When all directed cycles have the same weight

Eli Berger
University of Haifa, Haifa, Israel

Daniel Carter[1]
Princeton University, Princeton, USA

Paul Seymour[2]
Princeton University, Princeton, USA

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Abstract

A digraph G is *weightable* if its edges can be weighted with real numbers such that the total weight in each directed cycle equals 1. There are several equivalent conditions: that G admits a 0/1-weighting with the same property, or that G contains no subdivided "double-cycle" as a subdigraph, or that for every triple of vertices, all directed cycles containing all three pass through them in the same cyclic order. And there is quite a rich supply of such digraphs: for instance, any digraph drawn in the plane such that each of its directed cycles rotates clockwise around the origin is weightable (let us call such digraphs "circular"), and there are weightable planar digraphs with much more complicated structure than this.

Until now the general structure of weightable digraphs was not known, and that is our objective in this paper. We will show that:

- there is a construction that builds every planar weightable digraph from circular digraphs; and
- there is a (different) construction that builds every weightable digraph from planar ones.

We derive a poly-time algorithm to test if a digraph is weightable.

1 Introduction

Graphs and digraphs in this paper are finite, and may have loops or parallel edges. Let G be a digraph drawn in the plane (without crossings), where the origin belongs to one of the regions. Each edge e of G subtends an angle at the origin (a $w(e)$ fraction of a full rotation, say). If the drawing has the property that $w(e) > 0$ for every edge, then every directed cycle clearly must wind around the origin. But more than that; every directed cycle must wind around the origin exactly once, because curves that wind more than once intersect themselves. Let us call such a drawing *circular.*

This is an intriguing property of digraphs. It is related to a theorem of Thomassen [5, 9], that says in one form:

1.1. *Let G be a digraph with no loops or parallel edges such that every vertex has in-degree and out-degree at least two, and suppose there are $a, b \in V(G)$ such that every directed cycle contains at least one of a, b. Then there is no directed cycle containing both a, b if and only if G admits a circular drawing.*

This version of Thomassen's result looks like it ought to be made somehow more general, and that brings us to the question that was the starting point of the research in this paper: is there a theorem that says "every appropriately connected digraph G contains no thing of type X if and only if G admits a circular drawing"?

Let us say a *weighting* of a digraph G is a real-valued function $w : E(G) \to \mathbb{R}$, such that $w(C) = 1$ for every directed cycle C, where $w(C)$ means $\sum_{e \in E(C)} w(e)$, and if G admits a weighting, we say G is *weightable.* In a circular drawing, the function w defined earlier is a weighting, and so digraphs with circular drawings are weightable. For some time we hoped that for a converse, at least for sufficiently well-connected digraphs, that, say, every strongly 2-connected, weakly 3-connected weightable digraph admits a circular drawing (we will define these terms later). But this turns out to be false – see Figure 1.

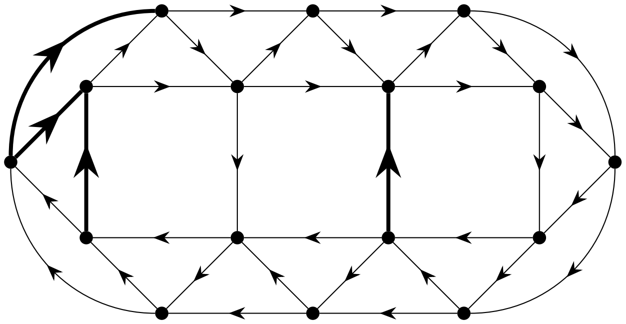


Figure 1: A weightable digraph with no circular drawing. The four thick edges have weight one, and the others have weight zero.

We still have not come up with the characterization we hoped for of the digraphs with circular drawings, but we now know which digraphs are weightable, and that is what we will explain in this paper.

An answer to the question "which digraphs are weightable?" could mean several things:

1. A characterization of the minimal digraphs (under subdigraph containment) that are not weightable.
2. A poly-time algorithm to test whether an input digraph is weightable.
3. A poly-time algorithm to find a weighting if there is one.
4. A poly-time algorithm to test if a given function on $E(G)$ is a weighting.
5. A method of construction that will build all (ond only) weightable digraphs by piecing together well-understood ones in prescribed ways.

The first of these was already known [6], but all the others are new and obtained in this paper. We will show in the next section that the third and fourth are both consequences of the second. The second is a consequence of the fifth, the construction method, which is our main theorem. We discuss this algorithm at the end of the paper.

Curiously, the construction method breaks into two parts: roughly, we will show how to build all planar weightable digraphs, by piecing together those with circular embeddings; and then show how to build general weightable digraphs, by piecing together planar ones (with a different construction).

Combining all of our constructions together yields the following theorem (we use some terms defined at the start of the next section):

1.2. *Let G be a 1-strong weightable digraph. Then at least one of the following is true:*

1. *G is not strongly 2-connected, and is obtained from two smaller 1-strong weightable digraphs by the construction described in 6.1 and depicted in Figure 5.*
2. *G is not 3-weak, and is obtained from two smaller 1-strong weightable digraphs by the construction described in 6.2 and depicted in Figure 6.*
3. *G is planar and admits a circular drawing.*
4. *G is planar, does not admit a circular drawing, and is obtained from two smaller 1-strong planar weightable digraphs by the construction described in 7.1 and depicted in Figure 7.*
5. *G is nonplanar, and there is a subset $Y \subseteq V(G)$ with $|Y| \in \{3,4\}$ such that $G \setminus Y$ has at least two weak components, and G is obtained from two or three smaller 1-strong weightable digraphs by one of the constructions described in 9.1, 9.4, or 9.5 and depicted in Figures 11, 14, or 15, respectively.*

There is another way to describe the 1-strong planar weightable digraphs, as planar digraphs that admit a kind of tree-decomposition called a “bond carving” of “diwidth two”. This is discussed in Section 8.

2 Preliminaries

First, let us state some standard definitions. A graph G is *k-connected* if it has at least $k+1$ vertices and $G \setminus X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. The *underlying graph* G^- of a digraph G is the graph obtained for forgetting the direction of all edges. A digraph is *weakly connected* if its underlying graph is connected, and *weakly k-connected* or *k-weak* if its underlying graph is k-connected. (We usually write “1-weak” for “weakly connected”.) A *weak component* of a digraph is a maximal 1-weak subdigraph.

A *dipath* is a directed path, and a *dicycle* is a directed cycle. A digraph G is *strongly connected* if for every two vertices u, v there is a dipath from u to v. (This is equivalent to saying that G is 1-weak and every edge is in a dicycle.) A digraph G is *strongly k-connected* or *k-strong* if $G \setminus X$ is strongly connected for every $X \subseteq V(G)$ with $|X| < k$.

Let us say a drawing of a digraph G (in a plane or a 2-sphere) is *diplanar* if for every vertex $v \in V(G)$, the edges of G with head v form an interval in the cyclic ordering of edges incident with v determined by the drawing (and a digraph is *diplanar* if it admits a diplanar drawing). For instance, the digraph in Figure 1 is diplanar. (This was called “strongly planar” in [4], but this seems a confusing name since we also need to talk about 1-strong digraphs in the sense of being strongly connected.)

We need “ears”. Let H be a 1-strong subdigraph of a 1-strong digraph G. If $H \neq G$, there is an *ear* for H in G, that is, either

- a dipath of G with length at least one, with both ends in $V(H)$ and with no edge or internal vertex in H; or
- a dicycle of G with exactly one vertex in H.

(This is a standard, elementary result.) The point of ears is that if P is an ear as above, then $H \cup P$ is also 1-strong, and either $H \cup P = G$ or we can choose an ear for $H \cup P$ in G, and so on. Thus every 1-strong subdigraph of a 1-strong digraph G can be grown by adding ears one at a time until it becomes G. More exactly, let H be a 1-strong subdigraph of a 1-strong digraph G, and let $P_1, \dots, P_n$ be a sequence of subdigraphs of G such that $H \cup P_1 \cup \cdots \cup P_n = G$, and for $1 \leq i \leq n$, P_i is an ear for $H \cup P_1 \cup \cdots \cup P_{i-1}$ in G. We call the sequence $P_1, \dots, P_n$ an *ear sequence* for H in G. Then there is an ear sequence for every 1-strong subdigraph of a 1-strong digraph. (We mention that if we were working with 1-strong, 2-weak digraphs, we could change the definition of “ear” to exclude the dicycle case in the second bullet, and the same result is true. This is proved in [6].)

If C is a (not necessarily directed) cycle of a digraph G, and we select one of the two cyclic orientations of C, let $\mathbf{c}$ be the map from $E(G)$ to $\mathbb{R}$ defined by $\mathbf{c}(e) = 1$ if $e \in E(C)$ in the direction of the chosen orientation, $\mathbf{c}(e) = -1$ if $e \in E(C)$ is in the other direction, and $\mathbf{c}(e) = 0$ if $e \notin E(C)$. We call $\mathbf{c}$ a *characteristic vector* of C. (Thus, C has two characteristic vectors, negations of each other.) If C is a dicycle, it has a non-negative characteristic vector. Let C_0 be a dicycle of a 1-strong digraph G, and let $P_1, \dots, P_n$ be an ear sequence for C_0 in G. For $1 \leq i \leq n$, there is a dicycle C_i consisting of P_i together with a dipath of $C_0 \cup P_1 \cup \cdots \cup P_{i-1}$ between the ends of P_i. We call the sequence of nonnegative characteristic vectors of $C_0, \dots, C_n$ an *ear-basis*. Thus, if G is 1-strong with $E(G) \neq \emptyset$, then it has an ear-basis. It is easy to see (and is proved in [7]) that for every cycle C, its characteristic vectors are integer linear combinations of the members of any ear-basis. This has two consequences that we will need later:

2.1. *Let w be a weighting of a 1-strong digraph G, and let C be a cycle of G (not necessarily directed). If $\mathbf{c}$ denotes a characteristic vector of C, then the scalar product $w \cdot \mathbf{c}$ is an integer.*

Proof. Let $\mathbf{c}_0, \dots, \mathbf{c}_n$ be an ear-basis. Then there are integers $\lambda_0, \dots, \lambda_n$ such that $\sum_{0 \leq i \leq n} \lambda_i \mathbf{c}_i = \mathbf{c}$. But since $w \cdot \mathbf{c}_i = 1$ for $0 \leq i \leq n$, it follows that $w \cdot \mathbf{c}$ is an integer. This proves 2.1. ∎

2.2. *Let G be a weightable 1-strong digraph, and let $w : E(G) \to \mathbb{R}$ be some function. Let $\mathbf{c}_0, \ldots, \mathbf{c}_n$ be an ear-basis. If $w \cdot \mathbf{c}_i = 1$ for $0 \le i \le n$ then w is a weighting.*

Proof. Let w' be a weighting of G, and let C be a dicycle of G, with characteristic vector $\mathbf{c}$. Then there are integers $\lambda_0, \ldots, \lambda_n$ such that $\sum_{0 \le i \le n} \lambda_i \mathbf{c}_i = \mathbf{c}$. Since $(w - w') \cdot \mathbf{c}_i = 0$ for $0 \le i \le n$, it follows that $(w - w') \cdot \mathbf{c} = 0$, and so $w(C) = w'(C) = 1$. This proves 2.2. ▮

At the end of the previous section we listed five possible meanings of "which digraphs are weightable?" Let us prove that solving the second would solve the third and fourth. We claim that:

2.3. *There are poly-time algorithms that, given a weightable digraph G as input:*

- *finds a weighting of G, and*
- *tests whether a given function on the edge set of G is a weighting.*

Proof. In both cases, we may assume that the input digraph G is 1-strong. First, choose an ear-basis $\mathbf{c}_0, \ldots, \mathbf{c}_n$.

To find a weighting, just find a function $w : E(G) \to \mathbb{R}$ that satisfies $w \cdot \mathbf{c}_i = 1$ for $0 \le i \le n$. (This can be done in poly-time in general by linear programming, but for an ear-basis it is particularly easy, since the corresponding matrix is upper triangular.) By 2.2, w is a weighting.

To test if a given function w is a weighting, just test that $w \cdot \mathbf{c}_i = 1$ for $0 \le i \le n$. If so then w is a weighting by 2.2, and if not then it is not. This proves 2.3. ▮

3 More examples

As we said, the digraph of Figure 1 disproved our original conjecture about the structure of 2-strong weightable digraphs, so what could be the true structure? All dicycles of the drawing in Figure 1 are clockwise (it is important that we are talking about plane drawings here, not drawings in the sphere, so that "clockwise" makes sense. We hoped this was a clue, because one can show if a digraph admits a diplanar drawing in the plane in which every dicycle is clockwise, then it is weightable. So our next hope was the conjecture that every 2-strong, 3-weak weightable digraph admits a diplanar drawing in which every dicycle is clockwise. But this was disproved by the digraph in Figure 2.

So both our conjectures were false, and we fell back to the question: is it at least true that every 2-strong, 3-weak weightable digraph is planar? But that too is false, disproved by the digraph in Figure 3.

4 Known results

Let us observe first that, to understand the weightable digraphs G, we may assume that G is 1-weak (since weightings for different weak components may be considered separately), and every edge is in a dicycle (since edges not in a dicycle may be given any weight and deleted). It follows that G is 1-strong. We will often assume that the digraphs we are considering are 1-strong, without further explanation.

There are some results about weightable digraphs that were already known. It was shown in [6] that:

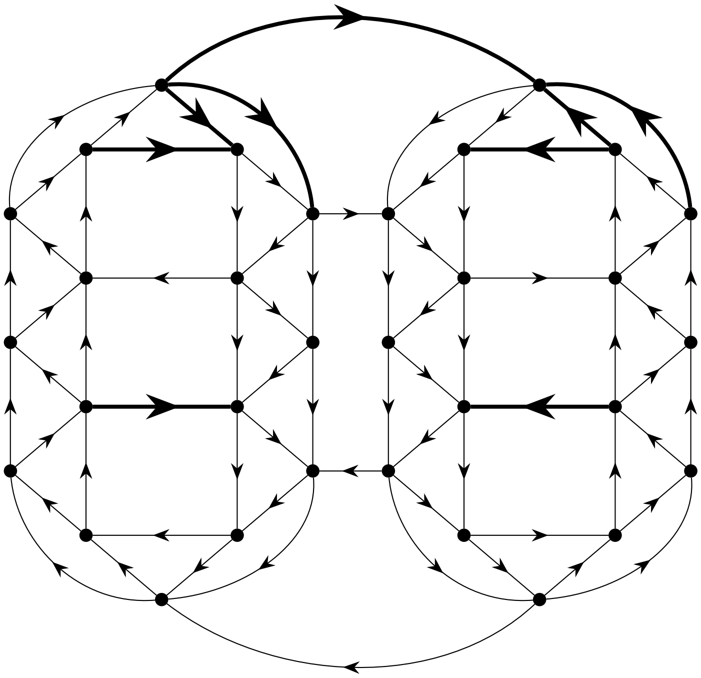


Figure 2: A diplanar, 2-strong, weightable digraph, that cannot be drawn in the plane such that all its dicycles are clockwise. The thick edges have weight one.

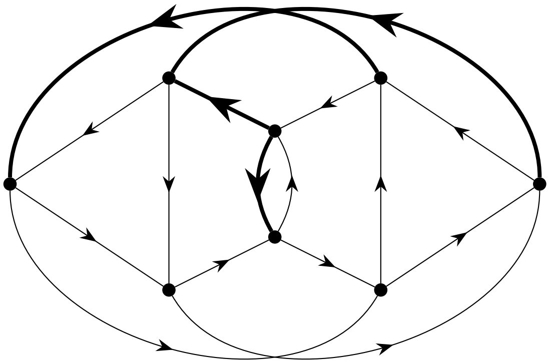


Figure 3: A nonplanar, 2-strong, weightable digraph. Again, the thick edges have weight one.

4.1. *If G admits a weighting then it admits a 0/1-valued weighting.*

Proof. We may assume that G is 1-strong. For every set X of vertices, since w is a weighting, so is $w + c_X$, where $c_X(e) = 1$ for edges e from X to $V(G) \setminus X$, $c_X(e) = -1$ for edges from $V(G) \setminus X$ to X, and $c_X(e) = 0$ otherwise. By adding multiples of the functions c_X to w for appropriate choices of X, we can obtain a weighting which is zero on every edge of some spanning tree T. For $e \in E(G) \setminus E(T)$, let C be the cycle that contains e and is otherwise in T, and let $\mathbf{c}$ be its characteristic vector. By 2.1, $w \cdot \mathbf{c}$ is an integer and so $w(e)$ is an integer. Consequently this weighting is integer-valued on every edge of G.

Having obtained an integer weighting, now let us choose an integer weighting w that minimizes the sum of $|w(e)|$ over all edges e with $w(e) < 0$. Suppose there is an edge $e = uv$ with $w(e) < 0$. Let X be the set of all vertices x such that there is a dipath P of G from v to x where $w(e) \leq 0$ for all edges $e \in E(P)$. Then $w - c_X$ is a better choice than w, a contradiction. So $w(e) \geq 0$ for all edges e, and the result follows. ▮

Henceforth in the paper we will only work with 0/1-valued weightings. Let $k \geq 3$. A *weak k-double-cycle* is a digraph formed by the union of k dicycles $C_1, \ldots, C_k$, where each vertex belongs to at most two of $C_1, \ldots, C_k$, and $C_i \cap C_{i+1}$ is a dipath for $1 \leq i \leq k$ (reading subscripts modulo k), and C_i is vertex-disjoint from C_j if $j \not\equiv i-1, i, i+1 \bmod k$. (In some earlier papers $k = 2$ is permitted, but we do not need that here.) An example is shown in Figure 4.

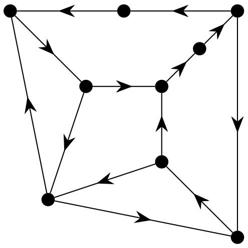


Figure 4: An example of a weak 4-double cycle.

It was shown in [6] that:

4.2. *A digraph G is weightable if and only if for all $k \geq 3$, no subdigraph is a weak k-double-cycle.*

A digraph G is *odd-weightable* if there is a function $w : E(G) \to \{0, 1\}$ such that $w(C)$ is odd for every dicycle of G. Thus, by 4.1, every weightable digraph is odd-weightable. The proof given in [6] for 4.2 was obtained from a similar proof in [7], where it was shown that:

4.3. *A digraph G is odd-weightable if and only if for all odd $k \geq 3$, no subdigraph is a weak k-double-cycle.*

There is another set of older results we need, not really about weightable digraphs, but relevant. Let H be a graph with a bipartition (A, B), and let M be a perfect matching of H; we call the pair (H, M) a *bisource*. Direct all the edges of H from A to B, except for the edges in M, and then contract all the edges in M. This produces some digraph, called a *collapse* of (H, M). (It also depends on the choice of the bipartition (A, B), so if H is connected, (H, M) has two collapses, one obtained from the other by reversing all edges.) Conversely, every digraph is a collapse of some (unique) bisource. There is a remarkable theorem:

4.4. *Let H be a bipartite graph, let M, M' be perfect matchings of H, and let G, G' be corresponding collapses. Then G is odd-weightable if and only if G' is odd-weightable.*

This is proved for "two-extendible" bipartite graphs in [4], although it seems to be implicit in earlier papers. We do not prove it here because the result is just for background. But the moral is that to understand odd-weightable digraphs, it is better to understand the bipartite graphs of the corresponding bisouces, because the choice of perfect matching is irrelevant. It would have been nice if the same simplification held for the property of being weightable, but it is not; we shall see that whether a digraph is weightable or not depends on both terms of its bisource.

