$$

where R is the Riemann curvature tensor of Σ. Is this calculation it is not relevant that Σ is a submanifold of M.

Proof. First, it is important to note that the second derivative of T in the formula means

$$\begin{aligned} \nabla_V\nabla_W T(X_1, \ldots, X_r) &= \nabla_V(\nabla T)(X_1, \ldots, X_r, W) \\ &= \nabla(\nabla T)(X_1, \ldots, X_r, W, V). \end{aligned}$$

By definition of covariant derivative,

$$\begin{aligned}
\nabla_V \nabla_W T(X_1, \ldots, X_r) &= V((\nabla T)(X_1, \ldots, X_r, W)) \\
&\quad - \sum_{i=1}^{r} (\nabla T)(X_1, \ldots, \nabla_V X_i, \ldots, X_r, W) \\
&\quad -(\nabla T)(X_1, \ldots, X_r, \nabla_V W) \\
&= V(W(T(X_1, \ldots, X_r))) \\
&\quad - \sum_{j=1}^{r} V(T(X_1, \ldots, \nabla_W X_j, \ldots, X_r)) \\
&\quad - \sum_{i=1}^{r} W(T(X_1, \ldots, \nabla_V X_i, \ldots, X_r)) \\
&\quad + \sum_{i=1}^{r} \sum_{k \neq i} T(\ldots, \nabla_V X_i, \ldots, \nabla_W X_k \ldots) \\
&\quad + \sum_{i=1}^{r} T(X_1, \ldots, \nabla_W \nabla_V X_i, \ldots, X_r) \\
&\quad -(\nabla_V W) T(X_1, \ldots, X_r) \\
&\quad + \sum_{j=1}^{r} T(X_1, \ldots, \nabla_{(\nabla_V W)} X_j, \ldots, X_r).
\end{aligned}$$

Observe the second and third terms together form a symmetric expression on V and W. The fourth term has also this symmetry. The first and sixth terms do not appear on the formula because of the symmetry of Riemannian connexion, $[V, W] = \nabla_V W - \nabla_W V$. $\square$

In the index notation, we have

$$T_{i_1 \ldots i_r; jk} = T_{i_1 \ldots i_r; kj} + \sum_{p=1}^{n-1} \sum_{l=1}^{r} R_{p i_l jk} T_{i_1 \ldots p \ldots i_r}, \tag{45}$$

where the p index on T is on the l-th entry and $R(e_i, e_j)e_k = \sum_{p=1}^{n-1} R_{ijkp} e_p$.

Exercise: Prove that

$$R^M_{nijk;k} = R^M_{nijk;\underline{k}} + Rm * h,$$

where the covariant derivative on the right hand side is on M.

Proof of Simons' equation. Start with

$$\begin{aligned}
\Delta|A|^2 &= \sum_k \big(\sum_{i,j} h_{ij}^2\big)_{;kk} \\
&= 2\sum_{i,j,k}(h_{ij}h_{ij;k})_{;k} \\
&= 2\sum_{i,j,k} h_{ij;k}^2 + 2\sum_{i,j,k} h_{ij}h_{ij;kk}.
\end{aligned}$$

Keep the first term and apply Codazzi to the second,

$$\begin{aligned}
\sum_{i,j,k} h_{ij}h_{ij;kk} &= \sum_{i,j,k} h_{ij}(h_{ij;k})_{;k} \\
&= \sum_{i,j,k} h_{ij}(h_{ik;j} - R^M_{nijk})_{;k} \\
&= \sum_{i,j,k} h_{ij}h_{ik;jk} - \sum_{i,j,k} h_{ij}R^M_{nijk;k}.
\end{aligned}$$

All derivatives here are intrinsic of Σ, including the one on the Riemann curvature tensor. Can think of R^M_{nijk} as a tensor of order 3 on Σ, then the derivative $R^M_{nijk;k}$ on the formula is also intrinsic. Use the exercise to get

$$\sum_{i,j,k} h_{ij}R^M_{nijk;k} = Rm * h * h + \nabla Rm * h.$$

Use the symmetry of h, the formula to commute the second derivatives, (45), and Gauss equation to obtain

$$\begin{aligned}
\sum_{i,j,k} h_{ij}h_{ik;jk} &= \sum_{i,j,k} h_{ij}h_{ki;jk} \\
&= \sum_{i,j,k} h_{ij}h_{ki;kj} + \sum_{i,j,k} h_{ij}\sum_{p=1}^{n-1}(R^\Sigma_{pikj}h_{pk} + R^\Sigma_{pkkj}h_{ip}) \\
&= \sum_{i,j,k} h_{ij}h_{ki;kj} \\
&\quad - \sum_{i,j,k,p} h_{ij}((h_{pk}h_{ij} - h_{pj}h_{ik})h_{pk} + (h_{pk}h_{kj} - h_{pj}h_{kk})h_{ip}),
\end{aligned}$$

up to some $Rm * h * h$ term. Simplifying, using Codazzi and the exercise again, we have

$$\begin{aligned}\sum_{i,j,k} h_{ij}h_{ik;jk} &= \sum_{i,j,k} h_{ij}(h_{kk;i} + R^M_{nkki})_{;j} - |A|^4 + \sum_{i,j,k,p} h_{ij}h_{pj}h_{kk}h_{ip} \\ &= \sum_{i,j,k} h_{ij}h_{kk;ij} - |A|^4 + \sum_{i,j,k,p} h_{ij}h_{pj}h_{kk}h_{ip} \\ &= \sum_{i,j} h_{ij}\nabla_i\nabla_j H - |A|^4 + H\mathrm{tr}A^3\end{aligned}$$

up to $\nabla Rm * h + Rm * h * h$ terms. Put everything together to conclude

$$\begin{aligned}\Delta|A|^2 &= 2|\nabla A|^2 + Rm * h * h + \nabla Rm * h + 2\sum_{i,j,k} h_{ij}h_{ik;jk} \\ &= 2|\nabla A|^2 + 2\sum_{i,j} h_{ij}\nabla_i\nabla_j H - 2|A|^4 + 2H\mathrm{tr}A^3,\end{aligned}$$

up to $\nabla Rm * h + Rm * h * h$ terms. This concludes the proof.

11. Schoen-Simon-Yau Theorem

In this section we present the following result:

11.1. **Theorem.** [Schoen-Simon-Yau] *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be an oriented complete stable minimal hypersurface such that*

$$\sup_{R>0} \frac{Vol(\Sigma \cap B_R)}{R^{n-1}} < C < \infty \tag{46}$$

and $n \leq 6$. Then Σ is a hyperplane.

In order to prove this we need to develop two preliminary steps, the first is Simons' Inequality and the second is the application of stability inequality for $|A|^{1+q} \cdot f$, where f is any test function $f \in C_c^\infty(\Sigma)$.

Simons' Inequality. Recall Simons' equation $\Delta|A|^2 = 2|\nabla A|^2 - 2|A|^4$, where $|\nabla A|^2$ and $|A|^2$ can be written, in an orthonormal basis $\{e_i\}_{i=1}^{n-1} \subset T\Sigma$, as

$$|\nabla A|^2 = \sum_{i,j,k} h_{ij;k}^2 \quad \text{and} \quad |A|^2 = \sum_{i,j} h_{i,j}^2,$$

and $h_{ij} = \langle B(e_i, e_j), N\rangle$.

Observe $\nabla\left(|A|^2\right) = 2|A|\nabla|A|$, this gives

$$\begin{aligned}4|A|^2|\nabla|A||^2 &= |\nabla(|A|^2)|^2 \\ &= \sum_k \left(\left(\sum_{i,j} h_{ij}^2\right)_{;k}\right)^2 \\ &= 4\sum_k \left(\sum_{i,j} h_{ij} \cdot h_{ij;k}\right)^2.\end{aligned}$$

Choose $\{e_i\}_{i=1}^{n-1}$ such that at p we have $h_{ij}(p) = \lambda_i \delta_{ij}$. Then, can rewrite the expression above as

$$\begin{aligned} |A|^2 |\nabla |A||^2 &= \sum_k \left(\sum_i \lambda_i \cdot h_{ii;k} \right)^2 \\ &\leq \sum_k \left(\sum_i \lambda_i^2 \right) \cdot \left(\sum_i h_{ii;k}^2 \right) \\ &= |A|^2 \cdot \sum_{i,k} h_{ii;k}^2, \end{aligned}$$

In particular, we have

$$|\nabla |A||^2 \leq \sum_{i,k} h_{ii;k}^2. \tag{47}$$

We analyze separately the right-hand side of last expression, using the symmetries of second fundamental form $h_{ij;k} = h_{ji;k}$ and $h_{ij;k} = h_{ik;j}$, and minimality condition $\sum_j h_{jj} = 0$:

$$\begin{aligned} \sum_{i,k} h_{ii;k}^2 &= \sum_{i \neq k} h_{ii;k}^2 + \sum_i h_{ii;i}^2 \\ &= \sum_{i \neq k} h_{ii;k}^2 + \sum_i \left(-\sum_{j \neq i} h_{jj;i} \right)^2 \\ &\leq \sum_{i \neq k} h_{ii;k}^2 + \sum_i (n-2) \sum_{j \neq i} h_{jj;i}^2 \\ &\leq (n-1) \sum_{i \neq k} h_{ii;k}^2 \\ &\leq \frac{(n-1)}{2} \sum_{i \neq k} h_{ik;i}^2 + \frac{(n-1)}{2} \sum_{i \neq k} h_{ki;i}^2 \\ &\leq \frac{(n-1)}{2} \sum_{i \neq k} h_{ik;i}^2 + \frac{(n-1)}{2} \sum_{i \neq k} h_{ik;k}^2. \end{aligned}$$

Hence, we are able to conclude that

$$\begin{aligned} \left(1 + \frac{2}{n-1} \right) |\nabla |A||^2 &\leq \sum_{i,k} h_{ii;k}^2 + \sum_{i \neq k} h_{ik;i}^2 + \sum_{i \neq k} h_{ik;k}^2 \\ &\leq \sum_{i,j,k} h_{ij;k}^2 = |\nabla A|^2. \end{aligned}$$

Exercise. Prove equality holds if $\Sigma^2 \subset \mathbb{R}^3$ is minimal.

11.2. **Proposition.** [Simons' Inequality] *If* $\Sigma^{n-1} \subset \mathbb{R}^n$ *is a minimal hypersurface, then*

$$|A|\Delta|A| \geq \frac{2}{n-1}|\nabla|A||^2 - |A|^4.$$

Proof. By Simons' equation and previous analysis, we have

$$\begin{aligned} 2|A|\Delta|A| + 2|\nabla|A||^2 &= 2|\nabla A|^2 - 2|A|^4 \\ &\geq 2\left(1 + \frac{2}{n-1}\right)|\nabla|A||^2 - 2|A|^4. \end{aligned}$$

Simplifying we obtain the above inequality. □

11.2.1. *Remark.* This makes sense only in the weak sense.

Stability inequality to $|A|^{1+q}f$. Apply stability inequality to $|A|^{1+q}f$, where $f \in C_c^\infty(\Sigma)$:

$$\begin{aligned} \int_\Sigma |A|^{4+2q} f^2 d\Sigma &\leq \int_\Sigma |\nabla(|A|^{1+q}f)|^2 d\Sigma \\ &= \int_\Sigma |(1+q)|A|^q(\nabla|A|)f + |A|^{1+q}\nabla f|^2 d\Sigma \\ &= (1+q)^2 \int_\Sigma |A|^{2q} f^2 |\nabla|A||^2 d\Sigma \\ &\quad + 2(1+q)\int_\Sigma |A|^{1+2q} f\langle\nabla|A|, \nabla f\rangle d\Sigma \\ &\quad + \int_\Sigma |A|^{2+2q}|\nabla f|^2 d\Sigma. \end{aligned}$$

We estimate the second term in the right-hand side of last expression using Young inequality with small δ:

$$\begin{aligned} \int_\Sigma |A|^{1+2q} f\langle\nabla|A|, \nabla f\rangle &= \int_\Sigma \langle |A|^q f\nabla|A|, |A|^{1+q}\nabla f\rangle \\ &\leq \delta\int_\Sigma |A|^{2q} f^2|\nabla|A||^2 + C(\delta)\int_\Sigma |A|^{2+2q}|\nabla f|^2. \end{aligned}$$

And the stability inequality becomes

$$\int_\Sigma |A|^{4+2q} f^2 \leq ((1+q)^2 + \delta)\int_\Sigma |A|^{2q} f^2 |\nabla|A||^2 + C(\delta)\int_\Sigma |A|^{2+2q}|\nabla f|^2.$$

The first term in the right-hand side of last expression is analyzed using Simons' Inequality multiplied by $|A|^{2q}f^2$:

$$\int_\Sigma |A|^{1+2q}(\Delta|A|) f^2 d\Sigma \geq \frac{2}{n-1}\int_\Sigma |A|^{2q} f^2 |\nabla|A||^2 d\Sigma - \int_\Sigma |A|^{4+2q} f^2 d\Sigma.$$

Integrating by parts the left-hand side, we have

$$\left(1+2q+\frac{2}{n-1}\right)\int_\Sigma |A|^{2q}|\nabla|A||^2 f^2 d\Sigma \;\leq\; \int_\Sigma |A|^{4+2q} f^2 d\Sigma - \int_\Sigma |A|^{1+2q} f\langle \nabla|A|, \nabla f\rangle.$$

Use Young with δ again to conclude

$$\left(1+2q+\frac{2}{n-1}-\delta\right)\int_\Sigma |A|^{2q}|\nabla|A||^2 f^2 d\Sigma \;\leq\; \int_\Sigma |A|^{4+2q} f^2 d\Sigma + C(\delta)\int_\Sigma |A|^{2+2q}|\nabla f|^2.$$

Putting everything together we obtain

$$\int_\Sigma |A|^{4+2q} f^2 \leq \frac{(1+q)^2+\delta}{1+2q+\frac{2}{n-1}-\delta}\int_\Sigma |A|^{4+2q} f^2 + \tilde{C}(\delta)\int_\Sigma |A|^{2+2q}|\nabla f|^2.$$

The next step is to observe the existence of small $\delta > 0$ with the property

$$\frac{(1+q)^2+\delta}{1+2q+\frac{2}{n-1}-\delta} < 1$$

is equivalent to $(1+q)^2 < 1+2q+\frac{2}{n-1}$. In particular, this is equivalent to

$$q < \sqrt{\frac{2}{n-1}}.$$

In this case we conclude there exists $C > 0$ for which

$$\int_\Sigma |A|^{4+2q} f^2 d\Sigma \leq C\int_\Sigma |A|^{2+2q}|\nabla f|^2, \tag{48}$$

for every $f \in C_c^\infty(\Sigma)$. Consider now, the following change of variables $f = \phi^p$. Then, $\nabla f = p\cdot\phi^{p-1}\cdot\nabla\phi$ and expression (48) gives

$$\int_\Sigma |A|^{4+2q}\phi^{2p} d\Sigma \;\leq\; C\int_\Sigma |A|^{2+2q}p^2\phi^{2p-2}|\nabla\phi|^2.$$

Observe $\frac{2+2q}{4+2q}+\frac{1}{2+2q} = 1$, and apply Holder inequality:

$$\int_\Sigma |A|^{4+2q}\phi^{2p} d\Sigma \;\leq\; C'\left(\int_\Sigma (|A|^{2+2q}\phi^{2p-2})^{\frac{4+2q}{2+2q}}\right)^{\frac{2+2q}{4+2q}}\left(\int_\Sigma (|\nabla\phi|^2)^{2+q}\right)^{\frac{1}{2+q}}.$$

Finally, choose $p = 2+q$ and observe this gives

$$\int_\Sigma |A|^{2p}\phi^{2p} d\Sigma \leq C_p\int_\Sigma |\nabla\phi|^{2p} d\Sigma, \tag{49}$$

for every $\phi \in C_c^\infty(\Sigma)$ and $p \in [2, 2+\sqrt{\frac{2}{n-1}})$. The constant $C_p > 0$ is not the same as in equation (48), but it is also independent of test function ϕ.

