Min-max theory for noncompact manifolds and three-spheres with unbounded widths

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pura e aplicada

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- Many others: Ketover, Pellandini, Tasnady, Li, Zhou, ...

Part 1: Minimal hypersurfaces in noncompact manifolds

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Need extra assumptions!

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  (Bangert, 1980)

Our assumptions: Let (N^n, g) be a complete non-compact Riemannian manifold.

- Bangert's condition: complete Riemannian surfaces of finite area have **bounded, mean-concave subsets**.
- We say that N has the $\star_k$**-condition** if there exist $p \in N$ and $R_0 > 0$, such that, for every $R \geq R_0$,

$$\sup_{q \in B(p,R)} |\mathrm{Sec}_N|(q) \leq R^k \quad \text{and} \quad \inf_{q \in B(p,R)} inj_N(q) \geq R^{-\frac{k}{2}}.$$

Main Result

Theorem A (—)

Let (N^n, g) be a complete non-compact Riemannian manifold of dimension $n \leq 7$. Suppose:

- N has a bounded and strictly mean-concave open subset Ω;
- N satisfies the $\star_k$-condition, for some $k \leq \frac{2}{n-2}$.

Then, there exists a closed embedded minimal hypersurface $\Sigma^{n-1} \subset N$ that intersects Ω.

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Remarks:

- Σ^{n-1} is a min-max minimal hypersurface.
- Collin, Hauswirth, Mazet and Rosenberg proved that any complete non-compact hyperbolic three-dimensional manifold of finite volume admits a closed, embedded minimal surface.

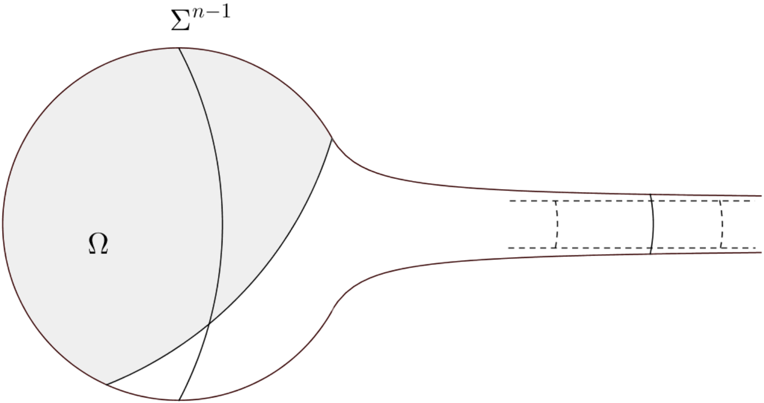
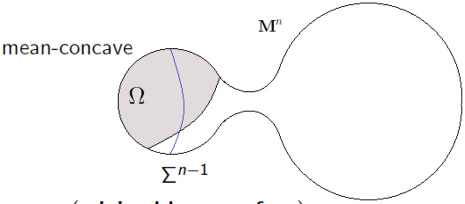


Figure: A non-compact Riemannian manifold, asymptotic to a cylinder and containing a mean-concave open set Ω. In this case, the theorem could be applied.

Theorem B (—)



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(possibly disconnected and with multiplicities)

$$||\Sigma||(M) = \mathbf{L}(\Pi, \Omega)$$

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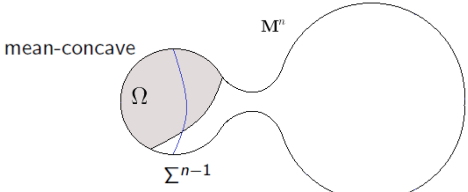

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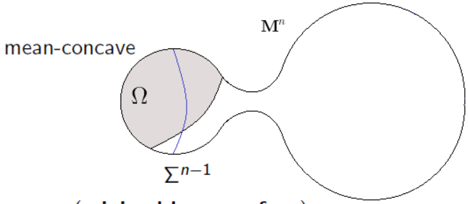

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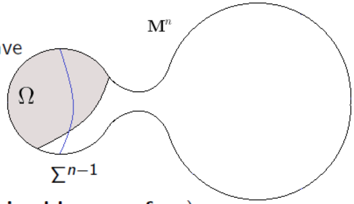

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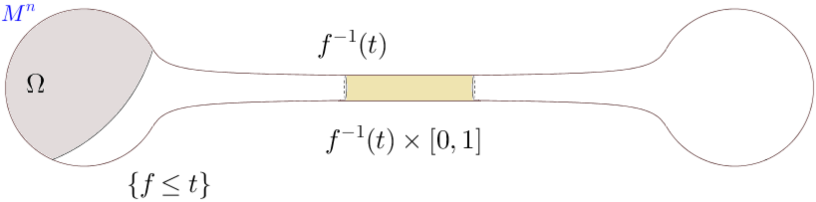
Remarks:

- $\mathbf{L}(\Pi, \Omega)$ is the new min-max invariant.
- $\mathbf{L}(\Pi, M) = \mathbf{L}(\Pi)$.
- Optimal with respect to the intersecting property.

(1) Let $f : N \to [0, \infty]$ be a Morse function and $t > 0$ such that $\Omega \subset \{f \leq t\}$. Let M^n be a closed manifold containing an isometric copy of $\{f \leq t\}$.