A graph is *k-extendible* if every matching of size at most k can be extended to a perfect matching; and a *brace* is a connected 2-extendible bipartite graph. A bipartite graph H with a perfect matching is a brace if and only if the collapse of (H, M) is 2-strong for some (or equivalently, every) choice of a perfect matching M. Let us say a bipartite graph H is *Pfaffian* if there is a perfect matching M of H such that the collapse of (H, M) is odd-weightable. (These are precisely the bipartite graphs that admit "Pfaffian orientations", a topic of interest in theoretical physics and other areas that we do not define here.) In [4], Robertson, Seymour and Thomas gave a construction for all Pfaffian bipartite graphs. Essentially, the problem can be reduced to constructing the Pfaffian braces; and they showed that a brace H is Pfaffian if and only if either H is planar, or H is the Heawood graph, or H admits a decomposition into three smaller Pfaffian braces that we will discuss in more detail later. And reversing this decomposition gives a way to piece together three Pfaffian braces to make a larger Pfaffian brace.

For our problem, we can reduce it to studying the 2-strong weightable digraphs, and such digraphs G are collapses of bisourses (H, M) for which H is a brace. Since, as we saw, every such G is odd-weightable, and therefore H is Pfaffian, we can apply the decomposition theorem of [4] to our problem, and deduce that either H is planar or admits some useful decomposition into three parts, and therefore the same applies to G. The problem is that the corresponding composition operation of gluing three Pfaffian braces together to make one larger Pfaffian brace does not preserve the property that the collapse is weightable, so this by itself does not reduce our problem to the planar case, and we will need to look carefully at the decomposition given by [4]. To illustrate: the graph Rotunda, shown in Figure 16, was fundamental in the result of [4]. It has only three perfect matchings that are not equivalent to one another under symmetries of the graph, and hence it only gives rise to three nonisomorphic collapses. All three are odd-weightable; but one is nonplanar and weightable (the digraph in Figure 3), one is nonplanar and not weightable, and one is planar and not weightable.

Nevertheless, by refining the Pfaffian brace decomposition theorem, we will able to reduce our problem to the planar case. And we can get a little more from it. When G is the collapse of (H, M), if H is planar, then G is not only planar but diplanar; and if H is not planar, then it is so far from planar that if G is weightable then it is also nonplanar. This will imply the convenient fact that if G is 2-strong, 3-weak, planar and weightable then it is diplanar.

5 Basic lemmas

If C is a dicycle of a digraph G, and $u, v, w \in V(C)$ are distinct, then C passes through these three vertices in some order, one of the two cyclic orders of the triple $\{u, v, w\}$. Let us say the ordered triple (u, v, w) is *in order* in C if the dipath of C from u to v does not pass through w. Thus (u, v, w) is in order in C if and only if (v, w, u) is in order in C. A triple $\{u, v, w\}$ of three distinct vertices is *bad* in a digraph G if there exist dicycles C, C' of G, both containing u, v, w, such that (u, v, w) is in order in C and (w, v, u) is in order in C'. We say such cycles C, C' *disagree* on $\{u, v, w\}$.

Here is a result that we will use very frequently:

5.1. *Let G be a digraph; then G is weightable if and only if there is no bad triple.*

Proof. First, we assume that G is weightable, and suppose that $\{u, v, w\}$ is a bad triple. By 4.1 we may choose a 0/1-valued weighting w. Let F be the set of edges e with $w(e) = 1$. So every dicycle has exactly one edge in F. Now choose dicycles C, C' of G, such that (u, v, w) is in order in C and (w, v, u) is in order in C'. Let $C(uv)$ be the subpath of C from u to v, and define $C(vw), C(wu), C'(uw), C'(wv), C'(vu)$ similarly. Since $|E(C) \cap F| = 1$ we may assume that $C(vw), C(wu)$ both have no edges in F, and similarly two of $C'(uw), C'(wv), C'(vu)$ have no edges in F. But $C(vw) \cup C'(wv)$ includes a dicycle, which has an edge in F, and so $F \cap C'(wv) \neq \emptyset$; and similarly $C(wu) \cup C'(uw)$ includes a dicycle and hence $C'(uw)$ has an edge in F, a contradiction.

For the converse, now we assume that G is not weightable, and therefore includes a weak k-double-cycle for some $k \geq 3$; let $C_1, \ldots, C_k$ be as in the definition of weak k-double-cycle. Choose $v_1 \in V(C_k \cap C_1)$, and $v_2 \in V(C_1 \cap C_2)$, and $v_3 \in V(C_2 \cap C_3)$. Then there is a dipath P_1 of C_1 from v_1 to v_2, and a dipath P_2 of C_1 from v_2 to v_3, and a dipath P_3 of $C_3 \cup \cdots \cup C_k$ from v_3 to v_1, and the union of these three paths is a dicycle in which (v_1, v_2, v_3) is in order. But similarly, there is a dipath Q_1 of C_1 from v_2 to v_1, and a dipath Q_2 of C_2 from v_3 to v_2, and a dipath Q_3 of $C_3 \cup \cdots \cup C_k$ from v_1 to v_3, giving a dicycle C' in which (v_3, v_2, v_1) is in order. So $\{v_1, v_2, v_3\}$ is a bad triple. This proves 5.1. ▮

We need a lemma which allows us to to convert a 0/1-valued weighting to one that is more convenient:

5.2. *Let G be a weightable digraph and let $u \in V(G)$. Then there is a 0/1-valued weighting w of G such that $w(e) = 1$ for every edge e with head u and $w(e) = 0$ for every edge with tail u.*

Proof. We may assume that every edge of G is in a dicycle (since edges not in a cycle can be given any weight we want). We can also assume that G is 1-weak, and hence 1-strong. By 4.1, there is a 0/1-valued weighting w. Let X_w be the set of all vertices v such that there is a dipath P of G from u to v where $w(e) = 0$ for each edge $e \in E(P)$, and choose w with X_w maximal. Let D^+ be the set of edges ab with $a \in X$ and $b \notin X$, and let D^- be the set of edges ab with $a \notin X$ and $b \in X$. From the definition of X_w, it follows that $w(e) = 1$ for each $e \in D^+$. Moreover, we claim that $w(e) = 0$ for each edge $e \in D^-$. To see this, observe that e is in a dicycle C, because G is 1-strong, and so C contains an edge in D^+; and since $w(C) = 1$, it follows that $w(e) = 0$, as we claimed.

Define w' by:

$$w'(e) = \begin{cases} w(e) - 1 & \text{if } e \in D^+, \\ w(e) + 1 & \text{if } e \in D^-, \\ w(e) & \text{otherwise.} \end{cases}$$

Then w' is a 0/1-valued weighting. But $X_w \subseteq X_{w'}$, and so $X_{w'} = X_w$ from the choice of w. Consequently $D^+ = \emptyset$, and therefore $X_w = V(G)$ since G is 1-strong. We deduce that $w(f) = 1$ for every $f = vu$ with head u since there is a dipath P from u to v with $w(e) = 0$ for each edge $e \in E(P)$, and adding f to P makes a dicycle. Moreover, for every edge f with tail u, since f belongs to a dicycle that contains an edge with head u, it follows that $w(f) = 0$. This proves 5.2. ▮

Similarly, we can obtain a 0/1-valued weighting such that $w(e) = 0$ for every edge e with head u, and $w(e) = 1$ for every edge with tail u. It is easy to convert the proof above to a poly-time algorithm that, given as input a 1-strong digraph, a 0/1-valued weighting, and a vertex u, outputs a weighting as in 5.2.

If e is an edge of a digraph G, we denote by G/e the digraph obtained by contracting e. If G is a digraph, we say an edge $e = uv$ is a *singular edge* of G if $u \neq v$ and either no edge different from e has head v or no edge different from e has tail u, and the operation of contracting this edge is called *singular contraction.* If H can be obtained from a subdigraph G' of a digraph G by repeated singular contraction, H is said to be a *butterfly minor* of G.

5.3. *If e is a singular edge of G, then G is weightable if and only if G/e is weightable. Consequently, if G is weightable and H is a butterfly minor of G, then H is weightable.*

Proof. Let $e = uv$ be a singular edge of G. We may assume that no edge different from e has tail u (the other case is the same, by reversing the direction of all edges). Suppose that G is weightable. By 5.2, there is a 0/1-valued weighting w of G with $w(e) = 0$. But then the restriction of w to $E(G')$ is a weighting of G/e, since for every dicycle C' of G', either C' is a dicycle of G, or there is a dicycle C containing e with $C/e = C'$.

Conversely, suppose that w' is a weighting of G/e, and define $w(e) = 0$ and $w(f) = w'(f)$ for each edge $f \neq e$ of G. For every dicycle C of G, either C is a dicycle of G/e, or $e \in E(C)$ and C/e is a dicycle of G/e, and it follows that w is a weighting of G. This proves 5.3. ▮

6 Some easy reductions

Let us say a digraph is *simple* if it has no loops or parallel edges (it might have directed cycles of length two). As we said before, to understand the weightable digraphs, it suffices to understand those that are 1-strong, and we can also assume that G is simple. It is slightly less clear that we may assume that G is 2-strong, so let us prove that. Suppose that G is a 1-strong digraph, and $c \in V(G)$, and there is a partition (A, B) of $V(G) \setminus \{c\}$ into two nonempty sets A, B, such that no edge of G is from B to A. Let G_1 be the digraph made from $G[A \cup \{c\}]$ by adding an edge from $a \in A$ to c for each $ab \in E(G)$ with $b \in B$, and similarly let G_2 be obtained from $G[B \cup \{c\}]$ by adding an edge from c to $b \in B$ for each such $ab \in E(G)$. Then remove any parallel edges in G_1 and G_2.

See Figure 5. There will be similar drawings for other constructions, so let us explain our conventions for drawing constructions/decompositions. Ovals indicate sets of vertices. Thick arrows indicate (possibly empty) *sets* of edges, all going in the direction of the arrow, and squiggly lines indicate sets of edges that may go in either direction. When the position of one end of an arrow or squiggly line matches in the drawing of G and G_i for some $i \in \{1, 2\}$, it indicates that those edges of G are to be “rerouted” in G_i in the way depicted; for instance, in Figure 5, all edges from A to B in G get rerouted so they now go from A to c in G_1 and from c to B in G_2. There are a few further conventions that we explain later.

6.1. *If G, c, A, B, G_1, G_2 are as above (or in Figure 5), then G is weightable if and only if G_1, G_2 are both weightable.*

Proof. Since G is 1-strong, it follows that for each $a \in A$ there is a dipath from c to a, and this path is a subpath of $G[A \cup \{w\}]$ since there is no edge from B to A. Consequently G_1 is 1-strong, and similarly so is G_2.

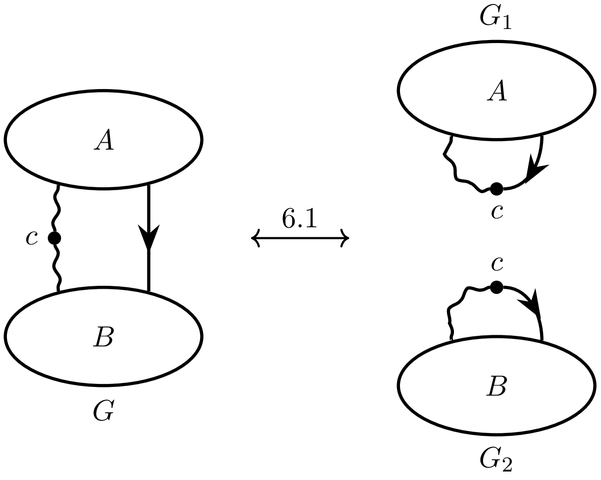


Figure 5: Construction for building non-2-strong weightable digraphs.

Suppose first that w_i is a 0/1-valued weighting of G_i for $i = 1, 2$. By 5.2, we may assume that $w_1(e) = 1$ for every edge of G_1 with head c, and $w_2(e) = 1$ for every edge of G_2 with tail c. For each $e \in E(G)$, define $w(e)$ by: $w(e) = w_i(e)$ if $e \in E(G_i)$ for $i = 1, 2$, and $w(e) = 1$ if e is from A to B. We claim w is a weighting of G. To see this, it suffices to check that $w(C) = 1$ for every dicycle C of G that is not a subdigraph of either of $G[A \cup \{c\}], G[B \cup \{c\}]$. Such a cycle C contains an edge ab from A to B, and since there is no edge from B to A, it follows that $c \in C$, and C consists of the union of a dipath P_1 of $G[A \cup \{c\}]$ from c to a, the edge ab, and a dipath P_2 of $G[B \cup \{c\}]$ from b to c. But adding the edge bc to P_1 makes a dicycle of G_1, and since $w_1(bc) = 1$, we deduce that $w_1(P_1) = 0$. Similarly $w_2(P_2) = 0$, and so $w(C) = 1$ as desired. Thus w is a weighting of G.

Conversely, suppose that one of G_1, G_2, say G_1, is not weightable, and let dicycles C, C' of G_1 disagree on some bad triple $\{u, v, w\}$ of G_1. Define a cycle D of G as follows. If every edge of C belongs to G let $D = C$. Otherwise, exactly one edge of C is not an edge of G, and any such edge is from some $a \in A$ to c where there is an edge $ab \in E(G)$ for some $b \in B$. Choose a dipath P of $G[B]$ from b to c, and let D be the dicycle of G made by the union of the dipath of C from c to a, the edge ab, and P. Define D' similarly, starting from C'. Then (u, v, w) is in order in D and (w, v, u) is in order in D', and so G is not weightable. This proves 6.1. ∎

In view of 6.1, it suffices to understand the 2-strong weightable digraphs. Let us see also that we can assume G is 3-weak. Suppose then that G is 2-strong and $|V(G)| \geq 4$, and so G is 2-weak, and assume G is not 3-weak. Choose distinct $c, d \in V(G)$ and a partition (A, B) of $V(G) \setminus \{c, d\}$ such that there is no edge of G between A, B in either direction. Let G_1 be obtained from $G[A \cup \{c, d\}$ by adding an edge cd and an edge dc if they are not already present, and define G_2 similarly from $G[B \cup \{c, d\}]$. (See Figure 6. In this figure and future (de)compositions, thin arrows between vertices represent single edges. Additionally, there may be edges between vertices outside of the labeled vertex sets that are not shown in the figure. For example, there may be an edge from c to d in G.)

6.2. *With G, c, d, A, B, G_1, G_2 as above (or in Figure 6), then G is weightable if and only if G_1, G_2 are weightable.*

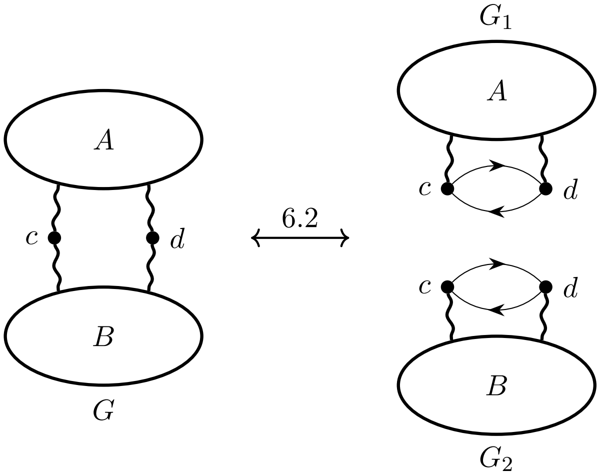


Figure 6: Construction for building non-3-weak weightable digraphs.

Proof. Suppose first that w_i is a 0/1-valued weighting of G_i for $i = 1, 2$. By 5.2, we may assume that $w_1(cd) = w_2(cd)$. Since $w_i(cd) = 1 - w_i(dc)$ for $i = 1, 2$, it follows that $w_1(dc) = w_2(dc)$. For each edge $e \in E(G)$, define $w(e) = w_i(e)$ where $e \in E(G_i)$. If C is a dicycle of G that is not a cycle of either of G_1, G_2, then, by exchanging c, d if necessary, we may assume that C consists of a dipath P_1 of G_1 from c to d and a dipath P_2 of G_2 from d to c. Since adding dc to P_1 makes a dicycle of G_1, it follows that $w(P_1) = w_1(P_1) = 1 - w_1(dc)$, and similarly $w(P_2) = 1 - w_2(cd)$. Since $w_1(dc) + w_2(cd) = w_1(dc) + w_1(cd) = 1$, it follows that $w(C) = 1$, as desired. Thus w is a weighting of G.

Conversely, suppose that one of G_1, G_2, say G_1, is not weightable, and let C, C' disagree on $\{u, v, w\}$ in G_1. We claim that there is a dipath P_2 of $G[B \cup \{c, d\}]$ from c to d. To see this, let $b \in B$. Since G is 2-strong, there is a dipath from c to b in $G \setminus \{d\}$, and this is therefore a path of $G[B \cup \{c, d\}]$. Similarly there is a dipath of $G[B \cup \{c, d\}]$ from b to d, and the union of these paths includes a dipath of $G[B \cup \{c, d\}]$ from c to d, as claimed. Let $P(cd)$ be such a path, and similarly let $P(dc)$ be a dipath of $G[B \cup \{c, d\}]$ from d to c. If C contains one of cd, dc, say cd, let D be the cycle consisting of the path of C from d to c together with $P(cd)$, and define D similarly if $dc \in E(C)$. (Not both $cd, dc \in E(C)$, so this is well-defined.) If $cd, dc \notin E(C)$ let $D = C$. Define D' similarly, starting from C'. Then D, D' are dicycles of G that disagree on $\{u, v, w\}$, and so G is not weightable. This proves 6.2. ▮

In view of 6.2, it suffices to understand the simple, 2-strong, 3-weak, weightable digraphs. More exactly, we have shown so far that:

- every weightable digraph G can be built from simple, 2-strong, 3-weak, weightable digraphs by operations that preserve being weightable; and
- if we have a poly-time algorithm to decide whether any 2-strong, 3-weak digraph is weightable, then in poly-time we can decide whether a general digraph is weightable, and if so, find a weighting of it.

7 The planar decomposition

Let G be a digraph drawn in a plane or 2-sphere; then each edge e is an open line segment, and we speak of *points* of e to refer to points in this line segment. Let F be a simple closed curve F, such that F passes through no vertex of G, and passes through at most one point of the interior of each edge, and crosses each edge that it intersects. Let us call such a curve F a *cut-curve*. Let us say a *gap* of F is a line segment in F with both ends in the drawing and no internal point in the drawing. Thus, its ends necessarily belong to the interiors of distinct edges. A *change* in F is a gap with ends in two edges e, f, such that exactly one of e, f has head inside the disc bounded by F (that is, e, f cross F in opposite directions). It follows that there is an even number of changes, and we call this number the *change number* of F. We are interested in cut-curves with change number two; they will give the construction we need to build all weightable diplanar digraphs.

Some terminology: if G is a digraph drawn in a 2-sphere, and F is a cut-curve, and A is the set of vertices drawn within one of the two discs defined by F, we want to consider the digraph and drawing obtained by *squishing* A, that is, deleting all edges with both ends in A and then identifying all the vertices in A into one vertex, forming a digraph G_1. (For the moment, $G[A]$ might not be 1-weak, so this is not the same as contracting the edges of $G[A]$.) This operation might introduce parallel edges, but not loops. Thus, the edges of G after squishing are the edges of G before squishing that have at least one end not in A, but the incidence relation between edges and vertices has changed. In particular, if in G an edge has head in A and tail in $V(G) \setminus A$, then in G_1, its head is the new vertex a and its tail is the same as before. So, for clarity, we speak of the *G-head* or *G_1-head* of edges, and similarly speak of *G-tail* and *G_1-tail*.