11.2.2. *Remark.* We have not use $n \leq 6$ yet.

Proof of 11.1. Use r to denote the intrinsic distance function to $x_0 \in \Sigma$. Fix $R > 0$ and let $\phi = \phi(r)$ be such that

(a) $\phi(r) = 1$, if $0 \leq r \leq R$;
(b) $0 \leq \phi(r) \leq 1$ and $|\phi'(r)| \leq R^{-1}$, everywhere;
(c) $\phi(r) = 0$, if $2R \leq r$.

Put ϕ on equation (49) to get

$$\begin{aligned}\int_{\Sigma\cap B_R} |A|^{2p} d\Sigma &\leq C_p \cdot \int_{\Sigma\cap B_{2R}} \left(\frac{1}{R}\right)^2 p d\Sigma \\ &\leq C \cdot \mathrm{Vol}(\Sigma \cap B_{2R}) \cdot R^{-2p} \\ &\leq C \cdot R^{n-1-2p}.\end{aligned}$$

To make sure we can choose p so that $n - 1 - 2p < 0$ we need the following dimension restriction:

$$2 + \sqrt{\frac{2}{n-1}} > \frac{n-1}{2}.$$

One can easily check this holds if $n \leq 6$. □

Open problems. The same result is still open without the density bound (46). For dimension $n = 7$ it is open even with the density bound.

11.2.3. *Remark.* The result is known to be false for $n = 8$, the counterexample is known as Simons' cone.

We have seen that minimal graphs are area minimizing, in particular, stable and have polynomial volume growth.

11.3. **Bernstein Theorem.** *Suppose $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a solution to the minimal surface equation and $n \leq 8$. Then u is a affine function.*

11.3.1. *Remark.* There exists non-trivial entire minimal graph in $\mathbb{R}^9$. The example is given by Bombieri, De Giorgi and Giusti.

Fleming's approach to Bernstein Theorem. Let $\Sigma = \mathrm{graph}(u)$ be a minimal entire graph. It is already known Σ is an area minimizing boundary $\Sigma = \partial U$. Choose a sequence $r_j > 0$, $r_j \to 0$. We consider then the blow-down of Σ

$$\Sigma_j = r_j \Sigma \quad \text{and} \quad \Sigma_j = \partial U_j. \tag{50}$$

Σ_j is also area minimizing. Moreover, if $R > 0$ if fixed we have volume and area bounds

$$\mathrm{Vol}(U_j \cap B_R) \leq \omega_n R^n \quad \text{and} \quad \mathrm{area}(\Sigma_j \cap B_R) \leq \sigma_{n-1} R^{n-1}. \tag{51}$$

Apply Compactness Theorem for integral currents, to obtain a subsequence U_j converging to U_∞ in the sense of volumes. Furthermore, there exists

$\Sigma_\infty = \partial U_\infty = \lim_j \Sigma_j$, and Σ_∞ is again area minimizing. About the area of the limit, we have

$$\begin{aligned}\text{area}(\Sigma_\infty \cap B_R) &= \lim_{j\to\infty} \text{area}(\Sigma_j \cap B_R)\\ &= \lim_{j\to\infty} r_j^{n-1}\text{area}\left(\Sigma \cap B_{\frac{R}{r_j}}\right).\end{aligned}$$

Hence, we obtain information about the density of area of Σ on each ball of radius R:

$$\begin{aligned}\Theta(\Sigma_\infty, R) &= \frac{\text{area}(\Sigma_\infty)\cap B_R}{R^{n-1}}\\ &= \lim_{j\to\infty} \frac{\text{area}\left(\Sigma \cap B_{\frac{R}{r_j}}\right)}{\left(\frac{R}{r_j}\right)^{n-1}}\\ &= \lim_{j\to\infty} \frac{\text{area}(\Sigma\cap B_s)}{s^{n-1}},\end{aligned}$$

i.e., $\Theta(\Sigma_\infty, R)$ coincide with the density of Σ at infinity for every $R > 0$. By monotonicity formula we infer Σ_∞ has to be a cone, and this cone is area minimizing. Use now the following result by Simons:

Simons: There is no non-trivial stable minimal cones in $\mathbb{R}^n$ for $n \leq 7$.

Then, our Σ_∞ is a hyperplane, possibly with multiplicities. But here the multiplicity is one because Σ_∞ is an area minimizing boundary $\Sigma_\infty = \partial U_\infty$. So, $\Theta(\Sigma, \infty) = 1$. Monotonicity formula again imply Σ is a cone with vertex p, for some $p \in \Sigma$. Since Σ is smooth, it must be a hyperplane.

11.3.2. *Remark.* The Simons' cone is the cone over $\mathbb{S}^3(\frac{1}{\sqrt{2}}) \times \mathbb{S}^3(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^7$. It is a stable minimal hypersurface that is area-minimizing. The same holds for $\mathbb{R}^{2m}$, as long as $m \geq 4$.

12. Pointwise curvature estimates

The curvature estimates for stable surfaces in 3-manifolds were developed by Heinz, Osserman and Schoen. The basic steps are the mean-value inequality and ε-regularity. The goal of this section is to introduce those curvature estimates. It is also possible to obtain curvature estimates for constant mean-curvature surfaces, as done by Bérard and Hauswirth. Another method to obtain curvature bounds is the blow-up technique developed by Rosenberg, Souam and Toubiana.

The geometric significance of curvature estimates is that bounds on $|A|^2$ imply graphical representation, as in the following result:

12.1. **Lemma.** *Let $\Sigma^2 \subset \mathbb{R}^3$ be an immersed surface such that:*

- $\sup_\Sigma |A|^2 \leq \frac{1}{16}$;
- $d_g(x, \partial\Sigma) \geq 2$.

Then,

(i) *$B_2^\Sigma(x)$ is a graph of some function u over $T_x\Sigma$ and $|\nabla u| \leq 1$, $|\nabla^2 u| \leq 1/\sqrt{2}$;*
(ii) *the connected component of $B_1(x) \cap \Sigma$ that contains x is contained in $B_2^\Sigma(x)$.*

Proof of (i). Use N to denote the unit normal vector field along Σ. Observe $|DN| \leq |A| \leq 1/4$. Suppose γ is a minimizing geodesic connecting x to y in Σ. Integrating $d/dtN(\gamma(t))$, we obtain

$$|N(x) - N(y)| \leq \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

In particular, $d_{\mathbb{S}^2}(N(x), N(y)) < \frac{\pi}{4}$ and $B_2^\Sigma(x)$ is a graph of some function u over $T_x\Sigma$. Hence, N can be written as

$$N = \frac{(\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

This implies gradient estimates, because

$$1 + |\nabla u|^2 = \langle N(x), N(y)\rangle^{-2} = (\cos(d_{\mathbb{S}^2}(N(x), N(y))))^{-2} \leq 2.$$

Also, the control on $|\nabla^2 u|$ comes from

$$\frac{|\nabla^2 u|^2}{(1 + |\nabla u|^2)^3} \leq |A|^2.$$

□

12.1.1. *Remark.* Quasilinear elliptic PDE theory imply bounds on all derivatives of u, precisely, $|\nabla^k u| \leq C_k$ for every $k \geq 3$.

Proof of (ii). Consider a minimizing geodesic $\gamma : [0, 2] \to \Sigma$, $\gamma(0) = x$ and $\gamma(2) \in \partial B_2^\Sigma(x)$ and parametrized by arc length. Observe $B(\gamma', \gamma') = D_{\gamma'}\gamma' - \nabla^\Sigma_{\gamma'}\gamma' = D_{\gamma'}\gamma'$, then

$$\left|\frac{d}{dt}\langle \gamma'(t), \gamma'(0)\rangle\right| = \left|\langle D_{\gamma'}\gamma'(t), \gamma'(0)\rangle\right| \leq |A| \leq \frac{1}{4}.$$

In particular, $\langle \gamma'(t), \gamma'(0)\rangle \geq 1/2$ and we conclude

$$\left|\frac{d}{dt}\langle \gamma(t) - \gamma(0), \gamma'(0)\rangle\right| = |\langle \gamma'(t), \gamma'(0)\rangle| \geq \frac{1}{2}.$$

Finally, this gives us $\langle \gamma(2) - \gamma(0), \gamma'(0)\rangle \geq 1$ and, then, $|\gamma(2) - x| \geq 1$. □

Mean-value Inequality. Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold and f be a function on Σ, such that $f \geq 0$ and $\Delta_\Sigma f + cf \geq 0$, for some constant $c > 0$. Define

$$g(t) := t^{-k} \int_{\Sigma \cap B_t} f d\Sigma. \tag{52}$$

Use d to denote the extrinsic distance. We have

$$g'(t) = -\frac{k}{t^{k+1}} \int_{\Sigma \cap B_t} f d\Sigma + \frac{1}{t^k} \int_{\Sigma \cap \partial B_t} \frac{f}{|\nabla_\Sigma d|} d\Sigma. \tag{53}$$

Recall that minimality of Σ^k imply $\Delta_\Sigma |x - x_0|^2 = 2k$, then

$$\begin{aligned}
2k \int_{\Sigma \cap B_t} f &= \int_{\Sigma \cap B_t} (\Delta_\Sigma |x - x_0|^2) \cdot f \\
&= \int_{\Sigma \cap B_t} \Delta_\Sigma (|x - x_0|^2 - t^2) \cdot f \\
&= \int_{\Sigma \cap B_t} (|x - x_0|^2 - t^2) \cdot \Delta_\Sigma f + \int_{\Sigma \cap \partial B_t} f(\partial_\nu |x - x_0|^2) \\
&\quad - \int_{\Sigma \cap \partial B_t} (|x - x_0|^2 - t^2)(\partial_\nu f),
\end{aligned}$$

where ν is the outward pointing unit normal vector to $\Sigma \cap B_t$. The third term in the right-hand side is clearly zero, because $|x - x_0| = t$ on $\Sigma \cap \partial B_t$. Since $|x - x_0|^2 - t^2 \leq 0$ on $\Sigma \cap B_t$, we can use the estimate $\Delta_\Sigma f \geq -cf$ and the fact $\nabla |x - x_0|^2 = 2|x - x_0| \nabla |x - x_0|$, to obtain

$$2k \int_{\Sigma \cap B_t} f \leq -\int_{\Sigma \cap B_t} (|x - x_0|^2 - t^2) cf + 2t \int_{\Sigma \cap \partial B_t} f. \tag{54}$$

From expression (53), we infer

$$g'(t) \geq -\frac{k}{t^{k+1}} \int_{\Sigma \cap B_t} f d\Sigma + \frac{1}{t^k} \int_{\Sigma \cap \partial B_t} f d\Sigma. \tag{55}$$

Combine (54) and (55) to conclude

$$\begin{aligned}
2t^{k+1} g'(t) &\geq -2k \int_{\Sigma \cap B_t} f + \int_{\Sigma \cap B_t} (|x - x_0|^2 - t^2) cf + 2k \int_{\Sigma \cap B_t} f \\
&\geq -ct^2 \int_{\Sigma \cap B_t} f = -ct^{k+2} g(t).
\end{aligned}$$

This analysis gives us the following inequality $g'(t) \geq -\frac{c}{2} t g(t)$. Finally,

$$e^{\frac{c}{4} t^2} g(t) \quad \text{is non-decrasing,}$$

and we have

$$f(x_0) \leq e^{\frac{c}{4} s^2} \cdot \frac{\int_{\Sigma \cap B_s} f d\Sigma}{\omega_k s^k}. \tag{56}$$

we refer to this expression as the mean-value inequality.

ε-regularity. The second step is developed in this section, the main result is the following Theorem by Choi and Schoen that roughly says that small total curvature implies curvature estimates.

12.2. **Theorem.** [Choi and Schoen] *Let (M^3, g) be a Riemannian manifold. There exists $\varepsilon, \rho > 0$, depending only on the geometry of M, such that for $p \in M$, $0 < r_0 < \rho$ and $\Sigma^2 \subset M$ minimal surface with*

- $\partial\Sigma \cap B_{r_0}(p) = \emptyset$;
- $\int_{\Sigma \cap B_{r_0}} |A|^2 d\Sigma \leq \delta \cdot \varepsilon$,

then, we have

$$\sigma^2 \cdot \sup_{\Sigma \cap B_{r_0 - \sigma}} \leq \delta.$$

Proof. This argument uses mean-value inequality and Simons' equation. Here we consider $\Sigma^2 \subset \mathbb{R}^3$ and $B_{r_0} = B_{r_0}(p)$ a euclidean ball of radius r_0 centered at p. Define

$$F(x) = (r_0 - r(x))^2 \cdot |A|^2(x),$$

where we use $r(x) = |x - p|$ to denote the distance function to p in $\mathbb{R}^3$. Observe F vanishes on $\Sigma \cap \partial B_{r_0}$ and choose $x_0 \in \Sigma$, such that x_0 is contained in the interior of $\Sigma \cap B_{r_0}$ and

$$F(x_0) = \sup_{\Sigma \cap B_{r_0}} F(x).$$

Claim 1. $F(x_0) \leq \delta$.

If this is true, then $(r_0 - r(x))^2 \cdot |A|^2(x) \leq \delta$, and we are done.