Proof of Theorem A

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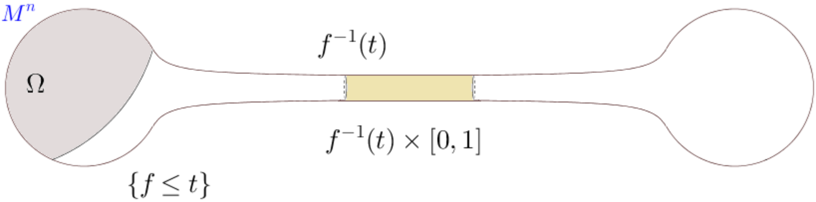


(2) Apply **Theorem B** to obtain a closed, embedded, minimal hypersurface $\Sigma^{n-1} \subset M$, that intersects Ω and with

$$\mathcal{H}^{n-1}(\Sigma) \leq \mathbf{L}(\Pi, \Omega).$$

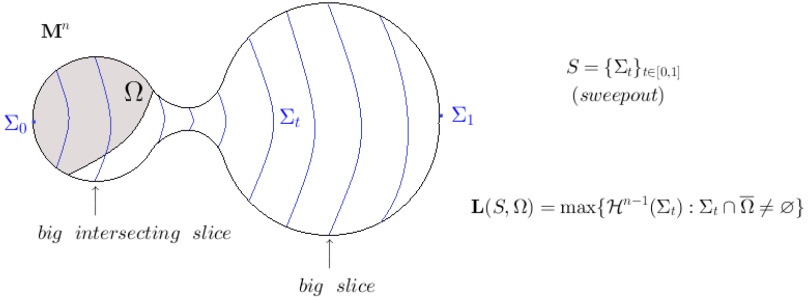
Moreover, $\mathbf{L}(\Pi, \Omega)$ has an upper bound which does not depend on t.

(3) If t is large, the **polynomial decay of the geometry at infinity** implies that

$$\Sigma \subset \{f \leq t\}.$$

In particular, Σ is a minimal hypersurface in the original manifold N.

Sweepout: Continuous one-parameter family of closed, oriented hypersurfaces (possibly with finitely many singularities).



Width with respect to Ω: Let Π be a homotopy class of sweepouts of M. Define:

$$\mathbf{L}(\Pi,\Omega) = \inf\{\mathbf{L}(S,\Omega) : S \in \Pi\}.$$

1. (Existence of critical sequences)
We can find $\mathfrak{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ such that
$$\lim_{k\to\infty} \mathbf{L}(S^k,\Omega) = \mathbf{L}(\Pi,\Omega)$$
and
$$\sup\{\mathcal{H}^{n-1}(\Sigma) : \Sigma \text{ is a slice of } S^k, \text{ for some } k \in \mathbb{N}\} < \infty.$$

Proof of Theorem B

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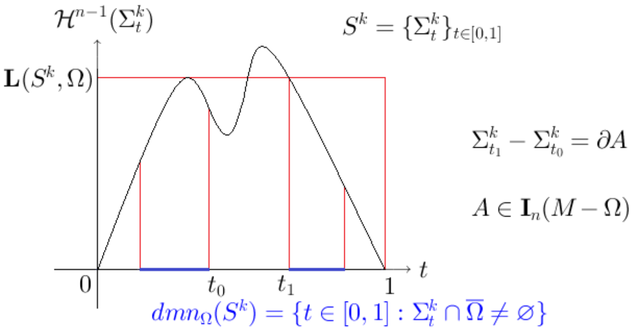

Lemma

We can find a continuous path

$$\phi : [0,1] \to \mathcal{Z}_{n-1}(M - \Omega)$$

such that

(i) $\phi(0) = \partial A$ and $\phi(1) = 0$;

(ii) $\mathcal{H}^{n-1}(\phi(t)) \leq C \cdot \mathcal{H}^{n-1}(\partial A)$, for every $t \in [0,1]$,

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Idea: In $M - \Omega$, we use the gradient flow of a Morse function without interior local maxima.

Min-max sequences and Critical sets

Let $\mathfrak{S} = \{S^k\}_{k \in \mathbb{N}} \subset \Pi$ be a critical sequence.

Min-max sequence: sequence of intersecting slices $\Sigma_{t_j}^{k_j}$ of S^{k_j} satisfying

$$\lim_{j \to \infty} \mathcal{H}^{n-1}(\Sigma_{t_j}^{k_j}) = \mathbf{L}(\Pi, \Omega).$$

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2. Can find $\mathfrak{S} \in \Pi$ critical and such that

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3. Given $\mathfrak{S} \in \Pi$ as in the previous item, we can find $\Sigma \in \mathcal{C}(\mathfrak{S}, \Omega)$ such that

$$\operatorname{spt}(\Sigma) \cap \Omega \neq \varnothing$$

and

$$\Sigma \text{ is } \textit{almost minimizing in small annuli.}$$

4. Apply *Pitts' Regularity Theorem*

$$(\text{stationary}) + (\text{a.m. in small annuli}) \Rightarrow \text{smooth}.$$

Existence of intersecting a.m. varifolds

- Suppose that no $V \in \mathcal{C}(\mathfrak{S}, \Omega)$ satisfies our assumption. Then, big intersecting slices of $\mathfrak{S}$ are close either
  (1) to intersecting non-almost minimizing varifolds
  (2) or to non-intersecting varifolds.
- We deform S^k to obtain better competitors $\tilde{S}^k \in \Pi$ such that

$$\mathbf{L}(\tilde{S}^k, \Omega) < \mathbf{L}(S^k, \Omega) - \rho,$$

  for some uniform $\rho > 0$.
- Therefore, $\mathbf{L}(\tilde{S}^k, \Omega) < \mathbf{L}(\Pi, \Omega)$ for large k. Contradiction!

The deformation from S^k to $\tilde{S}^k$ is done in two steps.

If V is type (1), there exist

small annuli $a(V)$ and $\varepsilon(V) > 0$ such that:

Given: $\Sigma_t^k \in S^k$ close to V and $\eta > 0$,
Can find: $\{\Sigma_t^k(s)\}_{s \in [0,1]}$ with the following properties:

- continuous
- $\Sigma_t^k(0) = \Sigma_t^k$
- $\Sigma_t^k(s) \cap (M - a(V)) = \Sigma_t^k \cap (M - a(V))$
- $\mathcal{H}^{n-1}(\Sigma_t^k(s)) \leq \mathcal{H}^{n-1}(\Sigma_t^k) + \eta$
- $\mathcal{H}^{n-1}(\Sigma_t^k(1)) < \mathcal{H}^{n-1}(\Sigma_t^k) - \varepsilon(V)$.