We observe, first:

7.1. *Let G be a digraph drawn in a 2-sphere, and let F be a cut-curve with change number two. Let A be the set of vertices of G inside one of the discs bounded by F, and let $B = V(G) \setminus A$. Let G_1 be obtained from G by squishing B into a vertex b, and define G_2, a similarly. (See Figure 7.) If G_1, G_2 are weightable and 1-strong then G is weightable.*

Proof. Suppose G_1, G_2 are weightable and 1-strong. By 5.2, there is a 0/1-valued weighting w_1 of G_1 such that $w_1(e) = 1$ for every edge of G_1 with head b and $w_1(e) = 0$ for every edge with tail b. Similarly, there is a 0/1-valued weighting w_2 of G_2 such that $w_2(e) = 1$ for every edge of G_2 with tail a and $w_1(e) = 0$ for every edge with head a. For each edge $e \in E(G)$, choose $i \in \{1, 2\}$ with $e \in E(G_i)$ and let $w(e) = w_i(e)$ (if e belongs to both G_1, G_2 then it crosses F and $w_1(e) = w_2(e)$, so this is well-defined). We claim that w is a weighting of G. Let C be a dicycle of G, we may assume that C is not a cycle of G_1 or of G_2, so C crosses F at least twice. But if we enumerate the edges of C that cross F in their cyclic order in F, then every consecutive pair cross F in opposite directions, and since F has change number two, it follows that C crosses F exactly twice. Hence there are two edges e, f of C that cross F such that the component P of $C \setminus \{e, f\}$ from the head of e to the tail of f is a path of G_1 and the component Q of $C \setminus \{e, f\}$ from the head of f to the tail of e is a path of G_2. The edges of P, together with e, f, make a dicycle of G_1, and since $w_1(e) = 0$ and $w_1(f) = 1$, if follows that $w_1(P) = w(P) = 0$. Similarly $w(Q) = 0$, and so

$$w(C) = w(P) + w(Q) + w(e) + w(f) = 1$$

as required. This proves 7.1. ∎

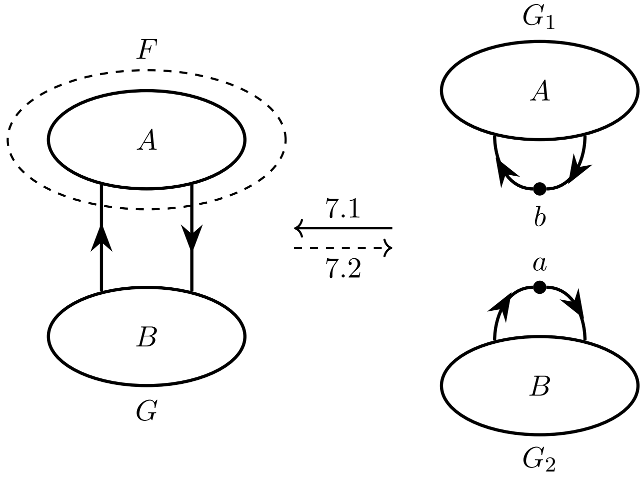


Figure 7: Construction for building planar weightable digraphs. Here, the pictures represent drawings of the relevant graphs in the plane or 2-sphere. It is important that a closed curve F separating A and B has change number two, as depicted. Also note that this construction admits only a partial converse, described in 7.2.

Thus, to give a construction for all planar 1-strong weightable digraphs, it would suffice to show that each such digraph (except some small ones that we would consider "building blocks") admits a cut-curve with change number two such that the digraphs G_1, G_2 as in 7.1 are weightable and smaller than G. But this needs some care. It is not enough to find a cut-curve with change number two such that $|A|, |B| \geq 2$ (where A, B are as in 7.1), because the digraphs G_1, G_2 might not be weightable. Indeed, even if G is diplanar and $G[A], G[B]$ are 1-weak, the digraphs G_1, G_2 still might not be weightable. This is shown by the digraph in Figure 8.

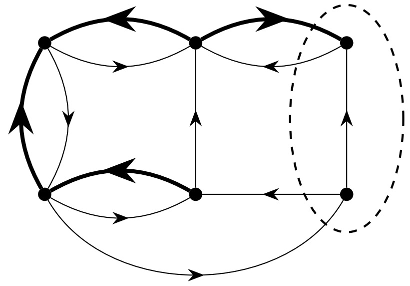


Figure 8: A cut-curve with change number two in a 1-strong diplanar weightable digraph; contracting the edge inside the cut-curve makes the digraph not weightable.

So what condition on G and F do we need to ensure that G_1, G_2 are weightable? The following is the most general we have found. (If P is a dipath and $u, v \in V(P)$, and u is earlier than v in P, then $P[u, v]$ denotes the subpath of P from u to v.)

7.2. *Let G be a digraph drawn in a 2-sphere, and let F be a cut-curve with change number two. Suppose that for every two edges e, f crossing F in opposite directions, there is a dicycle of G that* *contains both e, f. Then G_1, G_2 (defined as in 7.1) are weightable.*

Proof. Let A, B, a, b be as in 7.1. We will show that G_2 is weightable. Suppose not; then are two dicycles C_1, C_2 of G_2 that disagree on some triple $\{x, y, z\}$ of G_2. For $i = 1, 2$, if $a \notin V(C_i)$ let $C'_i = C_i$. If $a \in C_i$, let e_i, f_i be the edges of C_i that have G_2-head a and G_2-tail a respectively, let P_i be a dipath of $G[A]$ from the G-head of e_i to the G-tail of f_i, let $Q_i = C_i \setminus a$, and let C'_i be the dicycle of G formed by the union of P_i, Q_i, e_i and f_i. (Such a path P_i exists since there is a dicycle of G containing e_i, f_i, and since F has change number two, this cycle only crosses F twice.) Since G is weightable, C'_1, C'_2 do not disagree on $\{x, y, z\}$ in G, and so one of x, y, z equals a, say z. If some vertex $w \in A$ belongs to both $V(P_1), V(P_2)$, then the order of x, y, z in C_i is the same as the order of x, y, w in C'_i, for $i = 1, 2$, so C'_1, C'_2 disagree on $\{x, y, w\}$, a contradiction. Thus, P_1, P_2 are vertex-disjoint, and so we may assume that e_1, e_2, f_2, f_1 appear in this order in the cyclic order of the edges that cross F.

Now $x, y \in V(Q_1) \cap V(Q_2)$, and since C_1, C_2 disagree on $\{x, y, a\}$ in G_2, we may assume that x is before y in Q_1 and y is before x in Q_2. Since y is before x in Q_2, there is a minimal dipath $Q_2[r, s]$ of Q_2, such that $r, s \in V(Q_1)$, and s is strictly before r in Q_1. It follows that no edge or internal vertex of $Q_2[r, s]$ belongs to Q_1 (because if some t belongs to $V(Q_1)$ and the interior of $Q_2[r, s]$, then either t is after s or before r in Q_1, and in either case the minimality of $Q_2[r, s]$ is contradicted). Let R_2 be the subpath of Q_2 from the G-head of f_2 to r, and let S_2 be the subpath from s to the G-tail of e_2. Thus Q_2 is the concatenation of $R_2, Q_2[r, s]$, and S_2. Let S_1 be the subpath of Q_1 from the G-head of f_1 to s, and let R_1 be the subpath from r to the G-tail of e_1. Thus Q_1 is the concatenation of S_1, $Q_1[s, r]$, and R_1. (See Figure 9.)

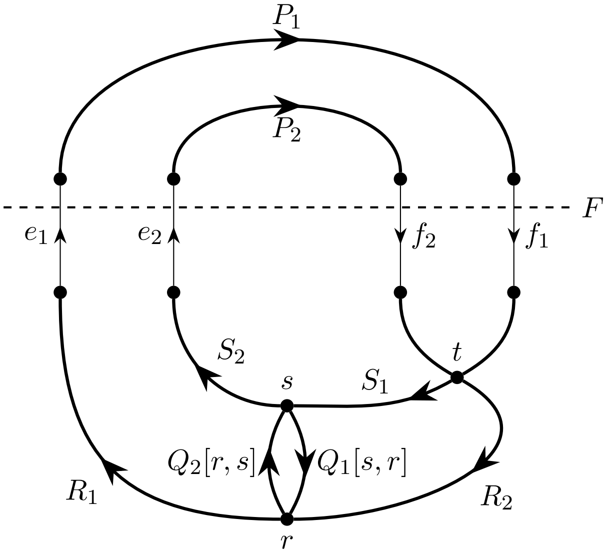


Figure 9: One of the cases in the proof of 7.2, where R_2 and S_1 intersect. Here, the arrows, with the exceptions of e_1, e_2, f_1, f_2, represent paths (possibly of length 0), not edges. There may be other intersections between some of the paths; the simplest possibility is shown. The other cases correspond to other ways that $R_1 \cup R_2$ and $S_1 \cup S_2$ might intersect. (There must be some intersection since e_1, e_2, f_2, f_1 appear in that cyclic order on the cut F.)

If some vertex $t \in V(R_2)$ also belongs to R_1 with $t \neq r$, then $t \neq s$, and t, r, s appear in this order in Q_2, and in the order s, r, t in Q_1, so C_1', C_2' disagree on $\{r, s, t\}$, a contradiction. If there is some $t \in V(R_2) \cap V(S_1)$ then $t \neq r, s$ and again C_1', C_2' disagree on $\{r, s, t\}$. (This is the case depicted in Figure 9.) Similarly, $V(S_1 \cap S_2) = \{s\}$ and $V(S_2 \cap R_1) = \emptyset$. Consequently $R_1 \cup R_2$ and $S_1 \cup S_2$ are vertex-disjoint dipaths, which contradicts that e_1, e_2, f_2, f_1 appear in this order in the cyclic order of the edges that cross F.

Hence there are no such C_1, C_2, and so G_2 is weightable, and similarly G_1 is weightable. This proves 7.2. ▮

In the same notation, we say $w \in A$ is a *centre* for A if for each edge uv with $u \in B$ and $v \in A$ there is a dipath of $G[A]$ from v to w, and for each edge uv with $u \in A$ and $v \in B$ there is a dipath of $G[A]$ from w to u. We define a centre for B similarly. An easy way to arrange that a cut-curve with change number two has the property of 7.2 is to ensure that A, B have centres. A *central cycle* for A is a dicycle of $G[A]$ such that all its vertices are centres for A.

In view of 7.2 and 7.1, we might now look for a theorem that if G is 1-strong, weightable, and drawn in the plane, then it admits a cut-curve as in 7.2, unless G is already suffiently simple to be understood. But we don't really need that, because we already reduced the general problem of constructing all weightable digraphs to constructing those that are 2-strong and 3-weak; and we will show later that if G is 2-strong, 3-weak, planar and weightable then it is diplanar. So we could confine ourselves to finding cut-curves in 2-strong 3-weak diplanar weightable digraphs if we wanted, and this extra information will be helpful.

But that turns out to be *too much* information. We can build such digraphs from smaller digraphs, but the smaller digraphs need not be 2-strong, even if the digraph we are building is 2-strong. That looks like a difficulty; but, fortunately, the same building method also serves to build all strong diplanar weightable digraphs rather than just those that are 2-strong. Thus, to get a result saying that we can build all digraphs with property X from smaller digraphs with property X, we will take property X to be "strong, diplanar, and weightable".

Let S be a 2-sphere, and let C_1, C_2 be dicycles, drawn in S, and bounding closed discs with disjoint interiors. Fix an orientation "clockwise" of the 2-sphere. The rotation of C_1 around its disc defines an orientation of the 2-sphere, and if C_1, C_2 define the same orientations of the 2-sphere, we say they are *similarly-oriented*. A great merit of working with diplanar drawings is that, if C_1, C_2 are similarly-oriented cycles that bound discs in the 2-sphere with disjoint interiors, then C_1, C_2 are vertex-disjoint, as is easily seen.

Let us assign an orientation "clockwise" to the plane. In a digraph drawn in the plane, each dicycle rotates clockwise or counterclockwise in the natural sense; a *clockwise cycle* means a dicycle that rotates clockwise, and a *counterclockwise cycle* is defined similarly. (It is important that we are working with drawings in the plane rather than in a 2-sphere.)

We extend the "similarly-oriented" terminology to planar drawings in the natural way; that is, we say two dicycles in a planar drawing are similarly-oriented if they are similarly-oriented in the 2-sphere obtained by one-point compactification of the plane. Thus, if C_1, C_2 are similarly-oriented vertex-disjoint dicycles in a planar drawing, then each bounds a unique disc in the plane, and these discs might be disjoint, or one might contain the other. In the first case, either both C_1, C_2 are clockwise in the plane, or both are counterclockwise. In the second case, one is clockwise in the plane and the other is counterclockwise (which might seem paradoxical at first sight, since we called them "similarly-oriented"). We will prove:

7.3. *Let G be a 1-strong, weightable digraph with a diplanar drawing in the 2-sphere. Suppose that there are two vertex-disjoint similarly-oriented cycles D_1, D_2 in G, and choose D_1, D_2 such that the annulus between them is minimal. Then G admits a cut-curve F with change number two, such that, in the usual notation, D_1 is a central cycle for A and D_2 is a central cycle for B.*

Proof. With a given planar drawing, for each cycle C of G, let $\mathrm{ins}(C)$ be the closed disc in the plane bounded by C. Since G admits a diplanar drawing in a 2-sphere in which D_1, D_2 bound disjoint discs and D_1, D_2 are similarly-oriented, G also admits a diplanar drawing in the plane such that D_1 is clockwise, and D_2 is counterclockwise, and $\mathrm{ins}(D_1) \subseteq \mathrm{ins}(D_2)$. Fix such a drawing. Let Σ be the annulus in the plane between D_1, D_2 (including D_1, D_2). The choice of D_1, D_2 implies that:

(1) *There is no dicycle C in Σ different from D_1, D_2 such that* $\mathrm{ins}(D_1) \subseteq \mathrm{ins}(C) \subseteq \mathrm{ins}(D_2)$.

Let us say a set of dicycles of G is *free* if its members bound discs in the plane with disjoint interiors, and each of them is drawn in $\mathrm{ins}(D_2)$ and vertex-disjoint from D_2. If $\mathcal{C}$ is a free set of dicycles, let $U(\mathcal{C})$ denote the union of the members of $\mathcal{C}$, and let $I(\mathcal{C})$ be the subdigraph of G consisting of all vertices and edges that are drawn in or inside some $C \in \mathcal{C}$.

Choose a free set $\mathcal{C}$ of dicycles with $D_1 \in \mathcal{C}$ such that $U(\mathcal{C})$ is 1-strong and, subject to that, with $I(\mathcal{C})$ maximal. Thus $I(\mathcal{C})$ is 1-strong.

(2) *Every ear for $I(\mathcal{C})$ has a vertex in D_2.*

Let P be an ear for $I(\mathcal{C})$, and suppose that $V(P \cap D_2) = \emptyset$. Consequently P is drawn in Σ, and either P is a dipath with both ends in $U(\mathcal{C})$ and no internal vertex or edge in $I(\mathcal{C})$, or P is a dicycle with one vertex in $U(\mathcal{C})$ and with no other vertex or edge in $I(\mathcal{C})$. Since $U(\mathcal{C})$ is 1-strong, there is a dipath Q (possibly of length zero) of $U(\mathcal{C})$ such that $P \cup Q$ is a dicycle. Since Q intersects $I(\mathcal{C})$ only in $U(\mathcal{C})$, it follows that for each $C \in \mathcal{C}$, either $\mathrm{ins}(C) \subseteq \mathrm{ins}(P \cup Q)$, or the interiors of $\mathrm{ins}(C), \mathrm{ins}(P \cup Q)$ are disjoint. Let $\mathcal{C}'$ be the set consisting of $P \cup Q$ and all $C \in \mathcal{C}$ such that $\mathrm{ins}(C), \mathrm{ins}(P \cup Q)$ are disjoint. From (1), $D_1 \in \mathcal{C}'$, and $\mathcal{C}'$ is free, and $I(\mathcal{C})$ is a proper subset of $I(\mathcal{C}')$, a contradiction. This proves (2).

(3) *For every edge e with one end in $V(U(\mathcal{C}))$ that does not belong to $I(\mathcal{C})$, there is a dipath containing e with one end in $V(U(\mathcal{C}))$, the other end in $V(D_1)$, and with no internal vertex in either set.*

Let $e = ab$, say. From the symmetry, we may assume that $a \in V(U(\mathcal{C}))$. Since G is 1-strong, e belongs to a dicycle C of G. Let P be the minimal subpath of C from b to $V(U(\mathcal{C})) \cup V(D_2)$ (this exists since $a \in V(U(\mathcal{C}))$. Let the ends of P be b, p. If $p \in V(U(\mathcal{C}))$ then the union of P and e is an ear violating (2), so $p \in V(D_2)$, and the union of P and e satisfies (3). This proves (3).

By (2), every edge with both ends in $V(U(\mathcal{C}))$ belongs to $I(\mathcal{C})$. Consequently, there is a cut-curve F, obtained by closely following the outer boundary of $U(\mathcal{C})$, such that the edges that cross F are precisely the edges e with exactly one end in $V(U(\mathcal{C}))$. If F has change number two, we are done, so we assume for a contradiction that F has change number at least four. Hence there are edges e_1, e_2, e_3, e_4, each crossing F and numbered according to the clockwise cyclic order defined by F, such that e_1, e_3 have tail in $V(U(\mathcal{C}))$ and e_2, e_4 have head in $V(U(\mathcal{C}))$. By (2), there are dipaths $P_1, \ldots, P_4$ such that e_i is an edge of P_i for $1 \le i \le 4$, and such that P_1, P_3 are from the tail of e_i to

D_2 and P_2, P_4 are from D_2 to the head of e_i, and for $1 \le i \le 4$, no internal vertex of P_i belongs to $V(U(\mathcal{C})) \cup V(D_2)$. Let P_i be from a_i to b_i for $1 \le i \le 4$.

(4) *For $1 \le i \le 4$, P_i, P_{i+1} are internally disjoint. (Here P_5 means P_1.)*

If w belongs to the interiors of P_i, P_{i+1}, then $P_i \cup P_{i+1}$ includes an ear for $I(\mathcal{C})$ violating (2). This proves (4).

Please refer to Figure 10 for the end of the proof.