Proof of Claim. Suppose, by contradiction, $F(x_0) > \delta$. Choose σ so that $\sigma^2 |A|^2(x_0) = \frac{\delta}{4}$. Observe σ has to be less than $(r_0 - r(x_0))/2$. About the second fundamental form on $\Sigma \cap B_\sigma(x_0)$, we have: if $y \in \Sigma \cap B_\sigma(x_0)$, we use $F(y) \leq F(x)$, to obtain

$$|A|^2(y) \leq \frac{(r_0 - r(x_0)^2)}{(r_0 - r(y))^2} \cdot |A|^2(x_0) \leq 4|A|^2(x_0).$$

Scale $\Sigma \cap B_\sigma(x_0)$ to radius 1 and let $\tilde{\Sigma}$ be the scaled surface and $\tilde{A}$ its second fundamental form. This scale yields, for every $y \in \tilde{\Sigma}$,

$$|\tilde{A}|^2(x_0) = \frac{\delta}{4} \quad \text{and} \quad |\tilde{A}|^2(y) \leq 4|\tilde{A}|^2(x_0) = \delta.$$

Hence, Simons' inequality gives

$$\Delta|\tilde{A}|^2 \geq -|\tilde{A}|^4 \geq -\delta|\tilde{A}|^2,$$

i.e., in the scaled region $|\tilde{A}|^2$ behaves like a subharmonic function. By mean-value inequality,

$$|\tilde{A}|^2(x_0) \leq C \cdot \int_{\tilde{\Sigma} \cap B_1} |\tilde{A}|^2,$$

where $C > 0$ is an universal constant. The key point is that $\int |A|^2 d\Sigma$ is scale invariant, then

$$\int_{\tilde{\Sigma}\cap B_1} |\tilde{A}|^2 d\tilde{\Sigma} = \int_{\Sigma\cap B_\sigma(x_0)} |A|^2 d\Sigma < \delta \cdot \varepsilon.$$

Putting everything together we obtain

$$\frac{\delta}{4} = |\tilde{A}|^2(x_0) \leq C\delta \cdot \varepsilon.$$

Let ε be sufficiently small such that $8C\varepsilon < 1$. This yields a contradiction and this finishes the proof of Claim. ☐

12.2.1. *Remark.* In case of a compact 3-manifold M, we obtain a similar subharmonic behavior of $|A|^2$. More precisely, we obtain $\Delta|A|^2 \geq -(\eta^2 + |A|^2)$, where η comes from curvature. To deal with this difficulty, observe that η becomes small after the scaling.

We are now ready to prove Schoen's curvature estimates.

12.3. **Theorem.** *Let $\Sigma^2 \subset \mathbb{R}^3$ be an immersed oriented stable minimal surface with boundary $\partial\Sigma$. Then, there exists $C > 0$ such that*

$$|A|^2(x) \leq \frac{C}{d_\Sigma(x, \partial\Sigma)^2}.$$

12.3.1. *Remark.* If we change $\mathbb{R}^3$ with a 3-manifold M such that $|Rm| + |\nabla Rm| \leq \Lambda$, and $\Sigma^2 \subset M$ is an immersed oriented stable minimal surface with $B^\Sigma_{r_0} \subset \Sigma \setminus \partial\Sigma$ and $r_0 \leq \rho(\Lambda)$, then

$$\sup_{\Sigma\cap B_{r_0-\sigma}} \leq C \cdot \sigma^{-2}. \tag{57}$$

Proof. We prove the case of a surface inside $\mathbb{R}^3$. If $r_0 = d(x, \partial\Sigma)$, we have $B^\Sigma_{r_0}(x) \subset \Sigma \setminus \partial\Sigma$. Recall the stability condition gives

$$\int_\Sigma |A|^2 f^2 \leq \int_\Sigma |\nabla f|^2, \quad \text{for all } f \in C^\infty_c(\Sigma).$$

By passing to the universal cover $\tilde{\Sigma}$, that is also stable, we have $B^{\tilde{\Sigma}}_{r_0}(\tilde{x})$ is a disk. Hence, we can apply Colding-Minicozzi estimates, and conclude $\text{area}(\Sigma \cap B_r) \leq Cr^2$, for some universal constant $C > 0$. Consider a function $f = f(r)$ be defined by

- $f(r) = 1$, if $r \leq e^{-n} r_0$;
- $f(r) = -n^{-1}(\log r - \log r_0)$, if $e^{-n} r_0 \leq r \leq r_0$;
- $f(r) = 0$, if $r_0 \leq r$,

n to be chosen. Plugging this f in the stability inequality gives us

$$\begin{aligned}\int_{\Sigma\cap B_{e^{-n}r_0}} |A|^2 &\leq \frac{1}{n^2}\int_{B_{r_0}\setminus B_{e^{-n}r_0}} \frac{1}{r^2}\\ &= \frac{1}{n^2}\sum_{k=1}^{n}\int_{B_{e^{-k+1}r_0}\setminus B_{e^{-k}r_0}} \frac{1}{r^2}\\ &\leq \frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{e^{-2k}r_0^2}\operatorname{area}\big(\Sigma\cap B_{e^{-k+1}r_0}\big)\\ &\leq \frac{C}{n^2}\sum_{k=1}^{n} e^2 = \frac{Ce^2}{n}.\end{aligned}$$

If $\varepsilon > 0$ is given by ε-regularity Theorem 12.2, choose n so that $Ce^2 < n\varepsilon$. Therefore, $(e^{-n}r_0)^2|A|^2(x) \leq 1$, and we conclude the proof of Schoen's pointwise curvature estimates for stable minimal surfaces. □

13. Plateau problem

The Plateau problem in minimal surface theory is related to minimize the area among all immersed disks with a given boundary. More precisely, we consider a closed curve $\Gamma \subset \mathbb{R}^3$ and for each immersion $\Phi : \mathbb{D}^2 \to \mathbb{R}^3$ with $\Phi(\partial\mathbb{D}^2) = \Gamma$, we take its area area$(\Phi(\mathbb{D}^2))$. We say $x : \mathbb{D}^2 \to \mathbb{R}^3$ is a solution to Plateau problem if

$$\operatorname{area}(x(\mathbb{D}^2)) = \inf_{\Phi(\mathbb{D}^2)} \operatorname{area}(\Phi(\mathbb{D}^2)).$$

The question was first solved by Douglas and Radó in the early 1930's. Later, Morrey proved existence of Plateau solutions in homogeneously regular Riemannian manifolds (M, g), i.e., if there are constants $C_1, C_2 > 0$ such that for every $p \in M$, there exists a small neighborhood of p parametrized by the unit ball of same dimension as M, in such a way the metric g in that coordinates satisfies

$$C_2|\xi|^2 \leq \sum_{i,j} g_{ij}\xi_i\xi_j \leq C_1|\xi|^2.$$

In particular, a compact Riemannian manifold is homogeneously regular.

Difficulties.

(1) The first difficulty that arises is that the area is invariant under parametrization, i.e., given $u : \mathbb{D}^2 \to \mathbb{R}^3$ and a diffeomorphism $\Phi \in Diff(\mathbb{D}^2)$, we have area$(u)$ = area$(u \circ \Phi)$. This causes a problem for compactness.
(2) The tentacles are the second problem, those are long pieces of surfaces with very small area. This causes a problem of non-convergence in the usual sense.

We use $x : \Sigma^2 \to \mathbb{R}^3$ to denote an immersion of Σ inside $\mathbb{R}^3$, we also write $\Sigma^2 \subset \mathbb{R}^3$. Let g be the induced metric on Σ via x. It is already known that

$$\Delta_g x_i = \langle \overrightarrow{H}, e_i \rangle. \tag{58}$$

We say $x : \mathbb{D}^2 \to \mathbb{R}^3$*is a conformal immersion* if $g = e^{2\phi}\delta$, where δ is the standard metric of $\mathbb{D}^2$ and ψ is some function. In this case, we have $\Delta_g \phi = e^{-2\psi}\Delta_\delta \phi$, for every function ϕ on $\mathbb{D}^2$. In particular, if x is a conformal minimal immersion, then $\Delta_\delta x_i = 0$, for all i, and x is harmonic. Conversely, if $x : \mathbb{D}^2 \to \mathbb{R}^3$ is a conformal harmonic immersion, then x is minimal.

Weierstrass representation. Let $u : \mathbb{D}^2 \to \mathbb{R}^3$ is a conformal harmonic immersion. Consider the functions

$$f_j = \frac{\partial u_j}{\partial x} - i\frac{\partial u_j}{\partial y} \in \mathbb{C}, \quad \text{for all } j = 1, 2, 3.$$

Since u is harmonic, the functions f_j are holomorphic. Moreover, u being conformal imply

$$\sum_{j=1}^{3} \left|\frac{\partial u_j}{\partial x}\right|^2 = \sum_{j=1}^{3} \left|\frac{\partial u_j}{\partial y}\right|^2 \quad \text{and} \quad \sum_{j=1}^{3} \frac{\partial u_j}{\partial x} \cdot \frac{\partial u_j}{\partial y} = 0,$$

and this is equivalent to $(f_1)^2 + (f_2)^2 + (f_3)^2 = 0$. From f_1, f_2 and f_3, we can rewrite $u(z)$ as

$$u(z) = u(0) + \int_0^z (f_1, f_2, f_3) dw.$$

Can actually write

$$u(z) = u(0) + \int_0^z \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1\right) fg dw,$$

where

13.1. **Definition.** [Dirichlet Integral] Let $u : \mathbb{D}^2 \to \mathbb{R}^3$ be an immersion. We define the *Dirichlet integral of* u to be

$$E(u) := \frac{1}{2}\int_{\mathbb{D}^2} (|u_x|^2 + |u_y|^2) dxdy.$$

Recall the area of a map $u : \mathbb{D}^2 \to \mathbb{R}^3$ is given by

$$A(u) = \int_{\mathbb{D}^2} |u_x \wedge u_y| dxdy = \int_{\mathbb{D}^2} \sqrt{|u_x|^2 \cdot |u_y|^2 - \langle u_x, u_y \rangle^2} dxdy.$$

And it is possible to relate area and Dirichlet energy in a simple inequality:

13.2. **Proposition.** *If $u : \mathbb{D}^2 \to \mathbb{R}^3$, then $A(u) \leq E(u)$. The equality holds if and only if the map u is almost conformal, i.e.,*

$$\left|\frac{\partial u}{\partial x}\right|^2 = \left|\frac{\partial u}{\partial y}\right|^2 \quad \textit{and} \quad \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle = 0.$$

Proof.

$$\begin{aligned} A(u) &= \int_{\mathbb{D}^2} \sqrt{|u_x|^2 \cdot |u_y|^2 - \langle u_x, u_y \rangle^2} dxdy \\ &\leq \int_{\mathbb{D}^2} |u_x| \cdot |u_y| dxdy \\ &\leq \frac{1}{2} \int_{\mathbb{D}^2} (|u_x|^2 + |u_y|^2) dxdy = E(u), \end{aligned}$$

where the last estimate is Young' inequality. □

13.3. **Riemann Mapping Theorem.** Let $u : \mathbb{D}^2 \to \mathbb{R}^3$ be an immersion and consider in $\mathbb{D}^2$ the pull-back metric $u^*\delta$, where δ is the standard euclidean metric. Any disk can be parametrized by isothermal coordinates, let $\Phi : (\mathbb{D}^2, \delta) \to (\mathbb{D}^2, u^*\delta)$ be such a conformal map. Consider a piecewise C^1 Jordan curve $\Gamma \subset \mathbb{R}^3$ (simple and closed).

13.4. **Theorem.** *There exists $u : \mathbb{D}^2 \to \mathbb{R}^3$ such that*

(1) $u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$*;*
(2) $u|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \Gamma$ *is monotone and surjective;*
(3) $u \in C^\infty(\mathbb{D})$*;*
(4) *the image of u minimizes area among all parametrized disks with boundary Γ.*

Definitions, notations and useful formulas.

Sobolev space: The space $W^{1,2}(\mathbb{D})$ is the sobolev space of functions f on $\mathbb{D}$ with f and ∇f in L^2, provided with the norm

$$||f||_{W^{1,2}(\mathbb{D})} = \int_{\mathbb{D}} (|f|^2 + |\nabla f|^2).$$

More precisely, $W^{1,2}(\mathbb{D})$ is the completion of

$$\{\psi \in C^\infty(\mathbb{D}) : \int_{\mathbb{D}} (|\psi|^2 + |\nabla \psi|^2) < \infty\}$$

with respect to the norm $|| \cdot ||_{W^{1,2}}$. Similarly, $W_0^{1,2}(\mathbb{D})$ is the completion of

$$\{\psi \in C^\infty(\mathbb{D}) \cap C^0(\overline{\mathbb{D}}) : \psi = 0 \text{ on } \partial\mathbb{D} \text{ and } \int_{\mathbb{D}} (|\psi|^2 + |\nabla \psi|^2) < \infty\},$$

with respect to $||\cdot||_{W^{1,2}}$. An important property of these functions is they satisfy a Poincaré inequality, i.e., there exits a universal constant $C > 0$ such that

$$\int_{\mathbb{D}^2} u^2 dxdy \leq C \cdot \int_{\mathbb{D}^2} |\nabla u|^2, \quad \text{for all } u \in W_0^{1,2}(\mathbb{D}). \tag{59}$$

Monotone on Γ*:* $u : \partial\mathbb{D} \to \Gamma$ is said to be monotone if $u^{-1}(C)$ is connected for any connected subset $C \subset \Gamma$.

We use the following notation:

$$X_\Gamma = \{u : \mathbb{D}^2 \to \mathbb{R}^3 : u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D}) \text{ and } u|_{\partial\mathbb{D}} \text{ is monotone and onto}\}.$$

Minimum area and energy: We use A_Γ and E_Γ to denote the minimum area and energy for maps on X_Γ, respectively, i.e.,

$$A_\Gamma = \inf_{u \in X_\Gamma} A(u) \quad \text{and} \quad E_\Gamma = \inf_{u \in X_\Gamma} E(u).$$

Morrey's theorem: The following Theorem of Morrey plays an important role in the argument of Riemann mapping theorem:

13.5. **Theorem.** *Let* $u : \partial\mathbb{D} \to \Gamma$ *be so that* $u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$ *and let* $\varepsilon > 0$ *be given. There exists* $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ *such that*

$$E(u \circ \Phi) \leq (1 + \varepsilon) \cdot A(u).$$

Lets sketch the proof of Morrey's theorem. Suppose $u : \partial\mathbb{D} \to \Gamma$ is smooth, but not necessarily immersion. Consider

$$u_s(x, y) := (u(x, y), sx, sy) \in \mathbb{R}^5.$$

Let $g = u^*(\delta)$ be given by $g_{ij} = \langle X_i, X_j \rangle$, and $g_s = u_s^*(\delta)$ be so that $(g_s)_{ij} = g_{ij} + s^2\delta_{ij} > 0$, which is positive definite for $s > 0$. Choose a conformal map $\Phi_s : (\mathbb{D}^2, \delta) \to (\mathbb{D}^2, g_s)$, then $u_s \circ \Phi_s$ is also conformal and we have

$$E(u \circ \Phi_s) \leq E(u_s \circ \Phi_s) = A(u_s \circ \Phi_s) = A(u_s).$$

Moreover, the area of u_s can be written as

$$A(u_s) = \int_{\mathbb{D}^2} \sqrt{\det(g_s)} = \int_{\mathbb{D}^2} \left(\sqrt{\det(g)} + s^2 |\nabla u|^2 + s^4 \right),$$

then, it converges to $A(u)$ as $s \to 0$, and this concludes the argument, for s small, we have $E(u \circ \Phi_s) \leq (1 + \varepsilon)A(u)$.

Morrey's result imply that we can work with energy instead of area, i.e., 13.6. **Proposition.** $A_\Gamma = E_\Gamma$.

