MIN-MAX THEORY AND THE WILLMORE CONJECTURE

FERNANDO CODÁ MARQUES - IMPA

The content of these lecture notes are the applications of the min-max theory of Almgren and Pitts to the proof of the Willmore conjecture and the Freedman-He-Wang conjecture for links in euclidean three-space. These lecture notes were organized by Rafael Montezuma.

Organization

1. Introduction

Fix topological type, ask what is the best immersion in $\mathbb{R}^3$. For a genus $g = 0$ surface is the round sphere. What about the case of $g = 1$?

Let $\Sigma \subset \mathbb{R}^3$ closed, embedded, smooth surface of genus g,

- k_1 and k_2 the principal curvatures
- $H = \frac{1}{2}(k_1 + k_2)$ the mean curvature
- $K = k_1 k_2$ the Gauss curvature

The integral on Σ of the Gauss curvature K is a topological invariant,

(Gauss-Bonnet) $$\int_\Sigma K d\Sigma = 2\pi\chi(\Sigma).$$

Other quadratic integrand on Σ is H^2

(Willmore Energy) $$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\Sigma,$$

and this is scaling invariant.

Fact: $\mathcal{W}(\Sigma)$ is invariant under conformal transformations of $\mathbb{R}^3$.

(Blaschke, Thomsen 1920's)

Indeed,

$$\int_\Sigma H^2 d\Sigma = \int_\Sigma \Big\{ \frac{(k_1 - k_2)^2}{4} + K \Big\} d\Sigma = 2\pi\chi(\Sigma) + \frac{1}{2}\int_\Sigma |\mathring{A}|^2 d\Sigma,$$

where $\mathring{A}$ is the trace free second fundamental form.

Remark. $|\mathring{A}|^2 d\Sigma$ is pointwise conformally invariant object.

Therefore, $F : \mathbb{R}^3 \to \mathbb{R}^3$ conformal map $\Rightarrow \mathcal{W}(F(\Sigma)) = \mathcal{W}(\Sigma)$.

Willmore (1960's): $\mathcal{W}(\Sigma) \geq 4\pi = \mathcal{W}(\text{Round sphere})$.

Willmore Conjecture (1965): If $\Sigma \subset \mathbb{R}^3$ is a torus, then $\mathcal{W}(\Sigma) \geq 2\pi^2$.

Example 1.1. Let $\Sigma_{\sqrt{2}}$ be the torus obtained by the revolution of a circle with center at distance $\sqrt{2}$ of the axis of revolution and radius 1. Then

$$\mathcal{W}(\Sigma_{\sqrt{2}}) = 2\pi^2.$$

Let $\pi : S^3 \setminus \{p\} \to \mathbb{R}^3$ the stereographic projection, $\Sigma \subset S^3 \setminus \{p\}$ and $\tilde{\Sigma} = \pi(\Sigma) \subset \mathbb{R}^3$. By some calculations, we have

$$\int_{\tilde{\Sigma}} \tilde{H}^2 d\tilde{\Sigma} = \int_{\Sigma} (1 + H^2) d\Sigma.$$

Definition 1.2. For $\Sigma \subset S^3$ the Willmore energy is $\mathcal{W}(\Sigma) = \int_\Sigma (1+H^2) d\Sigma$.

Remark. If $\Sigma \subset S^3$ is minimal, then $\mathcal{W}(\Sigma) = \text{area}(\Sigma)$.

Example 1.3.

- The equator $S_1^2(0)$ is minimal and has area 4π.
- The Clifford Torus $\hat{\Sigma} = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3$ is minimal and has area $2\pi^2$. Moreover, $\pi(\hat{\Sigma}) = \Sigma_{\sqrt{2}}$.

Li and Yau (1982): If Σ is not embedded, then $\mathcal{W}(\Sigma) \geq 8\pi$.

Fix $v \in \mathbb{R}^4$ and consider the vector field $x \mapsto v^\perp(x)$, given by tangential projection on $T_x S^3$. It is a conformal killing vector field in S^3 and generates a flow ϕ_t of centered dilations. By conformal invariance of Willmore energy,

$$\mathcal{W}(\Sigma) = \mathcal{W}(\phi_t(\Sigma)) = \int_{\phi_t(\Sigma)} (1 + H^2_{\phi_t(\Sigma)}) d\phi_t(\Sigma) \geq \text{area}(\phi_t(\Sigma)).$$

If we choose $v \in \Sigma$ and let $t \to \infty$, we have $\mathcal{W}(\Sigma) \geq 4\pi$. In case $F : \Sigma \to S^3$ is immersion and $p \in \Sigma$ satisfies $\#F^{-1}(p) = k$, then $\mathcal{W}(\Sigma) \geq 4\pi k$.

Theorem A. *Let $\Sigma \subset S^3$ closed, embedded, genus $g \geq 1$. Then $\mathcal{W}(\Sigma) \geq 2\pi^2$, and $\mathcal{W}(\Sigma) = 2\pi^2$ if and only if Σ is the Clifford torus up to Conf(S^3).*

$\Rightarrow$ **the Willmore Conjecture!**

Theorem B. *Let $\Sigma \subset S^3$ closed, embedded, genus $g \geq 1$. If Σ is minimal, then* area$(\Sigma) \geq 2\pi^2$*, and* area$(\Sigma) = 2\pi^2$ *if and only if Σ is the Clifford torus up to Iso(S^3).*

Example 1.4 (Min-max example). Let $S \subset \mathbb{R}^n$ a submanifold and h its height function. Consider $\gamma_0 : [0,1] \to S$ with $\gamma_0(0) = p$ and $\gamma_0(1) = q$ and Π the homotopy class of γ_0 relative to $\{0,1\}$. We define the min-max invariant

$$c = \inf_{\gamma \in \Pi} \sup_{t \in [0,1]} h(\gamma(t)).$$

The aim of min-max theory is to detect a critical point x_0 for h of index 1 as $c = h(x_0)$, for some path γ_0.

Remark. In order to detect index k critical points we need to do min-max over k-parameters families.

Can think of the equator (great sphere) as a min-max minimal surface of 1-dimensional sweep out of S^3.

Question: Can we produce Clifford torus as a min-max minimal surface? (one need 5-parameters)

The main ingredients on the proof are

(I) For each $\Sigma \subset S^3$, there exists canonical family $\Sigma_{(v,t)} \subset S^3$, $(v,t) \in B^4 \times [-\pi, \pi]$, with $\Sigma_{(0,0)} = \Sigma$ and $\text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$.

(II) [Urbano, 1990] If $\Sigma \subset S^3$ minimal, $\text{index}(\Sigma) \leq 5$, $g \geq 0$, then Σ is a Clifford torus (index 5) or a great sphere (index 1).

(III) Min-max theory for the area functional (Almgren-Pitts).

2. Canonical Family and Degree Calculation

We begin this section by introducing some notation:

- $\Sigma \subset S^3$ surface
- $B^4 \subset \mathbb{R}^4$ open unit ball and $\partial B^4 = S^3$
- $B^4_R(Q) = \{x \in \mathbb{R}^4 : |x - Q| < R\}$
- $B_r(q) = \{x \in S^3 : d(x,q) < r\}$, d is spherical distance

For each $v \in B^4$ we consider $F_v : S^3 \to S^3$ the conformal map defined by

$$F_v(x) = \frac{1 - |v|^2}{|x - v|^2}(x - v) - v. \tag{1}$$

It is a centered dilation fixing $\pm v/|v|$. Note that if $v \to p \in S^3$, then $F_v(x) \to -p$, for every $x \in S^3$, $x \neq p$.