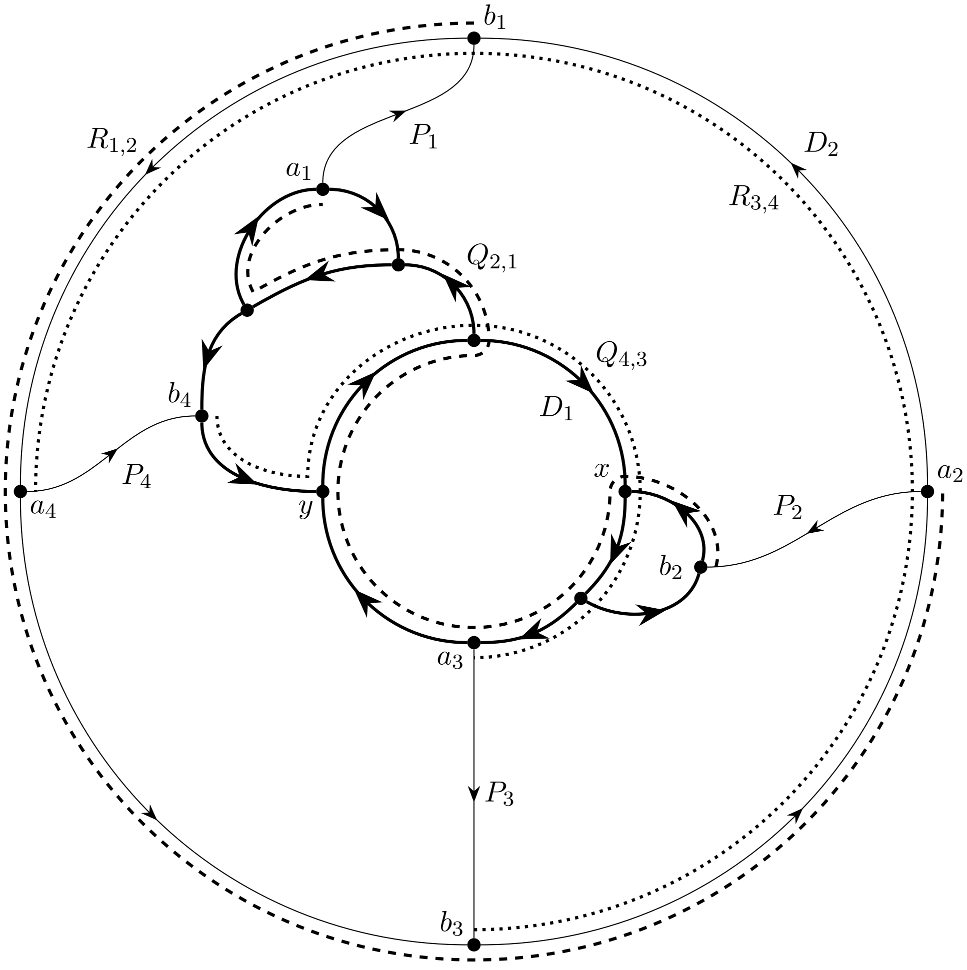


Figure 10: Part of the proof of 7.3. The innermost (clockwise) cycle is D_1, and the outermost (counterclockwise) cycle is D_2. The union of the thicker arrows in the middle of the figure is $U(\mathcal{C})$. The dashed lines mark $R_{1,2}$ and $Q_{2,1}$, and the dotted lines mark $R_{3,4}$ and $Q_{4,3}$.

Let $Q_{2,1}$ be a path of $U(\mathcal{C})$ from the last vertex of P_2 to the first vertex of P_1. Suppose that $b_1 = a_2$. Since the drawing of G is diplanar and D_2 is a counterclockwise circuit that intersects the dicycle $P_1 \cup P_2 \cup Q_{2,1}$, it follows that $P_1 \cup P_2 \cup Q_{2,1}$ is counterclockwise, and since e_1, e_2, e_3, e_4 are in clockwise order in F, it follows that b_1, a_2, b_3, a_4 are all equal, contradicting that the drawing is diplanar. Thus $b_1 \neq a_2$, and similarly $b_3 \neq a_4$. Let $R_{1,2}$ be the subpath of D_2 from the last vertex of P_1 to the first vertex of P_2. Then $P_2 \cup Q_{2,1} \cup P_1 \cup R_{1,2}$ is a counterclockwise dicycle, and by (1), $\text{ins}(D_1)$ is not a subset of $\text{ins}(P_2 \cup Q_{2,1} \cup P_1 \cup R_{1,2})$. So $\text{ins}(D_1)$ is a subset of $\text{ins}(P_2 \cup Q_{2,1} \cup P_1 \cup R_{2,1})$, where $R_{2,1}$ is the dipath of D_2 from a_2 to b_1. (This cycle is not a dicycle.) Similarly, $\text{ins}(D_1)$ is a subset of $\text{ins}(P_1 \cup Q_{4,3} \cup P_3 \cup R_{4,3})$, where $Q_{4,3}, R_{4,3}$ are respectively a dipath of $U(\mathcal{C})$ from b_4 to a_3 and a dipath of D_2 from a_4 to b_3. Consequently, $P_2 \cup Q_{2,1} \cup P_1$ and $P_4 \cup Q_{4,3} \cup P_3$ are not vertex-disjoint. Let x, y be the first and last vertices of the path $P_2 \cup Q_{2,1} \cup P_1$ that belong to $P_4 \cup Q_{4,3} \cup P_3$. Since the drawing is diplanar it follows that $x \neq y$, and hence from planarity, y is strictly earlier than x in the path $P_4 \cup Q_{4,3} \cup P_3$. But then the dicycles $P_2 \cup Q_{2,1} \cup P_1 \cup R_{1,2}$ and $P_4 \cup Q_{4,3} \cup P_3 \cup R_{3,4}$ disagree on $\{x, y, b_1\}$ (where $R_{4,3}$ is the dipath of D_2 from a_4 to b_3), a contradiction. This proves 7.3. ▮

A drawing in a 2-sphere can be converted to a drawing in the plane by removing from the 2-sphere one point in some region of the drawing. Let us call this *puncturing* the drawing. Let us say a drawing in the plane (without crossings) of a digraph G is *circular* if the origin belongs to one of the regions, and each edge of G is drawn as a curve that moves monotonically in a clockwise direction around the origin. (This is a more exact restatement of the definition of "circular" given in the introduction.) Theorem 4.2 of [1] implies that if a 1-strong digraph G admits a diplanar drawing in a 2-sphere with no two vertex-disjoint similarly-oriented cycles, then G admits a circular drawing in the plane. Here is a slight strengthening (we omit its proof): one can obtain a circular drawing by puncturing the 2-sphere drawing at some point inside a region bounded by a directed cycle.

Thus, from 7.3 we deduce our first main result:

7.4. *Let G be a 1-strong weightable digraph with a diplanar drawing in the 2-sphere. Then either:*

- *the drawing of G can be constructed by the construction of 7.1 from diplanar drawings of two smaller 1-strong weightable digraphs, in such a way that the sets A, B of 7.1 have similarly-oriented central cycles, or*
- *by puncturing the drawing one can obtain a circular drawing in the plane.*

8 Carvings of planar digraphs

There is another interesting way to view 1-strong diplanar weightable digraphs, following an approach in [8] for decomposing planar graphs. Let V be a finite set with $|V| \geq 2$. Two subsets $A, B \subseteq V$ *cross* if $A \cap B$, $A \setminus B$, $B \setminus A$ and $V \setminus (A \cup B)$ are all nonempty. A *carving* in V is a set $\mathcal{C}$ of subsets of V, such that:

- $\emptyset, V \neq \mathcal{C}$;
- no two members of $\mathcal{C}$ cross; and
- $\mathcal{C}$ is maximal with this property.

It follows that if $A \in \mathcal{C}$ then $V \setminus A \in \mathcal{C}$, and $\{v\} \in \mathcal{C}$ for each $v \in V$.

One can view a carving as arising from a tree, as follows. (The leaves of a tree are its vertices of degree 1.)

8.1. *Let V be a finite set with $|V| \geq 2$, let T be a tree in which every vertex has degree 1 or 3, and let τ be a bijection from V onto the set of leaves of T. For each edge e of T let $T_1(e), T_2(e)$ be the two components of $T \setminus e$, and let*

$$\mathcal{C} = \{\{v \in V : \tau(v) \in V(T_i(e))\} : e \in E(T), i = 1, 2\}.$$

Then τ is a carving in V. Conversely, every carving in V arises from some tree T and bijection τ in this way.

(This is theorem 1.1 of [8].)

The main result of [8] is a poly-time algorithm that, given as input some planar graph G with $|V(G)| \geq 2$, finds a carving $\mathcal{C}$ of $V(G)$ such that $\max_{C \in \mathcal{C}} |\delta(C)|$ is as small as possible, where $\delta(C)$ denotes the set of edges with an end in C and an end in $V(G) \setminus C$. (Its running time was $O((|V(G)| + |E(G)|)^2)$, where the multiplicative constant was reasonable.) But now we want to use carvings for planar digraphs, and instead of minimizing $\max_{C \in \mathcal{C}} |\delta(C)|$, we want to minimize something else, related to change number.

This can be done in a few different ways, but the neatest is only possible if we assume that the digraph G is 1-strong, 2-weak, and loopless. If G is drawn in the plane and $A \subseteq V(G)$ such that $G[A]$ and $G[V(G) \setminus A]$ are both nonnull and 1-weak, the set of edges between C and $V(G) \setminus C$ is a "bond" of the graph underlying G, that is, a minimal edge-cutset, and so it corresponds to a cycle of the dual graph. Hence there is a cut-curve F separating $C, V(G) \setminus C$, and the edges crossing F are the edges in $\delta(C)$. Let us say the *change number* of C is the change number of F.

Let us say a *bond carving* of a planar digraph G is a carving $\mathcal{C}$ of G such that $G[C]$ is 1-weak for all $C \in \mathcal{C}$. The *diwidth* of $\mathcal{C}$ is the maximum over all $C \in \mathcal{C}$ of the change number of C. We will prove:

8.2. *Let G be a 1-strong, 2-weak, loopless digraph with a diplanar drawing in the plane. Then G is weightable if and only if G admits a bond carving of diwidth two.*

To prove this we need a couple of lemmas. First:

8.3. *Let G be a 1-strong, 2-weak, loopless digraph drawn in the plane, and let A, B be a partition of $V(G)$ into two nonnull subsets such that $G[A], G[B]$ are both 1-weak. Let G_1, G_2 be the drawings obtained by contracting all the edges of $G[A]$ (respectively, all edges of $G[B]$). Then both G_1, G_2 are 1-strong, 2-weak, and loopless, and if they both admit bond carvings of diwidth two then so does G.*

Proof. Clearly G_1, G_2 are 1-strong and loopless. Let a be the vertex made by contracting $G[A]$ into a vertex, and define b similarly. To see that G_1, G_2 are 2-weak, note that $G_1 \setminus v$ is 1-weak for all $v \neq a$ since G is 2-weak, and $G_1 \setminus a = G[B]$ is 1-weak. So G_1 is 2-weak, and similarly so is G_2.

Suppose that for $i = 1, 2$, $\mathcal{C}_i$ is a bond carving of G_i with diwidth two. Thus $\{a\} \in \mathcal{C}_1$ and $\{b\} \in \mathcal{C}_2$. Let

$$\mathcal{C}_1' = \{C \in \mathcal{C}_1 : a \notin C\} \cup \{A \cup (C \setminus \{a\}) : C \in \mathcal{C}_1 \text{ with } a \in C\},$$

and define $\mathcal{C}_2'$ similarly. Thus A, B belong to both $\mathcal{C}_1, \mathcal{C}_2$. Let $\mathcal{C} = \mathcal{C}_1' \cup \mathcal{C}_2'$; then $\mathcal{C}$ is a bond carving of G with diwidth two. This proves 8.3. ∎

Second, we need:

8.4. *Let G be 1-strong, 2-weak, and loopless, and suppose G admits a diplanar drawing in the plane such that every dicycle is clockwise and bounds a disc including the origin. Then G admits a bond carving of diwidth two.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$. Thus, we may assume that G is simple. The boundary of the infinite region is a cycle (because G is a 2-weak graph drawn in the plane), and it is a clockwise dicycle (because each of its edges is in a directed cycle, since G is 1-strong, and this directed cycle is clockwise by hypothesis). Let D be this cycle.

(1) *We may assume that no vertex v of D has outdegree one and indegree one.*

Suppose that $v \in V(D)$ has outdegree one and indegree one. Let uv be the edge with head v. If D has length two, then either u is a 1-vertex cutset or $|V(G)| = 2$, in either case contradicting that G is 2-weak. So D has length at least three. If $|V(G)| = 3$, then $G = D$ and the theorem is true; so we assume that $|V(G)| \geq 4$. Hence the digraph G_1 obtained by contracting the edge uv is 1-strong and 2-weak. From the inductive hypothesis, G_1 admits a bond carving $\mathcal{C}_1$ with diwidth two. Let w be the vertex made by identifying u, v under contraction. Let

$$\mathcal{C} = \{C \in \mathcal{C}_1 : w \notin C\} \cup \{\{u, v\} \cup (C \setminus \{w\}) : C \in \mathcal{C}_1, w \in C\} \cup \{\{u\}, \{v\}\}.$$

Then $\mathcal{C}$ is a bond carving of G of diwidth two, as required. This proves (1).

(2) *We may assume that $G \setminus \{e, f\}$ is 1-weak for all distinct $e, f \in E(D)$.*

Suppose not, and let $e = a_1b_1$ and $f = b_2a_2$. Since G is 2-weak and by (1), it follows that a_1, b_1, b_2, a_2 are all distinct. Since $G \setminus \{e, f\}$ is not weakly connected, it has exactly two weak components, one (say A) including the dipath of D from a_2 to a_1, and the other (B) including the dipath from b_1 to b_2. Let G_1 be obtained from G by contracting all edges in $G[A]$, and G_2 by contracting the edges of $G[B]$. Since A, B have change number two, and hence G_1, G_2 admit diplanar drawings that satisfy the hypothesis of the theorem, we deduce from the inductive hypothesis that G_1, G_2 both admit bound carvings of diwidth two. But then so does G, by 8.3. This proves (2).

From (2) and theorem 2.1 of [2] (or just by choosing an edge e of D in as few directed cycles as possible), we deduce that there is an edge $e \in E(D)$ such that $G \setminus e$ is 1-strong. Let $e = ab$.

(3) *$G \setminus e$ is 2-weak.*

Suppose not; then there is a vertex w of the path $D \setminus e$ with $w \neq a, b$, such that a, b belong to different weak components (say, A, B respectively) of $(G \setminus e) \setminus w$. Since $G \setminus w$ is weakly connected, $(G \setminus e) \setminus w$ has at most two weak components, and so $A \cup B = (G \setminus e) \setminus w$. Since $G \setminus e$ is 1-strong, there is a directed path in $G \setminus e$ from a to w, which is therefore a dipath of A, and similarly there is a dipath of B from w to b. Consequently there is a directed cycle of A that contains the edge of D with tail w, and a directed cycle of B that contains the edge of D with head w. These two cycles are both clockwise, by hypothesis, and share exactly one vertex, and both contain an edge incident with the infinite region of G, which is impossible since each bounds an open disc including the origin.

From (3), the boundary of $G \setminus e$ is a directed cycle. The edge e is incident with two regions of the drawing of G, one the infinite region outside D, and the other, r say, inside D. The cycle of G

that forms the boundary of r consists of e and a path P joining the ends of e, and P is part of the boundary of the infinite region of $G \setminus e$. Hence P is a directed path from a to b. From the inductive hypothesis and (3), $G \setminus e$ admits a bond carving $\mathcal{C}$ of diwidth two. Hence $\mathcal{C}$ is also a bond carving of G, and we claim that it still has diwidth two. If not, then there exists $A \in \mathcal{C}$, containing exactly one of a, b, say a, such that in the cyclic order of edges in $\delta(A)$, the edges before and after e have head in A. But that is impossible, since one of these two edges belongs to P. This proves 8.4. ▮

Proof of 8.2. Let G be a 1-strong, 2-weak, loopless digraph with a diplanar drawing in the plane, and we assume first that G is weightable. We must show that G admits a bond carving of diwidth two, and we prove this by induction on $|V(G)|$.

(1) *We may assume that there is no partition (A, B) of $V(G)$ with $|A|, |B| \geq 2$ such that $G[A], G[B]$ are both 1-weak and A has change number two and such that the digraphs G_1, G_2 obtained by squishing A and squishing B, respectively, are weightable.*

Suppose that such A, B exist. Then G_1, G_2 are both loopless, 1-strong, and 2-weak, and admit diplanar drawings in the plane (since A has change number two). Since $|A|, |B| \geq 2$, we can apply the inductive hypothesis to G_1, G_2, and deduce that they both admit bond carvings of diwidth two. But then so does G, by 8.3. This proves (1).

Suppose that there are two vertex-disjoint similarly-oriented cycles in G. By 7.3, G admits a cut-curve F with change number two, such that both parts A, B of the corresponding partition have central cycles. But then $G[A], G[B]$ are 1-weak, and the correspond digraphs G_1, G_2 are weightable, by 7.2, contrary to (1). Thus there are no two vertex-disjoint similarly-oriented cycles in G. Hence G admits a diplanar drawing in the plane such that every directed cycle is clockwise and bounds an open disc containing the origin. But then the result holds by 8.4. This proves the "only if" part of the theorem.

For the "if" part of the theorem, assume now that G admits a bond carving $\mathcal{C}$ of diwidth two, and we must prove that G is weightable. We proceed by induction on $|V(G)|$. Suppose that there exists $A \in \mathcal{C}$ such that $|A|, |B| \geq 2$, where $B = V(G) \setminus A$. Let G_1, G_2 be obtained by squishing A and squishing B, respectively. Then G_1, G_2 both have diplanar drawings, both are 1-strong, 2-weak and loopless, and both admit bond carvings of diwidth two. From the inductive hypothesis both are weightable. But then, from 7.1, G is weightable. Thus we may assume that there is no such $A \in \mathcal{C}$. Hence $\mathcal{C}$ is the set of all singleton subsets of $V(G)$ and their complements. From the maximality condition in the definition of a carving, it follows that $|V(G)| \leq 3$. Since G is diplanar it follows that G is weightable. This proves the "if" part, and so proves 8.2. ▮

Could we extend this further? If we want a bond carving, G must be 2-weak, because planar digraphs that are not 2-weak do not admit bond carvings. But we could drop the "2-weak" hypothesis if we were willing to weaken the requirement that the corresponding edge-cutsets must be bonds. Instead of requiring $G[A]$ to be 1-weak for each $A \in \mathcal{C}$, we could just ask that for each $A \in \mathcal{C}$, there is a cut-curve with change number two that separates A and $V(G) \setminus A$ in the natural sense. Every diplanar 1-strong weightable digraph admits a carving with this property, but we omit the details.

9 Nonplanar compositions

We will prove that every 2-strong, 3-weak, weightable digraph can be built from planar ones by composition operations that preserve being weightable, and in this section we explain the compositions we will use.

For brevity, if G is a digraph and A is a subdigraph or a subset of $V(G)$, and v is a vertex of G not in A, we write $v \to A$ to mean that every edge of G between v and A is from v to A, and $v \leftarrow A$ to mean that every edge of G between v and A is from A to v. If G is a digraph and $Y \subseteq V(G)$, a *Y-wing* is a subdigraph W with $Y \subseteq V(W)$ such that every edge of G with an end in $V(W) \setminus Y$ belongs to W. (Edges with both ends in Y might or might not belong to W.) A Y-wing W is *non-trivial* if $V(W) \neq Y$ and *non-separable* if it is non-trivial and $W \setminus Y$ is 1-weak. Two Y-wings W_1, W_2 are *internally disjoint* if $V(W_1) \cap V(W_2) = Y$ and $E(W_1) \cap E(W_2) = \emptyset$. If W is a Y-wing, a *W-path* is a dipath of W with distinct ends both in Y such that none of its internal vertices belong to Y (it might have no internal vertices).

From now on, we use the convention in figures that an arrow drawn inside a vertex set indicates the existence of a path between the two attached vertices; for instance, in Figure 11, there must be a path from y_2 to y_1 in W_1.

9.1. *Let G be a digraph, let y_1, y_2, y_3 be distinct, and let $Y = \{y_1, y_2, y_3\}$. Let W_1, W_2 be internally disjoint Y-wings with union G. Suppose that:*

- *there is a W_1-path from y_2 to y_1, and y_1 is a source of W_2, and y_2 is a sink of W_2;*
- *the digraph G_1 obtained from W_1 by adding the edges y_1y_2, y_1y_3, y_3y_2 (if they are not already present) is 1-strong and weightable; and*
- *the digraph G_2 obtained from W_2 by adding the edges y_2y_1, y_1y_3, y_3y_2 (if they are not already present) is 1-strong and weightable (equivalently, by 5.3, the digraph obtained from G_2 by contracting y_2y_1 is 1-strong and weightable).*

See Figure 11. Then G is weightable.