Proof. $A_\Gamma \leq E_\Gamma$ is immediate. Given $\varepsilon > 0$, choose $u : \mathbb{D}^2 \to \mathbb{R}^3$ so that $A(u) \leq A_\Gamma + \varepsilon$. Find $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ with the property $E(u \circ \Phi) \leq (1+\varepsilon)A(u)$. Then, we conclude

$$(\mathbb{D}^2, \delta) \to (\mathbb{D}^2, u^*\delta) \leq (1+\varepsilon)(A_\Gamma + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, $E_\Gamma \leq A_\Gamma$ and this finishes the proof. $\square$

The first step in the proof of Riemann mapping theorem is to minimize energy among all disks that induce some fixed parametrization of Γ.

Dirichlet problem: Fix $w \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$. Consider the space of functions

$$X_w = \{u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D}) : u - w \in W_0^{1,2}(\mathbb{D})\}.$$

Minimize the energy in X_w.

13.7. **Theorem.** *There exists a unique $u \in X_w$, such that*

$$\int_{\mathbb{D}^2} |\nabla u|^2 \leq \inf_{v \in X_w} \int_{\mathbb{D}^2} |\nabla v|^2.$$

Moreover, $u \in C^\infty(\mathbb{D})$ and $\Delta u = 0$.

Proof. Choose a minimizing sequence $u_l \in X_w$, so that $\int |\nabla u_l|^2$ converges to the infimum of such integrals. In particular, u_l is bounded in $W_0^{1,2}(\mathbb{D})$. We invoke Poincaré's inequality to obtain

$$\int_{\mathbb{D}^2} (u_l - w)^2 \leq C \cdot \int_{\mathbb{D}^2} |\nabla(u_l - w)|^2 \leq C',$$

where $C' > 0$ is a constant not depending on l. In particular, $\{u_l\}_{l=1}^\infty$ is a bounded sequence in $W^{1,2}$-norm. By Rellich Compactness Theorem, up to subsequences, we can suppose

$$u_l \xrightarrow{L^2} u \quad \text{and} \quad \nabla u_l \xrightarrow{L^2} \nabla u.$$

Moreover, the limiting function u satisfies $u - w \in W_0^{1,2}(\mathbb{D})$ and its energy can be estimated by

$$\int_{\mathbb{D}^2} |\nabla u|^2 \leq \liminf_{l \to \infty} \int_{\mathbb{D}^2} |\nabla u_l|^2 = \inf_{v \in X_w} \int_{\mathbb{D}^2} |\nabla v|^2.$$

The convergence above is in the weak L^2-sense, i.e.,

$$\int_{\mathbb{D}^2} \langle X, \nabla u \rangle = \lim_{l \to \infty} \int_{\mathbb{D}^2} \langle X, \nabla u_l \rangle, \quad \text{for all } X \in L^2.$$

Then, u is a minimizer of energy. In particular, for every $\phi \in C_c^\infty(\mathbb{D})$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{D}^2} |\nabla(u + t\phi)|^2 = 0,$$

and this also means

$$\int_{\mathbb{D}^2} \langle \nabla u, \nabla \phi \rangle = 0, \quad \text{for all } \phi \in C_c^\infty(\mathbb{D}).$$

This condition is the same as saying $\Delta u = 0$ in the weak sense. The regularity theory for elliptic PDE imply $u \in C^\infty(\mathbb{D})$.

To prove $u \in C^0(\overline{\mathbb{D}})$, we use barriers. Let $p = (0,1) \in \partial\mathbb{D}$. Lets show that $\lim_{x \to p} u(x) = u(p)$. A function h is a barrier at p if:

- $h > 0$ on $\mathbb{D}^2 \setminus \{p\}$;
- $h(p) = 0$;
- $\Delta h \leq 0$.

In this case, one can choose $h(x,y) = 1 - x$. Given $\varepsilon > 0$, consider $\delta > 0$ so that $|w(x) - w(p)| \leq \varepsilon$, for every $x \in B_\delta(p) \cap \mathbb{D}^2$. Let w^+ and w^- be harmonic functions defined by

$$w^+ = w(p) + \varepsilon + k \cdot h \quad \text{and} \quad w^- = w(p) - \varepsilon - k \cdot h.$$

Choose k very large in such a way that $w^- \leq w \leq w^+$, on the whole $\mathbb{D}^2$. The maximum principle tells us $w^- \leq u \leq w^+$, and this implies

$$|u(x) - w(p)| \leq 2\varepsilon, \quad \text{if } x \in B_\delta(p) \cap \mathbb{D}^2.$$

□

The next thing we remark is that energy is conformally invariant.

13.8. **Proposition.** *Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a conformal diffeomorphism. Then, given $u : \mathbb{D}^2 \to \mathbb{R}^3$, we have $E(u \circ \Phi) = E(u)$.*

Proof. Let $z = (x,y) \in \mathbb{D}^2$, $\{e_1, e_2\}$ be an orthonormal basis at $\Phi(z)$ and $\lambda > 0$ be the conformal factor of Φ at z, so that $D\Phi(\partial_x) = \lambda e_1$ and $D\Phi(\partial_y) = \lambda e_2$. Observe

$$D(u \circ \Phi)_z(\partial_x) = Du_{\Phi(z)}(D\Phi(\partial_x)) = \lambda Du_{\Phi(z)}(e_1)$$

and, similarly,

$$D(u \circ \Phi)_z(\partial_y) = \lambda Du_{\Phi(z)}(e_2).$$

Then, we have

$$|D(u \circ \Phi)_z(\partial_x)|^2 + |D(u \circ \Phi)_z(\partial_y)|^2 = \lambda^2 \left(|Du_{\Phi(z)}(e_1)|^2 + |Du_{\Phi(z)}(e_2)|^2 \right).$$

But Jac $\Phi(z) = \lambda^2$, then

$$\begin{aligned} E(u \circ \Phi) &= \frac{1}{2} \int \left(|D(u \circ \Phi)_z(\partial_x)|^2 + |D(u \circ \Phi)_z(\partial_y)|^2 \right) dz \\ &= \frac{1}{2} \int |Du_{\Phi(z)}|^2 \cdot \text{Jac } \Phi(z) dz \\ &= \frac{1}{2} \int |\nabla u(w)|^2 dw = E(u). \end{aligned}$$

□

The orientation preserving maps in the conformal group of $\mathbb{D}^2$ are the maps given by the following expressions:

$$w \in \mathbb{D}^2 \mapsto e^{i\theta} \cdot \frac{a+w}{1-\bar{a}\cdot w}, \tag{60}$$

for fixed $\theta \in \mathbb{R}$ and $|a| < 1$. Observe this group is not compact.

Fix $p \in \mathbb{D}^2$. Use $C_\rho = \{q \in \mathbb{D}^2 : |q-p| = \rho\}$ to denote the points of $\mathbb{D}^2$ with distance $\rho > 0$ to p.

13.9. **Courant-Lebesgue's Lemma.** *If $u : \mathbb{D}^2 \to \mathbb{R}^3$, $u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$ has $E(u) \leq k$, then for all $\delta < 1$, there exists $\rho \in [\delta, \sqrt{\delta}]$ with*

$$d(C_\rho)^2 \leq \varepsilon_\delta = \frac{8\pi k}{-\log\delta}.$$

Here $d(C_\rho) = diam(u(C_\rho))$.

Remark. Observe $\varepsilon_\delta \to 0$ as $\delta \to 0$.

Proof. Observe

$$2k \geq \int_{\mathbb{D}} |\nabla u|^2 dxdy \geq \int_{\mathbb{D}\cap\left(B_{\sqrt{\delta}}(p)\setminus B_\delta(p)\right)} |\nabla u|^2 dxdy = \int_\delta^{\sqrt{\delta}} \int_{C_r} |\nabla u|^2 ds_r dr.$$

Consider the function $p(r)$ defined by

$$p(r) = r\int_{C_r} |\nabla u|^2 ds_r,$$

and observe

$$\int_\delta^{\sqrt{\delta}} p(r) d(\log r) \leq 2k.$$

Choose $\rho \in [\delta, \sqrt{\delta}]$ with the property

$$p(\rho) = \frac{\int_\delta^{\sqrt{\delta}} p(r)d(\log r)}{\int_\delta^{\sqrt{\delta}} d(\log r)}.$$

In conclusion,

$$\int_{C_\rho} |\nabla u|^2 ds_\rho \leq \frac{4k}{-\rho\log\delta}. \tag{61}$$

Let $L(\rho)$ be the length of $u(C_\rho)$, then

$$\begin{aligned} L(\rho) &\leq \left(\int_{C_\rho} |\nabla u|^2 ds_\rho\right)^2 \\ &\leq \left(\int_{C_\rho} |\nabla u|^2 ds_\rho\right) \cdot L(C_\rho) \\ &\leq 2\pi\rho \int_{C_\rho} |\nabla u|^2 ds_\rho. \end{aligned} \tag{62}$$

Put this and expression (61) together to obtain

$$L(\rho)^2 \leq \frac{8\pi k}{-\log\delta}.$$

To conclude, observe that the diameter of $u(C_\rho)$ is not greater than $L(\rho)$. $\square$

To overcome the difficulty that arises with the fact the conformal group of $\mathbb{D}^2$ is not compact, we fix three points p_1, p_2 and p_3 in $\partial\mathbb{D}^2$ and q_1, q_2 and q_3 in Γ. Given $u : \mathbb{D}^2 \to \mathbb{R}^3$, with $u|_{\partial\mathbb{D}}$ monotone and onto Γ, there exists $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ conformal and such that $(u \circ \phi)(p_i) = q_i$, for $i = 1, 2, 3$.

Given $k > 0$, consider the space $\mathcal{F}_k$ of maps $u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$, with the following properties:

- $E(u) \leq k$;
- $u|_{\partial\mathbb{D}}$ is monotone and onto Γ;
- $u(p_i) = q_i$, for all $i = 1, 2, 3$.

Remark. To prove $\mathcal{F}_k$ is non-empty, consider a piecewise C^1 parametrization $c(\theta)$ of Γ, $\theta \in [0, 1]$. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth map such that $\eta(\rho) = 0$, if $\rho \leq 2^{-1}$, and $\eta(\rho) = 1$ if $\rho \geq 1$. The class is non-empty because contains:

$$(\rho, \theta) \in [0, 1] \times [0, 1] \mapsto \eta(\rho)c(\theta).$$

13.10. **Theorem.** *$\mathcal{F}_k$ is equicontinuous on $\partial\mathbb{D}$.*

Proof. Let $\varepsilon > 0$ be such that $\varepsilon < \min_{i\neq j} |q_i - q_j|$. Choose $d > 0$, so that whenever $w, w' \in \Gamma$ with $0 < |w - w'| < d$, the complement $\Gamma \setminus \{w, w'\}$ contains exactly one component with diameter less than ε.

Given $p \in \partial\mathbb{D}$, choose $\delta > 0$ very small so that

- $B_{\sqrt{\delta}}(p)$ contains at most one of the p_i's;
- $\sqrt{\varepsilon_\delta} < \varepsilon$.

By Courant-Lebesgue's Lemma 13.9, given $u \in \mathcal{F}_k$ there exists $\rho \in [\delta, \sqrt{\delta}]$ such that $\operatorname{diam}(u(C_\rho)) \leq \sqrt{\varepsilon_\delta} < \varepsilon$. In particular, given $v, v' = \partial\mathbb{D} \cap C_\rho$ and $w = u(v)$, $w' = u(v')$, we have $|w - w'| < \varepsilon$.

Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be the components of $\Gamma \setminus \{w, w'\}$, being $\mathcal{A}^1$ the small component. Also, v and v' determine a small component A^1 of $\partial\mathbb{D}$, containing p, and a big component A^2, containing at least two of the p_i's. In consequence, $u(A^2)$ contains two of the q_i's. Hence $\operatorname{diam}(u(A^2)) > \varepsilon$ and this forces $u(A^1) \subset \mathcal{A}^1$. In other words, $x \in \partial\mathbb{D}$ with $|x - p| < \delta$ implies $|u(x) - u(p)| < \varepsilon$, for all $u \in \mathcal{F}_k$. $\square$

Solution to Plateau's Problem. Choose a minimizing sequence u_l inside $\mathcal{F}_k$. Each u_l being solution to some Dirichlet Problem, such that

$$E(u_l) = \frac{1}{2}\int |\nabla u_l|^2 \to E_\Gamma = A_\Gamma.$$

Up to subsequense, can suppose u_l converges uniformly on $\partial\mathbb{D}$ to a monotone and onto map. By maximum principle, for each i and j, we have

$$\max_{\overline{\mathbb{D}}} |u_i - u_j| = \max_{\partial\mathbb{D}} |u_i - u_j|.$$

Therefore, $\{u_l\}$ converges uniformly on $\overline{\mathbb{D}}$ to some function u. By Rellich Compactness Theorem, up to subsequences, we can suppose

$$u_l \xrightarrow{L^2} u \quad \text{and} \quad \nabla u_l \xrightarrow{L^2} \nabla u,$$

in the weak L^2-sense. We have

$$\frac{1}{2}\int |\nabla u|^2 \leq \liminf_{l\to\infty} \frac{1}{2}\int |\nabla u_l|^2 = E_\Gamma$$

and

$$E_\Gamma = A_\Gamma \leq \text{Area}(u) \leq E(u) \leq E_\Gamma.$$

Then u minimizes both area and energy. Moreover, u is almost conformal and harmonic. □

13.11. **Definition.** Let $u : \mathbb{D}^2 \to \mathbb{R}^3$ be an almost conformal and harmonic map. A branch point of u is a point $z \in \mathbb{D}^2$ such that $du(z)$ fails to be injective, i.e., $du(z) = 0$ since u is almost conformal.

13.12. **Proposition.** *The set of branch points is isolated.*

Proof. Branch points are the zeros of the maps

$$\frac{\partial u_j}{\partial x} - i\frac{\partial u_j}{\partial y},$$

for all $j = 1, 2, 3$. But those functions are holomorphic. □

13.13. **Theorem.** [Osserman and Gulliver] *The solution $u : \mathbb{D}^2 \to \mathbb{R}^3$ to Plateau's Problem has no interior branch points. In other words, u is an immersion.*

Open Problem. Whether there exists boundary branch points?

13.14. **Theorem.** [Nitsche] If $\Gamma \in C^{k,\alpha}$, $k \geq 1$ and $0 < \alpha < 1$, then $u \in C^{k,\alpha}(\overline{\mathbb{D}})$.

Exercise. Use complex analysis to show that Plateau's solution is a homeomorphism on $\partial\mathbb{D}$.

14. Harmonic maps

Let M^n be a compact Riemannian manifold. Suppose $M^n \subset \mathbb{R}^L$ isometrically. Let Σ be a compact Riemannian surface and $u : \Sigma \to M$ be given by $u = (u^1, \ldots, u^L)$. We define the energy of u to be

$$E(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 d\Sigma = \frac{1}{2}\sum_{i=1}^{L}\int_\Sigma |\nabla u^i|^2 d\Sigma. \tag{63}$$

Fact. If $\Phi : (\Sigma', g') \to (\Sigma, g)$ is conformal, then $E(u \circ \Phi) = E(u)$.

14.1. **Definition.** A map $u : (\Sigma, g) \to (M, \bar{g})$ is an *harmonic map* if u is a critical point for the energy with respect to compactly supported variations inside M.

