Some results and problems on tournament structure

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Abstract

This paper is a survey of results and problems related to the following question: is it true that if G is a tournament with sufficiently large chromatic number, then G has two vertex-disjoint subtournaments A, B, both with large chromatic number, such that all edges between them are directed from A to B? We describe what we know about this question, and report some progress on several other related questions, on tournament colouring and domination.

1 Introduction

There is an open question of El-Zahar and Erdős [6], the following:

1.1 Problem: *Is the following true? For all integers $t, c \geq 1$, there exists $d \geq 1$, such that if a graph G satisfies $\chi(G) \geq d$ and has no clique with t vertices, then there are subsets $A, B \subseteq V(G)$ with $\chi(G[A]), \chi(G[B]) \geq c$, such that there are no edges between A and B.*

($\chi(G)$ denotes the chromatic number of a graph G.) In this paper we consider an analogue of this for tournaments, and a number of related results and conjectures. (Henceforth we use "Conjecture" for open questions: we do not intend to imply that we believe them to be true.)

The *chromatic number* $\chi(T)$ of a tournament T is the minimum k such that $V(T)$ is the union of k subsets $A_1, \ldots, A_k$ that each induce an acyclic subtournament. If T is a tournament and $A \subseteq V(T)$, we denote by $T[A]$ the subtournament with vertex set A, and write $\chi(A)$ for $\chi(T[A])$. If A, B are disjoint, and there is no edge with tail in B and head in A, we say A is *complete to* B, and B is *complete from* A, and write $A \Rightarrow B$, and call (A, B) a *complete pair*.

In [11] we made the following conjecture, which implies 1.1:

1.2 Conjecture: *For all $c \geq 0$ there exists $d \geq 0$ such that if T is a tournament with $\chi(T) \geq d$, there is a complete pair (A, B) of T such that $\chi(A), \chi(B) \geq c$.*

Very recently (since an earlier draft of this paper was submitted for publication) Klingelhoefer and Newman [10] proved the equivalence of 1.1 and 1.2, and we have updated this paper accordingly. In particular, we give a different proof of the equivalence that is simpler and shorter than that in [10], although it uses some of the same ideas.

It seems to us that 1.2 is very strong, and may well be false, despite its equivalence with 1.1. Indeed, for several years it was an open conjecture even to show that a tournament with sufficiently large chromatic number has a vertex whose out-neighbour set has large chromatic number, and the proof of this [9] is highly non-trivial.

But there are some partial results in its favour. First, a *cyclic triangle* is a tournament with three vertices, each with out-degree one; and we use the same name for the vertex set of such a tournament. Thus, a tournament is transitive if and only if it contains no cyclic triangle. In defense of 1.2, we will prove:

1.3 *For all $c \geq 0$ there exists $d \geq 0$ such that if T is a tournament with $\chi(T) \geq d$, there is a complete pair (A, B) of T such that A is a cyclic triangle and $\chi(B) \geq c$.*

The *domination number* $\mathrm{dom}(T)$ of a tournament T is the size of the smallest set X of vertices such that every vertex in $V(T) \setminus X$ is adjacent from a vertex in X; it is always at most the chromatic number. A second result in defense of 1.2 is that it is true for tournaments with sufficiently large domination number. More exactly, we will show:

1.4 *For every integer $c \geq 1$, there exists $d \geq 1$ such that if T is a tournament with $\mathrm{dom}(T) \geq d$ then there is a complete pair (A, B) such that $\chi(A), \chi(B) \geq c$.*

This area seems to be relatively unexplored, and yet full of interesting, significant, interconnected questions. Indeed, our attempts to decide 1.2, while unavailing, led to progress on several other tournament problems, for instance:

- A *rebel* is a tournament H such that all H-free tournaments have bounded domination number. (A tournament is H*-free* if no subtournament is isomorphic to H.) Until now there was only one tournament with chromatic number more than two that was known to be a rebel, and the proof that it was a rebel was difficult [4]. We give a much more general and much simpler construction of rebels.
- A *hero* is a tournament H such that all H-free tournaments have bounded chromatic number. It was already known which tournament are heroes [3], but our methods give a simpler proof of this result.
- A *legend* is an ordered tournament that is contained in every ordered tournament with sufficiently large domination number. (An *ordered tournament* is a pair (T, τ), where T is a tournament and τ is a numbering of its vertex set; and its *domination number* is the domination number of T.) We will find all legends.
- Harutyunyan, Le, Thomassé, and Wu [9] proposed two open questions about domination. We will show that one implies the other.
- We give a new class of tournaments ("crossing tournaments") with surprising properties, that provide counterexamples to some of our wilder dreams.

This paper is a survey of what we know about 1.2, and its connections with some other questions. It is organized as follows:

- In the next section we survey a number of special cases and variants of 1.2, true, false and open.
- In section 3 we prove a result about "diamonds" that will have several applications later.
- The next three sections concern the equivalence of 1.2 with the problem of El-Zahar and Erdős.
- Sections 7 and 8 prove the result about legends, and make progress on understanding rebels.
- Section 9 proves an implication between two open conjectures of [9].
- Sections 10 and 11 give a proof of 1.3 and several related results.
- Sections 12 and 13 concern whether excluding two particular types of tournament forces a bound on "local chromatic number".
- Finally, section 14 applies some results of earlier sections to deduce an older theorem about how to construct heroes.

2 Complete pairs (A, B) with $\chi(A)$ small

The conjecture 1.2 says that if $\chi(T)$ is large then there is a complete pair (A, B) with both $\chi(A), \chi(B)$ large. Can we at least get a complete pair (A, B) with $\chi(A), \chi(B)$ small but nonzero? Or perhaps with one of them large?

It is not obvious even that there is a complete pair (A, B) with $|A| = 1$ and $\chi(B)$ large, and this was raised in [3] as a conjecture. But Harutyunyan, Le, Thomassé, and Wu proved the following fundamental result, in a breakthrough paper [9]:

2.1 *For every integer $c \geq 1$, there exist integers $K, k \geq 1$ such that every tournament T with* $\mathrm{dom}(T) \geq K$ *contains a subtournament on at most k vertices having chromatic number at least c.*

This has several useful consequences, including a proof of the conjecture of [3]:

2.2 *For all integers $c \geq 0$ there exists $d \geq 0$ such that for every tournament T with $\chi(T) \geq d$, there exists $v \in V(T)$ such that $\chi(N^+(v)) \geq c$.*

For a vertex v, we define $N^+[v] = N^+(v) \cup \{v\}$, and $N^-[v]$ is defined similarly. Here is another consequence of 2.1, more evidence for the truth of 1.2:

2.3 *For every integer $c \geq 1$, there exists $d \geq 1$ such that if T is a tournament with* $\mathrm{dom}(T) \geq d$, *then there is a complete pair (A, B) such that $\chi(A), \chi(B) \geq c$, and* $\mathrm{dom}(T[A]) \geq c$.

Proof. Choose K, k as in 2.1, and let $d = \max(K, k + c)$. Let T be a tournament with $\mathrm{dom}(T) \geq d$. Since $d \geq K$, there exists $B \subseteq V(T)$ with $|B| \leq k$ such that $\chi(B) \geq c$, by 2.1. Let X be the set of vertices of T that either belong to B or are adjacent from some vertex in B. Thus $\mathrm{dom}(T \setminus X) \geq c$, since $\mathrm{dom}(T) \geq k + c$ and $|B| \leq k$. Let $A = T \setminus X$; then every vertex in B is adjacent from every vertex in A, since $A \cap (B \cup X) = \emptyset$. This proves 2.3. ∎

This reduces proving 1.2 to proving it for tournaments in which every subtournament has bounded domination number.