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Remark: Pitts had only type (1) varifolds and he performed the desired deformation using the $\Sigma_t^k(s)$'s.

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Let $U \subset \Omega$ be an open set, with $\overline{U} \subset \Omega$ and such that $\Omega - U$ is inside a small tubular neighborhood of $\partial\Omega$.

If Σ_t^k is also close to a type (2) varifold, then it has small volume in U. Making $a(V)$ small enough, we have that

$$\mathcal{H}^{n-1}(\Sigma_t^k(s) \cap \overline{U}) \quad \textbf{is small} \text{ for every } s \in [0,1].$$

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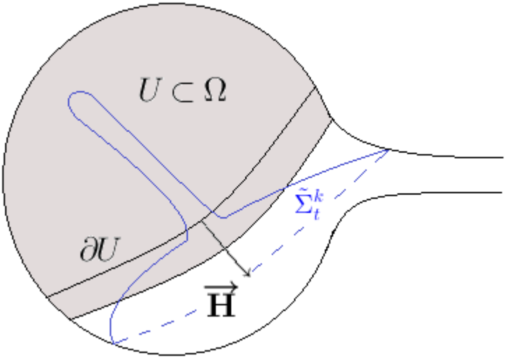
We obtain critical $\tilde{\mathfrak{S}} = \{\tilde{S}^k\}_{k\in\mathbb{N}}$ such that **the big intersecting slices $\tilde{\Sigma}_t^k$ have small mass inside $\overline{U}$.**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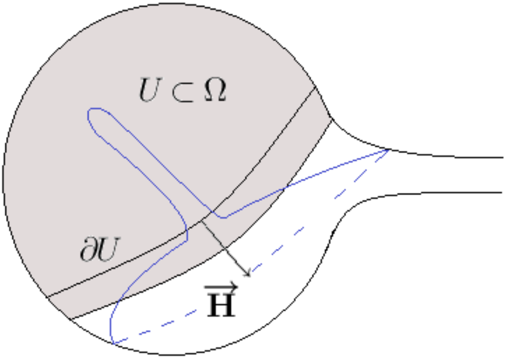
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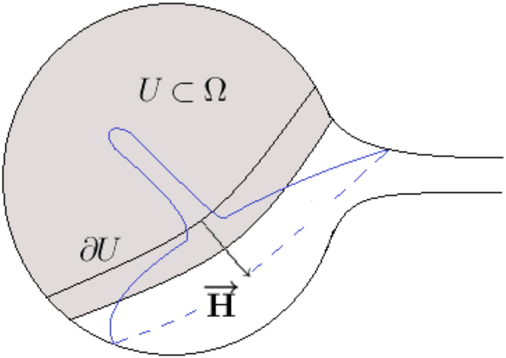
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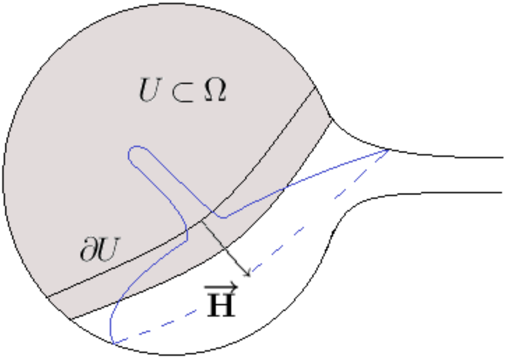
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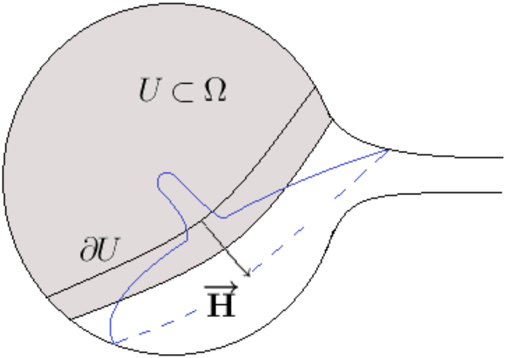

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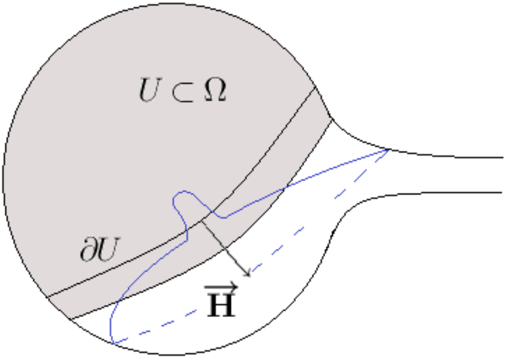
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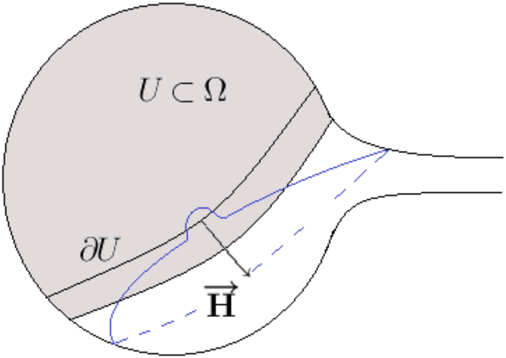
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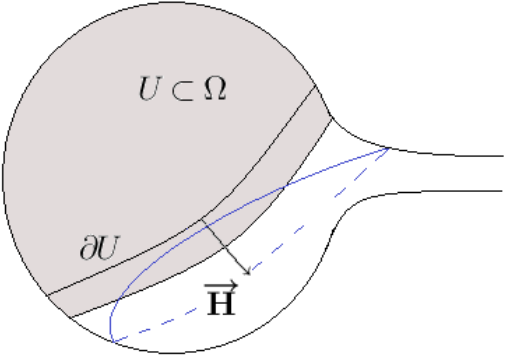
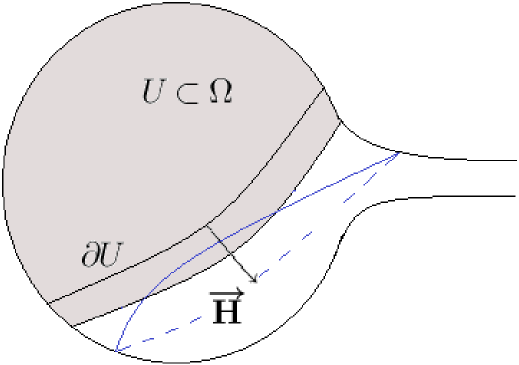


