Anticyclotomic Euler systems and diagonal cycles II

(joint work with F. Castellà and Ó. Rivero)

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Setting

- $g \in S_l(N_g, \chi_g)$, $h \in S_m(N_h, \chi_h)$ newforms with $l \equiv m \pmod 2$.
- $K/\mathbb{Q}$ imaginary quadratic field of discriminant $-D_K$ and with attached quadratic character ε_K.
- ψ Hecke character of K of conductor $\mathfrak{f}$, infinity type $(1-k, 0)$ for some even $k \geq 2$ and central character $\varepsilon_\psi = \varepsilon_K \bar{\chi}_g \bar{\chi}_h$.
- $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f}) D_K^2$, $N = \mathrm{lcm}(N_\psi, N_g, N_h)$.
- $p \geq 5$ prime with $(p, N) = 1$.

Setting

- V_g, V_h are the p-adic Galois representations attached g and h.
- $T_g \subseteq V_g$ and $T_h \subseteq V_h$ are Galois-stable lattices.
- $\psi_{\mathfrak{P}}$ is the p-adic avatar of ψ.
- We are interested in the G_K-representation

$$V := V_g \otimes V_h(\psi_{\mathfrak{P}}^{-1})(1-c),$$

  where $c = (k+l+m-2)/2$.
- $T = T_g \otimes T_h(\psi_{\mathfrak{P}}^{-1})(1-c)$ is a G_K-stable lattice in V.

Definition of Euler system

We follow the approach of Jetchev-Nekovář-Skinner:

- $\mathcal{S}$ is the set of squarefree products of primes q coprime with pN that split in K.
- For q split in K and $\mathfrak{q} \mid q$,

$$P_{\mathfrak{q}}(X) := \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | V^{\vee}(1)).$$

- An Euler system is a collection of cohomology classes

$$\boldsymbol{\kappa} = \left\{ \kappa_n \in H^1(K[n], T) : \; n \in \mathcal{S} \right\}$$

  such that

$$\mathrm{cor}_{K[n]}^{K[nq]}(\kappa_{nq}) \equiv P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\kappa_n \mod (q-1)H^1(K[n], T).$$

Construction for $k = l = m = 2$

- $Y_{10}(N, m)$ is the affine modular curve parameterizing triples (E, P, C), where
  - E is an elliptic curve,
  - P is a point in E of order N,
  - C is a cyclic subgroup of E of order Nm containing P.
- Two kinds of degeneracy maps $Y_{10}(N, mq) \longrightarrow Y_{10}(N, m)$:

$$\pi_1(mq, m) : (E, P, C) \longmapsto (E, P, qC)$$

$$\pi_2(mq, m) : (E, P, C) \longmapsto (E/NmC, P + NmC, C/NmC).$$

Construction for $k = l = m = 2$

- For each $n \in \mathcal{S}$, consider:

$$\begin{array}{ccc} Y_{10}(N, n^2) & \xhookrightarrow{\iota_n} & Y_{10}(N, n^2) \times Y_{10}(N, n^2) \times Y_{10}(N, n^2) \\ & & \big\downarrow{\scriptstyle (1,\pi_1,\pi_2)} \\ & & Y_{10}(N, n^2) \times Y_1(N) \times Y_1(N). \end{array}$$

- Let $\Delta_n = (1, \pi_1, \pi_2)_* \circ \iota_{n*}(Y_{10}(N, n^2))$, a codimension-2 cycle in $Y_{10}(N, n^2) \times Y_1(N) \times Y_1(N)$.

Construction for $k = l = m = 2$

- Let q be a prime such that $nq \in \mathcal{S}$.
- Consider the following maps $Y_{10}(N, n^2q^2) \longrightarrow Y_{10}(N, n^2)$:
  - $\pi_{11} = \pi_1(n^2q, n^2) \circ \pi_1(n^2q^2, n^2q)$,
  - $\pi_{12} = \pi_2(n^2q, n^2) \circ \pi_1(n^2q^2, n^2q)$,
  - $\pi_{22} = \pi_2(n^2q, n^2) \circ \pi_2(n^2q^2, n^2q)$.