If $u_t : \Sigma \to M$ is a compactly supported variation $u = u_0$, we have

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} E(u_t) &= \int_\Sigma \langle \nabla u, \nabla \left(\frac{\partial u_t}{\partial t} \right) \rangle dv_g \\ &= -\int_\Sigma \Delta u \cdot \frac{\partial u_t}{\partial t} dv_g. \end{aligned}$$

Then, the Euler-Lagrange equations for the energy is given by

$$(\Delta u)^T = 0, \tag{64}$$

where the left-hand side is the component of Δu tangential to M.

14.2. **Proposition.** *Let $u : (\mathbb{R}^2, \delta) \to (M, g)$ be an harmonic map with finite energy, $E(u) < \infty$. Then u is almost conformal and minimal.*

Proof of 14.2. Consider $\Phi : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\Phi = (|u_x|^2 - |u_y|^2) - 2i\langle u_x, u_y \rangle.$$

Exercise. Prove u harmonic imply Φ holomorphic.

Since Φ is holomorphic, we have

$$\frac{\partial \Phi}{\partial \bar{z}} = 0.$$

Using this, we can conclude

$$\Delta \left(\varepsilon + |\Phi|^2 \right)^{1/2} = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} (\varepsilon + |\Phi|^2)^{1/2} \geq 0.$$

Since the energy $E(u)$ is finite, we have $|\Phi| \in L^1(\mathbb{R}^2)$. Can apply the mean-value property to conclude

$$\left(\varepsilon + |\Phi|^2(p) \right)^{1/2} \leq \frac{\int_{D_R} (\varepsilon + |\Phi|^2)^{1/2}}{\pi R^2} \leq \frac{\int_{D_R} (\varepsilon + |\Phi|)}{\pi R^2} = \varepsilon + \frac{\int_{D_R} |\Phi|^2}{\pi R^2}.$$

Let $R \to \infty$, to conclude $\left(\varepsilon + |\Phi|^2(p) \right)^{1/2} \leq \varepsilon$. Later, let $\varepsilon \to 0$ to obtain $|\Phi|(p) = 0$, i.e., u is almost conformal.

Exercise. Check that u is minimal, i.e., $\overrightarrow{H}_{u(\mathbb{R}^2) \subset M} = 0$. Observe that given a map $x : \mathbb{R}^2 \to M \subset \mathbb{R}^L$, we have $\Delta u = \overrightarrow{H}_{u(\mathbb{R}^2) \subset \mathbb{R}^L}$.

14.3. **Example.**

15. Positive Mass Theorem

In this section we discuss the Positive Mass Theorem in the case of an isolated gravitational system.

15.1. **Definition.** (M^n, g) is *asymptotically flat* if there exists a compact set $K \subset M$, such that $M \setminus K = \cup_{k=1}^m E_k$, each E_k diffeomorphic to $\mathbb{R}^n \setminus B_1(0)$ and with

$$g_{ij} = \delta_{ij} + h_{ij},$$

so that

$$|h| \leq C|x|^{-p}, \quad |\partial h| \leq C|x|^{-p-1} \quad |\partial^2 h| \leq C|x|^{-p-2},$$

for some $p > \frac{n-2}{2}$, in these coordinates, and scalar curvature $R_g(x) = O(|x|^{-q})$, for some $q > n$. In particular, $R_g \in L^1(M)$.

15.2. **Examples.**

1. The euclidean space $(\mathbb{R}^n, \delta)$.

2. [Schwarzschild Metric] For each $m > 0$, consider the Riemannian manifold $(\mathbb{R}^n \setminus \{0\}, g_m)$, whose metric is given in coordinates by

$$(g_m)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}. \tag{65}$$

To check the scalar curvature decay, use the conformal law: if $\tilde{g} = u^{\frac{4}{n-2}} g$, with $u > 0$, the scalar curvature $R_{\tilde{g}}$ of the conformal metric is given by

$$R_{\tilde{g}} = -\frac{4(n-1)}{(n-2)} u^{-\frac{n+2}{n-2}} \left(\Delta_g u - \frac{n-2}{4(n-1)} R_g u\right). \tag{66}$$

Since $\Delta_{\mathbb{R}^n}\left(1 + \frac{m}{2|x|^{n-2}}\right) = 0$ and $R_\delta = 0$, we conclude g_m is also scalar flat. The term inside the parenthesis in the conformal law, expression (66) above, is called conformal Laplacian and denoted $L_g u$.

15.3. **Definition.** The mass of the end E in (M^n, g) is the number

$$m = \lim_{r \to \infty} a(n) \int_{S_r^{n-1}(0)} \sum_{i,j} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j dS_r, \tag{67}$$

where $E \approx \mathbb{R}^n \setminus B_1(0)$ that given local coordinates $(x_1, \ldots, x_n)$, $S_r^{n-1}(0)$ is the euclidean sphere, $\nu(x) = \frac{x}{|x|}$ and $a(n)$ is a dimensional constant.

The mass is well-defined. In fact, the expression of scalar curvature in local coordinates $(x_1, \ldots, x_n)$ is

$$R_g = \sum_{i,j} \left(\partial_i \partial_j g_{ij} - \partial_i \partial_i g_{jj}\right) + O(|x|^{-2p-2}),$$

with $-2p-2 < -n$. Integrating in $B_r \setminus B_{r_0}$ and using integration by parts, we conclude

$$\int_{B_r \setminus B_{r_0}} (R_g - O(|x|^{-2p-2})) = \int_{S_r} \sum_{i,j} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j - \int_{S_{r_0}} \sum_{i,j} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j.$$

Since $\int (R_g - O(|x|^{-2p-2})) < \infty$, the limit in (67) exists.

Remark. The mass of g_m is m.

15.4. **Positive Mass Conjecture.** Let (M^n, g) be an asymptotically flat manifold. If $R_g \geq 0$, then the mass of each end E of M is non-negative. If the mass of some end E is zero, then (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$.

Motivation. The motivation for the positive mass conjecture comes from General Relativity.

Let (V^4, g), g be a Lorentzian metric, i.e., at each point $p \in V$, there exists (e_0, e_1, e_2, e_3) basis of T_pV so that $g(e_i, e_j) = 0$, if $i \neq j$, $g(e_i, e_i) = 1$, if $1 \leq i \leq 3$ and $g(e_0, e_0) = -1$. We say $(M^3, g) \subset V$ is spacelike if the induced metric is Riemannian. In this case we have a normal vector field ν along M so that $g(\nu, \nu) = -1$.

It is required in the general relativity the spacelike hypersurface (M^3, g) to satisfy the Einstein Equations:

$$Ric_g - \frac{1}{2}R \cdot g = 8\pi T, \tag{68}$$

where T is the stress-energy tensor.‘It is also physically reasonable to ask the stress-energy tensor T to have the property

$$T(\nu, \nu) \geq 0. \tag{69}$$

This implies $R + (\text{Tr A})^2 - |A|^2 \geq 0$. If we assume time symmetry, i.e., $|A| = 0$, the Einstein Equations and the condition on the stress-energy tensor imply $R_g \geq 0$.

15.5. **Basic Methods.**

1) Schoen and Yau (1979), using stable minimal hypersurfaces. This proves the Positive mass conjecture in case $n \leq 7$;
2) Witten (1981), using harmonic spinors. This approach works for all n when the manifold M satifies a topological condition called spin.

Next, we present a nice consequence of Positive Mass Theorem, the rigidity of Euclidean Space.

15.6. **Corollary.** *Let $(\mathbb{R}^n, g)$ be the Euclidean space with a Riemannian metric g that coincides with the standard metric δ outside a compact set K and $R_g \geq 0$. Then $(\mathbb{R}^n, g)$ is isometric to $(\mathbb{R}^n, \delta)$.*

Remark. This consequence is true for all $n \geq 3$, because $\mathbb{R}^n$ is spin.

Remark. Some other remarks concerning the two methods:

- If $n = 3$, then M is spin and both methods work in this case;
- $\mathbb{CP}^{2n}$ is not spin for all $n \geq 1$, then Schoen-Yau's method works to $M^4 = \mathbb{CP}^2 \# \mathbb{R}^4$, but Witten's does not;
- $\mathbb{CP}^{2n+1}$ is spin, then Witten's method works to $M = \mathbb{CP}^{2n+1} \# \mathbb{R}^{4n+2}$, but Schoen-Yau's does not;
- If $M = \mathbb{CP}^{10} \# \mathbb{R}^{20}$, the question is still open.

Next, we sketch Schoen-Yau's proof of Positive Mass Theorem.

Let (M^3, g) be asymptotically flat with $R_g \geq 0$. The proof is by contradiction. Assume $m < 0$. Can assume

1) $g = u^4\delta$, outside a compact set with $u = 1 + \frac{m}{|x|} + O(|x|^{-2})$;
2) $R_g > 0$.

Let $\psi_\sigma : [0, +\infty) \to [0,1]$ be a smooth function such that $\psi_\sigma = 1$ in $[0, \sigma]$, and $\psi_\sigma = 0$ outside $[0, 2\sigma]$. Consider $g^{(\sigma)} = \psi_\sigma \cdot g + (1 - \psi_\sigma)\delta$, and note $g^{(\sigma)} \to g$ as $\sigma \to +\infty$.

Solve $L_{g^{(\sigma)}} u^{(\sigma)} = 0$, with $u^{(\sigma)} \to 1$ at infinity. This is possible because $R_g \geq 0$ implies $L_g = \Delta_g - c(n)R_g$ has a sign, then can find $u^{(\sigma)}$.

Define $\tilde{g}^{(\sigma)} = \left(u^{(\sigma)}\right)^4 g^{(\sigma)}$. This metric is asymptotically flat and has $R_{\tilde{g}^{(\sigma)}} = 0$.

15.7. **Claim.** If σ is large, $m_{\tilde{g}^{(\sigma)}} < 0$.

Proof. Observe

$$\tilde{g}^{(\sigma)} \to \tilde{g} = u^4 g, \text{ as } \sigma \to \infty,$$

where u satisfies $L_g u = 0$ and $u \to 1$ at infinity. Let $u = 1 + \frac{A}{|x|} + O(|x|^{-2})$. By maximum principle, we have $u < 1$ and so $A < 0$. In particular,

$$m_{\tilde{g}} = m_g + A < m_g < 0.$$

□

Observe $\tilde{g}$ is also scalar flat. Finally, solve $L_{\tilde{g}^{(\sigma)}} f = -\psi$ with $f \to 1$ at infinity, where $\psi > 0$ decay very rapidly. Put $\overline{g} = f\tilde{g}^{(\sigma)}$. This shows we can suppose assumptions 1) and 2) above.

Geometric Ingredient. $m < 0$ implies big slabs are mean convex.

If $h > 0$ is large, the unit normal to the slab $x_3 = h$ is given by $\eta = u^{-2}\frac{\partial}{\partial x_3}$. In this case, we have

$$\mathrm{div}_g \eta = u^{-3}\partial_{x_3}\left(u^3 \cdot u^{-2}\right) = u^{-3}\partial_{x_3} u = -u^{-3} m \frac{x_3}{|x_3|^3} > 0,$$

if $x_3 = h$ is very large. On the other hand, if e_1, e_2 is an orthonormal frame on $x_3 = h$ and $e_3 = \eta$, we can express the divergence of *eta* as

$$\mathrm{div}_g \eta = \sum_{i=1}^{3} \langle \nabla_{e_i} \eta, e_i \rangle = \sum_{i=1}^{2} \langle \nabla_{e_i} \eta, e_i \rangle,$$

and

$$\overrightarrow{H} = -\sum_{i=1}^{2} \langle \nabla_{e_i} \eta, e_i \rangle \eta.$$

In consequence, $\langle \overrightarrow{H}, \eta \rangle < 0$ on the slice $x_3 = h$, for sufficiently large $h > 0$. Analogously, $\langle \overrightarrow{H}, \eta \rangle > 0$, on the slice $x_3 = -h$.

15.8. **Goal.** The next step is the construction of an area-minimizing embedded minimal surface inside one of those mean-convex slabs.

Use C_σ to denote the circle $\{x_1^2 + x_2^2 = \sigma^2\} \cap \{x_3 = 0\}$. Find Σ_σ of least area with $\partial\Sigma_\sigma = C_\sigma$. This surface can be obtained in two ways. First, using Morrey's solution to Plateau's Problem, minimizing area among disks. In this case, Meeks and Yau proved the solution is an embedded disk. The second approach comes from Geometric Measure Theory, where we minimize area among all topological types.

In any case, we can guarantee, by Maximum Principle, that

$$\Sigma_\sigma \cap (M \setminus K) \subset \{-h \leq x_3 \leq h\},$$

where $h > 0$ is large enough, so that the slab $\{-h \leq x_3 \leq h\}$ is mean-convex, but does not depend on σ.

In case we have various ends and those C_σ are considered in a fixed end E of M, can also use Maximum Principle to prove the portion of Σ_σ on each different end is contained in a fixed large ball.

We explore now the area-minimizing property of Σ_σ to estimate its area inside the union of K and the cylinder $\mathrm{Cyl}(t) = \{x_1^2+x_2^2 \leq t^2\}$. We compare $\Sigma_\sigma \cap (K \cup \mathrm{Cyl}(t))$ with the union of the top disk $\mathrm{Cyl}(t) \cap \{x_3 = h\}$ and a subset of $\partial\mathrm{Cyl}(t)$ whose boundary is $(\Sigma_\sigma \cup \{x_3 = h\}) \cap \partial\mathrm{Cyl}(t)$. We obtain

$$\mathrm{Area}\,(\Sigma_\sigma \cap (K \cup \mathrm{Cyl}(t))) \leq \pi t^2 + O(t). \tag{70}$$

Consider a fixed curve γ joining $\{x_3 = -h\}$ and $\{x_3 = h\}$. If σ is large, γ and C_σ are linked and then $\Sigma_\sigma \cap \gamma \neq \emptyset$.

Next, we use Schoen's curvature estimates for stable surfaces to obtain uniform curvature bounds on compact sets. Precisely, we have

$$|A_{\Sigma_\sigma}| \leq C(t), \quad \text{in} \quad \Sigma_\sigma \cap (K \cup \mathrm{Cyl}(t))\,. \tag{71}$$

A diagonal argument gives a subsequence Σ_{σ_i} that converges smoothly to Σ in any compact set of M. We have then, Σ is an embedded and area-minimizing, in particular, stable. It is also possible to conclude outside a compact set Σ is a single valued graph over $\{x_3 = 0\}$.