Deformation used by Pitts in the construction of replacements.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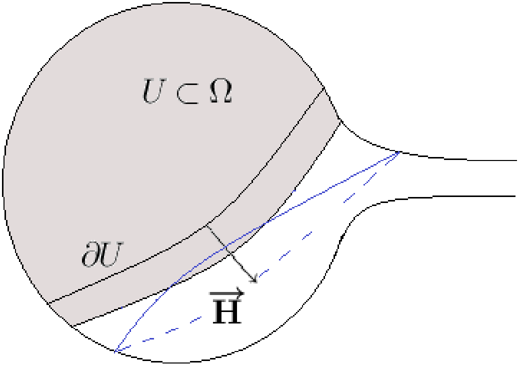
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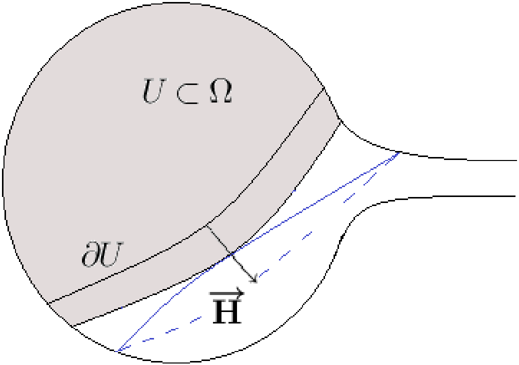
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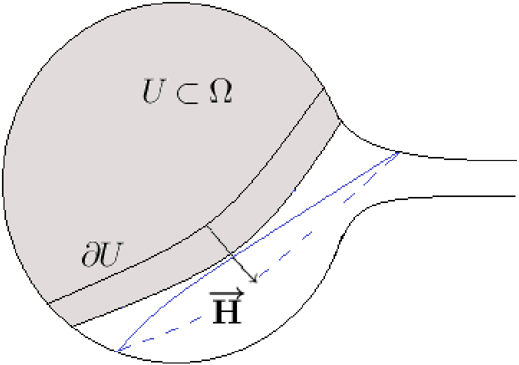
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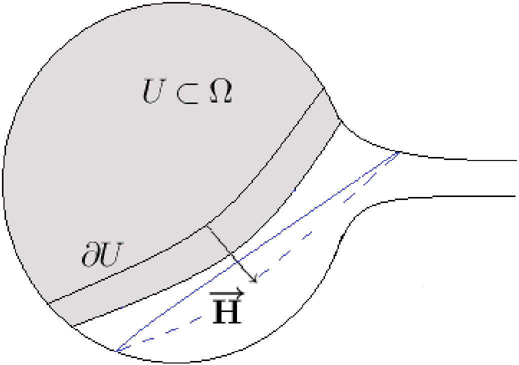
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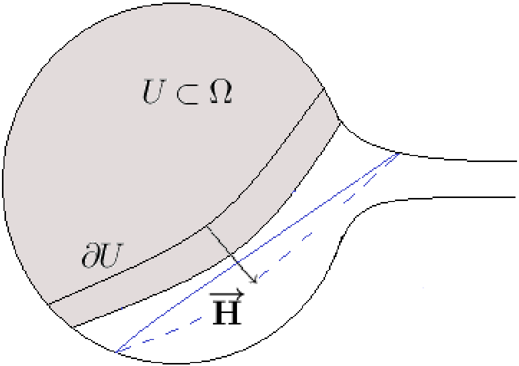

White's maximum principle for general varifolds.

- (S^3, g) with positive Ricci curvature in S^3 and scalar curvature $R \geq 6$, satisfies the upper bound

$$W(S^3, g) \leq 4\pi$$

and there exists an embedded minimal sphere Σ, of index one and surface area

$$area_g(\Sigma) = W(S^3, g).$$

In case of equality $W(S^3, g) = 4\pi$, the metric g has constant sectional curvature one.

(Marques - Neves)

Part 2: Motivation

- (S^3, g) with positive Ricci curvature in S^3 and scalar curvature $R \geq 6$, satisfies the upper bound

$$W(S^3, g) \leq 4\pi$$

and there exists an embedded minimal sphere Σ, of index one and surface area

$$area_g(\Sigma) = W(S^3, g).$$

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(Marques - Neves)

Question: Is there a scalar curv. rigidity version of this theorem?

Main result

Theorem C (—)

For any $m > 0$ there exists a Riemannian metric g on S^3, with scalar curvature $R \geq 6$ and width $W(S^3, g) \geq m$.