The proof of 2.2 in [9] gives d bounded by a very large, tower-type, function of c, but we do not know that it needs to be so large. In fact we do not have a counterexample for the following, although we find it hard to believe:

2.4 Conjecture: *For all integers $c \geq 1$ and every tournament T with $\chi(T) \geq 2c$, there exists $v \in V(T)$ such that $\chi(N^+(v)) \geq c$.*

Let us remark that if we replace chromatic number by "fractional chromatic number" in the usual sense, then 2.4 becomes true. More exactly, let us say a *k-multicolouring* of T is a list (possibly with repetition) of acyclic subsets of $V(T)$ such that every vertex is in at least k of them, and its *size* is the number of sets in the list (counting multiplicity). The *fractional chromatic number* $\chi_f(T)$ is the minimum of K/k, over all pairs of positive integers K, k such that there is a k-multicolouring of size K. Then:

2.5 *For all real numbers $c > 0$ and every tournament T with $\chi_f(T) > 2c$, there exists $v \in V(T)$ such that $\chi_f(N^+(v)) > c$.*

Proof. Suppose that $T[N^+[v]]$ has fractional chromatic number at most c, for every vertex v. We will prove that $\chi_f(T) \leq 2c$. For each v, there exist integers $K_v, k_v \geq 1$, with $K_v \leq ck_v$, and a k_v-multicolouring of $T[N^+[v]]$ of length K_v. Let K be a common multiple of all the numbers K_v. For each v, by replacing all the sets of the k_v-multicolouring of $T[N^+[v]]$ by $(K/K_v)k_v$ copies, we obtain a $k_v(K/K_v)$-multicolouring of $T[N^+[v]]$ with size K. Thus we may assume that all the numbers K_v are equal (to K). By replacing all the numbers k_v by $\min(k_v : v \in V(G))$, we may also assume that all the numbers k_v are equal (to some k say), where $K \le ck$. Thus, for each v, we have a k-multicolouring of $T[N^+[v]]$ of size K; let us call it M_v.

By a standard argument using linear programming duality, one can assign a non-negative integer weight $w(v)$ to each vertex v of T, totalling to some positive integer L, such that for every vertex v, the sum of the weights in $N^-[v]$ is at least $L/2$. For each $v \in V(G)$, let us take $w(v)$ copies of M_v, and take the union of all these lists, forming a list M say. Its size is KL. For each vertex v, since $\sum_{u \in N^-[v]} w(u) \ge L/2$, and v belongs to k members of M_u whenever $u \in N^-[v]$, it follows that M is a $(kL/2)$-multicolouring of T. Since its size is KL, it follows that $\chi_f(T) \le (KL)/(kL/2) = 2K/k \le 2c$. This proves 2.5. ∎

Two of us proposed [12] a strengthening of 2.2: that if G is a graph with large chromatic number, and T is a tournament with the same vertex set, then for some vertex v, the set of out-neighbours of v in T induces a subgraph of G with large chromatic number. But this has very recently been disproved, by Girão, Hendrey, Illingworth, Lehner, Michel, Savery and Steiner [7], who showed the following:

2.6 *For all $d \ge 0$, there is a graph G with chromatic number at least d, and a tournament T with vertex set $V(G)$, such that for each $v \in V(G)$, the set of out-neighbours of v in T induces a bipartite subgraph of G.*

Here is another possible strengthening of 2.2 in which we had some hope: that if A, B are disjoint subtournaments of a tournament T, both with sufficiently large chromatic number, then there is a vertex in one of A, B such that its set of out-neighbours in the other set has large chromatic number. But this too is false. By a simple modification of their example in 2.6, Girão et al. [7] showed:

2.7 *For all $d \ge 0$, there is a tournament with two disjoint subtournaments A, B, both with chromatic number at least d, such that $\chi(B[N^+(v)]) \le 2$ for each vertex $v \in V(A)$, and $\chi(A[N^+(v)]) \le 2$ for each vertex $v \in V(B)$.*

Indeed, we suspect that even the following is false, although it remains open for the moment:

2.8 Conjecture: *There exists $d \ge 0$ such that if T is a tournament, and A, B are disjoint subsets of $V(T)$ with $\chi(A), \chi(B) \ge d$, then there exist $A' \subseteq A$ and $B' \subseteq B$, both cyclic triangles, such that one of $(A', B'), (B', A')$ is a complete pair.*