Analysis near ∞. We analyze now the behavior of the constructed surface Σ near ∞. Since Σ is minimal, we have

$$\Delta_\Sigma x_3 = \mathrm{Hess}_M x_3(e_1, e_1) + \mathrm{Hess}_M x_3(e_2, e_2),$$

for some $\{e_1, e_2\} \subset T\Sigma$ orthonormal basis, where $\mathrm{Hess} f(X, Y)$ is given by the following expression

$$\mathrm{Hess} f(X, Y) = XY(f) - \nabla_X Y(f). \tag{72}$$

Since this involves covariant derivatives $\nabla_X Y$, we need the Christoffel Symbols to calculate $\Delta_\Sigma x_3$ following the above formula. The general expression of Christoffel Symbols is

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right). \tag{73}$$

In our case, we have $\Gamma_{ij}^k = O(|x|^{-2})$ and then,

$$\Delta_\Sigma x_3 = O\left((1+|x_3|)|x|^{-3}\right) = O\left(|x|^{-3}\right). \tag{74}$$

The linear theory of PDE says (74) implies

$$|x|\,|\nabla_\Sigma x_3| + |x|^2\left|\nabla_\Sigma^2 x_3\right| \le C.$$

Since Σ is a single valued graph over $\{x_3 = 0\}$, we can compare $|A_\Sigma|$ with $\left|\nabla_\Sigma^2 x_3\right|$ and conclude the following curvature estimates

$$|x|^4|A_\Sigma|^2 \le C.$$

But Σ has quadratic area growth, expression (70). Then, we have

$$\int_\Sigma |A_\Sigma|^2 d\Sigma < \infty.$$

Since the extrinsic curvature Rm of Σ is integrable and, by Gauss equation, K_Σ is given by Rm plus a quadratic term involving the second fundamental form of Σ, we obtain

$$\int_\Sigma |K_\Sigma|\, d\Sigma < \infty. \tag{75}$$

Stability Inequality. Recall the stability inequality

$$\int_\Sigma \left(|A_\Sigma|^2 + \mathrm{Ric}(N,N)\right) f^2 d\Sigma \le \int_\Sigma |\nabla_\Sigma f|^2\, d\Sigma, \quad \text{for all } f \in \mathrm{Lip}_c(\Sigma).$$

Using Gauss equation and the assumption on $R^M > 0$, we conclude

$$-\int_\Sigma K_\Sigma f^2 d\Sigma < \int_\Sigma \left(\frac{R^M}{2} - K_\Sigma + \frac{|A_\Sigma|^2}{2}\right) f^2 d\Sigma \le \int_\Sigma |\nabla_\Sigma f|^2\, d\Sigma.$$

Because of quadratic area growth, can apply the log cut-off trick to conclude

$$-\int_\Sigma K_\Sigma d\Sigma < 0 \tag{76}$$

Claim. $\int_\Sigma K_\Sigma \le 0$.

Let $\gamma_\sigma \subset \Sigma$ the curve with projection C_σ onto $\{x_3 = 0\}$. Let $\overline{g}$ denote the induced metric on Σ. Can put local coordinates in Σ using that outside a compact set it is the graph of a function f over $\{x_3 = 0\}$. In this local coordinates, we have

$$\overline{g}_{ij} = g_{ij} + u^4 f_i f_j = \delta_{ij} + O\left(|x|^{-1}\right).$$

Then, the geodesic curvature of C_σ with respect to $\overline{g}$ has the form $k_{g,\gamma_\sigma} = \sigma^{-1} + O\left(\sigma^{-2}\right)$. Integrate over the circle to get:

$$\int_{\gamma_\sigma} k_{g,\gamma_\sigma} \to 2\pi, \quad \text{as} \quad \sigma \to \infty. \tag{77}$$

Gauss-Bonnet Theorem to the domain D_σ in Σ bounded by γ_σ to conclude

$$\int_{D_\sigma} K_\Sigma d\Sigma + \int_{\gamma_\sigma} k_{g,\gamma_\sigma} = 2\pi\chi(D_\sigma) \le 2\pi. \tag{78}$$

The limit as $\sigma \to \infty$ gives $\int_\Sigma K_\Sigma \leq 0$. Contradiction!

Remarks. About higer dimensions, the proof is by induction. We still have the mean-convextity of the slabs. Then, use Geometric Measure Theory to produce Σ_σ of least volume, the regularity of Σ_σ is guaranteed if $n \leq 7$.

Open. Are singularities of area-minimizing hypersurfaces stable under perturbations?

We do not have the log cut-off trick any more, then we choose a different minimization problem. For each $a \in [-h, h]$, consider

$$C_{\sigma,a} = \{x_1^2 + x_2^2 = \sigma^2\} \cap \{x_3 = a\}.$$

Produce $\Sigma_{\sigma,a}$ of least volume with boundary $C_{\sigma,a}$. Choose $\Sigma_{\sigma,a(\sigma)}$ with least volume among all $\Sigma_{\sigma,a}$, $a \in [-h, h]$. This hypersurface satisfies a better stability inequality in which a test function f can be a non-trivial constant on $\partial\Sigma_{\sigma,a_\sigma}$.

We produce, as in the initial case, a volume-minimizing hypersurface Σ, which is again the graph of a single valued function f outside a compact set. The estimates on the derivatives of f in this case is

$$|x|^{n-2}|\nabla f| + |x|^{n-1}|\nabla^2 f| \leq C.$$

The Riemannian metric g has the form $g = u^{\frac{4}{n-2}}\delta$ outside a compact set K, then, the induced metric $\overline{g}$ on Σ is given in local coordinates by

$$\overline{g}_{ij} = g_{ij} + u^{\frac{4}{n-2}} f_i f_j = \delta_{ij} + O(|x|^{2-n}) = \delta_{ij} + O(|x|^{1-\dim(\Sigma)})$$

Observe $1 - \dim(\Sigma) < 2 - \dim(\Sigma)$, then $m(\overline{g}) = 0$.

The stability inequality allows us to choose $v > 0$ solving $L_g v = 0$ and $v \to 1$ at infinity. Let $\tilde{g} = v^{\frac{4}{n-2}}\overline{g}$. This new metric is scalar flat and has negative mass. This finishes the induction argument. To prove the choice of v as above is possible we check that L_Σ has a sign. Back to the better stability inequality, we have

$$\int_\Sigma \left(\frac{R^M}{2} - \frac{R^\Sigma}{2} + \frac{|A_\Sigma|^2}{2}\right) f^2 d\Sigma \leq \int_\Sigma |\nabla_\Sigma f|^2 \, d\Sigma,$$

then

$$-\int_\Sigma \frac{R^\Sigma}{2} f^2 d\Sigma \leq \int_\Sigma |\nabla_\Sigma f|^2 \, d\Sigma. \tag{79}$$

Recall the expression of the conformal Laplacian of Σ, $L_\Sigma = \Delta_\Sigma - \frac{(n-3)}{4(n-2)} R^\Sigma$. By expression (79), we get

$$-\int_\Sigma \frac{R^\Sigma}{2} f^2 d\Sigma \leq \frac{2(n-2)}{(n-3)} \int_\Sigma |\nabla_\Sigma f|^2 \, d\Sigma,$$

and then

$$\int_\Sigma \left(|\nabla_\Sigma f|^2 + \frac{(n-3)}{4(n-2)} R^\Sigma f^2\right) d\Sigma \leq 0.$$

This proves L_Σ has a sign and finishes the argument.

Rigidity Case. If (M^n, g) is asymptotically flat, $R \geq 0$ and $m = 0$, then we argue in three steps that it must be isometric to euclidean space.

Step 1. Scalar flat. Solve $L_g u = 0$ with $u \to 1$ at infinity, and consider $\tilde{g} = u^{\frac{4}{n-2}} g$. If R is not identically zero, this new metric is scalar flat and has mass $m(\tilde{g}) < m(g) = 0$. Contradiction!

Step 2. Ricci flat. If $g(t)$ is a variation of g among asymptotically flat metrics with $g'(0) = h$, then

$$\frac{d}{dt}\bigg|_{t=0} \left(m(g(t)) - \int_M R_{g(t)} dv_{g(t)} \right) = \int_M \langle h, \mathrm{Ric} - \frac{R}{2} g \rangle dv_g. \tag{80}$$

Since $m(g) = 0$, g is a minimum point of the mass functional among metrics with $R \geq 0$. Fix h_0 of compact support and consider $g(t) = g + th_0$. Solve $L_{g(t)} u(t) = 0$ with $u(t) \to 1$ at infinity. Define $\overline{g}(t) = u(t)^{\frac{4}{n-2}} g(t)$, an asymptotically flat and scalar flat metric. Observe $u(0) \equiv 1$ and

$$\frac{d}{dt}\bigg|_{t=0} m(\overline{g}(t)) = 0.$$

This imply

$$0 = \int_M \langle \mathrm{Ric}, \frac{d}{dt}\bigg|_{t=0} \overline{g}(t) \rangle dv_g = \int_M \langle \mathrm{Ric}, v \cdot g(0) + h_0 \rangle dv_g,$$

where $v = \frac{d}{dt}\big|_{t=0} u(t)^{\frac{4}{n-2}}$. In particular, since $\langle \mathrm{Ric}, g \rangle = R_g$ and $R_{g(0)} \equiv 0$, we get

$$\int_M \langle \mathrm{Ric}, h \rangle dv_g = 0,$$

for all h with compact support. This implies g is Ricci flat.

Step 3. Flat. This follows from volume comparison

16. Harmonic Maps - Part 2

Let $M^n \subset \mathbb{R}^N$ be a compact Riemannian manifold. Recall we proved in Proposition 14.2, that harmonic maps $u : (\mathbb{R}^2, \delta) \to (M, g)$ with finite energy are almost conformal and minimal.

Use $\pi : \mathbb{S}^2 \setminus \{p\} \to \mathbb{R}^2$ to denote the stereographic projection from the 2-sphere minus a point $p \in \mathbb{S}^2$, which is a conformal map. If $u : \mathbb{R}^2 \to M$ is an harmonic map with $E(u) < \infty$, then $u \circ \pi : \mathbb{S}^2 \setminus \{p\} \to M$ is again harmonic and has finite energy, in particular, it is almost conformal.

16.1. **Theorem.** *Let $u : \mathbb{D}^2 \setminus \{p\} \to M$ be an harmonic map with $E(u) < \infty$. Then u extends smoothly to $u : \mathbb{D}^2 \to M$ harmonic.*

Following the previous analysis, the map $u \circ \pi$ extends to an harmonic map $v : S^2 \to M$, which is almost conformal and branched minimal.

Open. It is an open question if $\Sigma^2 \subset B^3 \setminus \{0\}$ embedded minimal surface with $\partial\Sigma \subset \partial B^3$, imply Σ is an embedded minimal surface in B^3. It is an exercise to prove Σ has finite area. What can happen in the negative direction is an accumulation of small necks near the origin.

We describe now the Sacks and Uhlenbeck blow-up argument. Let $M^n \subset \mathbb{R}^L$ be a compact submanifold such that $\pi_2(M) \neq 0$. There exists $f : \mathbb{S}^2 \to M$ that is not homotopically trivial, then can find a non-trivial harmonic map $u : \mathbb{S}^2 \to M$.

For each $\alpha > 1$ we consider the perturbed functional

$$E_\alpha(u) = \int_{\mathbb{S}^2} \left(1 + |\nabla u|^2\right)^\alpha dv. \tag{81}$$

16.2. **Definition.** A critical point of E_α is called α-harmonic map.

Given $\beta \in \pi_2(M)$, $\beta \neq 0$, and each $\alpha > 1$, we minimize E_α in β. The idea is to use $W^{1,2\alpha}(\mathbb{S}^2) \subset C^{0,\gamma}$, if $\alpha > 1$. Find u_α minimizer, for each $\alpha > 1$ and try to take a limit as $\alpha \to 1$. We have two possibilities:

1) either $\sup_{\mathbb{S}^2} |\nabla u_\alpha| \leq C$, then u_α converges to a non-trivial harmonic map $u : \mathbb{S}^2 \to M$;
2) or $\sup_{\mathbb{S}^2} |\nabla u_\alpha| \to \infty$. In this case, apply the blow-up technique, which we briefly discuss now. For each map $u_\alpha : \mathbb{S}^2 \to M$, consider $x_\alpha \in \mathbb{S}^2$ and $\lambda_\alpha > 0$, such that
$$|\nabla u_\alpha(x_\alpha)| = \sup_{\mathbb{S}^2} |\nabla u_\alpha| = \lambda_\alpha.$$
Using stereographic projection, we find maps $\tilde{u}_\alpha : \mathbb{R}^2 \to M$ with the following properties
$$\sup_{\mathbb{R}^2} |\nabla \tilde{u}_\alpha| \to \infty \quad \text{and} \quad |\nabla \tilde{u}_\alpha(0)| = \lambda_\alpha.$$
Rescaling to $\hat{u}_\alpha(w) = \tilde{u}_\alpha\left(\frac{w}{\lambda_\alpha}\right)$, we have $\sup_{\mathbb{R}^2} |\nabla \hat{u}_\alpha| \leq 1$ and can take the limit $\hat{u} : \mathbb{R}^2 \to M$, harmonic map with $|\nabla \hat{u}(0)| = 1$. This implies $\hat{u}$ is non-trivial and has finite energy, then gives a non-trivial harmonic map $u : \mathbb{S}^2 \to M$.

Example. Use the identification $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ to define $f_j : \mathbb{S}^2 \to \mathbb{S}^2$ by
$$f_j(z) = \frac{z^2 + 1/j}{z}.$$
This is an harmonic map. (2 bubble?)

17. Ricci Flow and Poincaré Conjecture

17.1. **Ricci Flow.** Let M^n be a compact manifold with a Riemannian metric g_0. In 1982, in the pioneer paper by R. Hamilton, he introduced the following equation
$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)} \\ g(0) = g_0. \end{cases}$$

17.2. **Theorem.** *For every n, the flow exists and is unique in some short time interval $[0,\varepsilon)$. Moreover, if $n=3$ and $Ric_{g_0}>0$, then $Ric_{g(t)}>0$ and the flow disappear in finite time becoming asymptotically round.*

17.3. **Poincaré Conjecture.** Let M^3 be a compact manifold such that $\pi_1(M)=\{1\}$. Then M is diffeomorphic to $\mathbb{S}^3$.

This question was solved by Perelman. The strategy is to understand the behavior of the Ricci flow for general metrics, without curvature assumptions. At each singular time, proceed a surgery near the singularity and keep flowing.

Basic question. Does the flow become singular (extinct) in finite time?

If so, the number of surgeries is finite, then conclude

$$M \approx S^3 \# S^3/\Gamma_1 \# \cdots \# S^3/\Gamma_k \# (S^2\times S^1) \# \cdots \# (S^2 \times S^1).$$

Because $\pi_1(M)=\{1\}$, M is diffeomorphic to S^3.

To prove the basic question, Colding and Minicozzi use min-max minimal surfaces.

17.4. **Min-max Construction.**

17.4.1. *Curves on surfaces.* In 1905, Poincaré introduced the question about existence of closed geodesics in (S^2,g). In 1917, Birkhoff answers positively Poincaré's question.

Let $f : S^2 \to (S^2,g)$ be a map of degree $\deg(f)=1$. Suppose the S^2 in the domain of f is the standard two-sphere in $\mathbb{R}^3$ and consider in $\mathbb{R}^3$ the usual coordinates (x_1,x_2,x_3). The map f naturally induces a sweepout $\{\gamma_t\}_{t\in[0,1]}$ in (S^2,g) by closed curves

$$\gamma_t = f\left(\{x_3 = 1-2t\}\right), \quad \text{for } 0\le t\le 1.$$

Then, we consider the following min-max invariant of (S^2,g)

$$L = \inf_{\{\gamma_t\}_t} \sup_{t\in[0,1]} \ell(\gamma_t), \tag{82}$$

where the infimum is considered among all sweepouts in (S^2,g). The idea is to realize $L=\ell(\gamma)$ as the length of a smooth closed geodesic. We describe now a key construction in this theory, the curve shortening map.

