Min-max minimal hypersurfaces in noncompact manifolds
Rafael Montezuma
Oberwolfach - Geometrie 2016

This is a report on a talk given in the Oberwolfach conference Geometrie on the summer of 2016. The main result mentioned in that talk was a theorem on existence of closed, embedded, smooth minimal hypersurfaces in certain noncompact spaces.

Minimal surfaces, the extremizers of the area functional, are among the most important topics in differential geometry. Euler and Lagrange were the first to consider minimal surfaces, proving that if the graph of a C^2 function u is minimal in the Euclidean space, then u satisfies a second order elliptic quasilinear partial differential equation. Later, Meusnier discovered a geometric characterization for these surfaces, by the vanishing of the mean-curvature. These two points of view explain why minimal surfaces are natural objects of study in geometric analysis.

Henceforth, many mathematicians contributed to the development of the theory and applied its ideas to settle deep problems and establish beautiful results in geometry. For instance, minimal surfaces have a strong link with problems involving the scalar curvature of three-manifolds, the proof of the positive mass conjecture in general relativity by Schoen and Yau [20] is one of the most important examples of this connection. Some other important recent works using minimal surfaces are the proof of the finite time extinction of the Ricci flow with surgeries starting at a homotopy 3-sphere by Colding and Minicozzi [6] and [7], the proof of the Willmore Conjecture and the existence of infinitely many closed minimal hypersurfaces in closed manifolds with positive Ricci curvature by Marques and Neves [11] and [12].

The existence of minimal submanifolds with some specific properties plays a fundamental role in the development of the theory. The most natural way to produce minimal surfaces is by minimizing the area functional in a fixed class. This idea was applied in many contexts: in a class of surfaces with same boundary, known as the Plateau's Problem; or in a homology class; or even in the class of surfaces $\Sigma \subset \Omega$, with $\partial\Sigma \subset \partial\Omega$, the free boundary problem. It was proved, by Schoen and Yau [19], Sacks and Uhlenbeck [17] and Freedman, Hass and Scott [10], that if Σ is a closed incompressible surface in a 3-manifold M, then there is a least area immersion with the same action in the π_1 level.

In the topology of 3-manifolds, there are two important types of surfaces, namely the Heegaard splittings and the incompressible surfaces. The results above show that this second type occur as minimal surfaces produced by minimization processes, and for this reason, they are stable, i.e., the Morse index is equal to zero. It is also possible to apply variational methods to construct higher index minimal surfaces. There are two basic approaches: applying Morse theory to the energy functional on the space of maps from a fixed surface, such as in the works of Sacks and Uhlenbeck [16], Micallef and Moore [13] and Fraser [9], or via a min-max argument for the area functional over classes of sweepouts. In some cases, these methods can be applied to realize Heegaard splitings as embedded minimal surfaces, see [15] and [5].

This min-max technique was inspired by the work of Birkhoff [4] on the existence of simple closed geodesics in Riemannian 2-spheres, a question posed by Poincaré. Looking for closed geodesics, he interpreted the geometric point of view suggested by the Principle of Least Action as the iteration of a specific curve shortening process. This process is continuous with respect to variations of the beginning curve, so it can be applied to whole families of closed curves sweeping out a given Riemannian 2-sphere at the same time. Considering the longest curve in the family after each step of the shortening process, a subsequence of these converges to a closed geodesic.

In higher dimensions, the original method was introduced in [2] and [14] between the 1960's and 1980's. It has been used recently by Marques and Neves to answer deep questions in geometry, see [11] and [12]. The method consists of applications of variational techniques for the area functional. It is a powerful tool in the production of unstable minimal surfaces in closed manifolds. For instance, Marques and Neves, in the proof of the Willmore conjecture, proved that the Clifford Torus in the three-sphere is a min-max minimal surface. The min-max technique for the area functional appear also in a different setting, as introduced by Simon and Smith [18], and revisited by Colding and De Lellis [5].

In the presented talk, we dealt with a new min-max construction of minimal hypersurfaces and applied the technique to obtain existence results in noncompact manifolds.

There is no immersed closed minimal surface in the Euclidean space $\mathbb{R}^3$. This fact illustrates the existence of simple geometric conditions creating obstructions for a Riemannian manifold to admit closed minimal surfaces. In the Euclidean space, we can see the obstruction coming in the following way: by the Jordan-Brouwer separation theorem every connected smooth closed surface $\Sigma^2 \subset \mathbb{R}^3$ divides $\mathbb{R}^3$ in two components, one of them bounded, which we denote Ω. Start contracting a large Euclidean ball containing Ω until it touches Σ the first time. Let $p \in \Sigma$ be a first contact point, then the maximum principle says that the mean curvature vector of Σ at p is non-zero and points inside Ω. In particular, $\Sigma^2 \subset \mathbb{R}^3$ is not minimal.