A digression: how much does it matter that we are concerned with *chromatic number* in 2.4? If T is a tournament, a *submeasure* on T is a function $\mu : 2^{V(T)} \to \mathbb{R}^+$ (we use $\mathbb{R}^+$ to denote the set of nonnegative real numbers), such that $\mu(\emptyset) = 0$, and μ is increasing and subadditive (that is, $\mu(A) \le \mu(B)$ when $A \subseteq B$, and $\mu(A \cup B) \le \mu(A) + \mu(B)$ for all A, B). Thus chromatic number is a submeasure. One might hope that we could extend 2.2, to general submeasures instead of chromatic number, but that is false, because of the following, due to Noga Alon:

2.9 *For every tournament T, there is a submeasure μ on T such that $\mu(T)$ is the domination number of T, and $\mu(N^+(v)) = 1$ for every vertex v.*

Proof. For each $X \subseteq V(T)$, let $\mu(X)$ be the cardinality of the smallest subset $Y \subseteq V(T)$ such that every vertex in X either belongs to Y or is adjacent from a member of Y. Then μ is a submeasure with the desired properties. This proves 2.9. ∎

Let H be a tournament, and for a tournament T define $\chi_H(T)$ to be the minimum k such that $V(T)$ can be partitioned into k subsets each inducing an H-free tournament. Thus, $\chi(T)$ is the same as $\chi_H(T)$ when H is a cyclic triangle. Perhaps one can extend 2.2 to:

2.10 Conjecture: *For every tournament H with $\chi(H) \geq 2$ and every integer $c \geq 0$, there exists $d \geq 0$ such that for every tournament T with $\chi_H(T) \geq d$, there exists $v \in V(T)$ such that $\chi_H(N^+(v)) \geq c$.*

Another digression: the domination number of a tournament is at most its chromatic number. One might ask, does 2.2 work with chromatic number replaced by domination number? This is false, as a neat example due to Noga Alon shows:

2.11 *For all $d \geq 0$, there is a tournament T with domination number at least d, such that $T[N^+(v)]$ has domination number one for every vertex v.*

Proof. Take a tournament H with domination number at least d, and replace each vertex x with a cyclic triangle T_x, such that $T_x \Rightarrow T_y$ if y is adjacent from x in T. This forms a larger tournament T say. The domination number of T is at least d; but for every vertex v of T , $\mathrm{dom}(T[N^+(v)]) = 1$ (because if $v \in T_x$, the out-neighbour of v in T_x belongs to and dominates $N^+(v)$). This proves 2.11. ∎

Domination number is very interesting, and we will discuss it more later. The *reverse* of a tournament T is obtained from it by reversing the direction of all its edges. It is convenient at this point to prove a slight extension of 2.2, the following:

2.12 *There is an integer-valued function ϕ such that for every integer $c \geq 0$, if T is a tournament with $\chi(T) \geq \phi(c)$ then there exists $v \in V(T)$ such that $\chi(N^+(v)) \geq c$ and $\chi(N^-(v)) \geq c$.*

Proof. By 2.2, there is an integer-valued function ψ such that for every integer $c \geq 0$, if T is a tournament with $\chi(T) \geq \psi(c)$, then there exists $v \in V(T)$ such that $\chi(N^+(v)) \geq c$. Define $\phi(c) = 2\psi(c)$ for $c \geq 0$. We claim that ϕ satisfies the theorem.

