A SUPPLEMENTAL PROBLEM

PETER OZSVÁTH

Here's a 3-part supplemental problem inspired by a question in class (yet another variation on the "mean value theorem" theme).

(1) Let $U \subset \mathbb{R}^n$ be a non-empty, bounded subset. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Suppose that f vanishes on $\overline{U} \setminus U$. Prove that there is some point $p \in U$ so that $df(p)$ (in the sense of p. 138 of Gunning) vanishes.

(2) Let $F\colon U \to \mathbb{R}^3$ be a differentiabl function, and suppose that the restriction of F to $\overline{U}\setminus U$ lies in some fixed two-dimensional plane P inside $\mathbb{R}^3$. Prove that there is some point $p \in U$ so that the image of the Jacobian derivative (the linear map which I denoted $D_p f\colon \mathbb{R}^n \to \mathbb{R}^3$; it is the linear map represented by the matrix of partial derivatives which Gunning denotes $f'(p)$) lies in P.

(3) Do the conclusions for the above two exercise remain true without the hypothesis that U is bounded? (I.e. either improve your proofs to that case, or give a counterexample.)

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