Proof. By 5.2, there is a 0/1-valued weighting w_1 of G_1 such that $w_1(y_1y_2) = w_1(y_1y_3) = 1$. Since there is a W_1-path from y_2 to y_1, it follows that $w_1(y_3y_2) = 0$. By 5.2, there is a 0/1-valued weighting w_2 of G_2 such that $w_2(e) = 1$ for every edge e with tail y_1, and we have $w_2(y_2y_1) = 0$. It follows that $w_2(y_3y_2) = 0$ since y_1-y_3-y_2-y_1 is a dicycle. For each $e \in E(G)$, define $w(e) = w_i(e)$ where $i \in \{1, 2\}$ and $e \in E(W_i)$ (i exists and is unique since $W_1 \cup W_2 = G$ and W_1, W_2 are internally disjoint).

For every W_2-path P from y to y' with $y, y' \in Y$, it follows that (y, y') is one of (y_1, y_3), (y_3, y_2), (y_1, y_2) since $y_1 \to A_i$ and $y_3 \leftarrow A_i$, and so yy' is an edge of G_1 that we call the *P-substitute*. We claim that:

(1) *For every W_2-path P, $w_2(P) = w_1(yy')$ where yy' is the P-substitute.*

Let P be from y to y'. Then (y, y') is one of $(y_1, y_3), (y_3, y_2), (y_1, y_2)$. If $(y, y') = (y_1, y_3)$, then P can be completed to a dicycle of G_2 by adding the edges y_3y_2 and y_2y_1. Since $w_2(y_3y_2), w_2(y_2y_1) = 0$, it follows that $w_2(P) = 1 = w_1(y_1y_3)$. Similarly, if $(y, y') = (y_3, y_2)$ then $w_2(P) = 0 = w_1(y_3y_2)$ (by

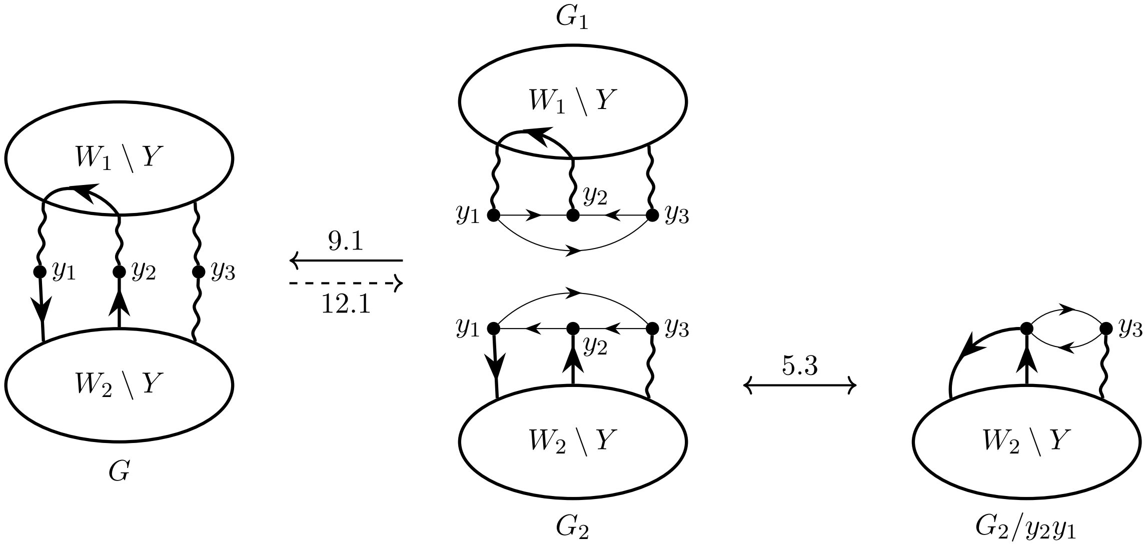


Figure 11: The first nonplanar construction.

adding y_2y_1 and y_1y_3), and if $(y, y') = (y_1, y_2)$ then $w_2(P) = 1 = w_1(y_1y_2)$ (by adding y_2y_1). This proves (1).

We claim that w is a weighting of G. To see this, let C be a dicycle of G. The W_2-paths included in C are pairwise edge-disjoint and include all edges of C not in W_1. By replacing each such W_2-path P by the P-substitute, we obtain a dicycle C' of G_1 such that $w_1(C') = w(C)$, and consequently $w(C) = 1$. This proves that w is a weighting, and so proves 9.1. ▮

For the second construction, it is easier to break it into two parts. First, we have:

9.2. *Let G be a digraph, let y_1, y_2, y_3, y_4 be distinct, and let $Y = \{y_1, \dots, y_4\}$. Let W_1, W_2 be internally disjoint Y-wings with union G. Suppose that:*

- *y_1 is a source of W_1, and y_2 is a sink of W_1, and y_3 is a source of W_2, and y_4 is a sink of W_2;*
- *there is a dipath of W_1 from y_1 to y_2, and there is a dipath of W_2 from y_3 to y_4;*
- *the digraph G_1 obtained from W_1 by adding a new vertex v_1 and the edges*

  $$v_1y_1, y_2v_1, y_3v_1, v_1y_4, y_2y_1$$

  is 1-strong and weightable; and
- *the digraph G_2 obtained from W_2 by adding a new vertex v_2 and the edges*

  $$y_1v_2, v_2y_2, v_2y_3, y_4v_2, y_4y_3$$

  is 1-strong and weightable.

See Figure 12. Then G is weightable.

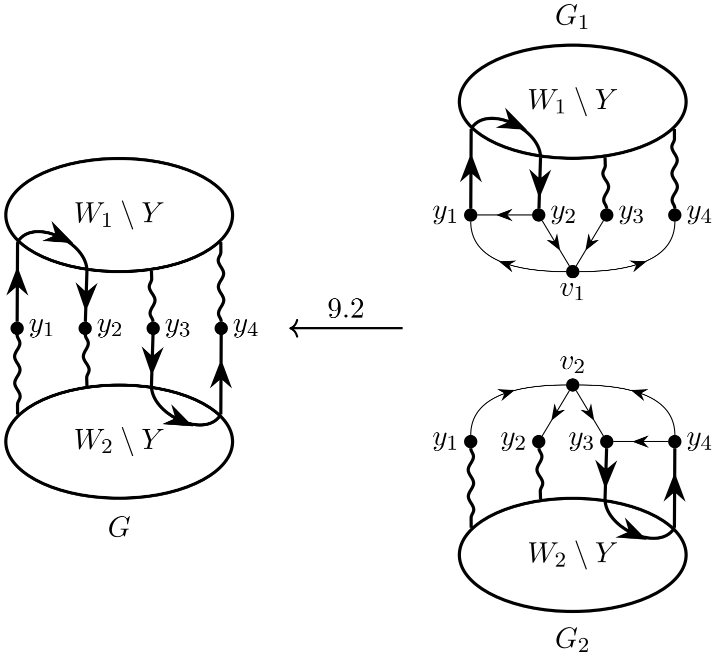


Figure 12: The first constituent of the second nonplanar construction.

Proof. By 5.2, there is a 0/1-valued weighting w'_1 of G_1 such that $w'_1(e) = 1$ for $e \in \{v_1y_1, v_1y_4\}$ and $w'_1(e) = 0$ for $e \in \{y_2v_1, y_3v_1\}$. Since there is a dipath of W_1 from y_1 to y_2, which can be completed to a dicycle of G_1 via y_2-y_1 or via y_2-v_1-y_1, it follows that $w'_1(y_2y_1) = 1$. Since G_1 is 1-strong, every edge e of G_1 with tail y_1 belongs to a dicycle which therefore contains one of v_1y_1, y_2y_1, and so $w'_1(e) = 0$. Consequently, since all in-edges of y_1 have weight one, and all out-edges of y_1 have weight zero, there is a 0/1-valued weighting w_1 of G_1 such that $w_1(v_1y_4) = 1$, $w_1(e) = 0$ for each of the other three edges incident with v_1, $w_1(y_2y_1) = 0$, and $w_1(e) = 1$ for every edge e of G_1 with tail y_1. Similarly, there is a 0/1-valued weighting w_2 of G_2 such that $w_2(y_1v_2) = 1$, $w_2(e) = 0$ for each of the other three edges incident with v_2, $w_2(y_4y_3) = 0$, and $w_2(e) = 1$ for every edge e of G_2 with head y_4. For each edge e of G, let $w(e) = w_i(e)$ where $i \in \{1, 2\}$ satisfies $e \in E(W_i)$. We claim that w is a weighting of G. To see this, let C be a dicycle of G. We may assume that C is not a dicycle of G_1 or of G_2. Thus either $C \cap W_1$ and $C \cap W_2$ are both dipaths, or $C \cap W_1$ is the disjoint union of two dipaths and so is $C \cap W_2$.

Suppose first that $C \cap W_1$ is a dipath P from y to y' with $y, y' \in Y$. Thus y, y' are distinct, and $C \cap W_2$ is a dipath Q from y' to y. Since y_1 is a source of W_1, it follows that $y' \neq y_1$, and similarly $y \neq y_2$, $y' \neq y_3$ and $y \neq y_4$. So, of the twelve pairs (y, y') where $y, y' \in Y$ are distinct, only four possibilities remain: the pairs $(y_1, y_3), (y_1, y_2), (y_4, y_3), (y_4, y_2)$. If $(y, y') = (y_1, y_3)$, then $w_1(P) = 1$, since P can be completed to a dicycle of G_1 via y_3-v_1-y_1, and $w_2(Q) = 0$, since Q can be completed to a dicycle of G_2 via y_1-v_2-y_3, and so $w(C) = 1$. The other three cases are can all be handled similarly, and we omit the details. (See the table below.)

(y,y')	$w_1(P)$	$w_2(Q)$	$w(C)$
(y_1,y_3)	1	0	1
(y_1,y_2)	1	0	1
(y_4,y_3)	0	1	1
(y_4,y_2)	0	1	1

Now we assume (for a contradiction) that $C \cap W_1$ is the disjoint union of two dipaths P_1, P_2, and so $C \cap W_2$ is also the union of two dipaths Q_1, Q_2. Since y_1 is a source of W_1, neither of P_1, P_2 has last vertex y_1, and so neither of Q_1, Q_2 has first vertex y_1. By the same arguments for y_2, y_3, y_4, it follows that P_1, P_3 are both from $\{y_1, y_4\}$ to $\{y_2, y_3\}$, and therefore Q_1, Q_2 are both from $\{y_2, y_3\}$ to $\{y_1, y_4\}$. Suppose first that P_1 is from y_1 to y_3, and so P_2 is from y_4 to y_2. We recall that there is a dipath R in W_1 from y_1 to y_2. But then

$$v_1\text{-}y_1\text{-}R\text{-}y_2\text{-}v_1,$$

$$v_1\text{-}y_4\text{-}P_2\text{-}y_2\text{-}y_1\text{-}P_1\text{-}y_3\text{-}v_1$$

are dicycles of G_1 that disagree on $\{v_1, y_1, y_2\}$, a contradiction. So neither of P_1, P_2 is from y_1 to y_3; and similarly, neither of Q_1, Q_2 is from y_3 to y_1, which is impossible. Thus there is no such cycle C, and consequently w is a weighting. This proves 9.2. ▮

We also need:

9.3. *Let G be a digraph, let y_1, y_2, y_3, y_4 be distinct, and let $Y = \{y_1, \dots, y_4\}$. Suppose that:*

- *y_1, y_3 are sources of G, and y_2, y_4 are sinks of G; and*
- *the digraph G_0 obtained from G by adding the edges y_1y_3, y_3y_1 and making the identifications $y_1 = y_2$ and $y_3 = y_4$ is 1-strong and weightable.*

Let H be obtained from G by adding two new vertices v_1, v_2 and the edges

$$y_2y_1, y_4y_3, v_1y_1, y_2v_1, y_3v_1, v_1y_4, y_1v_2, v_2y_2, v_2y_3, y_4v_2$$

(see Figure 13) is weightable.

Proof. Let G' be obtained from G by adding two new vertices v_1, v_2 and the edges

$$y_1v_2, v_2y_3, v_1y_1, y_3v_1, y_2v_1, y_4v_2.$$

Let $F = \{y_4y_3, v_1y_4, v_2y_2, y_2y_1\}$; thus, G' is obtained from H by deleting the edges in F. The edges $v_1y_1, v_2y_3, y_2v_1, y_4v_2$ are all singular edges of G' (because y_2, y_4 are sinks of G), and G_0 is obtained by contracting these edges. Consequently, G' is 1-strong, and weightable by 5.3. By 5.2 we may choose a 0/1-valued weighting w' of G' such that $w'(e) = 1$ for all edges e with tail y_1 and $w'(v_1y_1) = 0$. Since $y_1\text{-}v_2\text{-}y_3\text{-}v_1\text{-}y_1$ is a dicycle of G', it follows that $w'(e) = 0$ for $e \in \{v_2y_3, y_3v_1, v_1y_1\}$. Since G_0 is 1-strong, and y_3 is a source and y_2 a sink of G, there is a dipath R of G from y_1 to y_4. Consequently, $w'(y_4v_2) = 0$ (because R can be extended to a dicycle via $y_4\text{-}v_2\text{-}y_3\text{-}v_1\text{-}y_1$). Similarly, $w'(y_2v_1) = 0$. For each edge e of H, define $w(e) = w'(e)$ if $e \in E(G')$, and $w(y_4y_3) = 0$, $w(v_1y_4) = 1$, $w(v_2y_2) = 0$ and $w(y_2y_1) = 0$. We claim that w is a weighting of H. To show this, suppose that there is a dicycle

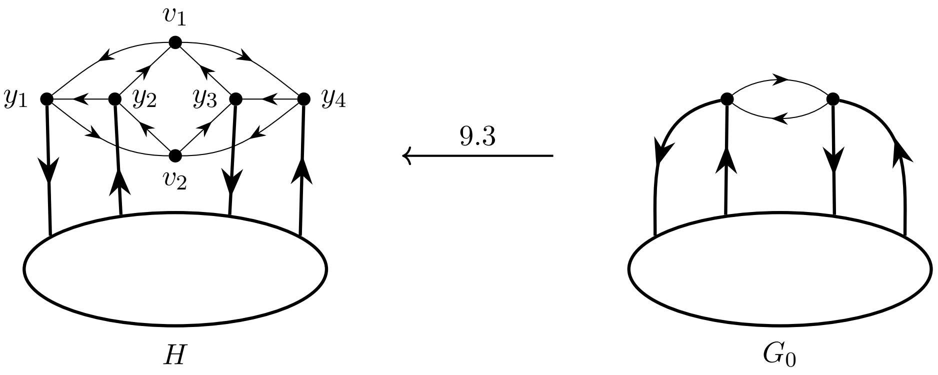


Figure 13: The second constituent of the second nonplanar construction.

C of H with $w(C) \neq 1$, and choose it with $E(C) \cap F$ minimal. Since w' is a weighting of G', it follows that $E(C) \cap F \neq \emptyset$.

Suppose that $y_4y_3 \in E(C)$. If $v_2 \notin V(C)$, then we can replace the edge y_4y_3 of C by the path y_4-y_2-y_3 and obtain another dicycle C' say, with $w(C') = w(C) \neq 1$, contrary to the minimality of $E(C) \cap F$. Thus, $v_2 \in V(C)$. Hence C is the concatenation of three dipaths: from y_4 to y_3 (of length one); a dipath P, say, from y_3 to v_2; and a dipath Q, say, from v_2 to y_4. Since $y_3 \notin V(Q)$, it follows that y_2 is the second vertex of Q, and since $y_2, y_4 \notin V(P)$, it follows that y_1 is the penultimate vertex of P. Hence $y_1 \notin V(Q)$, and so v_1 is the third vertex of Q, but then both in-neighbours of y_1 belong to Q, and yet one of them belongs to P, a contradiction. Thus $y_4y_3 \notin E(C)$, and similarly $y_2y_1 \notin E(C)$.

Suppose that $v_1y_4 \in E(C)$. Then $y_4v_2 \in E(C)$ (because $y_4y_3 \notin E(C)$), and since we cannot replace the subpath v_1-y_4-v_2 in C by v_1-y_1-v_2 (because these two paths have the same total weight and the latter has smaller intersection with F) it follows that $y_1 \in V(C)$. But there is no dipath in H from v_2 to v_1 passing through y_1 and not containing y_4, a contradiction. So $v_1y_4 \notin E(C)$, and similarly $v_2y_2 \notin E(C)$. This proves that w is a weighting of H, and this proves 9.3. ▌

By combining 9.2 and 9.3, we obtain what we really wanted:

9.4. *Let $y_1, \ldots, y_4$ be distinct vertices of a digraph G, and let $Y = \{y_1, \ldots, y_4\}$. Let W_1, W_2, W_3 be pairwise internally disjoint Y-wings with union G. Suppose that:*

- *y_1 is a source of $W_1 \cup W_3$, and y_2 is a sink of $W_1 \cup W_3$, and y_3 is a source of $W_2 \cup W_3$, and y_4 is a sink of $W_2 \cup W_3$;*
- *there is a dipath of W_1 from y_1 to y_2, and there is a dipath of W_2 from y_3 to y_4;*
- *the digraph G_1 obtained from W_1 by adding a new vertex v_1 and the edges*

  $$v_1y_1, y_2v_1, y_3v_1, v_1y_4, y_2y_1$$

  is 1-strong and weightable;
- *the digraph G_2 obtained from W_2 by adding a new vertex v_2 and the edges*

  $$y_1v_2, v_2y_2, v_2y_3, y_4v_2, y_4y_3$$

is 1-strong and weightable; and

- *the digraph G_0 obtained from W_3 by adding the edges y_1y_3, y_3y_1 and making the identifications $y_1 = y_2$ and $y_3 = y_4$ is 1-strong and weightable.*

See Figure 14. Then G is weightable.

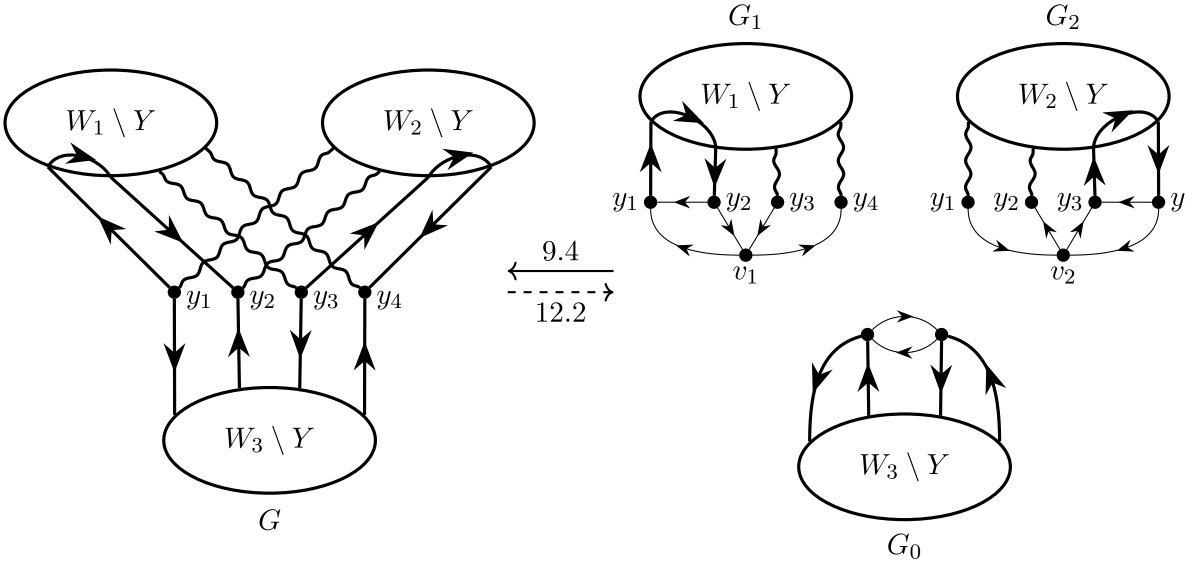


Figure 14: The second nonplanar construction.