Let T be a tournament with $\chi(T) \geq \phi(c)$, and let X be the set of vertices of T such that $\chi(N^+(v)) \geq c$. It follows that $\chi(T \setminus X) < \psi(c)$, and so $\chi(X) \geq \chi(T) - \psi(c) \geq \psi(c)$. Hence (by the property of ψ, applied to the reverse of $T[X]$), there exists $v \in X$ such that $\chi(X \cap N^-(v)) \geq c$. But then $\chi(N^+(v)) \geq c$ and $\chi(N^-(v)) \geq c$. This proves 2.12. ∎

Let us return to 2.2 and weakenings of 1.2. It is a consequence of a result of [3] that every tournament with sufficiently large chromatic number contains a complete pair (A, B) where A, B are both cyclic triangles. We will prove a strengthening of this which also is a strengthening of 2.2, and implies 1.3:

2.13 *For all $c \geq 0$ there exists $d \geq 0$ such that if T is a tournament with $\chi(T) \geq d$, then there exist disjoint sets $P, A, Q \subseteq V(T)$ such that $P \Rightarrow A \Rightarrow Q$, and A is a cyclic triangle, and $\chi(P), \chi(Q) \geq c$.*

Indeed, this partially extends to χ_H. We will show:

2.14 *For every tournament H with $\chi(H) \geq 2$, and all $c \geq 0$, there exists $d \geq 0$ such that if T is a tournament with $\chi_H(T) \geq d$, then there exist disjoint sets $P, A, Q \subseteq V(T)$ such that $P \Rightarrow A \Rightarrow Q$, where $T[A]$ is isomorphic to H, and $\chi(P), \chi(Q) \geq c$.*

We have not been able to show the same with the stronger conclusion that $\chi_H(P), \chi_H(Q) \geq c$, though this would be true if conjecture 2.10 is true.

3 Diamonds

A *diamond* in a tournament T is a quadruple (a, b, P, Q), where $a, b \in V(T)$ are distinct, and P, Q are disjoint subsets of $V(T) \setminus \{a, b\}$, such that $a \Rightarrow P \Rightarrow b \Rightarrow Q \Rightarrow a$. The *chromatic number* of a diamond is the minimum of $\chi(P), \chi(Q)$. Diamonds with large chromatic number are valuable, so in this section we explore tournaments that contain no such diamonds.

Fix a numbering $v_1, \ldots, v_n$ of the vertex set of a tournament T. For $1 \le i, j \le n$, we say v_j is a *right out-neighbour* of v_i if $j > i$ and v_j is adjacent from v_i; and v_j is a *right in-neighbour* of v_i if $j > i$ and v_i is adjacent from v_j. Similarly, we say v_j is a *left out-neighbour* of v_i if $j < i$ and v_j is adjacent from v_i; and and v_j is a *left in-neighbour* of v_i if $j < i$ and v_i is adjacent from v_j. The *local chromatic number* of the numbering $v_1, \ldots, v_n$ is the maximum over $1 \le i \le n$ of the chromatic number of the set of all v_j such that v_j is either a left out-neighbour or a right in-neighbour of v_i. (Thus, it is the chromatic number of the tournament induced on the set of all neighbours of v_i in the backedge graph under the given numbering. "Backedge graph" is defined in section 4.) Let us show a result that will have several applications, to 3.2, 12.2, 13.3, 14.2, and to the equivalence of 1.1 and 1.2:

3.1 *If a tournament T admits a numbering with local chromatic number at most c, then it contains no diamond with chromatic number more than $2c$. Conversely, for all $c \ge 0$ there exists $d \ge 0$ such that if a tournament T contains no diamond with chromatic number more than c, then it admits a numbering with local chromatic number at most d.*

Proof. For the first statement, let $v_1, \ldots, v_n$ be a numbering of $V(T)$ with local chromatic number at most c, and let (a, b, C, D) be a diamond. From the symmetry we may assume that $a = v_i$ and $b = v_j$ where $i < j$ (exchanging a, b and C, D if necessary). If $v_k \in D$, then either $k < j$ and so v_k is a left out-neighbour of v_j), or $k > i$ (and then v_k is a right in-neighbour of v_i); and so $\chi(D) \le 2c$. This proves the first statement.

For the second, by 2.2 there exists $d \ge 0$ such that if T is a tournament with $\chi(T) > d$, then there exists $v \in V(T)$ such that $\chi(N^+(v)) \ge 2c + 2$. Let T be a tournament that contains no diamond with chromatic number more than c. We will show that T admits a numbering with local chromatic number at most $2d$. Let H be the digraph with vertex set $V(T)$, in which for all distinct $a, b \in V(T)$, b is adjacent from a in H if $\chi(T[N_T^+(a) \cap N_T^-(b)]) \ge 2c + 2$.