Let $S^3 \setminus \Sigma = A \cup A^*$, the connected components, and N the unit normal vector to Σ pointing out A. Consider the images by F_v

$$A_v = F_v(A), \quad A^*_v = F_v(A^*) \quad \text{and} \quad \Sigma_v = F_v(\Sigma) = \partial A_v.$$

The unit normal vector to Σ_v is given by $N_v = DF_v(N)/|DF_v(N)|$. We also consider in S^3 the signed distance function to Σ_v

$$d_v(x) = \begin{cases} d(x, \Sigma_v) & \text{if } x \notin A_v \\ -d(x, \Sigma_v) & \text{if } x \in A_v. \end{cases}$$

This is a Lipschitz function, smooth near Σ_v.

Definition 2.1 (Canonical Family)**.** For $v \in B^4$ and $t \in [-\pi, \pi]$, define $\Sigma_{(v,t)} = \partial A_{(v,t)}$, where $A_{(v,t)} = \{x \in S^3 : d_v(x) < t\}$ are open subsets of S^3.

Remark. Consider $\psi_{(v,t)} : \Sigma_v \to S^3$ given by $\psi_{(v,t)}(y) = exp_y(tN_v(y))$, i.e., $\psi_{(v,t)}(y) = \cos(t)y + \sin(t)N_v(y)$. Then $\Sigma_{(v,t)} \subset \psi_{(v,t)}(\{\text{Jac } \psi_{(v,t)} \geq 0\})$. In particular, we conclude that $\Sigma_{(v,t)}$ are 2-rectifiable subsets of S^3.

Area Estimate of $\Sigma_{(v,t)}$.

Theorem 2.2. *We have, for every $(v,t) \in B^4 \times [-\pi,\pi]$,*

$$\text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma). \tag{2}$$

Moreover, if Σ is not a geodesic sphere and

$$\text{area}(\Sigma_{(v,t)}) = \mathcal{W}(\Sigma),$$

then $t = 0$ and Σ_v is a minimal surface.

Proof. Consider $\{e_1, e_2\} \subset T_y\Sigma_v$ orthonormal basis such that $DN_v|_y = -k_i(v)e_i, i = 1,2$, where $k_i(v)$ are the principal curvatures of Σ_v. Since $D\psi_{(v,t)}|_y(e_i) = (\cos(t) - k_i(v)\sin(t))e_i$, we have

$$\begin{aligned} \text{Jac } \psi_{(v,t)} &= (\cos(t) - k_1(v)\sin(t))(\cos(t) - k_2(v)\sin(t)) \\ &= \cos^2(t) - (k_1 + k_2)\cos(t)\sin(t) + k_1k_2\sin^2(t). \end{aligned}$$

By a simple calculation, this gives

Lemma 2.3. *If $H(v)$ is the mean curvature of Σ_v, we have*

$$\text{Jac } \psi_{(v,t)} = (1 + H(v)^2) - (\sin(t) + H(v)\cos(t))^2 - \frac{(k_1(v) - k_2(v))^2}{4}\sin^2(t).$$

Using this lemma, the area formula, and the conformal invariance of the Willmore energy we obtain

$$\begin{aligned} \text{area}(\Sigma_{(v,t)}) &\leq \text{area}\big(\psi_{(v,t)}(\{\text{Jac } \psi_{(v,t)}(p) \geq 0\})\big) \\ &\leq \int_{\{\text{Jac } \psi_{(v,t)} \geq 0\}} (\text{Jac } \psi_{(v,t)})\, d\Sigma_v \\ &\leq \int_{\Sigma_v} (1 + H(v)^2)\, d\Sigma_v - \sin^2 t \int_{\Sigma_v} \frac{(k_1(v) - k_2(v))^2}{4}\, d\Sigma_v \\ &= \mathcal{W}(\Sigma) - \frac{\sin^2 t}{2} \int_\Sigma |\mathring{A}|^2 d\Sigma. \end{aligned}$$

Suppose that equality holds for some (v,t) and that Σ is not a geodesic sphere. Then from conformal invariance we have

$$0 \neq \int_\Sigma \frac{(k_1 - k_2)^2}{4}\, d\Sigma = \int_{\Sigma_v} \frac{(k_1(v) - k_2(v))^2}{4}\, d\Sigma_v \quad \text{for all } v \in B^4$$

and so $t = 0$, $t = \pi$, or $t = -\pi$. The last two cases are impossible because $\Sigma_{(v,\pi)} = \emptyset$ and $\Sigma_{(v,t)} = \emptyset$ and thus $t = 0$. This means $\text{area}(\Sigma_v) = \mathcal{W}(\Sigma_v)$ and so Σ_v is a minimal surface. ∎

Blow-up Argument. Next we analyse what happens to $\Sigma_{(v,t)}$ as $v \to S^3$. The notation for symmetric difference here is, as usual, $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Assume $v_n \in B^4$ and $t_n \in [-\pi, \pi]$, $(v_n, t_n) \to (v, t) \in \overline{B}^4 \times [-\pi, \pi]$. The case $v \notin \Sigma$ is divided in three:

Claim 1. *If $v \in B^4$, $\Sigma_{v_n} \to \Sigma_v$ smoothly and*

$$vol\ (A_{(v_n,t_n)} \Delta A_{(v,t)}) \to 0.$$

Claim 2. *If $v \in A$, $\Sigma_{v_n} \to -v$, $A_{v_n} \to B_\pi(v)$ and*

$$vol\ (A_{(v_n,t_n)} \Delta B_{\pi+t}(v)) \to 0.$$

Claim 3. *If $v \in A^*$, $\Sigma_{v_n} \to -v$, $A_{v_n} \to B_0(-v)$ and*

$$vol\ (A_{(v_n,t_n)} \Delta B_t(-v)) \to 0.$$

Problem: If $v_n \to v = p \in \Sigma$. For example, if $v_n = (1 - 1/n)v$, we have $A_{v_n} \to B_{\pi/2}(-N)$. But this convergence only happens because the angle that v_n makes with $N(p)$ also converge. In fact, the limit of $F_{v_n}(A)$ depends only on the angle of convergence.

Let $D^2_+(r) = \{s = (s_1, s_2) \in \mathbb{R}^2 : |s| < r, s_1 \geq 0\}$. Consider $\varepsilon > 0$ small so that $\Lambda : \Sigma \times D^2_+(3\varepsilon) \to \overline{B}^4$ is a diffeomorphism to a tubular neighbourhood of Σ in $\overline{B}^4$, given by $\Lambda(p, s) = (1 - s_1)(\cos(s_2)p + \sin(s_2)N(p))$. We also use Ω_r for $\Lambda(\Sigma \times D^2_+(r))$.

Conformal images of balls and sectors.