Proof. Let W_3^+ be obtained from W_3 as in 9.3; that is, by adding two new vertices v_1, v_2 and the edges

$$y_1v_2, v_2y_3, v_1y_1, y_3v_1, y_2v_1, y_4v_2.$$

Then W_3^+ is 1-strong and weightable, by 9.3. By 9.2 applied to W_1 and W_3^+, we deduce that the digraph obtained from $W_1 \cup W_3$ by adding a new vertex v_1 and the edges $v_1y_1, y_2v_1, y_3v_1, v_1y_4, y_2y_1$ is strong and weightable. By 9.2, applied to $W_1 \cup W_3$ and W_2, it follows that G is weightable. This proves 9.4. ▮

Finally, we need:

9.5. *Let $y_1, \dots, y_4$ be distinct vertices of a digraph G, and let $Y = \{y_1, \dots, y_4\}$. Let W_1, W_2 be internally disjoint Y-wings with union G. Suppose that:*

- *y_1, y_3 are sources of W_1 and y_2, y_4 are sinks of W_1;*
- *there are dipaths of W_1 from y_1 to y_4 and from y_3 to y_2, and there are dipaths of W_2 from y_2 to y_1 and from y_4 to y_3;*
- *the digraph G_1 obtained from W_1 by adding the edges $y_2y_1, y_4y_3, y_1y_3, y_3y_1$ is 1-strong and weightable (equivalently, by 5.3, the digraph obtained from G_1 by contracting y_2y_1, y_4y_3 is 1-strong and weightable);*

- *the digraph G_2 obtained from W_2 by adding two new vertices v_1, v_2 and the edges*

$$y_1v_1, y_3v_2, v_1y_4, v_2y_2, y_3y_4, v_1v_2, v_2v_1$$

  is 1-strong and weightable.

See Figure 15. Then G is weightable.

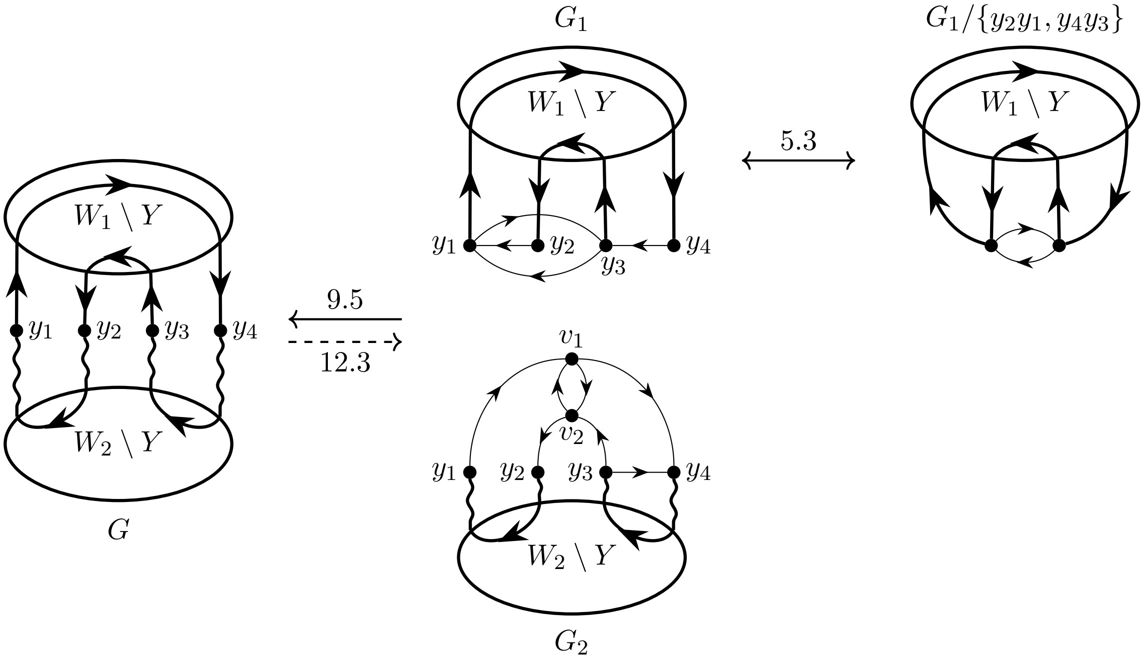


Figure 15: The third nonplanar construction. Note that the pair of paths depicted in $W_1 \setminus Y$ need not be disjoint, and likewise for $W_2 \setminus Y$.

Proof. By 5.2, there is a 0/1-valued weighting w_1 of G_1 such that $w_1(e) = 1$ for the edges of G_1 with tail y_1 and $w_1(e) = 0$ for those with head y_1. Since there are dipaths of W_1 from y_1 to y_4 and from y_3 to y_2, it follows that $w_1(y_4y_3) = w_1(y_2y_1) = 0$. By 5.2, there is a 0/1-valued weighting w_2 of G_2 such that $w_2(e) = 1$ for the edges with head v_1 and $w_2(e) = 0$ for the edges with tail v_1. Since there are dipaths of W_2 from y_2 to y_1 and from y_4 to y_3, it follows that $w_2(v_2y_2) = w_2(y_3y_2) = 0$. Let P be a path in W_2 from y_4 to y_3; we have $w_2(P) = 0$ since P completes to a cycle in G_2 via y_3-v_2-v_1-y_4. Therefore $w_2(y_3y_4) = 1$, since this edge completes to a cycle via P.

For each edge $e \in E(G)$, define $w(e) = w_i(e)$ where $e \in E(G_i)$; note that w_1 and w_2 agree on any edges in common between G_1, G_2 (namely edges with both ends in Y). We claim that w is a weighting of G. To show this, let C be a dicycle of G. We may assume that C is not a cycle of W_1 or of W_2, so either $C \cap W_1$, $C \cap W_2$ are both paths, or $C \cap W_1$, $C \cap W_2$ are both the disjoint union of two paths. Suppose first that $C \cap W_1$ is a path P from y to y', where $y, y' \in Y$ are distinct, so $C \cap W_2$ is a path Q from y' to y. Since y_1, y_3 are sources of W_1, and y_2, y_4 are sinks of W_1, it follows that (y, y') is one of the pairs $(y_1, y_2), (y_1, y_4), (y_3, y_2), (y_3, y_4)$. If $(y, y') = (y_1, y_2)$, then $w_1(P) = 0$ (because P

can be completed to a dicycle of G_1 via y_3-y_2), and $w_2(Q) = 0$ (because Q can be completed via y_1-v_1-v_2-y_2), and so $w(C) = 1$. The argument is similar in the other three cases and we omit the details. (See the table below.)

(y, y')	$w_1(P)$	$w_2(Q)$	$w(C)$
(y_1, y_2)	1	0	1
(y_1, y_4)	1	0	1
(y_3, y_2)	0	1	1
(y_3, y_4)	1	0	1

Now we assume that $C \cap W_1$, $C \cap W_2$ are both the disjoint union of two paths. Since y_1, y_3 are sources of W_1, and y_2, y_4 are sinks of W_1, the two components of $C \cap W_1$ are both from $\{y_1, y_3\}$ to $\{y_2, y_4\}$, so there are two possibilities:

- $C \cap W_1$ is the disjoint union of P_1 from y_1 to y_4 and Q_1 from y_3 to y_2, and $C \cap W_2$ is the disjoint union of P_2 from y_2 to y_1 and Q_2 from y_4 to y_3; or
- $C \cap W_1$ is the disjoint union of P_1 from y_1 to y_2 and Q_1 from y_3 to y_4, and $C \cap W_2$ is the disjoint union of P_2 from y_2 to y_3 and Q_2 from y_4 to y_1.

In the first case, $w_1(P_1) = 1$ (by completing via y_4-y_3-y_1), and similarly $w_1(Q_1) = 0$, $w_2(P_2) = 0$, and $w_2(Q_2) = 0$, so $w(C) = 1$ as required. In the second case, since P_2, Q_2 are disjoint, it follows that

$$y_2\text{-}P_2\text{-}y_3\text{-}y_4\text{-}Q_2\text{-}y_1\text{-}v_1\text{-}v_2\text{-}y_2$$

is a dicycle of G_2. But $w_2(P_2) = 1$ (by completing via y_3-v_2-y_2) and $w_2(y_1v_1) = 1$, so this second case cannot occur. This proves that w is a weighting, and so proves 9.5. ∎

10 Decompositions in the nonplanar case

If a 2-strong weightable graph is nonplanar, we can deduce from the main theorem of [4] that it has a 2-, 3-, or 4-vertex cutset X such that its deletion from G makes at least three weak components, and this will allow us to construct G from smaller weightable 1-strong digraphs by the constructions of the previous section.

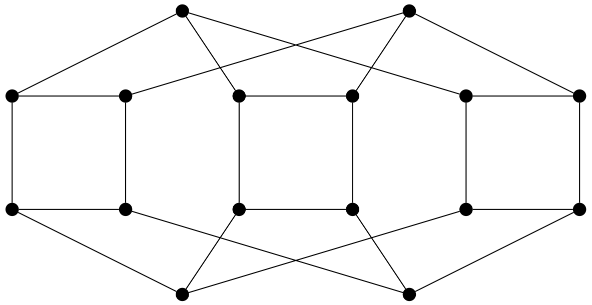


Figure 16: Rotunda.

The graph "Rotunda" is shown in Figure 16. Note that Rotunda is bipartite. There are four vertices of Rotunda (two at the top and two at the bottom of the figure) such that deleting them makes a graph with three components, each a cycle of length four. We call the set of these four vertices the *join* of Rotunda. By an *odd subdivision* of Rotunda, we mean a graph R obtained from a copy of Rotunda by replacing each edge by a path of odd length, all internally disjoint. Such a graph R is bipartite, and the set of four vertices of R that corresponds to the join of Rotunda is called the *join* of R. (It is uniquely defined by R.)

The *Heawood graph* is shown in Figure 17. For every perfect matching M of the Heawood graph H, the collapse of (H, M) is isomorphic to the digraph F_7 in Figure 17.

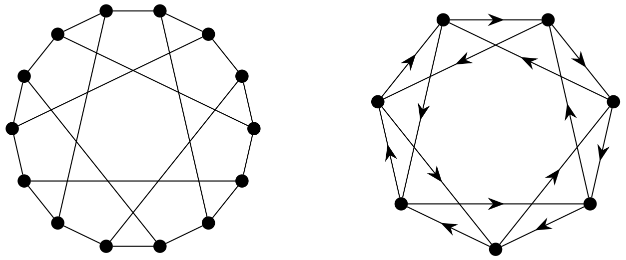


Figure 17: The Heawood graph (left) and its unique collapse F_7 (right).

The following two results are proved in [4]:

10.1. *Let H be a Pfaffian brace. Then either H is planar, H is the Heawood graph, or there is a subgraph R of H that is an odd subdivision of Rotunda such that $H \setminus V(R)$ has a perfect matching.*

10.2. *Let H be a Pfaffian brace, and suppose that R is a subgraph of H that is an odd subdivision of Rotunda such that $H \setminus V(R)$ has a perfect matching. Let X be the join of R. Then the three components of $R \setminus X$ are contained in three distinct components of $H \setminus X$.*

We will deduce from 10.1 and 10.2 that:

10.3. *Let G be a 2-strong, 3-weak odd-weightable digraph that is not diplanar. Then either G is F_7 (shown in Figure 17), or there is a set $Y \subseteq V(G)$ such that one of the following holds (shown in Figure 18):*

1. *$|Y| = 3$, $Y = \{y_1, y_2, y_3\}$ say, and $G \setminus Y$ has the following properties:*
   - *(a) It has at least three weak components $D_1, D_2, \ldots$.*
   - *(b) For $i = 1, 2$, $|V(D_i)| \geq 2$, $y_1 \to D_i$, and $y_2 \leftarrow D_i$.*
2. *$|Y| = 4$, $Y = \{y_1, y_2, y_3, y_4\}$ say, where $y_4y_1, y_2y_3 \notin E(G)$, and $G \setminus Y$ has the following properties:*
   - *(a) It has at least three weak components $D_1, D_2, \ldots$.*
   - *(b) $y_1 \to D_1$ and $y_2 \leftarrow D_1$.*
   - *(c) $y_3 \to D_2$ and $y_4 \leftarrow D_2$,*

(d) *For* $i > 2$, $y_1, y_3 \to D_i$ *and* $y_2, y_4 \leftarrow D_i$.

3. $|Y| = 4$, $Y = \{y_1, y_2, y_3, y_4\}$ *say, and* $G \setminus Y$ *has the following properties:*
   (a) *It has at least two weak components* $D_1, D_2, \ldots$.
   (b) *For* $i = 1, 2$, $|V(D_i)| \geq 2$, $y_1, y_3 \to D_i$, *and* $y_2, y_4 \leftarrow D_i$.

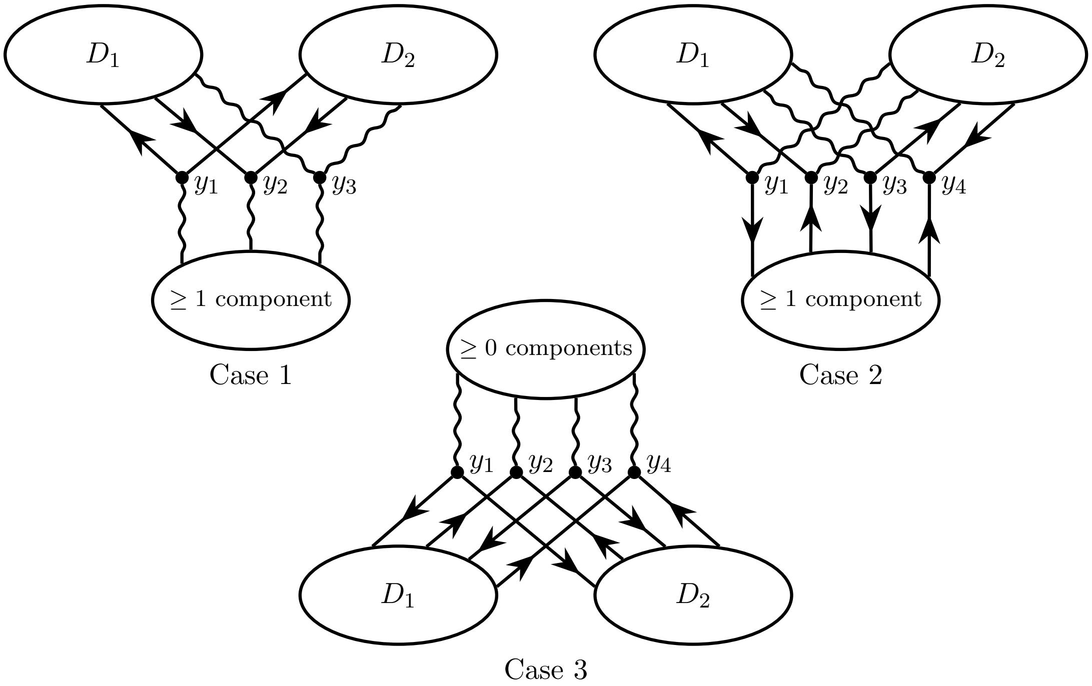


Figure 18: The three decompositions of 10.3. Note that some of the ovals may correspond to more than one weak component or may be empty, as labeled. Also, edges with both ends in Y are not shown, and there are some additional size constraints on certain D_i; check the theorem statement for details.

Proof. Let (H, M) be the bisource for G. Since G is 2-strong and odd-weightable, it follows that H is a Pfaffian brace, by theorems of [4]. Since G is not diplanar, it follows that H is not planar, so we can apply 10.1 and 10.2. If H is the Heawood graph, then G is the digraph F_7 of Figure 17, so we assume not. Hence by 10.1 and 10.2, there is a subgraph R of H that is an odd subdivision of Rotunda, with join X say, such that:

- $H \setminus V(R)$ has a perfect matching, and
- the three components of $R \setminus X$ are contained in three distinct components of $H \setminus X$.

Let (A, B) be a bipartition of R. Let R_1, R_2, R_3 be the three components of $R \setminus X$. Since R is an odd subdivision of Rotunda, it follows that $|A \cap X| = 2$. Let $X = \{x_1, \ldots, x_4\}$ where $X \cap A = \{x_1, x_3\}$.

Let $C_1, \dots, C_k$ be the components of $H \setminus X$, and let $R_i \subseteq C_{t_i}$ for $i = 1, 2, 3$. We have $k \geq 3$, and by 10.2, we have that t_1, t_2, t_3 are all different.

(1) $|V(C) \cap A| = |V(C) \cap B|$ *for each component* C *of* $H \setminus X$*, and therefore the number of edges in* M *between* $X \cap A$ *and* $V(C) \cap B$ *equals the number between* $X \cap B$ *and* $V(C) \cap A$*.*

By relabeling the components of $H \setminus X$ if necessary, we may assume $t_i = i$ for $i = 1, 2, 3$. We may also assume $C \neq C_1, C_2$ by the symmetry between C_1, C_2, C_3. Choose edges e_1, e_2 between x_1 and $V(C_1)$ and between x_3 and $V(C_2)$, respectively. Since H is 2-extendible, the matching $\{e_1, e_2\}$ can be extended to a perfect matching of H, and consequently $|V(C) \cap A| \leq |V(C) \cap B|$. Similarly, $|V(C) \cap B| \leq |V(C) \cap A|$, and so equality holds. This proves the first statement of (1), and the second follows since M is a perfect matching. This proves (1).

For $1 \leq i \leq k$, let M_i be the set of edges in M that have both ends in $V(C_i)$. We produce G from H by directing the edges of H from A to B, and then contracting the edges of M; for each edge $e \in M$, let $\phi(e)$ be the vertex of G made by contracting e, and for $N \subseteq M$ let $\phi(N) = \{\phi(e) : e \in N\}$. Let Y be the set of vertices $\phi(e)$ such that $e \in M$ has an end in X. For each i, let $D_i = G[\phi(M_i)]$; then the nonnull digraphs among $D_1, \dots, D_k$ are precisely the weak components of $G \setminus Y$. Since G is 3-weak, $3 \leq |Y| \leq 4$, so at most one edge of M has both ends in X.

The relationship between the matching M and the vertices X determines the outcome of the theorem. The three cases are depicted in Figure 19.