The main result presented in that talk was:

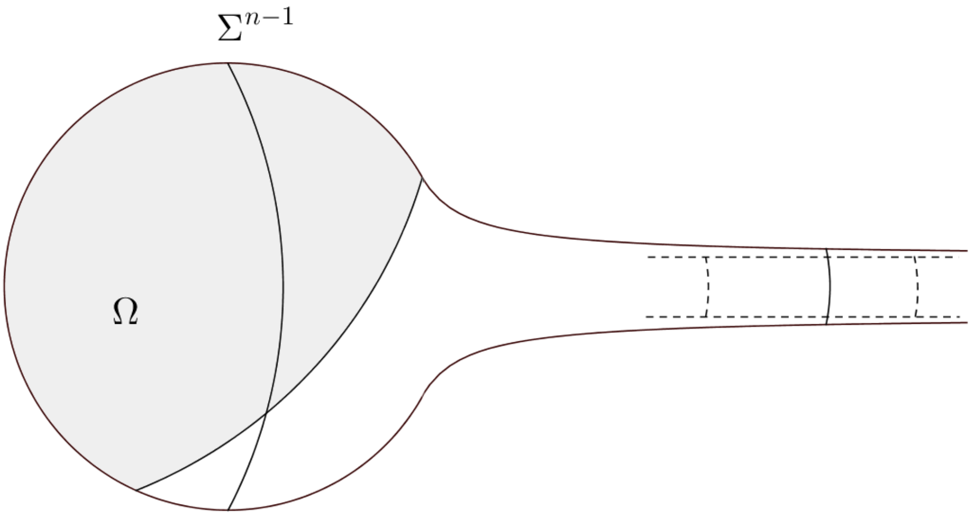

Figure 1: A complete non-compact Riemannian manifold, asymptotic to a cylinder and containing a mean-concave open set Ω. In this case, Theorem 1 could be applied.

Theorem 1. *Let (N^n, g) be a complete non-compact Riemannian manifold of dimension $n \leq 7$. Suppose:*

- *N contains a bounded open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary, and*
- *N is thick at infinity.*

Then, there exists a closed embedded minimal hypersurface $\Sigma^{n-1} \subset N$ that intersects Ω.

The thickness assumption can be loosely phrased as: "the decay of the geometric objects at infinity is at most polynomial". For instance, manifolds of bounded geometry and manifolds with ends asymptotic to right cylinders are thick at infinity, see figure 1.

In the recent paper [8], Collin, Hauswirth, Mazet and Rosenberg prove that any complete non-compact hyperbolic three-dimensional manifold of finite volume admits a closed embedded minimal surface. These manifolds have a different behavior at infinity from those considered in Theorem 1, their ends are all thin hyperbolic cusps.

The hypothesis involving the mean-concave bounded domain Ω comes from the theory of closed geodesics in non-compact surfaces. In 1980, Bangert proved the existence of infinitely many closed geodesics in a complete Riemannian surface M of finite area and homeomorphic to either the plane, or the cylinder or the Möbius band, see [3]. The first step in his argument is to prove that the finite area assumption implies the existence of locally convex neighborhoods of the ends of M.

To prove the Theorem 1, we developed a min-max method that is adequate to produce minimal hypersurfaces with intersecting properties. Let (M^n, g) be a closed Riemannian manifold and Ω be an open subset of M. Consider a homotopy class Π of one-parameter families of codimension-one submanifolds sweeping M out. For each given sweepout $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we consider the number

$$L(S, \Omega) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \Sigma_t \cap \overline{\Omega} \neq \varnothing\},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated with the Riemannian metric. Define the width of Π with respect to Ω to be

$$L(\Pi, \Omega) = \inf\{L(S, \Omega) : S \in \Pi\}.$$

Then, we prove:

Theorem 2. *Let (M^n, g) be a closed Riemannian manifold, $n \leq 7$, and Π be a non-trivial homotopy class of sweepouts. Suppose that M contains an open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary. There exists a stationary integral varifold Σ whose support is a smooth embedded closed minimal hypersurface intersecting Ω and with $||\Sigma||(M) = \boldsymbol{L}(\Pi, \Omega)$.*

The intersecting condition in Theorem 2 is optimal in the sense that it is possible that the support of the minimal surface Σ is not entirely in $\overline{\Omega}$. We illustrate this with two examples of mean-concave subsets of the unit three-sphere $S^3 \subset \mathbb{R}^4$ containing no great sphere. The first example is the complement of three spherical geodesic balls, which can be seen in Figure 2.