Let $p, N \in S^3$ and $r > 0$, so that $\langle p, N\rangle = 0$. In this section we study the conformal images of sectors of S^3,

$$\Delta(p, N, r) = S^3 \setminus \{B_r(\cos(r)p + \sin(r)N) \cup B_r(\cos(r)p - \sin(r)N)\}, \tag{3}$$

and of balls

$$B_{\pi/2}(-N) = B^4_{\sqrt{2}}(-N) \cap S^3. \tag{4}$$

Consider the following notation. For sufficient small ε_0, $v = (1-s)(\cos(t)p + \sin(t)N)$ with $0 < s \leq \varepsilon_0$ and $|t| \leq \varepsilon_0$,

$$\overline{Q} = -\frac{t/s}{\sqrt{1 + (t/s)^2}}p - \frac{1}{\sqrt{1 + (t/s)^2}}N \in S^3$$

and

$$\overline{R} = \sqrt{2\left(1 - \frac{t/s}{\sqrt{1 + (t/s)^2}}\right)}.$$

Proposition 2.4. *Then, there exists $C > 0$ so that:*

(i) $$B^4_{\overline{R}-C\sqrt{|(s,t)|}}(\overline{Q}) \cap S^3 \subset F_v(B^4_{\sqrt{2}}(-N) \cap S^3) \subset B^4_{\overline{R}+C\sqrt{|(s,t)|}}(\overline{Q}) \cap S^3$$

(ii) $$F_v(\Delta(p, N, r)) \subset \overline{B}^4_{\overline{R}+C\sqrt{|(s,t)|}}(\overline{Q}) \setminus \overline{B}^4_{\overline{R}-C\sqrt{|(s,t)|}}(\overline{Q}).$$

If $v_n \to v = p \in \Sigma$ we can write $v_n = \Lambda(p_n, (s_{n1}, s_{n2}))$, with $p_n \to p$ and $|s_n| \to 0$. Let $B_{p_n} = B^4_{\sqrt{2}}(-N) \cap S^3$. Suppose also the angle convergence

$$\lim_{n\to\infty} \frac{s_{n2}}{s_{n1}} = k \in [-\infty, \infty]. \tag{5}$$

Remark. Choose $r_0 > 0$ sufficient small for which

$B_{r_0}(\cos(r_0)p - \sin(r_0)N(p)) \subset A$, and $\overline{A} \subset S^3 \backslash B_{r_0}(\cos(r_0)p + \sin(r_0)N(p))$.

Since then, we have $A\Delta B_{p_n} \subset \Delta(p_n, N(p_n), r_0)$, and thus $F_{v_n}(A\Delta B_{p_n}) \subset F_{v_n}(\Delta(p_n, N(p_n), r_0))$. Using the above result,we have

$$F_{v_n}(A)\Delta F_{v_n}(B_{p_n}) \subset \overline{B}^4_{\overline{R}_n + C\sqrt{|s_n|}}(\overline{Q}_n) \setminus \overline{B}^4_{\overline{R}_n - C\sqrt{|s_n|}}(\overline{Q}_n),$$

where

$$\overline{Q}_n \to -\frac{k}{\sqrt{1+k^2}}p - \frac{1}{\sqrt{1+k^2}}N(p) \in S^3$$

and

$$\overline{R}_n \to \sqrt{2\left(1 - \frac{k}{\sqrt{1+k^2}}\right)}.$$

Then $\mathrm{vol}(F_{v_n}(A)\Delta F_{v_n}(B_{p_n})) \to 0$. But $F_{v_n}(B_{p_n})$ converges if we have angle convergence $s_{n2}/s_{n1} \to k \in [-\infty, \infty]$. Summarizing,

Proposition 2.5. *Consider a sequence $(v_n, t_n) \in B^4 \times [-\pi, \pi]$ converging to $(v, t) \in \overline{B}^4 \times [-\pi, \pi]$.*

(i) *If $v \in B^4$ then*

$$\lim_{n\to\infty} vol\left(A_{(v_n,t_n)}\Delta A_{(v,t)}\right) = 0.$$

(ii) *if $v \in A$ then*

$$\lim_{n\to\infty} vol\left(A_{(v_n,t_n)}\Delta B_{\pi+t}(v)\right) = 0$$

and, given any $\delta > 0$,

$\Sigma_{(v_n,t_n)} \subset \overline{B}_{\pi+t+\delta}(v) \setminus B_{\pi+t-\delta}(v)$ for all n sufficiently large.

(iii) *if $v \in A^*$ then*

$$\lim_{n\to\infty} vol\left(A_{(v_n,t_n)}\Delta B_t(-v)\right) = 0$$

and, given any $\delta > 0$,

$\Sigma_{(v_n,t_n)} \subset \overline{B}_{t+\delta}(-v) \setminus B_{t-\delta}(-v)$ for all n sufficiently large.

(iv) *If $v = p \in \Sigma$ and*

$$v_n = \Lambda(p_n, (s_{n1}, s_{n2})) \text{ with } \lim_{n\to\infty} \frac{s_{n2}}{s_{n1}} = k \in [-\infty, \infty],$$

then

$$\lim_{n\to\infty} vol\left(A_{(v_n,t_n)}\Delta B_{\overline{r}_k+t}(\overline{Q}_{p,k})\right) = 0$$

and, given any $\delta > 0$,

$\Sigma_{(v_n,t_n)} \subset \overline{B}_{\overline{r}_k+t+\delta}(\overline{Q}_{p,k}) \setminus B_{\overline{r}_k+t-\delta}(\overline{Q}_{p,k})$ for all n sufficiently large.

Reparametrizing Map $T : \overline{B}^4 \to \overline{B}^4$.

Take a smooth function $\phi : [0, 3\varepsilon] \to [0, 1]$ such that $\phi([0, \varepsilon]) = 0$, strictly increasing in $[\varepsilon, 2\varepsilon]$ and $\phi([2\varepsilon, 3\varepsilon]) = 1$. Define $T : \overline{B}^4 \to \overline{B}^4$ by

$$T(v) = \begin{cases} v & \text{if } v \in \overline{B}^4 \setminus \Omega_{3\varepsilon} \\ \Lambda(p, \phi(|s|)s) & \text{if } v = \Lambda(p, s) \in \Omega_{3\varepsilon}. \end{cases}$$

T is continuous and collapses Ω_ε onto Σ preserving s_{n2}/s_{n1}. Moreover, $T : B^4 \setminus \overline{\Omega}_\epsilon \to B^4$ is a homeomorphism.

This map allows us to make a blow up argument and reparametrize the canonical family in such a way it can be continuously extended to the whole $\overline{B}^4 \times [-\pi, \pi]$. For each $v \in B^4 \setminus \overline{\Omega}_\varepsilon$, put $C(v, t) = \Sigma_{(T(v),t)}$. Using the results in the previous section we know that as $v \to \partial(B^4 \setminus \overline{\Omega}_\varepsilon)$, $C(v, t)$ converges to some geodesic sphere (round sphere). Extend C to Ω_ε being constant radially, along $s_2 =$ constant. By construction, we have

$$C(v, t) \text{ is continuous in volume in } \overline{B}^4 \times [-\pi, \pi] \tag{6}$$

and

$$C(v, t) = \partial B_{\overline{r}(v)+t}(\overline{Q}(v)), \text{ for } v \in \partial B^4, \tag{7}$$

where the center map $\overline{Q} : S^3 \to S^3$ is given by

$$\overline{Q}(v) = \begin{cases} -T(v) & \text{if } v \in A^* \setminus \overline{\Omega}_\epsilon \\ T(v) & \text{if } v \in A \setminus \overline{\Omega}_\epsilon \\ -\frac{s}{\varepsilon}p - \frac{\sqrt{\varepsilon^2 - s^2}}{\varepsilon} N(p) & \text{if } v = \cos(s)p + \sin(s)N(p), s \in [-\varepsilon, \varepsilon]. \end{cases}$$

Theorem 2.6. $\deg(\overline{Q}) = \text{genus}(\Sigma)$.