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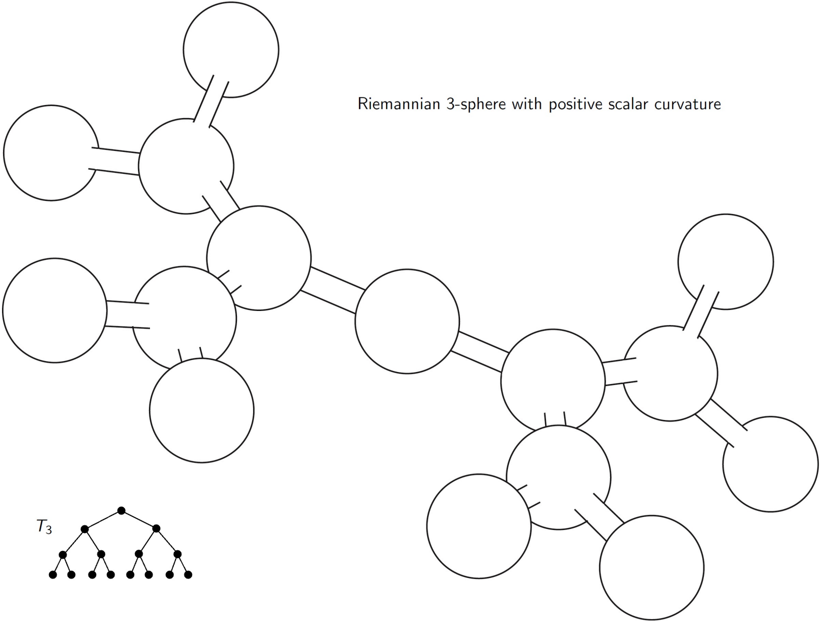
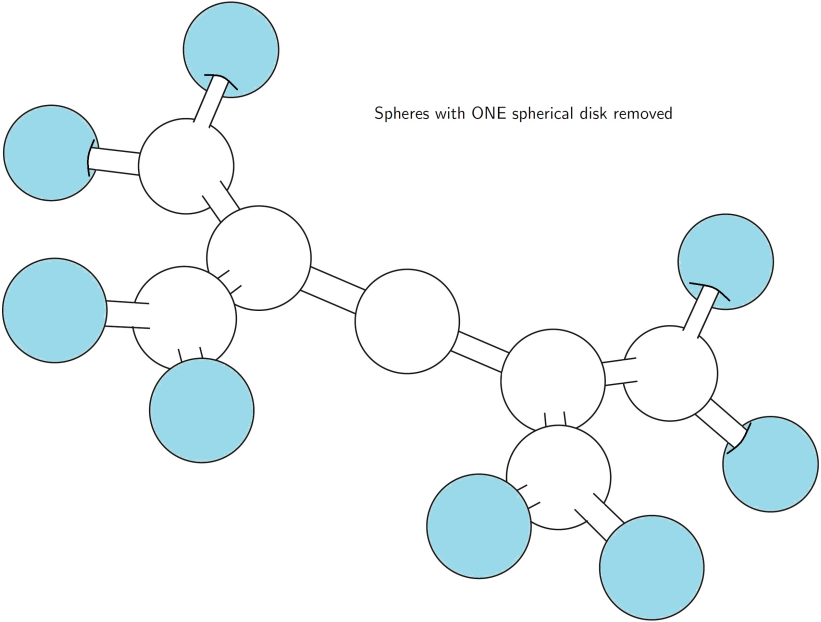
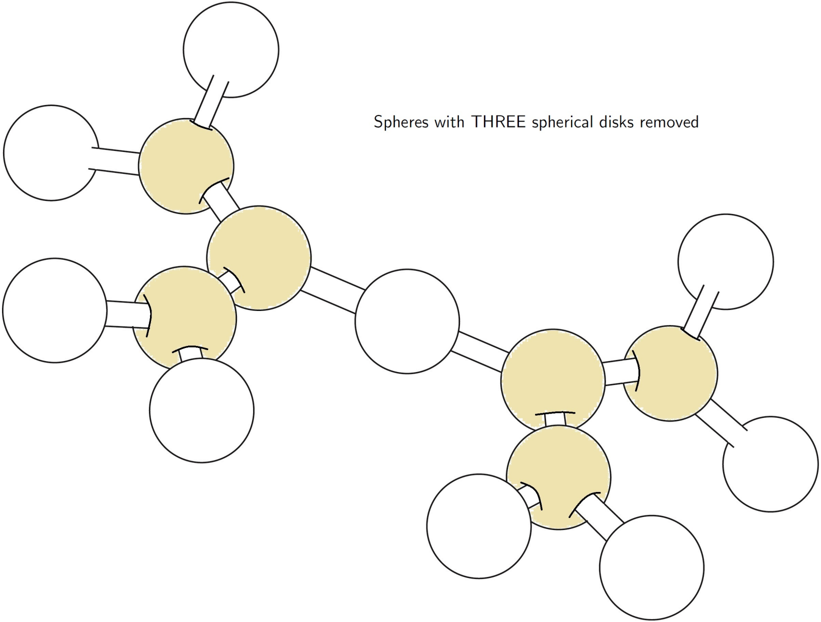
Remark: It is interesting to stress that the Riemann curvature tensors of the examples that we construct are uniformly bounded.

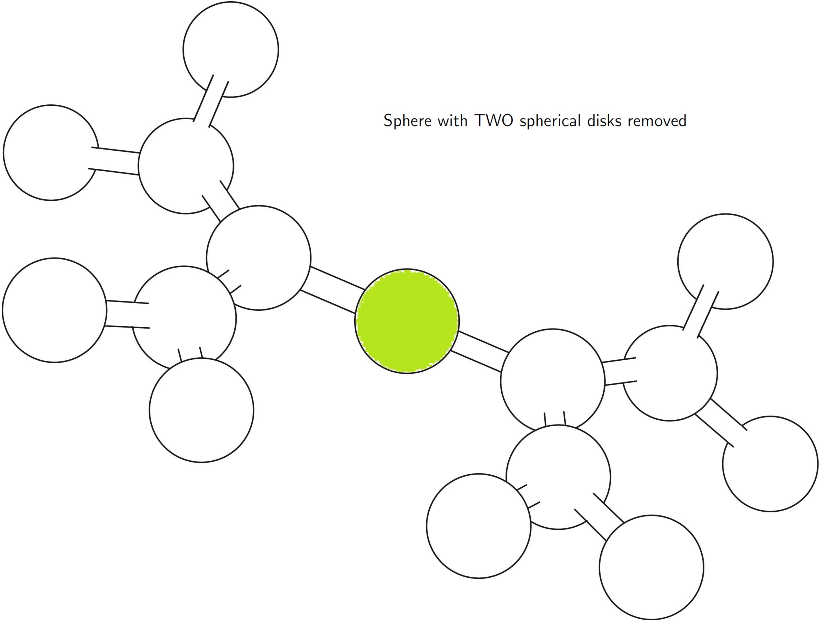
Riemannian 3-sphere with positive scalar curvature




