INTEGRATING VECTOR FIELDS

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1. Disclaimer:

This is not a textbook. For example, if I were writing a textbook, I'd have stated Corollary 10 as the main result and Theorem 3 as a workhorse proposition.

More importantly, there may well be inaccuracies in these notes. Please feel free to give me feedback!

In these notes, I'm mostly following Chapter 5 of Spivak's *Comprehensive Introduction to Differential Geometry, Volume I*, which I recommend heartily for further reading.

2. Integrating vector fields

We fix a domain $U \subset \mathbb{R}^n$ and a function $F\colon U \to \mathbb{R}^n$ (which we can think of as a vector field over U). Fix any point $x \in U$. We wish to solve the ordinary differential equation (equation for a differentiable curve c in U):

$$\frac{dc}{dt} = F(c(t))$$

with specified initial condition

$$c(0) = x.$$

A word on notation. The function $F\colon U \to \mathbb{R}^n$ has n can be written in its component functions:

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n)).$$

The curve c can be written in its components

$$c(t) = (c_1(t), \dots, c_n(t)).$$

Thus, the differential equation above is shorthand for the n equations

$$\frac{dc_i(t)}{dt} = F_i(c_1(t), \dots, c_n(t)).$$

Sometimes we don't have a solution c defined for all $t \in \mathbb{R}$. Moreover, sometimes, we have more than one solution with given initial conditions.

Consider:

$$\frac{dc}{dt} = c^2.$$

The general solution is

$$c(t) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{(1/x)-t} & \text{if } x \neq 0. \end{cases}$$

The solution does not exist for all time t: the curve runs off the real line when $t = 1/x$.

Next, consider the ODE:

$$\frac{dc}{dt} = c^{2/3}.$$

There are two solutions with $c(0) = 0$. $c(t) \equiv 0$ and $c(t) = \frac{1}{27}t^3$. (The non-uniqueness is caused by the fact that $y \mapsto y^{2/3}$ is not a smooth function of y.)

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We will give conditions on F that ensure existence (for small time) for c and uniqueness.

Definition 1. *We say that F is* Lipschitz on U (with Lipschitz constant K) *if there is a K so that*

$$|F(x) - F(y)| \leq K|x - y|$$

for all $x, y \in U$.

Exercise 2. (1) *Show that if $U \subset \mathbb{R}^n$ is a bounded open set $F\colon \mathbb{R}^n \to \mathbb{R}$ C^1, then F is Lipschitz in U.*
(2) *Show that if $F\colon X \to Y$ is a Lipschitz function between two metric spaces, then F is continuous.*

We will make use of the *contraction mapping principle*. Recall that we applied that in the case of a closed Euclidean ball, to prove the inverse function theorem. We will apply it in more abstraction, in the following *Banach fixed point theorem*: (A *Banach space* is a complete normed vector space.)

2.1. The Banach fixed point theorem, and spaces of functions.

Theorem 3. *Suppose that (X, d) is a non-empty complete metric space, and $\Psi\colon X \to X$ is a contraction; i.e. there is some $C < 1$ so that*

$$d(\Psi(x), \Psi(y)) \leq Cd(x, y)$$

for all $x, y \in X$. Then, there is a unique $x \in X$ so that $\Psi(x) = x$.

Proof. The Cauchy sequence is obtained by iterating Ψ on any point $x_0 \in X$. Let $x_i = \Psi(x_{i-1})$. It's Cauchy, since

$$d(x_n, x_{n+1}) \leq C^n \cdot d(x_0, \Psi(x_0)),$$

so

$$\begin{aligned} d(x_n, x_{n+i}) &\leq (C^n + C^{n+1} + \cdots + C^{n+i-1}) \cdot d(x_0, x_1) \\ &< \frac{C^n}{1 - C} \\ &\mapsto 0. \end{aligned}$$

Q: Why does that imply this is a Cauchy sequence? Finish off the argument as if you were writing a textbook.

Let $x = \lim_{n\mapsto\infty} x_i$. By continuity

$$\Psi(x) = \lim_{n\mapsto\infty} \Psi(x_n) = \lim_{n\mapsto\infty} x_{n+1} = x.$$

This proves existence.

Uniqueness is easy: if $\Psi(x) = x$ and $\Psi(y) = y$, then $d(x, y) = d(\Psi(x), \Psi(y)) \leq C \cdot d(x, y)$ implies $d(x, y) = 0$. □

A metric space (M, d) is called *bounded* if there is constant C so that $d(x, y) \leq C$; equivalently, the whole space is contained in some large enough ball.

Suppose that (X, d) is a metric space and V is a normed vector space. Then the set of bounded, continuous functions from X to V, written $C_b(X, V)$ can be equipped with a norm defined by

$$\|f\|_{L^\infty} = \sup_{x\in X} \|f(x)\|. \tag{1}$$

If V is complete, so is $C_b(X, V)$ since the uniform limit of continuous functions is continuous. The metric defined in Equation (1) is often called the *supremum norm* or the *L^∞ norm*.