Proof. Let dV and $d\text{vol}$ be the volume forms of S^3 and $\mathbb{R}^4$, respectively. First, regard that $\overline{Q}$ is piecewise smooth. The idea is to calculate the degree by integrating $\overline{Q}^*(dV)$ on S^3, and using the formula

$$\int_{S^3} \overline{Q}^*(dV) = \deg(\overline{Q}) \int_{S^3} dV.$$

In order to proceed the calculation, divide S^3 on $A^* \setminus \Omega_\varepsilon$, $A \setminus \Omega_\varepsilon$ and $\Omega_\epsilon \cap S^3$. in the first two we have

$$\int_{A^* \setminus \Omega_\varepsilon} \overline{Q}^*(dV) = \int_{A^* \setminus \Omega_\varepsilon} (-T)^*(dV) = \int_{A^*} dV = \text{vol}(A^*)$$

and, analogously,

$$\int_{A \setminus \Omega_\varepsilon} \overline{Q}^*(dV) = \text{vol}(A).$$

Let $G : \Sigma \times [-\varepsilon, \varepsilon] \to S^3 \cap \Omega_\varepsilon$ be the diffeomorphism $G(p, t) = \cos(t)p + \sin(t)N(p)$ and define $Q = \overline{Q} \circ G : \Sigma \times [-\varepsilon, \varepsilon] \to S^3$, so that

$$Q(p, t) = -\frac{t}{\varepsilon}p - \frac{\sqrt{\varepsilon^2 - t^2}}{\varepsilon} N(p).$$

In this reparametrization, the third integral can be calculated as

$$\int_{\Omega_\varepsilon \cap S^3} \overline{Q}^*(dV) = \int_{\Sigma\times[-\varepsilon,\varepsilon]} Q^*(dV).$$

If $\{e_1, e_2\} \subset T_p\Sigma$ orthonormal basis with $DN|_p(e_i) = -k_i e_i$, we have

$$\begin{aligned} Q^*(dV)(e_1,e_2,\partial_t) &= dV(DQ(e_1), DQ(e_2), DQ(\partial_t)) \\ &= d\mathrm{vol}(DQ(e_1), DQ(e_2), DQ(\partial_t), Q) \\ &= \Big(-\frac{t}{\varepsilon} + \frac{\sqrt{\varepsilon^2-t^2}}{\varepsilon}k_1\Big)\Big(-\frac{t}{\varepsilon} + \frac{\sqrt{\varepsilon^2-t^2}}{\varepsilon}k_2\Big)\frac{(-1)}{\sqrt{\varepsilon^2-t^2}}. \end{aligned}$$

The Gauss curvature of Σ satisfies $K = 1 + k_1k_2$, the Gauss equation, hence

$$\begin{aligned} \int_{\Sigma\times[-\varepsilon,\varepsilon]} Q^*(dV) &= -\iint \frac{1}{\varepsilon^2}\Big(k_1k_2\sqrt{\varepsilon^2-t^2} - (k_1+k_2)t + \frac{t^2}{\sqrt{\varepsilon^2-t^2}}\Big)dt d\Sigma \\ &= -\frac{\pi}{2}\int_\Sigma (K-1)d\Sigma - \frac{\pi}{2}\int_\Sigma d\Sigma = \pi^2(2g-2). \end{aligned}$$

Adding the results,

$$\int_{S^3} \overline{Q}^*(dV) = \mathrm{vol}(S^3) + \pi^2(2g-2) = 2\pi^2 g = g\int_{S^3} dV.$$

□

3. Geometric Measure Theory

The intent of this section is to introduce a few geometric measure theory notions that will appear latter. Let (M^3, g) be an orientable Riemannian 3-manifold isometrically embedded in $\mathbb{R}^L$. Consider the notation $I_k(M)$ for the space of k-dimensional integral currents in $\mathbb{R}^L$ with support in M, i.e., k-dimensional rectifiable sets with integer multiplicities and orientations chosen H^k-almost everywhere. A k-dimensional integral current T can also be interpreted as a continuous functional that operates in $\Omega^k_c(\mathbb{R}^L)$, the space of compactly supported differential k-forms provided with the comass norm, by integration

$$T(\phi) = \int_M \langle \phi, \xi\rangle \theta dH^k, \quad \phi \in \Omega^k_c(\mathbb{R}^L),$$

where θ and ξ are the multiplicity and orientation maps of T, respectively. The boundary operator $\partial : I_k(M) \to I_{k-1}(M)$ is defined as

$$\partial T(\phi) = T(d\phi).$$

The set of currents without boundary is denoted by $Z_k(M)$.

Example 3.1. Oriented closed k-dimensional surfaces are the basic examples of k-dimensional integral currents.

The notion of volume extends to currents. For each $T \in I_k(M)$ we define its mass to be

$$\mathbf{M}(T) = \sup\{T(\phi) : \phi \in \Omega_c^k(\mathbb{R}^L) \text{ and } ||\phi|| \leq 1\},$$

where $||\phi||$ is the comass norm. We can introduce then the mass norm

$$\mathbf{M}(S_1, S_2) = \mathbf{M}(S_1 - S_2).$$

The flat metric defined below in the space of currents has the important property that its induced topology is the weak topology

$$\mathcal{F}(S_1, S_2) = \inf\{\mathbf{M}(S_1 - S_2 - \partial Q) + \mathbf{M}(Q) : Q \in I_{k+1}(M)\}.$$

We use also the notation $\mathcal{F}(T) = \mathcal{F}(T, 0)$. Note that $\mathcal{F}(T) \leq \mathbf{M}(T)$.

Theorem 3.2. *The map $C : \overline{B}^4 \times [-\pi, \pi] \to Z_2(S^3)$ is well defined and continuous in the flat topology.*

In order to prove the map is well defined recall that

$$C(v,t) = \partial U(v,t) \Rightarrow \partial C(v,t) = 0.$$

And for the continuity claim use

$$\begin{aligned} \mathcal{F}(C(v_1,t_1), C(v_2,t_2)) &= \mathcal{F}(\partial(U(v_1,t_1) - U(v_2,t_2))) \\ &= \text{vol}(U(v_1,t_1) \Delta U(v_2,t_2)). \end{aligned}$$

4. Min-max Theory

Theorem 4.1 (Almgren-Pitts)**.** *Every compact Riemannian manifold (M^n, g), $n \leq 7$, contains a smooth embedded minimal hypersurface $\Sigma \subset M$.*

Those are obtained by min-max methods with only 1-parameter and the proof can be founded in "Existence and regularity of minimal surfaces in Riemannian manifolds", by Pitts.