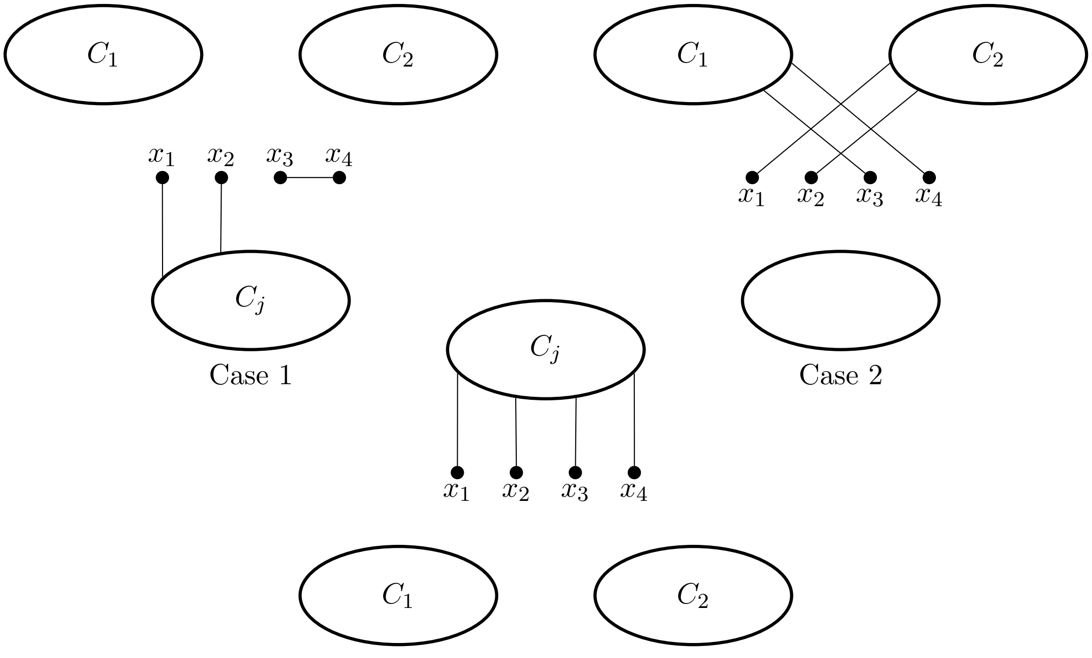
Suppose first that there is an edge of M with both ends in X, say $x_3x_4 \in M$. In this case, we may assume that $t_i = i$ for $i = 1, 2, 3$. Let e_1, e_2 be the edges in M incident with x_1, x_2 respectively. By (1), there exists $j \in \{1, \dots, k\}$ such that e_1, e_2 both have an end in $V(C_j)$. From the symmetry between C_1, C_2, C_3, we may assume that $j \neq 1, 2$. Since C_1, C_2 each have at least four vertices, it follows that D_1, D_2 each have at least two vertices. Moreover, for $i = 1, 2$, all edges of H between $V(C_i)$ and an end of e_1 are incident with x_1 (since the other end of e_1 is in $V(C_j)$), and so in G, all edges between $V(D_i)$ and y_1 are directed from y_1 to $V(D_i)$, that is, $y_1 \to D_i$. Similarly, $y_2 \leftarrow D_i$ for $i = 1, 2$. Since $|V(C_3)| \geq 4$, it follows that $|D_3| \geq 1$ (even if $j = 3$), so $G \setminus Y$ has at least three weak components, and therefore case 1 theorem holds.

Thus we may assume that no edge of M has both ends in X. For $1 \leq i \leq 4$, let e_i be the edge in M incident with x_i, and let $\phi(e_i) = y_i$.

Next, we assume that there exist distinct $j_1, j_2 \in \{1, \dots, k\}$ such that e_3, e_4 have an end in $V(C_{j_1})$ and e_1, e_2 each have an end in $V(C_{j_2})$; and so, by relabeling the components of $H \setminus X$ if necessary, we may assume $j_1 = 1$ and $j_2 = 2$. It follows that $y_4y_1 \notin E(G)$ (because the end of e_1 in B and the end of e_4 in A belong to different components of $H \setminus X$), and similarly $y_2y_3 \notin E(G)$.

If no edge in M joins X and $V(C_i)$, then $y_1, y_3 \to D_i$ and $y_2, y_4 \leftarrow D_i$. This is the case for all i except for $i = 1, 2$. Additionally, we have $y_1 \to D_1$, $y_2 \leftarrow D_1$, $y_3 \to D_2$, and $y_4 \leftarrow D_2$. Also, since $|V(C_{t_i})| \geq 4$ for $i = 1, 2, 3$, the corresponding subdigraphs D_{t_i} are nonnull (even if $t_i \in \{j_1, j_2\}$). Therefore case 2 of the theorem holds.

Finally, we assume that for some $j \in \{1, \dots, k\}$, $e_1, \dots, e_4$ all have an end in $V(C_j)$. In this case, we may again assume that $t_i = i$ for $i = 1, 2, 3$, and we may further assume that $j \neq 1, 2$. Then for $i = 1, 2$, we have $y_1, y_3 \to D_i$ and $y_2, y_4 \leftarrow D_i$. Also, C_1, C_2 both have at least four vertices, so D_1, D_2 both have at least two vertices (possibly $V(D_3) = \emptyset$). Thus case 3 of the theorem holds. This proves 10.3. ∎



Case 1

Case 2

Case 3

Figure 19: The cases of $M \subseteq E(H)$ that lead to the three outcomes of 10.3. Here we just draw the edges in M and the components of $H \setminus X$ that play an important role in the proof. The labels of the elements drawn are consistent with the proof of 10.3, and their positions are consistent with Figure 18.

This result tells us about decompositions in odd-weightable digraphs. Since weightable digraphs are odd-weightable, 10.3 applies also to weightable digraphs, but for weightable digraphs we can refine these decompositions so that they become the reverse of constructions, as we show in the next few sections.

11 Path lemmas

We need some lemmas about dipaths. We recall that if P is a dipath and $u, v \in V(P)$, and u is earlier than v in P, then $P[u, v]$ denotes the subpath of P from u to v.

11.1. *Let G be a digraph, and let $a_1, a_2, b_1, b_2 \in V(G)$ be distinct. Suppose that:*

- $|V(G)| \geq 5$*;*
- *no edge has head a_1 or a_2 and no edge has tail b_1 or b_2;*
- *for each vertex $v \in V(G) \setminus \{b_1, b_2\}$ and $i = 1, 2$, there is a dipath from v to b_i; and*
- *for every vertex $v \in V(G) \setminus \{a_1, a_2\}$ and $i = 1, 2$, there is a dipath from a_i to v.*

Then there are two dipaths from $\{a_1, a_2\}$ to $\{b_1, b_2\}$ that are not vertex-disjoint, such that the ends of these paths are all distinct.

Proof. We begin with:

(1) *There is a dipath from a_1 to b_1 with length at least two.*

Since $|V(G)| \geq 5$, there is a vertex v different from a_1, a_2, b_1, b_2; by the third and fourth bullets, there is a dipath from a_1 to v and from v to b_1. Therefore the union of these paths includes a dipath from a_1 to b_1 of length at least two. This proves (1).

Let us say a *fork* means a triple (P_1, P_2, Q_1), where P_1, P_2, Q_1 are dipaths, pairwise vertex-disjoint except that they have a common end $v \neq a_1, a_2, b_1, b_2$, and P_1 is from a_1 to v, P_2 is from a_2 to v, and Q_1 is from v to b_1.

(2) *There is a fork.*

By (1), there is a dipath from a_1 to b_1 with non-null interior. By the fourth bullet of the theorem, there is a dipath from a_2 to the interior of P_1; choose a minimal such path P_2, and let v be the end of P_2 in the interior of R. Let P_1, Q_1 be the subpaths of R from a_1 to v and from v to b_1, respectively. Then (P_1, P_2, Q_1) is a fork. This proves (2).

Let us choose a fork (P_1, P_2, Q) with Q maximal, and let v be the common end of the three paths.

(3) *There is no dipath between $V(P_1)$, $V(P_2)$ that is vertex-disjoint from $V(Q_1)$.*

Suppose there is a such a path from $V(P_1)$ to $V(P_2)$, say, and let R be a minimal such path, with ends $r_i \in V(P_i)$ for $i = 1, 2$. Then

$$(P_1[a_1, r_1] \cup R, P_2[a_2, r_2], P_2[r_2, v] \cup Q_2)$$

is a fork, contrary to the maximality of Q_1. This proves (3).

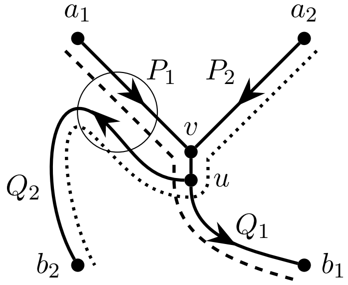


Figure 20: The end of the proof of 11.1. Inside the circle, the paths P_1 and Q_2 may intersect in an arbitrary way; none of the other paths may intersect. The dashed and dotted paths are the ones we desired.

There is a dipath from $V(Q_1)$ to b_2; let Q_2 be a minimal such path, with ends $u \in V(Q_1)$ and b_2. Thus, u is the only vertex of Q_2 in $V(Q_1)$, and $u \neq b_1$. By (3), Q_2 does not meet both $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{v\}$, and so, exchanging a_1, a_2 if necessary, we may assume that $V(Q_2) \cap V(P_2) \subseteq \{u\} \cap \{v\}$. But then the dipaths $P_2 \cup Q_1[v, u] \cup Q_2$ and $P_1 \cup Q_1$ have nonempty intersection and satisfy the theorem. (See Figure 20.) This proves 11.1. ∎

Let G be a digraph, let $a_1, a_2, b_1, b_2 \in V(G)$, and for $i = 1, 2$ let P_i be a dipath from a_i to b_i, chosen such that P_1, P_2 intersect, and subject to that their union is minimal. We say that P_1, P_2 are in *bubble form.*

11.2. *Let $G, a_1, a_2, b_1, b_2, P_1, P_2$ be as above, where P_1, P_2 are in bubble form. Then $P_1 \cap P_2$ is the disjoint union of dipaths $Q_1, \ldots, Q_k$ for some $k \geq 1$, such that $Q_1, \ldots, Q_k$ are in this order in P_1, and in the reverse order in P_2. (See Figure 21.)*

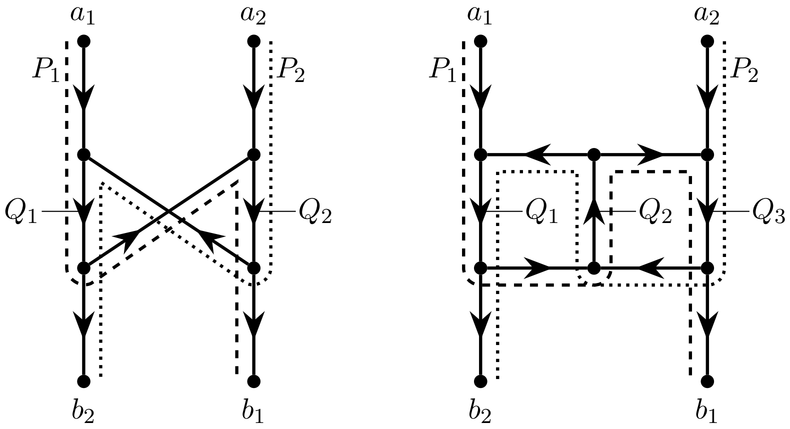


Figure 21: Two examples of paths P_1 (dashed) and P_2 (dotted) in bubble form as described in 11.2, here with $k = 2$ (left) or 3 (right) paths in $P_1 \cap P_2$.

Proof. We claim that:

(1) *If $u, v \in V(P_1 \cap P_2)$, then either u is before v in one of P_1, P_2 and after v in the other, or the subpaths of P_1, P_2 joining u, v are equal.*

To see this, suppose that u is before v in both P_1, P_2. There is a dipath P_2' from a_2 to b_2, included in $P_2[a_2, u] \cup P_1[u, v] \cup P_2[v, b_2]$. Since $P_2' \subseteq P_1 \cup P_2$, the minimality of $P_1 \cup P_2$ implies that $P_2 \subseteq P_1 \cup P_2'$, and in particular every edge of $P_2[u, v]$ belongs to $E(P_1 \cup P_2')$ and hence to $E(P_1)$ (because it cannot belong to P_2' unless it is in $P_1[u, v]$). So $P_2[u, v]$ is a subpath of P_1, and therefore $P_2[u, v] = P_1[u, v]$. This proves (1).

Certainly $P_1 \cap P_2$ is the disjoint union of some number of dipaths $Q_1, \ldots, Q_k$; let us number them in their order they appear in P_1. By (1), these subpaths appear in P_2 in reverse order. This proves 11.2. ▮

A similar statement appears as Lemma 4.3 in [3]. If P_1, P_2 are in bubble form, we call the number of components of $P_1 \cap P_2$ their *intersection number*, and we say P_1, P_2 *make a bubble* if they have intersection number two.

12 Refining the outcomes of 10.3

Now we will start to convert the outcomes of 10.3 to reversible decompositions for weightable digraphs. We begin with the the first outcome of 10.3. Two dipaths are *internally disjoint* if every vertex in their intersection is an end of each of them.

12.1. *Let y_1, y_2, y_3 be distinct vertices of a 2-strong, 3-weak, weightable digraph G, and let $Y = \{y_1, y_2, y_3\}$. Suppose that $G \setminus Y$ has at least three weak components D_i, and for $i = 1, 2$, $y_1 \to D_i$ and $y_2 \leftarrow D_i$. Then G is nonplanar and can be built from two smaller weightable digraphs by an application of the construction of 9.1.*

Proof. Since G is 3-weak, each component of $G \setminus Y$ has an edge to each of y_1, y_2, y_3, so the underlying graph G^- has a $K_{3,3}$ minor and is not planar. In this proof, let $A = D_1$ and $B = D_2$. Let $C = G \setminus V(A \cup B)$. For distinct $y_i, y_j \in Y$, we mean by $A[i, j]$ a dipath from y_i to y_j with all internal vertices in $V(A)$, and the same for B. By $C[i, j]$ we mean a dipath of C from y_i to y_j with no internal vertex in Y.

(1) $C[2, 1], C[3, 2]$ *exist.*

Since G is 2-strong, there are two internally disjoint dipaths from $\{y_2, y_3\}$ to y_1, and since $y_1 \to A \cup B$, it follows that each of these dipaths has no vertex in $V(A \cup B)$. This proves (1).

(2) $A[1, 3], A[3, 2], B[1, 3], B[3, 2]$ *exist.*

Choose $v \in V(A)$. Since G is 2-strong, there are two internally disjoint dipaths of G from v to Y, and since $y_1 \to A$, all their vertices belong to $V(A) \cup \{y_2, y_3\}$. Similarly, there are dipaths from y_1, y_3 to v such that all their vertices belong to $\{y_1, y_3\} \cup V(A)$. Consequently, $A[1, 3], A[3, 2]$ exist, and the same holds for B. This proves (2).

(3) $B[1, 3], B[3, 2]$ *are internally disjoint, for every choice of* $B[1, 3], B[3, 2]$.

Suppose not, and choose $b \in V(B)$ that belongs to both paths. Then the dicycles $B[1, 3] \cup C[3, 1]$ and $B[3, 2] \cup C[2, 1] \cup A[1, 3]$ disagree on $\{b, y_3, y_1\}$, contrary to 5.1. This proves (3).

Let $W_1 = G \setminus V(A)$, and let W_2 be the subdigraph obtained from A by adding Y and all edges of G between $Y, V(A)$. Thus, W_1, W_2 are internally disjoint Y-wings with union G. We claim:

- There is a W_1-path from y_2 to y_1. This is true because $C[2, 1]$ is such a path.
- y_1 is a source of W_2 and y_2 is a sink of W_2. This is true because $y_1 \to A$ and $y_2 \leftarrow A$.
- The digraph G_1 obtained from W_1 by adding the edges y_1y_2, y_1y_3, y_3y_2 is weightable and 1-strong. This is true because G_1 is clearly 1-strong, and it is weightable by 5.3 and (3) since it can be obtained as a butterfly minor from G by contracting singular edges of $W_1 \cup A[1, 2] \cup B[1, 3] \cup B[3, 2]$.
- The digraph G_2 obtained from W_2 by adding the edges y_2y_1, y_1y_3, y_3y_2 is weightable and 1-strong. This is true because G_2 is clearly 1-strong, and it is weightable by 5.3 and (3) since it can be obtained as a butterfly minor from G by contracting singular edges of $W_2 \cup C[2, 1] \cup B[1, 3] \cup B[3, 2]$.

This proves 12.1. ▮

Now we handle the second outcome of 10.3:

12.2. *Let $y_1, \dots, y_4$ be distinct vertices of a 2-strong, 3-weak, weightable digraph G, and let $Y = \{y_1, \dots, y_4\}$. Suppose that $G \backslash Y$ has at least three weak components D_i such that the four assumptions of 10.3 case 2 hold. Then G is nonplanar and can be constructed from three smaller weightable digraphs by the construction of 9.4.*

Proof. For each weak component D of $G \setminus Y$ and all distinct $i, j \in \{1, \dots, 4\}$, $D[i,j]$ denotes a dipath of G from y_i to y_j with all internal vertices in $V(D)$, and D^+ is the digraph formed by adding to D the vertices of Y and all edges between D and Y. Let $A = D_1$, $B = D_2$, and $C = D_3$.

(1) *For each $i > 2$, $D_i[1,2], D_i[1,4], D_i[3,2], D_i[3,4]$ exist, and every choice of $D_i[3,2]$ is vertex-disjoint from every choice of $D_i[1,4]$.*

It suffices to prove this for $i = 3$ (so $D_i = C$). Choose $v \in V(C)$. Since G is 2-strong, there are two dipaths from v to Y, vertex-disjoint except for v, and since $y_1, y_3 \to C$, it follows that these paths are from v to y_2, y_4 respectively. Similarly, there are dipaths from y_1, y_3 to v, and consequently $C[1,2], C[1,4], C[3,2], C[3,4]$ exist.

Suppose that there is a vertex c in some $C[3,2]$ and in some $C[1,4]$. There are two vertex-disjoint dipaths from y_4 to y_1, say P, Q. Since each weak component D of $G \setminus Y$ has either $y_1 \to D$ or $y_4 \leftarrow D$ (or both), each of P and Q contains one of y_2, y_3, so we may assume that P contains y_2 and Q contains y_3. Since $y_4 \leftarrow D_i$, $y_1 \to D_i$, and $y_2 \notin V(Q)$, it follows that $V(Q) \cap V(C) = \emptyset$. But then $C[1,4] \cup Q$ is a dicycle containing c, y_3, y_1 in this order.

Similarly, there are two internally disjoint paths P', Q' from y_2 to y_3. These each must contain one of p_1, p_4; we may assume Q' contains p_1. Then $V(Q') \cap V(C) = \emptyset$ and $C[3,2] \cup Q'$ contains c, y_1, y_3 in this order, so $\{c, y_1, y_3\}$ is a bad triple, contradicting 5.1. This proves (1).

(2) *In G, every dipath from y_1 to y_3 intersects every dipath from y_4 to y_2.*

Suppose that P, Q are dipaths from y_1 to y_3 and from y_4 to y_2, respectively, that are vertex-disjoint. In particular, no internal vertex of P or Q belongs to Y. Hence, either P has length one or P is a path of A^+. Similarly, either Q has length one or it is a path of A^+. There is a dipath R from y_2 to y_1 in $G \setminus \{y_4\}$. Either R has length one, or its second vertex belongs to $V(B)$; if the latter, then R is a path of B^+ since $y_3 \to D_1$. In particular, R is internally disjoint from P, Q.