Spheres with ONE spherical disk removed

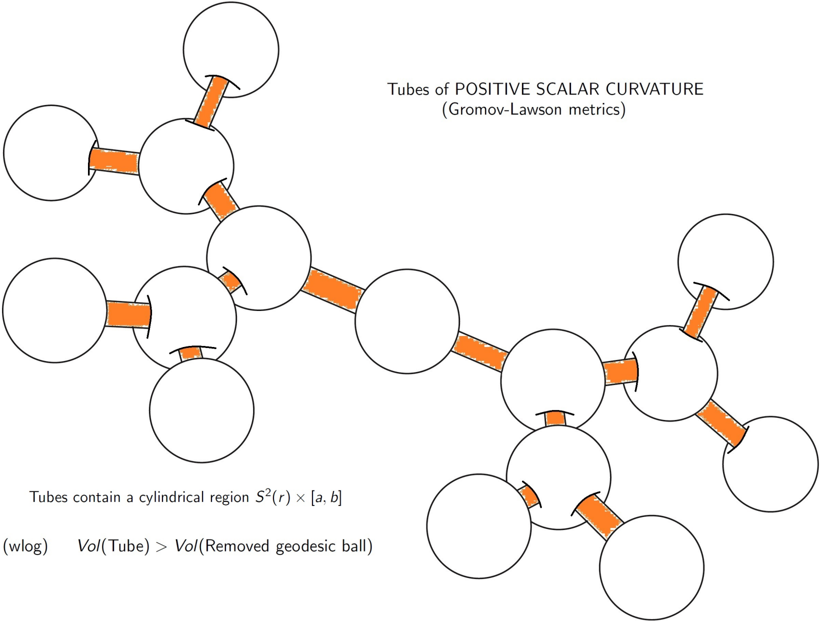
Spheres with THREE spherical disks removed





Sphere with TWO spherical disks removed

Tubes of POSITIVE SCALAR CURVATURE
(Gromov-Lawson metrics)



Tubes contain a cylindrical region $S^2(r) \times [a, b]$

(wlog) $\quad Vol(\text{Tube}) > Vol(\text{Removed geodesic ball})$

Main Idea: Choose $\Omega \subset (S^3, g_m)$ so that we can obtain

$$area(\partial\Omega \cap \mathcal{A}) \geq C, \quad \text{for MANY disjoint regions } \mathcal{A}.$$

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Step 1:("Local lower bounds")

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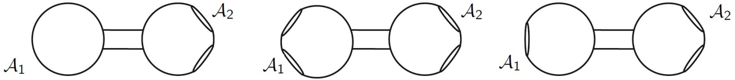

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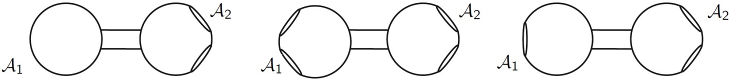
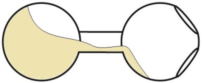


$\mathcal{A} = \mathcal{A}_1 \cup \text{Tube} \cup \mathcal{A}_2$

- Fix $\alpha > 0$. We say a pair $(\mathcal{A}, \alpha)$ is compatible with Ω if



$$\alpha \leq vol(\Omega \cap \mathcal{A}) \leq vol(\mathcal{A}) - \alpha$$
$$\Downarrow$$
Lower bound on $area(\partial\Omega \cap int(\mathcal{A}))$ (depending on $\mathcal{A}$ and α only)

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Need lower bounds for $\mathbf{L}(\{\Sigma_t\})$. (ANY sweepout of S^3 w.r.t. g_m)

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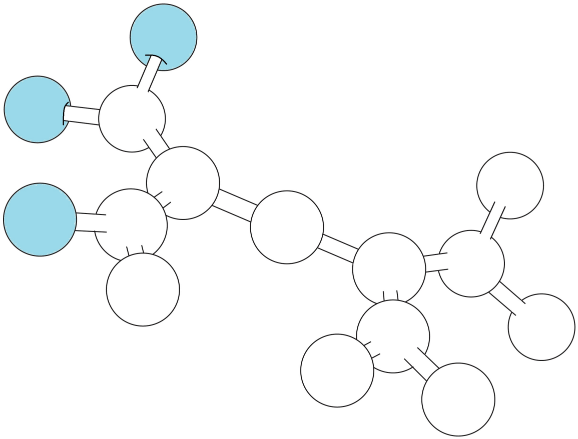
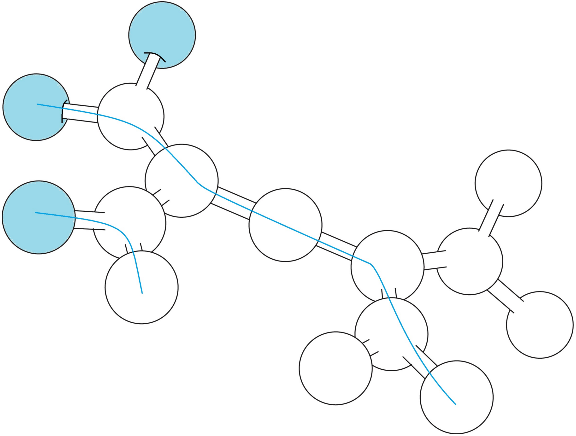
Claim.