The min-max theory we would like to have.

We would like to run min-max with maps $\Phi : I^n \to Z_2(M^3)$ continuous in the flat topology, defined on the n-dimensional cube $I^n = [0,1]^n$. If Π is the homotopy class of Φ relative the boundary, define the width of Π by

$$L(\Pi) = \inf\{L(\Phi') : \Phi' \in \Pi\}, \tag{8}$$

where $L(\Phi') = \sup\{\mathbf{M}(\Phi'(x)) : x \in I^n\}$. In this context, the result we would like to have is

Min-max Theorem: If $L(\Pi) > \sup\{\mathbf{M}(\Phi(x)) : x \in \partial I^n\}$, then there exists a smooth embedded minimal surface Σ (possibly disconnected, with integer multiplicities), in M so that $L(\Pi) = \text{area}(\Sigma)$. Moreover, if $\{\phi_i\}_i \subset \Pi$ is such that $L(\phi_i) \to L(\Pi)$, then there exists $x_i \in I^n$ with $\mathbf{F}(\Sigma, \phi_i(x_i)) \to 0$.

Almgren-Pitts min-max theory.

We briefly describe Almgren-Pitts min-max theory comparing this with the continuous theory. We need the following notation.

- $I(1,k)$ denotes the cell complex on I^1 whose 1-cells and 0-cells are, respectively,

$$[0,3^{-k}],[3^{-k},2\cdot 3^{-k}],\ldots,[1-3^{-k},1], \text{ and } [0],[3^{-k}],\ldots,[1-3^{-k}],[1].$$

- $I(n,k)$ is the n-dimensional cell complex on I^n given by

$$I(n,k) = I(1,k)\otimes\ldots\otimes I(1,k) \ (n \text{ times}).$$

- $I(n,k)_0$ is the subset of all 0-cells, or vertices, of $I(n,k)$.

Continuous	**Discrete**
$\Phi : I^n \to Z_2(M)$	$S = \{\phi_i\}_i, \phi_i : I(n,k_i)_0 \to Z_2(M)$
Φ continuous in flat topology	$\mathbf{M}(\phi_i(x)-\phi_i(y)) < \delta_i, \forall [x,y]\in I(n.k_i)_1$
$L(\Phi) = \sup\{\mathbf{M}(\Phi(x)) : x\in I^n\}$	$L(S) = \lim_i \sup\{\mathbf{M}(\phi_i(x)) : x \in I(n,k_i)_0\}$
$\Phi \simeq \Phi'$ homotopy	$\phi_i \simeq_{\delta_i} \phi_i'$ and $\delta_i \to 0$
$L(\Pi) = \{L(\Phi') : \Phi' \in \Pi\}$	$L(\Pi) = \{L(S') : S' \in \Pi\}$

5. Ruling out great spheres (Topological Argument)

The min-max family $C : \overline{B}^4 \times [-\pi,\pi] \to Z_2(S^3)$ constructed above gives, up to a reparametrization, a map Φ defined on the 5-cube.

Theorem 5.1. *Let $\Sigma \subset S^3$ embedded, closed, genus g surface. Then, the map $\Phi : I^5 \to Z_2(S^3)$ satisfies the following*

(i) *Φ is continuous in the flat topology.*
(ii) $\Phi(I^4\times 0) = \Phi(I^4 \times 1) = 0$.
(iii) *for every $x \in \partial I^4$, $\{\Phi(x,t)\}_{t\in[0,1]}$ is a standard sweep out of S^3 by oriented geodesic spheres.*
(iv) *there exists a map $Q : \partial I^4 \to S^3$, called the center map, so that*

$$\Phi(x,1/2) = \partial B_{\pi/2}(Q(x)),$$

for all $x \in \partial I^4$. This map has degree g.

Also, $\sup\{\boldsymbol{M}(\Phi(x)) : x \in I^5\} \le \mathcal{W}(\Sigma)$.

Consider Π the homotopy class of Φ. This homotopy class is considered in the discrete min-max context, but here we explain the arguments using the continuous language, supposing we have the continuous version of Min-max Theorem stated in the previous section.

Remark. $L(\Pi) \ge 4\pi =$ least area a minimal surface can have in S^3.

Theorem 5.2. *If $g\ge 1$, then $L(\Pi) > 4\pi$.*

Before proving that, we introduce some notation. We denote by $\tau \subset \mathcal{V}_2(S^3)$ the space of unoriented great spheres. For each element in τ we can associate its center, this gives a diffeomorphism between τ and $\mathbb{RP}^3$. This equivalence has the property that if we push the F-metric of $\tau \subset \mathcal{V}_2(S^3)$ to $\mathbb{RP}^3$, the obtained metric is equivalent to the standard metric of $\mathbb{RP}^3$.

Associated with Φ by forgetting the orientations of currents in $Z_2(S^3)$ we have a map $|\Phi| : \partial I^4 \times \{1/2\} \to \mathbb{RP}^3 \simeq \tau$, given by $|\Phi|(x, 1/2) = |\Phi(x, 1/2)|$. The degree calculation gives $|\Phi|_*(\partial I^4 \times \{1/2\}) = 2g \in H_3(\mathbb{RP}^3)$.

Proof. Suppose, by contradiction, that $L(\Pi) = 4\pi$. Then, there exists a sequence $\{\phi_i\}_i \subset \Pi$ so that

$$\sup_{x \in I^5} \operatorname{area}(\phi_i(x)) \leq 4\pi + \frac{1}{i}.$$

Recall $\phi_i = \Phi$ on ∂I^5. Let $\varepsilon_0 > 0$ small (to be chosen later) and take $\delta > 0$, such that

$$y = (x, t) \in J_\delta = \partial I^4 \times [1/2 - \delta, 1/2 + \delta] \Rightarrow F(|\Phi(y)|, |\Phi(x, 1/2)|) \leq \varepsilon_0.$$

Consider $0 < \varepsilon_1 \leq \varepsilon_0$ with the property

$$y \in \partial I^4 \times I, \mathbf{F}(|\Phi(y)|, \tau) \leq \varepsilon_1 \Rightarrow y \in J_\delta.$$

If we could find a 4-manifold R so that $\partial R = \partial I^4 \times \{1/2\}$ and a continuous function $f : R \to \mathbb{RP}^3$ with $f|_{\partial R} = |\Phi|$, then we would have

$$\Phi|_\sharp(\partial I^4 \times \{1/2\}) = f_\sharp(\partial R) = \partial f(R) = 0 \text{ in } H_3(\mathbb{RP}^3, \mathbb{Z}).$$

Instead of that, we first construct a sequence of 4-dimensional $R(i) \subset I^5$ with support$(\partial R(i)) \subset \partial I^4 \times I$. It is constructed so that $|\phi_i(x)|$ is sufficient close to τ for any $x \in R(i)$, when i is large. We begin with

$$\overline{A}(i) = \{x \in I^5 : \mathbf{F}(|\phi_i(x)|, \tau) \geq \varepsilon_1\},$$

$$A(i) \subset \overline{A}(i) \text{ the connected component of } I^4 \times 0.$$