By 11.1, for $i > 2$ there are two dipaths L, M in C^+ from $\{y_1, y_3\}$ to $\{y_2, y_4\}$ that are not vertex-disjoint, and such that their ends are all distinct. By (1), it must be that L is from y_1 to y_2 and M is from y_3 to y_4. If L, M have intersection number one, let $u \in L \cap M$; then $L[y_1, u] \cup M[u, y_4]$ and $M[y_3, u] \cup L[u, y_2]$ are $C[1,4]$ and $C[3,2]$ paths that intersect, contradicting (1). Thus L, M do not have intersection number one, so there exist a, b such that y_1, a, b, y_2 are in order in L and y_3, b, a, y_4 are in order in M. But then $R \cup L$ and $P \cup M \cup Q \cup R$ are both dicycles, and they disagree on $\{a, b, y_1\}$, a contradiction. (See Figure 22.) This proves (2).

Consequently, $A[1,3], A[4,2]$ exist and intersect, and therefore $A[1,2], A[4,3]$ exist. Similarly, $B[3,1], B[2,4]$ exist and intersect, and $B[3,4], B[2,1]$ exist. Thus, each vertex in Y has a neighbour

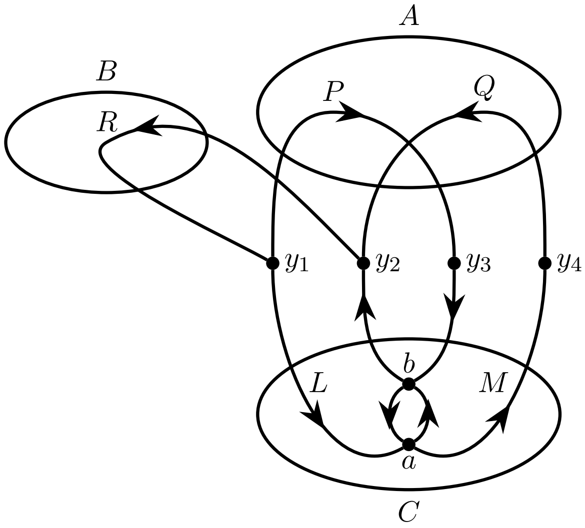


Figure 22: End of the proof of (2). Note that P, Q, R could each be a single edge.

in $V(B)$ and a neighbour in $V(A)$, and at least three vertices in Y have a neighbour in $V(C)$ since G is 3-weak. Thus the underlying graph G^- contains a $K_{3,3}$ minor, so G is nonplanar.

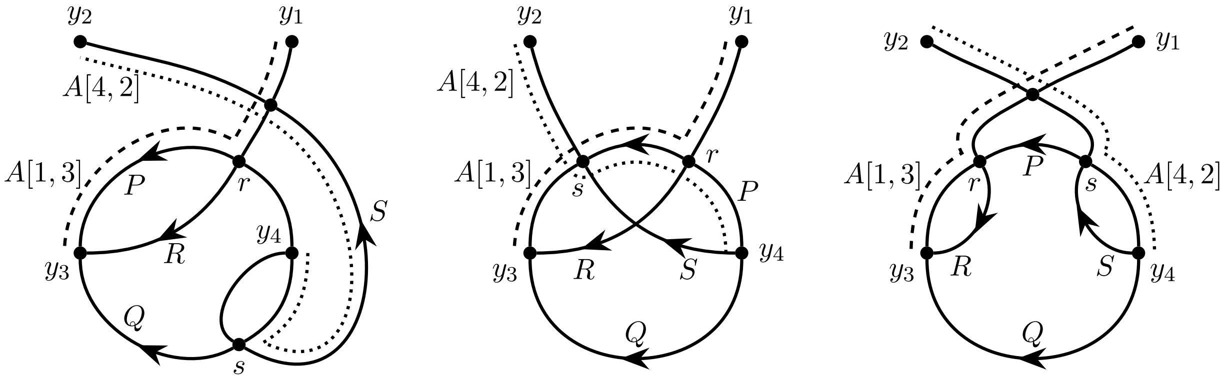
(3) *There do not exist distinct vertices u, v in both $A[1,3], A[4,2]$ such that y_1, u, v, y_3 are in order in $A[1,3]$ and y_4, v, u, y_2 are in order in $A[4,2]$. Consequently, if $A[1,3], A[4,2]$ are in bubble form, they have intersection number one, and the same for $B[3,1], A[2,4]$.*

Suppose that u, v belong to both $A[1,3], A[4,2]$ such that y_1, u, v, y_3 are in order in $A[1,3]$ and y_4, v, u, y_2 are in order in $A[4,2]$. Choose w in both $B[3,1], B[2,4]$. Then $A[1,3] \cup B[3,1]$ and $A[4,2] \cup B[2,4]$ are dicycles that disagree on $\{u, v, w\}$, a contradiction. This proves (3).

(4) *There is a choice of the three dipaths $A[1,3], A[4,2], A[4,3]$ such that the intersection of each pair of them is a path and the intersection of all three is null. Similarly there is a choice of $B[3,1], B[2,4], B[2,1]$ with the same property.*

If $y_4y_3 \in E(G)$ then the claim is true by (3), so we assume that $y_4y_3 \notin E(G)$. There are two internally disjoint dipaths P, Q in G from y_4 to y_3, both of length at least two. By (2), the second vertex of P is not equal to y_2, and so it belongs to $V(A)$. Similarly, the penultimate vertex of P belongs to $V(A)$. Suppose that P has some internal vertex not in $V(A)$. Since $y_1 \to A$, $P \cap A$ contains two disjoint subpaths, one from y_4 to y_2, and the other from y_1 to y_3, contrary to (2). Hence all vertices of P belong to $V(A) \cup Y$, and so $y_1, y_2 \notin V(P)$ (because $y_1 \to A$ and $y_2 \leftarrow A$, and so neither is an interior vertex of a dipath of A^+). Thus P is a choice for $A[4,3]$, and so is Q.

Let R be a choice of $A[1,3]$ and S a choice of $A[4,2]$, and, let us choose R, S with $P \cup Q \cup R \cup S$ minimal. Let R_1 be the minimal path of R from y_1 to $V(P \cup Q)$, and we may assume that R_1 has ends y_1 and $r \in V(P)$. (Possibly $r = y_3$, but if not then R_1 is disjoint from Q.) Similarly, let S_1 be the minimal subpath of S from $V(P \cup Q)$ to y_2, with ends s, y_2. Thus s belongs to one of P, Q.



$s \in V(Q)$
$A[4,3] = P$

$s \in V(P[r, y_3]) \setminus \{r\}$
$A[4,3] = Q$

$s \in V(P[y_4, r])$
$A[4,3] = Q$

Figure 23: The three cases in the proof of (4).

Suppose first that $s \in V(Q)$. By (3) applied to $R_1 \cup P[r, y_3]$ and $Q[y_4, s] \cup S_1$, it follows that R_1, S_1 have intersection number one, and by the minimality of $P \cup Q \cup R \cup S$ it follows that $R_1 \cap S_1$ is a path. But then the claim holds, taking

$$\begin{aligned} A[1,3] &= R_1 \cup P[r, y_3], \\ A[4,2] &= Q[y_4, s] \cup S_1, \\ A[4,3] &= P. \end{aligned}$$

Thus we may assume that $s \notin V(Q)$, and so $s \in V(P)$. Next suppose that s belongs to $P[r, y_3]$ and $s \neq r$. Then from (3) applied to $R_1 \cup P[r, y_3]$ and S, it follows that R_1, S_1 are disjoint, and so the claim holds, taking

$$\begin{aligned} A[1,3] &= R_1 \cup P[r, y_3], \\ A[4,2] &= P[y_4, s] \cup S_1, \\ A[4,3] &= Q. \end{aligned}$$

So we may assume that s belongs to $P[y_4, r]$. By (3) applied to $R_1 \cup P[r, y_3]$ and $P[y_4, s] \cup S_1$, we have that R_1, S_1 have intersection number one, and so by the minimality of $P \cup Q \cup R \cup S$ it follows that $R_1 \cap S_1$ is a path. But then taking

$$\begin{aligned} A[1,3] &= R_1 \cup P[r, y_3], \\ A[4,2] &= P[y_4, s] \cup S_1, \\ A[4,3] &= Q \end{aligned}$$

again satisfies the claim. (See Figure 23.) This proves (4).

Let W_1 be the Y-wing obtained from A by adding Y, all edges between Y and $V(A)$, and the edge y_4y_3 if it exists. Let W_2 be the Y-wing obtained from B by adding Y, all edges between Y and $V(B)$, and the edge y_2y_1 if it exists. Let W_3 be the Y-wing such that W_1, W_2, W_3 are pairwise internally disjoint and have union G. Then we claim:

- y_1 is a source of $W_1 \cup W_3$, and y_2 is a sink of $W_1 \cup W_3$, and y_3 is a source of $W_2 \cup W_3$, and y_4 is a sink of $W_2 \cup W_3$. To see this, it is clear that y_1 is a source of W_1, but we must check that it is a source of W_3, and similarly we must check the other three statements for W_3. For each $i > 2$, we have $y_1, y_3 \to D_i$ and $y_2, y_4 \leftarrow D_i$, so we only need to check the edges of W_3 with both ends in Y. Let $yy' \in E(G)$, where $y, y' \in Y$. We must show that either $(y, y') = (y_4, y_3)$ (in which case this edge is included in W_1, not W_3), or $(y, y') = (y_2, y_1)$ (similarly), or $y \neq y_2, y_4$ and $y' \neq y_1, y_3$. Suppose that $y \in \{y_2, y_4\}$. By (2) and since $A[1,3], B[3,1]$ exist, it follows that $y' \notin \{y_2, y_4\}$. But there are no edges y_4y_1 or y_2y_3, by hypothesis, so we may assume that $y \notin \{y_2, y_4\}$, and similarly $y' \notin \{y_1, y_3\}$, as required.

- There is a dipath of W_1 from y_1 to y_2, and there is a dipath of W_2 from y_3 to y_4. This is true since $A[1,2]$ and $B[3,4]$ exist.

- The digraph G_1 obtained from W_1 by adding a new vertex v_1 and the edges

  $$v_1y_1, y_2v_1, y_3v_1, v_1y_4, y_2y_1$$

  is 1-strong and weightable. This is true by 5.3 since this digraph is a butterfly minor of G, obtained from $W_1 \cup B[3,1] \cup B[2,4] \cup B[2,1]$ by contracting singular edges, where $B[3,1], B[2,4]$, $B[2,1]$ are as in (4).

- The digraph G_2 obtained from W_2 by adding a new vertex v_2 and the edges

  $$y_1v_2, v_2y_2, v_2y_3, y_4v_2, y_4y_3$$

  is 1-strong and weightable. This is true for the same reason as the previous bullet.

- The digraph G_0 obtained from W_3 by adding the edges y_1y_3, y_3y_1 and making the identifications $y_1 = y_2$ and $y_3 = y_4$ is 1-strong and weightable. This is true by 5.3 since this digraph is a butterfly minor of G, obtained by contracting singular edges of $W_3 \cup A[1,3] \cup A[4,3] \cup B[3,1] \cup B[2,1]$, where $A[1,3], A[4,3], B[3,1], B[2,1]$ are as in (4).

This proves 12.2. ▮

Finally, we handle the third outcome:

12.3. *Let $y_1, \ldots, y_4$ be distinct vertices of a 2-strong, 3-weak, weightable digraph G, and let $Y = \{y_1, \ldots, y_4\}$. Suppose that $G \setminus Y$ has at least two weak components D_i such that $y_1, y_3 \to D_i$ and $y_2, y_4 \leftarrow D_i$ for $i = 1, 2$. Then G is nonplanar, and G can be built from two smaller weightable digraphs by the construction of 9.5.*

Proof. Let $A = D_1$, $B = D_2$, and $C = G \setminus V(A \cup B)$. For distinct $y_i, y_j \in Y$, $A[i,j]$ means a dipath from y_i to y_j with all internal vertices in $V(A)$, and $B[i,j]$ is defined analogously. By $C[i,j]$ we mean a dipath of C from y_i to y_j with no internal vertex in Y.

(1) *We may assume that there are choices of $C[2,1], C[4,3]$ that are vertex-disjoint.*

Since G is 2-strong, there are two vertex-disjoint dipaths from $\{y_2, y_4\}$ to $\{y_1, y_3\}$, and by exchanging y_1, y_3 if necessary, we may assume that there are disjoint dipaths from y_2 to y_1 and from

y_4 to y_3. Since neither path has any internal vertex in Y, and $y_1, y_3 \to A \cup B$, it follows that both paths are paths of C. This proves (1).

Since $V(A) \neq \emptyset$ and G is 2-strong, it follows as usual that $A[1,2], A[1,4], A[3,2], A[3,4]$ exist, and the same for B.

(2) *$A[1,4], A[3,2]$ are vertex-disjoint for every choice of $A[1,4], A[3,2]$, and the same for B.*

Suppose there is a vertex c in both $A[1,4], A[3,2]$. Then, choosing $C[2,1], C[4,3]$ as in (1),

$$A[1,4] \cup C[4,3] \cup B[3,2] \cup C[2,1],$$
$$A[3,2] \cup C[2,1] \cup B[1,4] \cup C[4,3]$$

are dicycles that disagree on $\{y_1, y_3, c\}$, contradicting 5.1. This proves (2).

(3) *There are choices of $A[1,2], A[3,4]$ that intersect, and the same for B.*

This is immediate from (2) and 11.1.

(4) *$C[4,1], C[2,3]$ intersect for every choice of $C[4,1], C[2,3]$, and hence G is nonplanar.*

Otherwise, choosing $A[1,2]$ and $A[3,4]$ as in (3) and c in both paths,

$$A[3,4] \cup C[4,1] \cup B[1,2] \cup C[2,3],$$
$$A[1,2] \cup C[2,3] \cup B[3,4] \cup C[4,1]$$

are dicycles that disagree on $\{y_1, y_3, c\}$, a contradiction. This proves the first claim. Since the common vertices of $C[4,1], C[2,3]$ are not in Y, the internal vertices of $C[4,1], C[2,3]$ all belong to a weak component D of $G \setminus Y$ and each of $y_1, \dots, y_4$ has a neighbour in $V(D)$. Since each of $y_1, \dots, y_4$ also has a neighbour in $V(A)$ and a neighbour in $V(B)$, the underlying graph G^- has a $K_{3,3}$ minor, so G is nonplanar. This proves (4).

(5) *There is a choice of $A[1,2]$ and $A[3,4]$ that make a bubble, and the same for B.*

By (3), we may choose $P = A[1,2]$ and $Q = A[3,4]$ in bubble form. If they have intersection number one, choose $v \in P \cap Q$. Then $P[y_1, v] \cup Q[v, y_4]$ and $Q[y_3, v] \cup P[v, y_2]$ are $A[1,4], A[3,2]$ paths that intersect, contradicting (2). So $P \cap Q$ consists of at least two disjoint paths $R_1, \dots, R_k$. Suppose that $k \geq 3$, and choose $p_i \in V(R_i)$ for $i = 1,2,3$. Then, by 11.2, the dicycles $P \cup C[2,1]$ and $Q \cup C[4,3]$ disagree on $\{p_1, p_2, p_3\}$, a contradiction. Thus $k = 2$, and this proves (5).

(6) *$C[2,1], C[4,3]$ are vertex-disjoint for every choice of $C[2,1], C[4,3]$.*

Suppose some vertex c belongs to both $C[2,1], C[4,3]$, and choose $A[1,2], A[3,4]$ as in(5). Choose p_1, p_2 in both $A[1,2], A[3,4]$ such that y_1, p_1, p_2, y_2 are in order in $A[1,2]$ and y_3, p_2, p_1, y_4 are in order in $A[3,4]$. Then $A[1,2] \cup C[2,1]$, $A[3,4] \cup C[4,3]$ are dicycles that disagree on $\{p_1, p_2, c\}$, a contradiction. This proves (6).

(7) *There is a choice of $C[4,1], C[2,3]$ that make a bubble.*

From (4) we may choose $C[4,1], C[2,3]$ in bubble form. By (6), their intersection is not one path, so it is the disjoint union of at least two. If it is the disjoint union of at least three paths, choose vertices c_1, c_2, c_3 from three of these paths; then $C[4,1] \cup A[1,4]$, $C[2,3] \cup A[3,2]$ disagree on $\{c_1, c_2, c_3\}$, a contradiction. So $C[4,1], C[2,3]$ make a bubble. This proves (7).

Let W_1 be the Y-wing obtained from $A \cup B$ by adding Y and all edges between $V(A \cup B)$ and Y, and let $W_2 = C$. Thus, W_1, W_2 are internally disjoint Y-wings with union G. We claim:

- y_1, y_3 are sources of W_1 and y_2, y_4 are sinks of W_1. This is true because $y_1, y_3 \rightarrow A \cup B$ and $y_2, y_4 \leftarrow A \cup B$.
- There are dipaths of W_1 from y_1 to y_4 and from y_3 to y_2, and there are dipaths of W_2 from y_2 to y_1 and from y_4 to y_3. This is true because $A[1,4], A[3,2], C[2,1], C[4,3]$ exist.
- The digraph G_1 obtained from W_1 by adding the edges $y_2y_1, y_4y_3, y_1y_3, y_3y_1$ is 1-strong and weightable. This is true by 5.3 since G_1 is a butterfly minor of G, obtained by contracting singular edges of $W_1 \cup C[4,1] \cup C[2,3]$, where $C[4,1], C[2,3]$ are chosen as in (7).
- The digraph G_2 obtained from W_2 by adding two new vertices v_1, v_2 and the edges

  $$y_1v_1, y_3v_2, v_1y_4, v_2y_3, y_3y_4$$

  is 1-strong and weightable. This is true by 5.3 since G_2 is a butterfly minor of G, obtained by contracting singular edges of $W_2 \cup A[1,2] \cup A[3,4] \cup B[3,4]$, where $A[1,2], A[3,4]$ are as in (5).

This proves 12.3. ▮

We deduce:

12.4. *Every 2-strong 3-weak weightable planar digraph is diplanar.*

Proof. We assume that G is 2-strong, 3-weak, weightable and not diplanar, and we need to show that G is not planar. Since G is weightable, $G \neq F_7$, and it follows from 10.3 that G satisfies one of the three outcomes of that theorem, and so G is nonplanar by 12.1, 12.2, or 12.3. This proves 12.4. ▮

And our second main theorem:

12.5. *Every 1-strong weightable digraph can be built by means of the constructions of Sections 6 and 9 from 1-strong diplanar digraphs.*

Proof. Let G be a weightable digraph; we prove that G can be so constructed by induction on $|V(G)|$. We can assume that G is simple. We may assume that G is 2-strong and 3-weak, because otherwise it can be built from smaller weightable digraphs by the constructions of Sections 6. If G is planar, then it is diplanar by 12.4 and we are done. Otherwise, by 10.3 and 12.1, 12.2, and 12.3, G can be built by the constructions of Section 9 from smaller 1-strong digraphs, and the result follows from the inductive hypothesis. This proves 12.5. ▮

Finally, let us derive an algorithm to test whether a digraph G is weightable. Let $n = |V(G)| + |E(G)|$; we will give an algorithm with running time $O(n^6)$. Given G, we can search for one of the decompositions listed in 1.2 (or deduce that G is circular) in time $O(n^5)$. The slowest case is testing for the second and third nonplanar decompositions, which can be done by testing each set of four vertices for the necessary properties to be the set Y (this test can be done in linear time in E, for a total running time of $O(n^5)$). If we find a decomposition, continue recursively, until either we have constructed G from circular digraphs (and it is therefore weightable), or one of the parts does not admit any further decomposition and yet is not circular (when G is not weightable). Then the induction adds one more factor of n to the total running time, since in the worst case every step of the decomposition splits the graph into one piece of size $n - c_1$ and one or two pieces of size c_2.

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