In order to introduce the second example, for each $0 < t < 1$, consider the subset of S^3 given by

$$\Omega(t) = \{(x, y, z, w) \in S^3 : x^2 + y^2 > t^2\}.$$

It is not hard to see that no $\Omega(t)$ contain great spheres. Moreover, the boundary of $\Omega(t)$ is a constant mean-curvature torus in S^3. If $0 < t < 1/\sqrt{2}$, the mean-curvature vector of $\partial\Omega(t)$ points outside $\Omega(t)$. In this case, $\Omega = \Omega(t)$ is a mean-concave subset of S^3 that contains no great sphere.

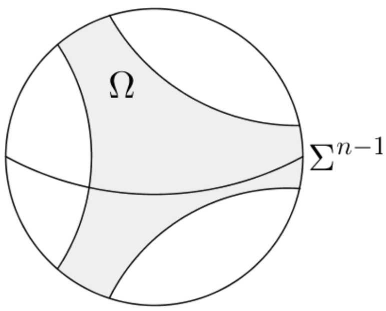

Figure 2: Example of mean-concave domain $\Omega \subset S^3$ for which Σ^{n-1} given by Theorem 2 is not entirely inside Ω.

Bibliography

[1] Almgren, F., *The homotopy groups of the integral cycle groups.* Topology (1962), 257–299.

[2] Almgren, F., *The theory of varifolds.* Mimeographed notes, Princeton (1965).

[3] Bangert, V., *Closed geodesics on complete surfaces.* Math. Ann. 251 (1980), no. 1, 83-96.

[4] Birkhoff, G. D., *Dynamical systems with two degrees of freedom.* Trans. Amer. Math. Soc. 18 (1917), no. 2, 199–300.

[5] Colding, T. and De Lellis, C., *The min-max construction of minimal surfaces,* Surveys in Differential Geometry VIII , International Press, (2003), 75–107.

[6] Colding, T. and Minicozzi, W., *Estimates for the extinction time for the Ricci flow on certain* 3*-manifolds and a question of Perelman.* J. Amer. Math. Soc. 18 (2005), 561–569.

[7] Colding, T. and Minicozzi, W., *Width and finite extinction time of Ricci flow.* Geom. Topol. 12 (2008), no. 5, 2537–2586.

[8] Collin, P., Hauswirth, L., Mazet, L. and Rosenberg, H., *Minimal surfaces in finite volume non compact hyperbolic* 3*-manifolds.* preprint, arXiv:1405.1324 [math.DG]

[9] Fraser, A., *On the free boundary variational problem for minimal disks.* Comm. Pure Appl. Math., 53 (8) (2000) 931–971.

[10] Freedman, M., Hass, J. and Scott, P., *Least area incompressible surfaces in 3-manifolds.* Invent. Math., 71 (3) (1983) 609–642.

[11] Marques, F. C. and Neves, A., *Min-max theory and the Willmore conjecture.* Ann. of Math. (2) 179 (2014), no. 2, 683–782.

[12] Marques, F. C. and Neves, A., *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature.* preprint, arXiv:1311.6501 [math.DG]

[13] Micallef, M. and Moore, J., *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes.* Ann. of Math. (2), 127(1) : 199–227, (1988).

[14] Pitts, J., *Existence and regularity of minimal surfaces on Riemannian manifolds,* Mathematical Notes 27, Princeton University Press, Princeton, (1981).

[15] Pitts, J. and Rubinstein, J., *Applications of minimax to minimal surfaces and the topology of* 3*-manifolds.* In Miniconference on geometry and partial differential equations, 2 (Canberra, 1986), volume 12 of Proc. Centre Math. Anal. Austral. Nat. Univ., p. 137-170. Austral. Nat. Univ., Canberra, 1987.

[16] Sacks, J. and Uhlenbeck, K., *The existence of minimal immersions of* 2*-spheres.* Ann. of Math. (2), 113 (1) : 1–24, (1981).

[17] Sacks, J. and Uhlenbeck, K., *Minimal immersions of closed Riemann surfaces.* Trans. Amer. Math. Soc. 271 (1982), no. 2, 639-652.

[18] Smith, F., *On the existence of embedded minimal 2-spheres in the 3–sphere, endowed with an arbitrary Riemannian metric*, supervisor L. Simon, University of Melbourne (1982).

[19] Schoen, R. and Yau, S.T., *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds of non-negative scalar curvature.* Ann. of Math. 110 (1979), 127–142.

[20] Schoen, R. and Yau, S.T., *On the proof of the positive mass conjecture in general relativity.* Comm. Math. Phys. 65, 45–76 (1979).