Claim 4. *$A(i) \cap (I^4 \times \{1\}) = \emptyset$, if i is large.*

Proof. If not, up to a subsequence, can find $\gamma_i : [0, 1] \to I^5$ with $\gamma_i(0) \in I^4 \times 0$, $\gamma_i(1) \in I^4 \times 1$ and $\gamma_i([0, 1]) \subset A(i)$. Since $\phi_i \in \Pi$, for every i, we have $[\phi_i \circ \gamma_i] = \Pi_1$ does not depend on i, and

$$4\pi \leq L(\Pi_1) \leq L(\phi_i \circ \gamma_i) \leq 4\pi + 1/i.$$

Then $L(\Pi_1) = 4\pi$ and the sequence $\{\phi_i \circ \gamma_i\}_i$ is optimal in Π_1. By *"Min-max theorem"*, we find a smooth embedded minimal surface $\Sigma \subset S^3$ and a sequence $x_i \in I$ so that

$$\operatorname{area}(\Sigma) = 4\pi \text{ and } \mathbf{F}(|(\phi_i \circ \gamma_i)(x_i)|, \Sigma) \to 0.$$

The only minimal surface in S^3 with area 4π are great spheres, so $\phi_i(\gamma_i(x_i))$ approximates to τ, and this is a contradiction, since $\gamma_i(x_i) \in A(i)$. □

In particular, $A(i) \cap \partial I^5 \subset (\partial I^4 \times I) \cup (I^4 \times 0)$. Define

$$R(i) = (\partial A(i)) \cap \operatorname{int}(I^5).$$

Therefore $\mathbf{F}(|\phi_i(x)|, \tau) = \varepsilon_1$ on $R(i)$ and, if i is large enough, $\partial R(i) \subset J_\delta$. Take $C(i) = \partial A(i) \cap (\partial I^4 \times I)$. Because $\partial(\partial A(i)) = 0$ and $\partial A(i) = R(i) \cup C(i) \cup (I^4 \times 0)$, we have

$$\partial C(i) = \partial R(i) \cup \partial(I^4 \times 0) \Rightarrow [\partial R(i)] = [\partial I^4 \times \{1/2\}] \text{ in } H_3(J_\delta, \mathbb{Z}).$$

Dealing with the discrete case we have some extra problems, for instance:

(1) surfaces along $R(i)$ are close to τ, but not exactly in τ.
(2) the cycles $\partial R(i)$ and $\partial I^4 \times \{1/2\}$ are homologous and close, but not necessarily equal.

Need a projection argument to conclude.

Definition 5.3. Let $\widehat{\Phi} : \partial I^4 \times I \to \mathbb{RP}^3 \simeq \tau$ be the continuous map given by $\widehat{\Phi}(x, t) = |\Phi(x, 1/2)|$. In particular, $\widehat{\Phi}(\partial I^4 \times \{1/2\}) = 2g$ in $H_3(\mathbb{RP}^3, \mathbb{Z})$.

Claim 5. *There exists a continuous $f_i : R(i) \to \tau$ with $f_i|_{\partial R(i)} = \widehat{\Phi}|_{\partial R(i)}$.*

Proof. We can choose a triangulation Δ of $R(i)$ so that if $[x, y] \in \Delta_1$ and $x, y \in \Delta_0$, then

$$\mathbf{F}(|\phi_i(x)|, |\phi_i(y)|) \leq \varepsilon_0.$$

First define $f_i : \Delta_0 \to \tau$ by $f_i(x) = \widehat{\Phi}(x)$, if $x \in \partial R(i)$, and $f_i(x) = y$, if $x \notin \partial R(i)$, so that $\mathbf{F}(|\phi_i(x)|, y) = \mathbf{F}(|\phi_i(x)|, \tau)$. If $x, y \in \Delta_0$ are adjacent, the triangle inequality gives

$$\mathbf{F}(f_i(x), f_i(y)) \leq 3\varepsilon_0,$$

If ε_0 is sufficient small (depending only on size of convex balls in $\mathbb{RP}^3$ with standard geometry), then the map f_i can be extended to $|\Delta| = R(i)$. □

In the end, we have

$$\widehat{\Phi}_*[\partial R(i)] = (f_i)_*[\partial R(i)] = [\partial (f_i)_\sharp(R(i))] = 0 \text{ in } H_3(\mathbb{RP}^3, \mathbb{Z}).$$

On the other hand, we already had used the degree calculation to obtain

$$\widehat{\Phi}_*[\partial R(i)] = \widehat{\Phi}_*[\partial I^4 \times \{1/2\}] = 2g \text{ in } H_3(\mathbb{RP}^3, \mathbb{Z}).$$

Contradiction if $g \geq 1$. Then $L(\Pi) > 4\pi$. □

6. Proof of Theorems(A and B)

Theorem B. $\Sigma \subset S^3$ *embedded, closed,* $g \geq 1$, *minimal. Then* $\operatorname{area}(\Sigma) \geq 2\pi^2$, *and* $\operatorname{area}(\Sigma) = 2\pi^2$ *if and only if* Σ *is isometric to the Clifford Torus.*

Proof. Consider

$$\mathfrak{F}_1 = \{S \subset S^3 : S \text{ is embedded, closed}, g \geq 1, \text{ minimal}\}.$$

There exists $\Sigma \in \mathfrak{F}_1$ (Appendix), satisfying

$$\operatorname{area}(\Sigma) = \inf_{S \in \mathfrak{F}_1} \operatorname{area}(S). \tag{9}$$

Urbano (1990): If $\Sigma \subset S^3, H = 0, \text{index}(\Sigma) \leq 5$, then Σ is the Clifford torus or the great sphere.

Claim 6. *Take* Σ *as* (**??**)*, then* $\text{index}(\Sigma) \leq 5$.

The general index bound is missing in current min-max theory: Σ min-max with k parameters, then $\text{index}(\Sigma) \leq k$.

If Σ_t is a variation of the minimal surface $\Sigma = \Sigma_0$, the second variation is given by

$$\frac{d^2}{dt^2}\text{area}(\Sigma_t)|_{t=0} = -\int_\Sigma \psi \mathrm{L}\psi d\Sigma, \tag{10}$$

where L is the Jacobi operator, ψN is the variational vector field and N is the unit normal vector to Σ. Recall we constructed a Canonical family $\Sigma_{(v,t)}$, $(v,t) \in \overline{B}^4 \times [-\pi, \pi]$, with the area estimate

$$\text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma) = \text{area}(\Sigma).$$

Consider the variational vector fields of $\Sigma_{(v,t)}$ at $(0,0)$:

$$\psi_i = \langle N, e_i \rangle, \quad i = 1,2,3,4 \text{ and } \psi_5 = 1,$$

where e_i are the canonical directions in $\mathbb{R}^4$ and let $\mathrm{E} = \text{span}\{\psi_1, \ldots, \psi_5\}$.