Let $C^0(X, V)$ denote the vector space of continuous functions. When X is compact, $C^0(X, V) = C_b(X, V)$.

2.2. **The fundamental theorem of calculus.** We will repeatedly use the following basic result. There are many versions (depending on the hypothesis on f); the following version will suffice for our present purposes:

Theorem 4. *(Fundamental theorem of calculus) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous, bounded function. Then the parent function*

$$F(t) = \int_0^t f(s)ds$$

is a continuous function which is differentiable everywhere, and with the property that

$$\frac{dF}{dt}(t) = f(t).$$

2.3. **Picard's existence theorem for ODE.** With the preliminaries in place, we can now state and prove Picard's theorem on the existence of solutions to ODE's.

Theorem 5. *(Theorem 5.2 of Spivak) Fix an open set $U \subset \mathbb{R}^n$, and let $F\colon U \to \mathbb{R}^n$ be a (continuous) function. Fix $x_0 \in U$ and r so that $\overline{B}_{2r}(x_0) \subset U$. Suppose that there are real numbers K and L so that*

1. *$\|F\|_{L^\infty(U,\mathbb{R}^n)} \leq L$ on $\overline{B}_{2r}(x_0)$*
2. *$\|F(x) - F(y)\|_{\mathbb{R}^n} \leq K\|x - y\|_U$ for all $x, y \in \overline{B}_{2r}(x_0)$.*

Choose $\epsilon > 0$ so that

$$\epsilon < r/L \tag{2}$$

$$\epsilon < 1/K \tag{3}$$

Then, for each $x \in \overline{B}_r(x_0)$, there is a unique $c_x\colon (-\epsilon, \epsilon) \to U$ so that

$$\frac{dc}{dt}(t) = F(c(t)) \tag{4}$$

$$c(0) = x. \tag{5}$$

Remark 6. We might as well use that L^∞ norm on U and $\mathbb{R}^n$; but the statement for those norms implies the same statement (with possibly different choices of constants) for any norm. Why?

Proof. By the fundamental theorem of caluculus, Equation (4) is equivalent to the following integral equation [1]:

$$c(t) = x + \int_0^t F(c(s))ds. \tag{6}$$

Next, note that if $c\colon (-\epsilon, \epsilon) \to U$ satisfies Equations (4) and (5) for ϵ as in Equation (2), then in fact c maps $(-\epsilon, \epsilon)$ into $B_r(x) \subset B_{2r}(x_0) \subset U$.

For our application of the Banach fixed point theorem, we will consider the space

$$\mathfrak{X} = \{c\colon (-\epsilon, \epsilon) \to \overline{B}_{2r}(x_0) \mid c \text{ is continuous}\}.$$

$\mathfrak{X}$ is a complete metric space, with its supremum norm, according to our earlier remarks. We interpret the right hand side of Equation (6) as an application of the operator

$$I_x\colon \mathfrak{X} \to \mathfrak{X},$$

[1] Equation (6) is shorthand for integrating out components; i.e. if $c(t) = (c_1(t), \ldots, c_n(t))$, then $\int_0^t F(c(s))ds$ is shorthand for the curve

$$t \mapsto \left(x_1 + \int_0^t F_1(c(s))ds, \ldots, x_n + \int_0^t F_n(c(s))ds\right).$$

specified by

$$I_x(c(t)) = x + \int_0^t F(c(s))ds.$$

The solution $c = c_x$ is a fixed point for the operator I_x.

We must check that I_x maps $\mathfrak{X}$ to itself. First, we argue that I_x maps $\mathfrak{X}$ to itself: this is a repetition of the claim at the beginning of the proof: for $t \in (-\epsilon, \epsilon)$,

$$|I_x(c(t)) - x| \leq \Big| \int_0^t F(c(s))ds \Big| \leq \epsilon L < r.$$

Since $|x - x_0| \leq r$, it follows that $|I_x(c(t)) - x_0| < 2r$ for all $t \in (-\epsilon, \epsilon)$; i.e.

$$I_x(c(t)) \in B_{2r}(x_0) \subset \overline{B}_{2r}(x_0) \tag{7}$$

for all $t \in (-\epsilon, \epsilon)$. Thus, $I_x \colon \mathfrak{X} \to \mathfrak{X}$, as needed.

Next, we verify that I_x is a contraction mapping. To this end, fix $\alpha, \beta \in \mathfrak{X}$, and note that

$$\begin{aligned} \|I_x(\alpha) - I_x(\beta)\| &\leq \sup_t \Big| \int_0^t (F(\alpha(s)) - F(\beta(s)))ds \Big| \\ &\leq \Big| \sup_t \int_0^t \Big| F(\alpha(s)) - F(\beta(s)) \Big| ds \Big| \\ &\leq \epsilon K \sup_{-\epsilon < s < \epsilon} |\alpha(s) - \beta(s)| \\ &= \epsilon K \|\alpha - \beta\|. \end{aligned}$$

So since $\epsilon K < 1$, I_x is a contraction.