Let $c : S^1 \to (S^2,g)$ be a closed curve, $\ell(c)\le 2L$ and suppose c is parametrized by arc length. Choose a fine partition $\{s_1,s_2,\dots,s_m=s_1\}$ of S^1 such that each pair $c(s_l),c(s_{l+1})$ can be joined by a unique length minimizing geodesic. The curve shortening applied to c is another closed curve $D(c)$ with smaller length. The definition of $D(c)$ has two steps:

- (i) let $c_1 : S^1 \to S^2$ be such that between s_l and s_{l+1} the curve c_1 is the minimizing geodesic arc joining $c(s_l)$ and $c(s_{l+1})$;
- (ii) fix the middle points of each geodesic arc of c_1 and repeat the procedure in step (i) for these points to obtain $D(c)$.

The key properties of this construction are the following:

1) $\ell(D(c)) \leq \ell(c)$;
2) $\ell(D(c)) = \ell(c)$ if and only if c is a smooth closed geodesic.

Observe that if $\{\gamma_t^i\}_t$ is a sequence of sweepouts with

$$\sup_{t\in[0,1]} \ell(\gamma_t^i) \to L,$$

then $\{D(\gamma_t^i)\}_t$ is also sequence of sweepouts with

$$\sup_{t\in[0,1]} \ell(D(\gamma_t^i)) \to L.$$

To conclude the argument, take $t_i \in [0,1]$ such that $\ell(D(\gamma_t^i)) \to L$, then there exists subsequence $\{j\} \subset \{i\}$ such that $\{D(\gamma_{t_j}^j)\}_j$ converges to a closed geodesic of length L. The idea is to take a subsequence such that both $\gamma_{t_j}^j$ and $D(\gamma_{t_j}^j)$ converge to γ and $D(\gamma)$, respectively (Arzela-Ascoli). Both $\ell(D(\gamma)) = L$ and $\ell(\gamma) = L$, and then $D(\gamma) = \gamma$ is a closed geodesic.

17.5. **Minimal spheres.** The idea now is to apply min-max methods to produce minimal surfaces, replace $u : S^1 \to M$ by a map $u : S^2 \to M$ and do min-max for area or energy. A topological fact, consequence of Hurewicz Theorem is that if M^3 has $\pi_1(M) = \{1\}$, then $\pi_3(M) = \mathbb{Z}$. Fix $\beta \in \pi_3(M)$, $\beta \neq 0$.

In this min-max setting, the sweepouts are the maps

$$\sigma : [0,1] \times S^2 \to M,$$

with the following properties:

1) $\sigma(t,\cdot) \in C^0 \cap W^{1,2}$;
2) $t \mapsto \sigma(t,\cdot)$ is continuous;
3) σ maps $\{0\} \times S^2$ and $\{1\} \times S^2$ to points;
4) the induced map $S^3 \to M$ belongs to β.

In this case, use the notation $\sigma \in \Omega_\beta$. The width can be considered with respect to the area or the energy:

$$W_A = \inf_{\sigma\in\Omega_\beta} \sup_{t\in[0,1]} A(\sigma(t,\cdot)) \quad \text{and} \quad W_E = \inf_{\sigma\in\Omega_\beta} \sup_{t\in[0,1]} E(\sigma(t,\cdot)).$$

In fact, the two above notions of width coincide.

Exercise. Prove $0 < W_A = W_E =: W$.

17.6. **Theorem.** [Colding and Minicozzi] *Let M^n be a compact manifold and $\beta \in \pi_3(M)$, $\beta \neq 0$. Then, there exists a sequence of sweepouts $\{\sigma_i\}_i \subset \Omega_\beta$ with*

$$\sup_{t\in[0,1]} E(\sigma_i(t,\cdot)) \to W$$

and such that for every sequence $t_i \in [0,1]$ with $A(\sigma_i(t_i,\cdot)) \to W$, there exists subsequence $\{j\} \subset \{i\}$ such that $\{\sigma_j(t_j,\cdot)\}$ converges (in sense of varifolds) to a finite union $\cup_k u_k$ of harmonic maps $u_k : S^2 \to M$.

A sequence of surfaces in M converge in sense of varifolds, $\Sigma_i \rightharpoonup \Sigma$, if the following holds

$$\int_{\Sigma_i} f(x, T_x\Sigma_i) d\Sigma_i \to \int_\Sigma f(x, T_x\Sigma) d\Sigma,$$

for every compactly supported continuous function f defined in the Grassmannian of 2-planes of M. This convergence imply that $A(\Sigma_i) \to A(\Sigma)$. Can summarize this notion saying that convergence in sense of varifolds means that the surfaces are close and their tangent spaces are also close.

In particular, follows from the convergence of $\{\sigma_j(t_j, \cdot)\}$ in the Colding-Minicozzi's result, that

$$\sum_k A(u_k) = W.$$

Moreover, each map u_k must be almost conformal and minimal. Hence,

$$\sum_k E(u_k) = W.$$

The proof of Colding-Minicozzi's Theorem use harmonic replacements, with estimates of Wente and Hélein, and a refined blow-up analysis.

17.7. **Proposition.** *Let (M^3, g_0) be a compact Riemannian manifold, $\Sigma \subset M$ be a branched minimal S^2 for $g_0 = g(t_0)$, where $g(t)$ is a solution to the Ricci Flow. Then*

$$\frac{d}{dt} area_{g(t)}(\Sigma)\Big|_{t=t_0} \leq -4\pi - \frac{area_{g_0}(\Sigma)}{2} \min_M R_{g_0}. \tag{83}$$

Proof. Recall that the area can be written as

$$\text{area}_{g(t)}(\Sigma) = \int_\Sigma \sqrt{\det g^\Sigma_{ij}(t)}\, dx. \tag{84}$$

Then, the first derivative of the area satisfies

$$\begin{aligned} \frac{d}{dt}\text{area}_{g(t)}(\Sigma) &= \frac{1}{2}\int_\Sigma \left(\text{tr}_\Sigma \ \frac{\partial g^\Sigma_{ij}}{\partial t}\right)\sqrt{\det g^\Sigma(t)}\, dx \\ &= -\int_\Sigma \big(\text{Ric}_{g(t)}(e_1, e_1) + \text{Ric}_{g(t)}(e_2, e_2)\big) d\Sigma, \end{aligned}$$

where $\{e_1, e_2\} \subset T\Sigma$ is an orthonormal basis. That can be rewritten as follows

$$\frac{d}{dt}\text{area}_{g(t)}(\Sigma) = -\int_\Sigma \big(R_{g(t)} - \text{Ric}_{g(t)}(N, N)\big) d\Sigma,$$

where N denote a choice of unit normal vector field for Σ. The Gauss equation can be applied to reformulate the above expression as

$$\begin{aligned}\frac{d}{dt}\mathrm{area}_{g(t)}(\Sigma) &= -\int_\Sigma K_\Sigma d\Sigma - \frac{1}{2}\int_\Sigma \big(R_{g(t)} + |A|^2\big) d\Sigma + \frac{1}{2}\int_\Sigma H^2 d\Sigma \\ &= -\Big(2\pi\chi(\Sigma) + 2\pi\sum_i b_i\Big) - \frac{1}{2}\int_\Sigma \big(R_{g(t)} + |A|^2\big) d\Sigma \\ &\leq -4\pi - \frac{\mathrm{area}_{g(t)}(\Sigma)}{2}\min_M R_{g(t)}.\end{aligned}$$

The evolution of the scalar curvature under the Ricci Flow is given by the following equation

$$\frac{\partial}{\partial t}R_{g(t)} = \Delta_{g(t)}R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2_{g(t)}. \tag{85}$$

In the case of 3-dimensional manifolds, we have $|\mathrm{Ric}|^2 = |\mathring{\mathrm{Ric}}|^2 + R^2/3$, where $\mathring{\mathrm{Ric}}$ is the trace-free Ricci tensor. In particular, we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{g(t)} &= \Delta_{g(t)}R_{g(t)} + 2|\mathring{\mathrm{Ric}}_{g(t)}|^2_{g(t)} + \frac{2}{3}R^2_{g(t)} \\ &\geq \Delta_{g(t)}R_{g(t)} + \frac{2}{3}R^2_{g(t)}.\end{aligned}$$

We can use the maximum principle to obtain

$$\frac{d}{dt}\Big(\min_M R_{g(t)}\Big) \geq \frac{2}{3}\Big(\min_M R_{g(t)}\Big)^2.$$

And that will imply

$$\min_M R_{g(t)} \geq \frac{\min_M R_{g(0)}}{1 - \frac{2}{3}t\min_M R_{g(0)}} = -\frac{3}{2(t+c)}, \tag{86}$$

for some constant c depending on the initial metric.

17.8. **Theorem.** [Colding and Minicozzi] *Let (M^3, g_0) be a compact Riemannian three-manifold and let $g(t)$ be a solution to the Ricci flow. Consider $\beta \in \pi_3(M)$, $\beta \neq 0$, and define $W(t) := W(g(t), \beta)$. Then,*

$$\frac{d}{dt}W(t) \leq -4\pi + \frac{W(t)}{4}\cdot\frac{3}{(t+c)}. \tag{87}$$

The estimate in expression (87), occurs in the sense of

$$\limsup_{h\to 0,\quad h>0} \frac{W(g(t+h)) - W(g(t))}{h}.$$

Outline of proof. Let $\{\gamma_s\}_{s\in[0,1]}$ be an almost optimal sweepout for the width $W(g(t),\beta)$. Since the two definitions of widths coincide, we have

$$W(g(t+h),\beta) \leq \max_{s\in[0,1]} \text{area}_{g(t+h)}(\gamma_s). \tag{88}$$

Then we use the estimate for the evolution of the area and the expression (86) to conclude the upper bounds for the rate of change of the width.

To see that Theorem 17.8 implies the finite extinction time, rewrite:

$$\frac{d}{dt}\big(W(t)\cdot(t+c)^{-3/4}\big) \leq -4\pi(t+c)^{-3/4}. \tag{89}$$

Then, integrating,

$$W(t)\cdot(t+c)^{-3/4} \leq W(0)\cdot c^{-3/4} - 16\pi\big((t+c)^{1/4}-c^{1/4}\big). \tag{90}$$

If t is sufficiently large, depending on c and $W(0)$, we have $W(t)<0$, what is a contradiction. Therefore, the flow becomes singular in finite time. It becomes also extinct.

18. The proof of Willmore Conjecture

In this section we discuss about the best realization of a torus in $\mathbb{R}^3$. To deal with this question, we consider the following functional:

18.1. **Definition.** Let $\Sigma^2\subset\mathbb{R}^3$ be a closed surface of any genus. We defined the *Willmore energy of* Σ to be

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\Sigma. \tag{91}$$

Observe that

$$\int_\Sigma H^2 d\Sigma = \int_\Sigma \Big\{\frac{(k_1-k_2)^2}{4} + K_\Sigma\Big\}d\Sigma = 2\pi\chi(\Sigma) + \frac{1}{2}\int_\Sigma |\mathring{A}|^2 d\Sigma.$$

The term $|\mathring{A}|^2 d\Sigma$ is pointwise conformally invariant. In particular, this says that for any $F\in\text{Conf}(\mathbb{R}^3)$, we have $\mathcal{W}(F(\Sigma)) = \mathcal{W}(\Sigma)$.

In the beginning of the 1960's, Willmore proved that

$$\mathcal{W}(\Sigma) \geq 4\pi = \mathcal{W}(round\ \ sphere),$$

and also the rigidity statement in case of equality. In 1965, he conjectured:

Willmore Conjecture. If Σ is a torus, then $\mathcal{W}(\Sigma)\geq 2\pi^2$.

The energy $2\pi^2$ is attained by a special torus of revolution $\Sigma_{\sqrt{2}}$, called the Clifford torus. If is obtained by the revolution of the unit circle

$$\{(x,0,z)\in\mathbb{R}^3 : (x-\sqrt{2})^2+z^2=1\}$$

with respect to the z-axis.

Using the stereographic projection $\pi : S^3 - \{p\} \to \mathbb{R}^3$, which is a conformal map, we can relate the Willmore energy of surfaces $\tilde{\Sigma} \subset \mathbb{R}^3$ with a very similar functional defined on surfaces $\Sigma \subset S^3$. If we have $\tilde{\Sigma} = \pi(\Sigma)$, then

$$\mathcal{W}(\tilde{\Sigma}) = \int_\Sigma (1 + H^2) d\Sigma. \tag{92}$$

This motivates the following definition:

18.2. **Definition.** The *Willmore energy of* $\Sigma \subset S^3$ is defined by

$$\mathcal{W}(\Sigma) = \int_\Sigma (1 + H^2) d\Sigma. \tag{93}$$

18.3. **Remarks.** Given $\Sigma^2 \subset S^3$, we have

(1) $\mathcal{W}(\Sigma) \geq \text{Area}(\Sigma)$;
(2) if Σ is minimal, then $\mathcal{W}(\Sigma) = \text{Area}(\Sigma)$.

18.4. **Examples.**

- The equator $S^2_1(0) = S^3 \cap \{x_4 = 0\}$ is minimal and has area 4π.
- The Clifford Torus $\hat{\Sigma} = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3$ is minimal and has area $2\pi^2$. Moreover, $\pi(\hat{\Sigma}) = \Sigma_{\sqrt{2}}$.

18.5. **Proposition.** [Li and Yau] *If* $\Sigma^2 \subset S^3$ *is an immersed closed surface that is not embedded, then its Willmore energy is at least* 8π.

Fix $v \in \mathbb{R}^4$ and consider the vector field $x \mapsto v^\perp(x)$, given by tangential projection on $T_x S^3$. It is a conformal killing vector field in S^3 and generates a flow ϕ_t of centered dilations. By conformal invariance of Willmore energy,

$$\mathcal{W}(\Sigma) = \mathcal{W}(\phi_t(\Sigma)) = \int_{\phi_t(\Sigma)} (1 + H^2_{\phi_t(\Sigma)}) d\phi_t(\Sigma) \geq \text{area}(\phi_t(\Sigma)).$$

If we choose $v \in \Sigma$ and let $t \to \infty$, we have $\mathcal{W}(\Sigma) \geq 4\pi$. In case $F : \Sigma \to S^3$ is immersion and $p \in \Sigma$ satisfies $\#F^{-1}(p) = k$, then $\mathcal{W}(\Sigma) \geq 4\pi k$.

18.6. **Theorem.** [Marques and Neves] *Let* $\Sigma \subset S^3$ *closed, embedded, genus* $g \geq 1$*. Then* $\mathcal{W}(\Sigma) \geq 2\pi^2$*, and* $\mathcal{W}(\Sigma) = 2\pi^2$ *if and only if* Σ *is a conformal image of the Clifford torus.*

In particular, this implies the Willmore Conjecture.