Proposition

- $(\pi_{11}, 1, 1)_*(\Delta_{nq}) = \left\{(1, 1, \langle q\rangle'^{-2} T_q'^2) - (q+1)(1, 1, \langle q\rangle'^{-1})\right\}\Delta_n$.
- $(\pi_{12}, 1, 1)_*(\Delta_{nq}) = \left\{(1, T_q', \langle q\rangle'^{-1} T_q') - (T_q', \langle q\rangle', 1)\right\}\Delta_n$.
- $(\pi_{22}, 1, 1)_*(\Delta_{nq}) = \left\{(1, T_q'^2, 1) - (q+1)(1, \langle q\rangle', 1)\right\}\Delta_n$.

Construction for $k = l = m = 2$

- Let κ_n^1 be the image of Δ_n under the following sequence of maps.

$$\mathrm{CH}^2\left(Y_{10}(N,n^2)\times Y_1(N)\times Y_1(N)\right)$$

$$\Big\downarrow \mathrm{AJ}_p$$

$$H^1\left(\mathbb{Q}, H^3_{\text{ét}}(\bar{Y}_{10}(N,n^2)\times \bar{Y}_1(N)\times \bar{Y}_1(N),\mathbb{Z}_p)(2)\right)$$

$$\Big\downarrow$$

$$H^1\left(\mathbb{Q}, H^1_{\text{ét}}(\bar{Y}_{10}(N,n^2),\mathbb{Z}_p)\otimes H^1_{\text{ét}}(\bar{Y}_1(N),\mathbb{Z}_p)\otimes H^1_{\text{ét}}(\bar{Y}_1(N),\mathbb{Z}_p)(2)\right)$$

$$\Big\downarrow$$

$$H^1\left(\mathbb{Q}, H^1_{\text{ét}}(\bar{Y}_{10}(N_\psi,n^2),\mathbb{Z}_p)\otimes H^1_{\text{ét}}(\bar{Y}_1(N_g),\mathbb{Z}_p)\otimes H^1_{\text{ét}}(\bar{Y}_1(N_h),\mathbb{Z}_p)(2)\right)$$

Construction for $k = l = m = 2$

- Let $\mathbb{T}'_{10}(N_\psi, n^2)$ be the $\mathbb{Z}_p$-algebra of endomorphisms of $H^1_{\acute{e}t}(\bar{Y}_{10}(N_\psi, n^2), \mathbb{Z}_p)$ generated by the operators T'_q and $\langle d \rangle'$.
- $K[n]$ is the maximal p-subextension of the ring class field of conductor n and $R_n := \mathrm{Gal}(K[n]/K)$.
- $E/\mathbb{Q}_p$ is a finite extension and $\mathcal{O}$ is the ring of integers of E.

Proposition (Lei-Loeffler-Zerbes)

There exists a homomorphism

$$\phi_n : \mathbb{T}'_{10}(N_\psi, n^2) \longrightarrow \mathcal{O}[R_n]$$

such that $\phi_n(T'_q) = \sum_{\mathfrak{q}} \psi(\mathfrak{q})[\mathfrak{q}]$, with the sum over prime ideals of norm q coprime to $\mathfrak{f}n$, and $\phi_n(\langle d \rangle') = \varepsilon_K \varepsilon_\psi(d)$.

Construction for $k = l = m = 2$

- For $n, nq \in \mathcal{S}$, we define maps

$$\begin{array}{c} \mathcal{O}[R_{nq}] \otimes_{(\mathbb{T}'_{10}(N_\psi, n^2q^2), \phi_{nq})} H^1_{\text{ét}}(\bar{Y}_{10}(N_\psi, n^2q^2), \mathbb{Z}_p) \\ \Big\downarrow \mathcal{N}_n^{nq} \\ \mathcal{O}[R_n] \otimes_{(\mathbb{T}'_{10}(N_\psi, n^2), \phi_n)} H^1_{\text{ét}}(\bar{Y}_{10}(N_\psi, n^2), \mathbb{Z}_p) \end{array}$$

by the following formula:

$$\mathcal{N}_n^{nq} = 1 \otimes \pi_{11*} + \left(\frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} + \frac{\psi(\bar{\mathfrak{q}})[\bar{\mathfrak{q}}]}{q} \right) \otimes \pi_{12*} + \frac{\varepsilon_\psi(q)}{q} \otimes \pi_{22*}.$$

Construction for $k = l = m = 2$

Theorem (Lei-Loeffler-Zerbes)