Lemma 6.1.

$$-\int_\Sigma \psi \mathrm{L}\psi d\Sigma < 0, \forall \psi \in \mathrm{E} \setminus \{0\}, \tag{11}$$

unless Σ *is totally geodesic.*

Suppose $\text{index}(\Sigma) \geq 6$. So, there exists $\varphi \in C^\infty(\Sigma)$ with

$$-\int_\Sigma \varphi \mathrm{L}\varphi d\Sigma < 0 \quad \text{and} \quad -\int_\Sigma \varphi \mathrm{L}\psi_i d\Sigma = 0, \text{ for all } i = 1, \ldots, 5.$$

Choose vector field X on S^3 with $X = \varphi N$ on Σ, and let Γ_s be its flow. Define $f(v,t,s) = \text{area}(\Gamma_s(\Sigma_{(v,t)}))$, for $|v| < \delta, |t| < \delta$ and $|s| < \delta$, then

$$f(0,0,0) = \text{area}(\Sigma), \quad Df(0,0,0) = 0, \quad D^2 f(0,0,0) < 0,$$

and then f has a strict local maximum at the origin. Replace $\Sigma_{(v,t)}$ with $\Sigma'_{(v,t)} = \Gamma_{\beta(v,t)}(\Sigma_{(v,t)})$, where $\beta(v,t)$ is chosen satisfying $\text{area}(\Sigma'_{(v,t)}) < \text{area}(\Sigma)$, if $|(v,t)| < \delta$ and $\Sigma'_{(v,t)} = \Sigma_{(v,t)}$, if $|(v,t)| \geq \delta$. Therefore,

$$\text{area}(\Sigma'_{(v,t)}) < \text{area}(\Sigma), \quad (v,t) \in B^4 \times [-\pi, \pi]. \tag{12}$$

Recall the equality in $\text{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma)$ implies $t = 0$ and Σ_v is minimal.

Remark. Σ minimal $\Rightarrow \text{area}(\Sigma_v) = \text{area}(\Sigma) - 4\int \frac{\langle v, N(x)\rangle^2}{|x-v|^4} d\Sigma$.

Since $\Sigma_0 = \Sigma$ is minimal, if Σ_v is minimal, $v \in B^4$, then $\text{area}(\Sigma_v) = \mathcal{W}(\Sigma_v) = \mathcal{W}(\Sigma) = \text{area}(\Sigma)$, and we conclude $\langle v, N\rangle = 0$ along Σ. Because of that, Σ is invariant under flow of $v^\perp$.

From the canonical family we constructed $C(v,t)$ via a blow-up argument and Φ defined on the 5-cube. Analogous considerations from $\Sigma'_{(v,t)}$ gives $C'(v,t)$ and $\Phi' : I^5 \to Z_2(S^3)$, with

$$\sup_{x\in I^5} \mathbf{M}(\Phi'(x)) \leq \sup\{\text{area}(\Sigma'_{(v,t)}) : (v,t) \in \overline{B}^4 \times [-\pi,\pi]\} < \text{area}(\Sigma).$$

Apply min-max theory to Π', the homotopy class of Φ', to obtain a minimal surface Σ' so that

$$4\pi < \text{area}(\Sigma') = L(\Pi') < \text{area}(\Sigma) \leq 2\pi^2 < 8\pi,$$

where the $2\pi^2$ estimate is the area of Clifford torus, as a surface in $\mathfrak{F}_1$. The multiplicity of Σ' is 1, otherwise would be $\text{area}(\Sigma') \geq 8\pi$. Also, Σ' has $\text{genus}(\Sigma') \geq 1$, because there is a theorem by Almgren that guarantees the only minimal surfaces of genus $g = 0$ in S^3 are great spheres. Contradiction, since $\Sigma' \in \mathfrak{F}_1$ and $\text{area}(\Sigma') < \text{area}(\Sigma)$. This implies $\text{index}(\Sigma) \leq 5$. By Urbano, Σ is a Clifford torus and Theorem B is proved. □

Theorem A. $\Sigma \subset S^3$ *closed, embedded, genus* $g \geq 1$*. Then* $\mathcal{W}(\Sigma) \geq 2\pi^2$ *and* $\mathcal{W}(\Sigma) = 2\pi^2$ *if and only if* Σ *is Clifford torus up to* $\text{Conf}(S^3)$*.*

Proof. Let $\Phi : I^5 \to Z_2(S^3)$ the min-max family constructed from Σ and Π its homotopy class. Applying min-max theory to Π, we obtain a minimal surface $\widehat{\Sigma}$ with

$$4\pi < \text{area}(\widehat{\Sigma}) = L(\Pi) \leq W(\Sigma).$$

If $\widehat{\Sigma}$ has multiplicity greater than 1, then $\text{area}(\widehat{\Sigma}) \geq 8\pi > 2\pi^2$. If not, the lower bound on the area implies $\text{genus}(\widehat{\Sigma}) \geq 1$. In any case, $\text{area}(\widehat{\Sigma}) \geq 2\pi^2$, and we have

$$W(\Sigma) \geq 2\pi^2.$$

Suppose $W(\Sigma) = 2\pi^2$. Recall $\text{area}(\Sigma_{(v,t)}) \leq W(\Sigma) = 2\pi^2$.

Claim 7. *For some* $(v,t) \in B^4 \times [-\pi,\pi]$*, we have* $\text{area}(\Sigma_{(v,t)}) = W(\Sigma)$*.*

Otherwise, the min-max family Φ constructed from Σ has $L(\Phi) < W(\Sigma) = 2\pi^2$. We can run min-max again and contradic Theorem B.

Because of equality $\text{area}(\Sigma_{(v,t)}) = W(\Sigma)$, we conclude $t = 0$ and Σ_v is a minimal surface. Then $\text{area}(\Sigma_v) = W(\Sigma_v) = W(\Sigma) = 2\pi^2$. By rigidity part of Theorem B, $\Sigma_v = F_v(\Sigma)$ must be isometric to the Clifford torus, with $F_v \in \text{Conf}(S^3)$. □

7. Min-max theory and the energy of links

A 2-component link in $\mathbb{R}^3$ is a pair of rectifiable closed curves $\gamma_i : S^1 \to \mathbb{R}^3$, $i = 1, 2$, so that $\gamma_1(S^1) \cap \gamma_2(S^1) = \emptyset$.

Möbius Cross Energy.

$$\mathrm{E}(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2} dsdt.$$

Freedman-He-Wang(1994).

$$F \in \mathrm{Conf}(\mathbb{R}^3) \Rightarrow \mathrm{E}(F \circ \gamma_1, F \circ \gamma_2) = \mathrm{E}(\gamma_1, \gamma_2).$$

In the same paper they also conjecture

Conjecture: The energy of any non-trivial link should be at least $2\pi^2$.

Example 7.1 (Standard Hopf link). The stereographic projection of $\widehat{\gamma}(s) = (\cos s, \sin s, 0, 0) \in S^3$ and $\tilde{\gamma}(t) = (0, 0, \cos t, \sin t) \in S^3$ is a 2-component link in $\mathbb{R}^3$, called standard Hopf link. Its Möbius energy is given by

$$\mathrm{E}(\widehat{\gamma}, \tilde{\gamma}) = 2\pi^2.$$

He: The minimizer is isotopic to standard Hopf link.