(1) *H has no directed cycle.*

Let v_1-v_2-$\cdots$-v_k-v_1 be the vertices in order of a directed cycle of H. Since T has no diamond of chromatic number more than c (and therefore, none of order more than $2c$) it follows that $k > 2$. For $1 \le i \le k$ let A_i be the set of vertices of T that are adjacent to v_{i+1} and from v_i, where v_{k+1} means v_1. Thus $\chi(A_i) \ge 2c + 2$ for $1 \le i \le k$. For each i, let B_i be the set of vertices in A_i adjacent from v_1, and C_i be the set of vertices in A_i adjacent to v_1. Thus $B_i \cup C_i = A_i$ if $v_1 \notin A_i$, and $B_i \cup C_i = A_i \setminus \{v_1\}$ if $v_1 \in A_i$. Let I be the set of $i \in \{1, \ldots, k\}$ such that $\chi(B_i) > c$, and J the set with $\chi(C_i) > c$. Since $\chi(B_i \cup C_i) \ge \chi(A_i) - 1 > 2c$, it follows that $i \in I \cup J$ for each i. But $1 \in I$ and $k \in J$, and so there exists i with $1 \le i < k$ such that $i \in I$ and $i + 1 \in J$. But then $i(v_1, v_{i+1}, B_i, C_{i+1})$ is a diamond with chromatic number more than c, a contradiction. This proves (1).

From (1), there is a numbering $v_1, \ldots, v_n$ of $V(T)$ such that for all i, j with $1 \le i < j \le n$, v_i is not adjacent from v_j in H. Let $1 \le i \le n$, and let A be the set of right in-neighbours of v_i, and let B be the set of left out-neighbours of v_i. For each $v_j \in A$, it follows that $j > i$, and so $v_j v_i \notin E(H)$, and therefore the set of vertices in A that are out-neighbours of v_j has chromatic number at most $2c+1$. From the choice of d, it follows that $\chi(A) \le d$, and similarly $\chi(B) \le d$ (using in-neighbours instead of out-neighbours). Thus $\{v_1, \ldots, v_n\}$ has chromatic number at most $2d$. This proves 3.1. ∎

For $k \ge 0$, let us say an edge uv of a tournament T is *k-heavy* if $\chi(X) \ge k$, where X is the set of all vertices that are adjacent from v and to u. Here is an application of 3.1, proved independently in [10]:

3.2 *For all $k \ge 0$ there exists $q \ge 0$ such that every tournament T with $\chi(T) \ge q$ has a k-heavy edge.*

Proof. By 3.1, there exists $d \ge 0$ such that if a tournament contains no diamond with chromatic number at least k, then it admits a numbering with local chromatic number at most d. Let $q = 2k + 4d + 1$, and let T be a tournament with $\chi(T) \ge q$. If T contains a diamond (a, b, P, Q) with chromatic number at least k, we may assume that $ab \in E(T)$, by exchanging a with b and P with Q if necessary; but then ab is k-heavy. So we assume that there is no such diamond. By 3.1, T admits a numbering $v_1, \ldots, v_n$ with local chromatic number at most d. Choose $i_1 \le n$ maximum such that $\chi(V_1) \le k + 2d$, where $V_1 = \{v_1, \ldots, v_{i_1}\}$; if $i_1 < n$, choose i_2 with $i_1 + 1 \le i_2 \le n$ maximum such that $\chi(V_2) \le k + 2d$, where $V_2 = \{v_{i_1+1}, \ldots, v_{i_2}\}$, and so on. This partitions $V(T)$ into intervals $V_1, \ldots, V_t$ say, for some $t \ge 1$; and $\chi(V_s) = k + 2d$ for $1 \le i < t$, and $\chi(V_t) \le k + 2d$. Choose two disjoint sets P, Q both of $k + 2d$ colours. For $1 \le s \le t$ with s odd, take a colouring of $T[V_i]$ using colours in P, and do the same for s even using colours in Q. Since $\chi(T) > 2k + 4d$, this is not a valid colouring of T, and so there exist r, s with $1 \le r, s \le t$ and $s \ge r + 2$, such that there is an edge from V_s to V_r. Consequently there exist i, j with $1 \le i < j \le n$, such that $v_j v_i \in E(T)$, and $\chi(X) \ge k + 2d$, where $X = \{v_{i+1}, \ldots, v_{j-1}\}$. Let X_1 be the set of vertices in X adjacent to v_i; let X_2 be the set adjacent from v_j; and let X_3 be the remainder. Since $\chi(X_1), \chi(X_2) \le d$ (because the numbering has local chromatic number at most d), it follows that $\chi(X_3) \ge k$, and so $v_j v_i$ is k-heavy. This proves 3.2. ∎