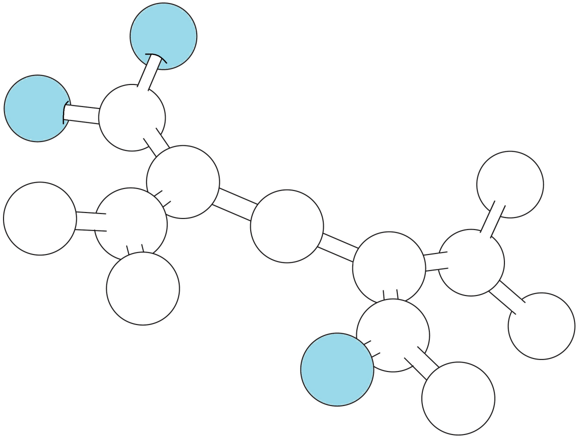
The first Ω_t for which

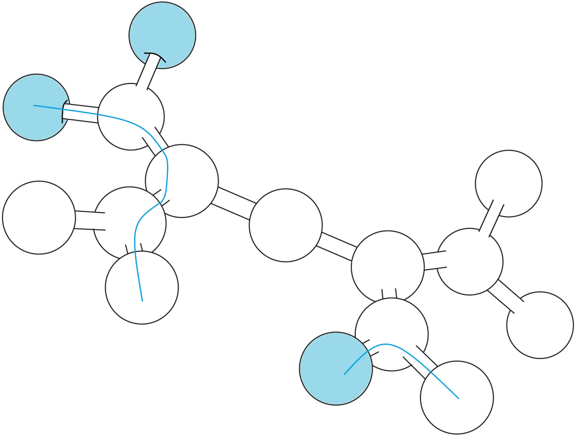
$$vol(\Omega_t \cap L) \geq \alpha, \text{ at } b(m) \text{ "spherical leaves" } L,$$

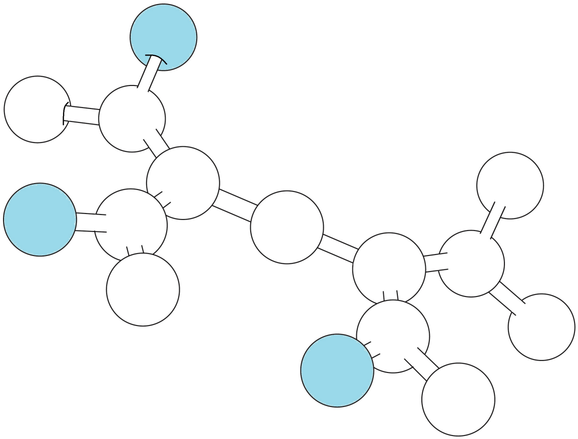
has at least $\frac{1}{5} \cdot \left\lceil \frac{m}{2} \right\rceil$ disjoint compatible pairs $(\mathcal{A}, \alpha)$. □

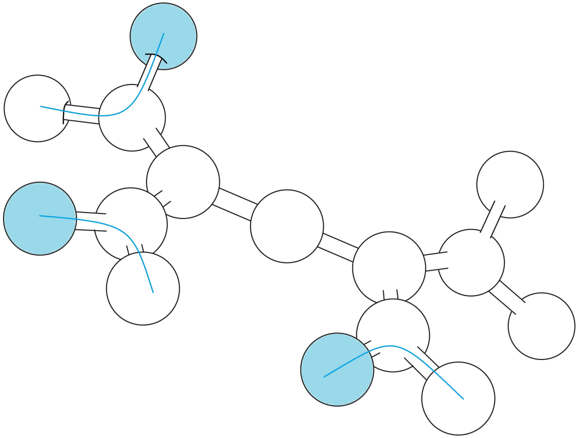












Isoperimetric profiles

Let $\Sigma_t = \partial\Omega_t$ be a sweepout of (S^3, g). For all $v \in [0, vol(S^3, g)]$, there exists $t \in [0, 1]$ so that $vol(\Omega_t) = v$. In particular,

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Theorem

$$\sup_{m\in\mathbb{N}}\left(\sup_v \mathcal{I}_{g_m}(v)\right) = +\infty.$$

The proof is by contradiction. Suppose there exists $L > 0$ so that

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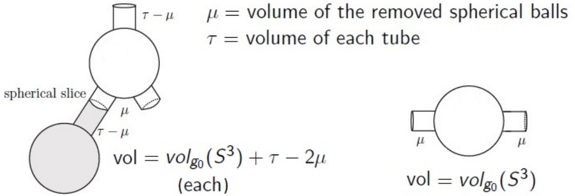

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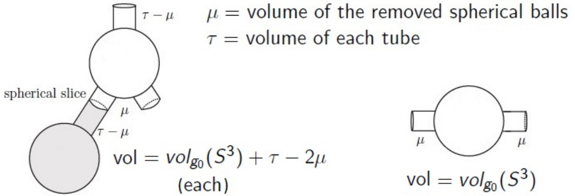

- Choice of Ω:

$$vol_{g_m}(\Omega_t) = \tilde{b}(k) \cdot (vol_{g_0}(S^3) + \tau - 2\mu) \quad \Rightarrow$$

$$\left| \# \left\{ \mathcal{A} : vol(\Omega_t \cap \mathcal{A}) \geq \frac{1}{2} vol(\mathcal{A}) \right\} - \tilde{b}(k) \right| \leq C(\mathcal{A}\text{-regions}, L, \tau, \mu).$$

Combinatorial results

For each $m \in \mathbb{N}$, T_m denotes the full binary tree of 2^m leaves. We consider 2-colorings of the nodes of T_m.