Thus, the Banach fixed point theorem (Theorem 3) guarantees the existence of a unique $c(t)$ $c \colon (-\epsilon, \epsilon) \to \overline{B}_{2r}(x_0)$ solving Equations (4) and (5). But that's not the uniqueness we want. We want to know that c is the unique solution among all $c \colon (-\epsilon, \epsilon) \to U$. □

We can improve the statement a little with following:

Definition 7. *If X and Y are metric spaces, a function $f \colon X \to Y$ is called* uniformly continuous *if for any $\epsilon > 0$ there is a δ so that $d(f(x), f(y)) < \epsilon$ if $d(x, y) < \delta$.*

Example 8. A Lipschitz function is uniformly continuous.

Proposition 9. *Let X, Y be metric spaces, suppose that Y is complete. If $U \subset X$ is an open set and $f \colon U \to Y$ is uniformly continuous, then f extends uniquely to a continuous map $\overline{f} \colon \overline{U} \to Y$.*

Proof. The point: the image of a convergent sequence is Cauchy. □

We can now give a slightly better statement of Picard's existence theorem:

Corollary 10. *Fix a locally Lipschitz function $F \colon U \to \mathbb{R}^n$ on U (i.e. U is covered by open sets for which the restriction of F is Lipschitz). Fix $x_0 \in U$ so that $B_{2r}(x_0) \subset U$, and choose L so that $\|F(x)\| \leq L$ for all $x \in B_{2r}(x_0)$. Then, for any $\epsilon < r/L$, for all $x \in B_r(x_0)$, there is a unique curve $c \colon (-\epsilon, \epsilon) \to U$ satisfying Equation* (4) *with initial condition* (5).

Proof. Theorem 5 gives η so that c is defined on $(-\eta, \eta)$. Clearly, c is Lipschitz with constant L. It follows that c is unformly continuous; hence it can be continuously extended to $[-\eta, \eta]$. We can now extend further using Theorem 5.

Said better; consider the supremum η so that $c|_{(-\eta,\eta)}$ solves the differential equation. We've proved that $\eta = \epsilon$. □

3. Further remarks

3.1. **Regularity.** One might ask how the solution to Equations (4) and (5) depend on x. For example:

Question 11. *Given $t \in (-\epsilon, \epsilon)$, is the map $x \mapsto c(t)$ a continuous function of x?*

The next exercise will address this question.

Exercise 12. *Fix a Banach space V, and the unit ball $B_1(0) \subset V$. Suppose that $I\colon B_1(0) \to B_1(0)$ is a contraction mapping for some $K < 1$ which fixes 0.*

- *Show there is a $\delta > 0$, so that $a \in B_\delta(0) \subset V$, the map $I_a(y) = a + I(y)$. is a contraction map from $B_1(0)$ to itself.*
- *Show that the map $B_\delta(0) \to B_1(0)$ which associates to x the fixed point I_a is a continuous function of a.*
- *Use this to show that, under the hypotheses of Theorem 5, the assignment $B_r(x_0) \to C^0((-\epsilon, \epsilon), U)$ which associates to $x \in B_r(x_0)$ the solution $c\colon (-\epsilon, \epsilon) \to U$ satisfying Equations (4) and (5) is a continuous function.*

More generally, Corollary 10 can be interpreted as defining a map

$$\Phi\colon (-\epsilon, \epsilon) \times B_r(x_0) \to B_r(x_0).$$

This map has the property that $\Phi(0, x) = x$ and $\Phi(s, \Phi(t, x)) = \Phi(s+t, x)$, provided that $s, t, s+t \in (-\epsilon, \epsilon)$.

One might ask about regularity properties of this map.

3.2. **Small generalizations.** There is a trick to reducing from a higher order to the first order case, as follows.

Rather than writing out the general case, let's consider a second order equation

$$\frac{d^2c}{dt^2}(t) = F(c(t))$$

(where F is Lipschitz). We can replace this by a system of first order equations, by introducing a new function v:

$$\begin{aligned} \frac{dc}{dt} &= v(t) \\ \frac{dv}{dt} &= F(c(t)). \end{aligned}$$

We can view this now as an ODE on $\mathbb{R}^n \times U$, with $(v, x) \mapsto (F(x), v)$.

Another (different) little improvement can be obtained by noting that Theorem 5 can be adapted to a system of the form

$$\begin{aligned} \frac{dc}{dt}(t) &= F(t, c(t)) \\ c(0) &= x, \end{aligned}$$

where now F also depends on t. (Now, of course, we require $F\colon U \times \mathbb{R} \to \mathbb{R}^n$ to be Lipschitz.)

There is another fundamental existence theorem for ODE's, dropping the Lipschitz hypothesis, which we will state and prove in the next handout.

Exercise 13. *Show that if F is a bounded, continuous function on of $\mathbb{R}^n$, which is locally Lipshitz, then the curves c given by Theorem 5 can be extended for all time; i.e. there is a unique solution $c\colon \mathbb{R} \to U$ to Equation (4) for any choce of initial condition $x \in \mathbb{R}^n$ (Equation (5)).*

4. AI statement

I ran an earlier version through AI to check for typeos. It found some, which I fixed; and it offered to write the follow-up handout, which I declined.