We introduce now the Gauss map and linking number of a link in $\mathbb{R}^3$. Let (γ_1, γ_2) be a 2-component link in $\mathbb{R}^3$. The Gauss map $g = G(\gamma_1, \gamma_2) : S^1 \times S^1 \to S^2$ of (γ_1, γ_2) is defined as

$$g(s, t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|} \in S^2.$$

The linking number of (γ_1, γ_2) is defined to be the degree of the Gauss map, and the notation is $\mathrm{lk}(\gamma_1, \gamma_2)$. This can also be given via the Gauss formula

$$\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\gamma_1'(s), \gamma_2'(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} dsdt.$$

We conclude then the first lower bound to the Möbius energy:

$$\mathrm{E}(\gamma_1, \gamma_2) \geq 4\pi |\mathrm{lk}(\gamma_1, \gamma_2)|.$$

In particular, if $\mathrm{lk}(\gamma_1, \gamma_2) = 1$ or -1, we have $\mathrm{E}(\gamma_1, \gamma_2) \geq 4\pi$.

Theorem 7.2. *If* $(\gamma_1, \gamma_2) \subset \mathbb{R}^3$ *has* $\mathrm{lk}(\gamma_1, \gamma_2) = 1$ *or* -1*, then* $\mathrm{E}(\gamma_1, \gamma_2) \geq 2\pi^2$*. Moreover, if* $\mathrm{E}(\gamma_1, \gamma_2) = 2\pi^2$ *then* (γ_1, γ_2) *is a conformal image of standard Hopf link.*

Remark. Analogy with the Willmore problem: $\Sigma \subset \mathbb{R}^3$ closed surface, genus g, $\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\Sigma$ is conformal invariant. In this case we know

Marques-Neves: $g \geq 1 \Rightarrow \mathcal{W}(\Sigma) \geq 2\pi^2$.

$2\pi^2$ **Theorem.** Let $\Phi : I^5 \to Z_2(S^3)$ a map with the following properties:

(1) Φ is continuous in the flat norm
(2) $\Phi(I^4 \times \{0\}) = \Phi(I^4 \times \{1\}) = 0$
(3) $x \in \partial I^4$, $\{\Phi(x,t)\}_{t\in[0,1]}$ is a standard foliation of S^3 by round spheres centered at $Q(x) \in S^3$
(4) $x \in \partial I^4$, $\Phi(x, 1/2) = \partial B_{\pi/2}(Q(x))$
(5) $Q : \partial I^4 \to S^3$ has $\deg(Q) \neq 0$.

Then, there exists $y \in I^5$ so that $\mathbf{M}(\Phi(y)) \geq 2\pi^2$.

Fix (γ_1, γ_2) with $\mathrm{lk}(\gamma_1, \gamma_2) = -1$. We produce $\Phi : I^5 \to Z_2(S^3)$ satisfying $(1) - (5)$ and $\mathbf{M}(\Phi(x)) \leq \mathrm{E}(\gamma_1, \gamma_2)$, for every $x \in I^5$.

Recall the notation $g = G(\gamma_1, \gamma_2)$ for the Gauss map of the link. The key calculation here is the following estimate of the jacobian of g:

$$|\mathrm{Jac}(g)|(s,t) \leq \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2}.$$

Consider $g_\sharp(S^1 \times S^1) \in Z_2(S^3)$ given by $g_\sharp(S^1 \times S^1)\phi = \int_{S^1\times S^1} g^*\phi$. The key calculation is useful to estimate the mass of this current

$$\mathbf{M}(g_\sharp(S^1 \times S^1)) \leq \int_{S^1\times S^1} |\mathrm{Jac}(g)| ds dt \leq \mathrm{E}(\gamma_1, \gamma_2).$$

We introduce some notation and observation before constructing Φ. Let $p \in \mathbb{R}^4$, $\lambda > 0$ and define $D_{\lambda,p}(x) = \lambda(x - p) + p$, the dilation of λ centered at p. If $(\gamma_1, \gamma_2) \subset S^3_r(p) \subset \mathbb{R}^4$, $\lambda > 0$ and $\tilde{g} = G(\gamma_1, D_{\lambda,p} \circ \gamma_2)$, then $\mathbf{M}(\tilde{g}_\sharp(S^1 \times S^1)) \leq \mathrm{E}(\gamma_1, \gamma_2)$. For every $v \in B^4$, we consider the conformal map $F_v : \mathbb{R}^4 \setminus \{v\} \to \mathbb{R}^4$ given as

$$F_v(x) = \frac{x - v}{|x - v|^2}.$$

Note that

$$F_v(S^3_1(0)) = S^3_{\frac{1}{1-|v|^2}}(c(v)), \quad \text{where} \quad c(v) = \frac{v}{1 - |v|^2}.$$

Finally, for every $v \in B^4$ and $z \in (0, 1)$ take

$$b(v, z) = \frac{2z - 1}{(1 - |v|^2 + z)(1 - z)} \in (0, \infty)$$

and

$$a(v, z) = 1 + (1 - |v|^2)b(v, z) \in \Big(- \frac{1}{1 - |v|^2}, \infty\Big).$$

Definition of the family: For every $(v, z) \in B^4 \times (0, 1)$ define

$$C(v, z) = g_{(v,z)\sharp}(S^1 \times S^1) \in Z_2(S^3), \tag{13}$$

where

$$g_{(v,z)} = G(F_v \circ \gamma_1, D_{c(v),a(v,z)} \circ F_v \circ \gamma_2). \tag{14}$$

By construction, we know the masses of this family has an upper bound $\mathbf{M}(C(v,z)) \leq \mathrm{E}(\gamma_1, \gamma_2)$, for every $(v,z) \in B^4 \times (0,1)$.

Remark. $C(v,0) = 0 = C(v,1)$.

Need to extend C to $\overline{B}^4 \times [0,1]$ and determine the boundary behaviour. Fix $v \in S^3 \setminus (\gamma_1(S^1) \cup \gamma_2(S^1))$ and observe that

$$F_v(S^3_1(0) \setminus \{v\}) = P_v = \{x \in \mathbb{R}^4 : \langle x, v \rangle = -1/2\}. \tag{15}$$

If $x \in \mathbb{R}^4 \setminus \{v\}$ and $w \in B^4 \to v \in S^3$, then

$$D_{a(w,z),c(w)} \circ F_w(x) = a(w,z)(F_w(x) - c(w)) + c(w) \to F_v(x) - b(z)v,$$

where $b(z) = (2z-1)/z(1-z) \in (-\infty, \infty)$.

The Gauss map $G(F_v \circ \gamma_1, F_v \circ \gamma_2)$ parametrizes $\partial B_{\pi/2}(-v)$ with degree -1, so we can write

$$C(v, 1/2) = G(F_v \circ \gamma_1, F_v \circ \gamma_2)_\sharp(S^1 \times S^1) = \mathrm{lk}(\gamma_1, \gamma_2) \partial B_{\pi/2}(-v) = \partial B_{\pi/2}(v).$$

Then the center map has degree 1. The problem we have to deal with is that $C(v,z)$ does not parametrize a round sphere if $z \neq 1/2$. But it is contained in a hemisphere, then it is possible to take a length decreasing retraction $R(v, \lambda, t)$ onto $\partial B_{r(z)}(v)$. By doing this it is possible to apply the $2\pi^2$-Theorem.

References

[1] F. Codá Marques and A. Neves, *Min-max theory and the Willmore conjecture.* To appear in Annals of Mathematics.

[2] I. Agol, F. Codá Marques and A. Neves, *Min-max theory and the energy of links.* preprint arXiv:1205.0825 [math.GT]