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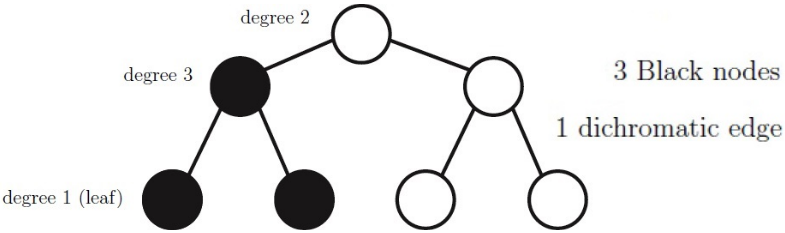

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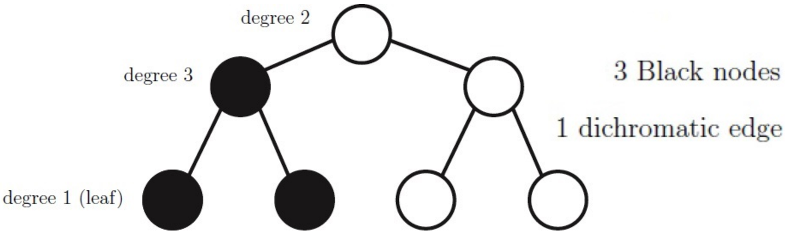

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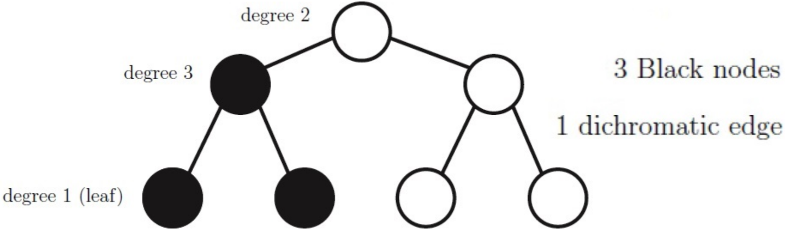

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Yes! (for both, nodes and leaves)

Theorem (prescribed number of black leaves)

Any 2-coloring of T_m with

$$b(m) = \begin{cases} 1 + 2 + 2^3 + \ldots + 2^{m-2}, & \text{if } m \text{ is odd} \\ 1 + 2^2 + 2^4 + \ldots + 2^{m-2}, & \text{if } m \text{ is even,} \end{cases}$$

black leaves exactly has at least

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Given $k \in \mathbb{N}$, there exist $m = m(k)$ and $\tilde{b}(k)$, such that any 2-coloring of T_m with b black nodes exactly has at least

$$\frac{k - |b - \tilde{b}(k)|}{5}$$

pairwise disjoint pairs of neighboring nodes with different colors.

Sketch of proof:(in case of leaves)

First step: Given a 2-coloring $\mathcal{C}$ in T_m with $b(m)$ black leaves exactly, we associate a 2-coloring of T_{m-1}.

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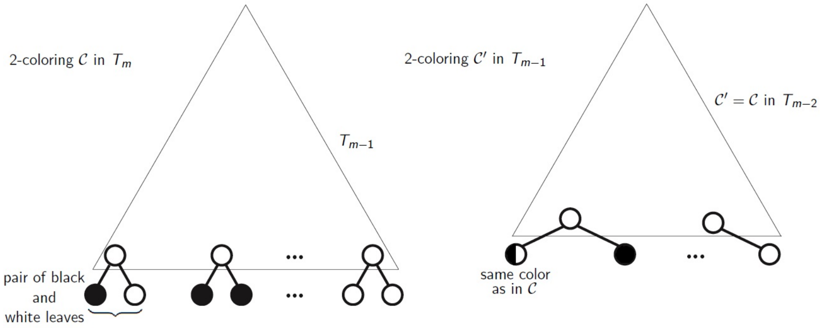

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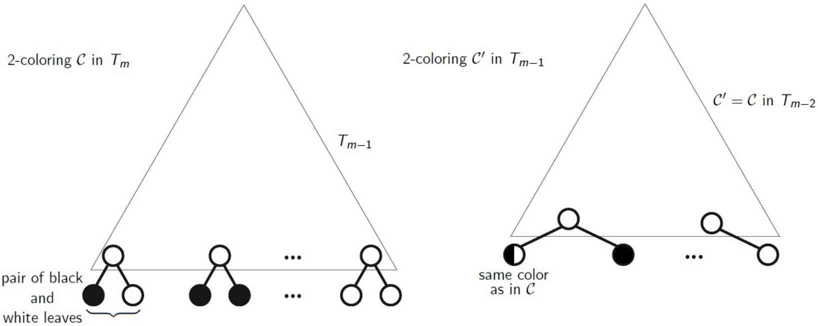

Analyzing $\#\{\text{black leaves of } \mathcal{C}'\}$ settles the induction step in case:

- m is even; or
- there are at least two pairs of black and white leaves of $\mathcal{C}$.

Second step: If m is odd and there is one pair of black and white leaves in $\mathcal{C}$ only, we induce a 2-coloring in T_{m-2} and analyze the size of its set of black leaves.

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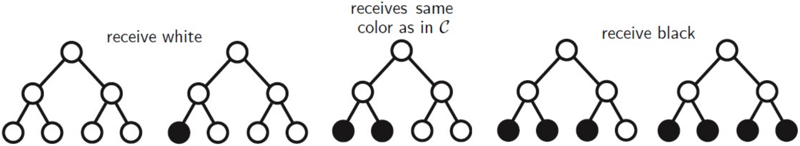

The color induced on each leaf of T_{m-2} depends only on the $\mathcal{C}$-colors of its four associates leaves of T_m.

Obrigado!