There exists a family of $G_{\mathbb{Q}}$-equivariant isomorphisms of $\mathcal{O}[R_n]$-modules

$$\mathcal{O}[R_n] \otimes_{(\mathbb{T}'_{10}(N_\psi, n^2), \phi_n)} H^1_{\text{ét}}(\bar{Y}_{10}(N_\psi, n^2), \mathbb{Z}_p(1)) \xrightarrow{\nu_n} \mathrm{Ind}^{\mathbb{Q}}_{K[n]} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

for $n \in \mathcal{S}$ such that if $n \mid n'$ we get a commutative diagram

$$\begin{array}{ccc} \mathcal{O}[R_{n'}] \otimes_{(\mathbb{T}'_{10}(N_\psi, n'^2), \phi_{n'})} H^1_{\text{ét}}(Y_{10}(N_\psi, n'^2), \mathbb{Z}_p(1)) & \xrightarrow{\nu_{n'}} & \mathrm{Ind}^{\mathbb{Q}}_{K[n']} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \downarrow \mathcal{N}_n^{n'} & & \downarrow \mathrm{Norm} \\ \mathcal{O}[R_n] \otimes_{(\mathbb{T}'_{10}(N_\psi, n^2), \phi_n)} H^1_{\text{ét}}(Y_{10}(N_\psi, n^2), \mathbb{Z}_p(1)) & \xrightarrow{\nu_n} & \mathrm{Ind}^{\mathbb{Q}}_{K[n]} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}). \end{array}$$

Construction for $k = l = m = 2$

- Let κ_n^2 be the image of κ_n^1 under the following sequence of maps.

$$H^1\left(\mathbb{Q}, H^1_{\text{ét}}(\bar{Y}_{10}(N_\psi, n^2), \mathbb{Z}_p) \otimes H^1_{\text{ét}}(\bar{Y}_1(N_g), \mathbb{Z}_p) \otimes H^1_{\text{ét}}(\bar{Y}_1(N_h), \mathbb{Z}_p)(2)\right)$$

$$\downarrow$$

$$H^1\left(\mathbb{Q}, \operatorname{Ind}_{K[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \otimes T_g \otimes T_h(-1)\right)$$

$$\downarrow \cong$$

$$H^1\left(K[n], T_g \otimes T_h(\psi_{\mathfrak{P}}^{-1})(-1)\right)$$

- Let $\kappa_n = \chi_h(n)\kappa_n^2 \in H^1(K[n], T)$.

Construction for $k = l = m = 2$

Theorem

The classes κ_n lie in $H^1_f(K[n], T)$ for all $n \in \mathcal{S}$ and satisfy

$$\mathrm{cor}^{K[nq]}_{K[n]}(\kappa_{nq}) = \Bigg\{\chi_g(q)\chi_h(q)q\left(\frac{\psi(\mathfrak{q})}{q}\,\mathrm{Fr}_{\mathfrak{q}}^{-1}\right)^2 - a_q(g)a_q(h)\left(\frac{\psi(\mathfrak{q})}{q}\,\mathrm{Fr}_{\mathfrak{q}}^{-1}\right) + \frac{a_q(g)^2}{\chi_g(q)q} + \frac{a_q(h)^2}{\chi_h(q)} - \frac{q^2+1}{q} - a_q(g)a_q(h)\left(\frac{\psi(\bar{\mathfrak{q}})}{q}\,\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-1}\right) + \chi_g(q)\chi_h(q)q\left(\frac{\psi(\bar{\mathfrak{q}})}{q}\,\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-1}\right)^2\Bigg\}\kappa_n$$

for all $n, nq \in \mathcal{S}$. Therefore

$$\mathrm{cor}^{K[nq]}_{K[n]}(\kappa_{nq}) \equiv \left(\frac{\psi(\mathfrak{q})}{q}\,\mathrm{Fr}_{\mathfrak{q}}^{-1}\right)^{-2} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\kappa_n \mod (q-1)H^1(K[n], T).$$

A Λ-adic Euler system

- Working with p-adic families of diagonal cycles, as constructed in the work of Darmon-Rotger and Bertolini-Seveso-Venerucci, we can obtain classes that vary along the anticyclotomic $\mathbb{Z}_p$-extension.

Theorem

Assume that

- p splits in K and $p \nmid h_K$,
- g and h are ordinary at p.

Then we obtain a collection of cohomology classes

$$\boldsymbol{\kappa} = \left\{ \kappa_n \in H^1_{\mathrm{Iw}}(K[np^\infty], T) \ : \ n \in \mathcal{S} \right\}$$

such that

$$\mathrm{cor}_{K[n]}^{K[nq]}(\kappa_{nq}) \equiv P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\kappa_n \mod (q-1)H^1_{\mathrm{Iw}}(K[np^\infty], T).$$