18.7. **Theorem.** [Marques and Neves] *Let* $\Sigma \subset S^3$ *closed, embedded, genus* $g \geq 1$*. If* Σ *is minimal, then* $\text{area}(\Sigma) \geq 2\pi^2$*, and* $\text{area}(\Sigma) = 2\pi^2$ *if and only if* Σ *is the Clifford torus up to Iso*(S^3).

The strategy of proof is as follows: first we prove Theorem 18.7 by min-max. Then we prove Theorem 18.6.

18.8. **Min-max theory for the area functional (Almgren-Pitts).** Lets start with the one-parameter sweepouts, following the work of Simon-Smith and the recent survey by Colding-De Lellis. Let M be a three-dimensional closed Riemannian manifold. One can think of a one-parameter sweepout of M as a family $\{\Sigma_t\}_{t\in[0,1]}$, where $\Sigma_t = f^{-1}(t)$ are the level sets of a Morse function $f : M \to [0,1]$. This particular sweepout generates a class of sweepouts

$$\Lambda = \{\{\tilde{\Sigma}_t = \psi(t, \Sigma_t)\}_{t\in[0,1]}\} \tag{94}$$

where $\psi : [0,1] \times M \to M$ is a C^∞ map such that $\psi(t,\cdot) \in \mathrm{Diff}_0(M)$, for every $t \in [0,1]$. Here $\mathrm{Diff}_0(M)$ denotes the connected component of the identity map in the space of diffeomorphisms of M.

Then, we consider *the width of* Λ defined by

$$\mathbf{L}(\Lambda) = \inf_{\{\tilde{\Sigma}_t\}\in\Lambda} \sup_{t\in[0,1]} \mathrm{area}(\tilde{\Sigma}_t). \tag{95}$$

The fact that we started with the sweepout of the level sets of a Morse function implies that $\mathbf{L}(\Lambda) > 0$.

The basic idea of the min-max theory is to realize $\mathbf{L}(\Lambda)$ as the area of some closed embedded minimal surface $\Sigma^2 \subset M$. In this case, we say that Σ is a min-max minimal surface.

18.9. **Theorem.** [Simon and Smith] *Any (S^3, g) admits a minimal embedded S^2.*

18.10. **Example.** Consider S^3 with the standard round metric. Let $\Sigma_t = \{x_4 = 2t - 1\}$, $t \in [0,1]$. This generates a class of sweepouts Λ whose width is equal 4π.

Moreover, it is a well known fact that any non-trivial sweepout of (S^3, g), with arbitrary Riemannian metric, whose width is at most 4π, must contain a great sphere.

Important question. Can we construct the Clifford torus by min-max?

In general, we expect that a min-max minimal surfaces produced by one-parameter sweepouts have Morse index at most 1. But Clifford torus has index 5.

To describe the min-max theory for k-parameter sweepouts it is convenient to use the language of the Geometric Measure Theory. Roughly speaking, instead of smooth surfaces, the role of the slices is played by integral cycles. We denote the space of cycles by

$$\mathcal{Z}_{n-1}(M^n, \mathbb{Z}) = \{(n-1)\text{-integral currents } T \text{ for which } \partial T = 0\}. \tag{96}$$

Intuitively, one can think of an integral cycle as a Lipschitz oriented submanifold with integer multiplicity and no boundary. Let I^k denote the k-dimensional unit cube.

Fixed a map $\Phi : I^k \to \mathcal{Z}_{n-1}(M^n, \mathbb{Z})$, we consider the homotopy class of Φ relative to ∂I^k, which we denote by Π. Then, we define the width of Π as

$$\mathbf{L}(\Pi) = \inf_{\Phi' \in \Pi} \sup_{x \in I^k} \text{area}(\Phi'(x)). \tag{97}$$

The main existence result of the Almgren-Pitts Min-max Theory is the following:

18.11. **Theorem.** *Suppose $3 \leq n \leq 7$. If $\boldsymbol{L}(\Pi) > \sup_{x\in\partial I^k} \text{area}(\Phi(x))$, then there exists smooth embedded closed minimal surface, maybe disconnected and with integer multiplicities, such that area$(\Sigma) = \boldsymbol{L}(\Pi)$.*

Lets summarize the main ingredients for the proof of Theorem 18.7.

(1) For each $\Sigma \subset S^3$, there exists canonical family $\Sigma_{(v,t)} \subset S^3$, $(v, t) \in B^4 \times [-\pi, \pi]$, with $\Sigma_{(0,0)} = \Sigma$ and $\text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$;
(2) [Urbano, 1990] If $\Sigma \subset S^3$ minimal, $\text{index}(\Sigma) \leq 5$, $g \geq 0$, then Σ is a Clifford torus (index 5) or a great sphere (index 1);
(3) Min-max theory for the area functional (Almgren-Pitts).

18.12. **Canonical Family and Area Estimate.** We begin this section by introducing some notation:

- let $\Sigma \subset S^3$ be a closed embedded surface;
- $B^4 \subset \mathbb{R}^4$ open unit ball and $\partial B^4 = S^3$;
- $B^4_R(Q) = \{x \in \mathbb{R}^4 : |x - Q| < R\}$;
- $B_r(q) = \{x \in S^3 : d(x, q) < r\}$, d is spherical distance.

For each $v \in B^4$ we consider $F_v : S^3 \to S^3$ the conformal map defined by

$$F_v(x) = \frac{1 - |v|^2}{|x - v|^2}(x - v) - v. \tag{98}$$

It is a centered dilation fixing $\pm v/|v|$. Note that if $v \to p \in S^3$, then $F_v(x) \to -p$, for every $x \in S^3$, $x \neq p$.

Let $S^3 - \Sigma = A \cup A^*$ be the connected components, and N be the unit normal vector to Σ pointing outside A. Consider the images by F_v

$$A_v = F_v(A), \quad A^*_v = F_v(A^*) \quad \text{and} \quad \Sigma_v = F_v(\Sigma) = \partial A_v.$$

The unit normal vector to Σ_v is given by $N_v = DF_v(N)/|DF_v(N)|$. We also consider in S^3 the signed distance function to Σ_v

$$d_v(x) = \begin{cases} d(x, \Sigma_v) & \text{if } x \notin A_v \\ -d(x, \Sigma_v) & \text{if } x \in A_v. \end{cases}$$

This is a Lipschitz function, smooth near Σ_v.

18.13. **Definition.** [Canonical Family] For $v \in B^4$ and $t \in [-\pi, \pi]$, define $\Sigma_{(v,t)} = \partial A_{(v,t)}$, where $A_{(v,t)} = \{x \in S^3 : d_v(x) < t\}$ are open subsets of S^3.

Reparametrizing the canonical family we can extend it to $\overline{B} \times [-\pi, \pi]$ in such a way that: if $|v| = 1$, the $\Sigma_{v,t}$ are round spheres centered at the same $Q(v) \in S^3$. The map $Q : S^3 \to S^3$ is called the center map. Next, we have the area estimate for the slices of the canonical family. This is due to A. Ros.

18.14. **Theorem.** *We have, for every* $(v,t) \in B^4 \times [-\pi,\pi]$,

$$\operatorname{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma_v) = \mathcal{W}(\Sigma). \tag{99}$$

Moreover, if Σ *is not a geodesic sphere and*

$$\operatorname{area}(\Sigma_{(v,t)}) = \mathcal{W}(\Sigma),$$

then $t = 0$ *and* Σ_v *is a minimal surface.*

18.15. **Key Calculation.** The center map $Q : S^3 \to S^3$ satisfies

$$\deg(Q) = \operatorname{genus}(\Sigma).$$

18.16. **Reparametrization.** We can reparametrize again to get

$$\Phi : I^5 \to \mathcal{Z}_2(S^3),$$

with the following properties:

(a) $\Phi(I^4 \times \{0\}) = \Phi(I^4 \times \{1\}) = 0$;
(b) for every $x \in \partial I^4$, $\{\Phi(x,t)\}_{t\in[0,1]}$ is the standard foliation of S^3 by round spheres centered at $Q(x) \in S^3$;
(c) for every $x \in \partial I^4$, $\Phi(x, 1/2) = \partial B_{\pi/2}(Q(x))$;
(d) $\deg(Q) = \operatorname{genus}(\Sigma)$.

Moreover, the area estimate provides the following upper bound

$$\sup_{x\in I^5} \operatorname{Area}(\Phi(x)) \leq \mathcal{W}(\Sigma). \tag{100}$$

Let Π denote the homotopy class of Φ relative to ∂I^5 and $\mathbf{L}(\Pi)$ denote its width.

18.17. **Theorem.** *If genus*$(\Sigma) \geq 1$, *then* $\mathbf{L}(\Pi) > 4\pi$. *In particular, the min-max can not be a great sphere.*

Outline of proof. The proof is by contradiction. Assume that $\mathbf{L}(\Pi) = 4\pi$ and that we have an optimal $\tilde{\Phi} : I^5 \to \mathcal{Z}_2(S^3)$ such that $\tilde{\Phi} = \Phi$ on ∂I^5 and

$$\sup_{x\in I^5} \operatorname{Area}(\tilde{\Phi}(x)) = 4\pi.$$

The we construct a 4-dimensional object $R^4 \subset I^5$ satisfying the following properties: $\partial R^4 = \partial I^4 \times \{1/2\}$ and $\tilde{\Phi}(x)$ is a great sphere, for every $x \in R^4$. The existence of such a R^4 can be seen in the following way: each path joining the bottom $I^4 \times \{0\}$ of I^5 to its top $I^4 \times \{1\}$, is a non-trivial one-parameter sweepout of S^3 whose areas of the slices are at most 4π. Then, there exists a great sphere on each such sweepout. Look at the center of the great spheres on R^4. This says that the center map $Q : \partial R \to S^3$ can be extended to $Q : R^4 \to S^3$. In homology, this implies that

$$\deg(Q) \cdot S^3 = Q(\partial R) = \partial Q(R) = 0 \text{ in } H_3(S^3, \mathbb{Z}).$$

This contradicts the fact that $\deg(Q) = \operatorname{genus}(\Sigma) \geq 1$.

18.18. **Proof of Theorem 18.7.** Let $\Sigma^2 \subset S^3$ be a minimal surface of least area among all minimal surfaces with genus greater or equal to one. Since the Clifford torus belongs to this class, we have that $\text{area}(\Sigma) \leq 2\pi^2$.

Claim. $\text{index}(\Sigma) \leq 5$.

Open. Understand the index of a min-max minimal surface and relate with the number of parameters.

Proof of Claim. Suppose, by contradiction, that $\text{index}(\Sigma) > 5$. Consider the canonical family $\{\Sigma_{(v,t)}\}_{(v,t)}$ generated by the chosen Σ. Recall that

$$\sup_{(v,t)} \text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma) = \text{area}(\Sigma).$$

Since the index of Σ is greater than the number of parameters, it is possible to perturb slightly $\{\Sigma_{(v,t)}\}_{(v,t)}$, not changing the boundary values, to get a new sweepout $\{\Sigma'_{(v,t)}\}_{(v,t)}$ with $\text{area}(\Sigma'_{(v,t)}) < \text{area}(\Sigma)$, for every (v,t). Next, apply Min-max theory to obtain a min-max minimal surface $\hat{\Sigma}$ such that

$$4\pi < \text{area}(\hat{\Sigma}) = \mathbf{L}(\Pi) < \text{area}(\Sigma) \leq 2\pi^2 < 8\pi.$$

That is a contradiction, because $\hat{\Sigma}$ would have $\text{genus}(\hat{\Sigma}) \geq 1$ and area lower than $\text{area}(\Sigma)$. This proves the Claim.

To conclude the proof of Theorem 18.7, we invoke Urbano's Theorem to conclude that $\text{index}(\Sigma) \leq 5$ imply $\text{index}(\Sigma) = 5$ and Σ is the Clifford torus.

18.19. **Proof of Theorem 18.6.** Let $\Sigma^2 \subset S^3$ be a closed embedded surface with genus $g \geq 1$. Consider the Canonical Family $\{\Sigma_{(v,t)}\}_{(v,t)}$ generated by Σ. Use it to produce a min-max minimal surface $\hat{\Sigma}$ such that

$$4\pi < \text{area}(\hat{\Sigma}) = \mathbf{L}(\Pi) \leq \mathcal{W}(\Sigma).$$

If $\hat{\Sigma}$ has multiplicity two somewhere, that would give $\text{area}(\hat{\Sigma}) \geq 8\pi$. Otherwise, since $\text{area}(\hat{\Sigma}) > 4\pi$ and the great spheres are the only minimal spheres in S^3, we have $\text{genus}(\hat{\Sigma}) \geq 1$. Then, we can use Theorem 18.7 to conclude that $\text{area}(\hat{\Sigma}) \geq 2\pi^2$. In any case, we have that $\mathcal{W}(\Sigma) \geq 2\pi^2$.

18.20. **Remark.** The fact we used in the above argument about the uniqueness of great spheres as minimal spheres in S^3 is due to Almgren.

About the rigidity statement of Theorem 18.6, suppose that $\mathcal{W}(\Sigma) = 2\pi^2$. Recall that $\text{area}(\Sigma_{(v,t)}) \leq W(\Sigma) = 2\pi^2$.

Claim. *For some* $(v,t) \in B^4 \times [-\pi, \pi]$*, we have* $\text{area}(\Sigma_{(v,t)}) = W(\Sigma)$.

Otherwise, the min-max family Φ constructed from Σ has $L(\Phi) < W(\Sigma) = 2\pi^2$. We can run min-max again and contradic Theorem 18.7.

Because of equality $\text{area}(\Sigma_{(v,t)}) = \mathcal{W}(\Sigma)$, we conclude $t = 0$ and Σ_v is a minimal surface, by Theorem 18.14. Then $\text{area}(\Sigma_v) = \mathcal{W}(\Sigma_v) = \mathcal{W}(\Sigma) = 2\pi^2$. By rigidity part of Theorem 18.7, $\Sigma_v = F_v(\Sigma)$ must be isometric to the Clifford torus, with $F_v \in \text{Conf}(S^3)$.

18.21. **Open Problems.** We end this section with some open problems related to the Proof of Willmore Conjecture.

(1) What about higher genus surfaces in $\mathbb{R}^3$?
(2) What about genus one surfaces in $\mathbb{R}^4$? In this case the Willmore functional is defined by
$$\mathcal{W}(\Sigma) = \int_\Sigma |\overrightarrow{H}|^2 d\Sigma,$$
where $\overrightarrow{H}$ denotes the mean curvature vector of Σ.
(3) What is the least volume for non-trivial minimal hypersurfaces in S^n? The natural candidate is the generalized Clifford torus
$$S^k\left(\sqrt{\frac{k}{n-1}}\right) \times S^l\left(\sqrt{\frac{l}{n-1}}\right).$$
(4) Characterize the Clifford torus by the index $n+2$.
(5) More recently, also using min-max theory, Marques and Neves proved the following result:

Theorem. *Let* (M^n, g) *be a closed Riemannian manifold with* $3 \leq n \leq 7$ *and* $Ric_g > 0$. *Then, there exist infinitely many embedded closed min-max minimal hypersurfaces in* M.

The related open question is to understand these minimal hypersurfaces. We expect the index and volume to go to infinity, and they should be equidistributed in M.