4 Equivalence with the problem of El-Zahar and Erdős: the easy half

In this section we will show that 1.2 implies 1.1, and we discuss the converse implication in the next section. There is a standard technique to go between graphs and tournaments, as follows. Let T be a tournament, and choose a numbering $v_1, \ldots, v_n$ of its vertex set. Let G be the graph with vertex set $V(T)$, in which for $1 \le i < j \le n$, the pair v_i, v_j are adjacent in G if and only if v_i is adjacent from v_j in G. We call G the *backedge graph* of T under the given numbering. The construction can evidently be reversed: given a graph G and a numbering, there is a tournament T such that G is the backedge graph of T under the numbering.

We begin with a standard result:

4.1 *Let G be the backedge graph of a tournament T under the numbering $v_1, \dots, v_n$. Let $\omega(G)$ be the size of the largest clique of G. Then*

$$\chi(T) \leq \chi(G) \leq \omega(G)\chi(T).$$

Proof. Every set that is stable in G is transitive in T, so $\chi(T) \leq \chi(G)$. Now let X be transitive in T, and let $<_P$ be the partially ordered set with element set X in which for $v_i, v_j \in X$, we say $v_i <_P v_j$ if $i < j$ and $v_j v_i \in E(T)$. This is indeed a poset, because if $i < j < k$ and $v_i, v_j, v_k \in X$, and $v_j v_i, v_k v_j \in E(T)$, then $v_k v_i \in E(T)$ (since X is transitive and so $v_i v_k \notin E(T)$). Every totally ordered subset of the poset is a clique of G and so has size at most $\omega(G)$; and hence by (the dual of) Dilworth's theorem, X can be partitioned into $\omega(G)$ subsets, each an antichain of the poset and hence each a stable set of G. Thus $\chi(G[X]) \leq \omega(G)$ when X is transitive in T; and so $\chi(G) \leq \omega(G)\chi(T)$. This proves 4.1. ∎

Proof of 1.1, assuming 1.2. We proceed by induction on t; so we may assume that $t \geq 3$, and there exists d_1 such that for every graph G with $\chi(G) \geq d_1$ and $\omega(G) < t - 1$, there are vertex-disjoint subsets A, B of $V(G)$, both inducing subgraphs with chromatic number at least c, with no edges between A and B. Let $c_2 = \max(2d_1, 2c)$; by the assumed truth of 1.2, there is an integer $d_2 \geq 1$ such that if T is a tournament and $\chi(T) \geq d_2$, there is a complete pair (A, B), where A, B both induce tournaments with chromatic number at least c_2. Let $d = td_2$. Let G be a graph with $\chi(G) \geq d$ and $\omega(G) < t$. We must show that there are disjoint subsets A, B of $V(G)$, both inducing subgraphs with chromatic number at least c, and with no edges beteeen A, B. We may therefore assume that for every vertex $v \in V(G)$, the subgraph induced on its neighbour set has chromatic number less than d_1.

Let $V(G) = \{v_1, \dots, v_n\}$, and let T be the tournament such that G is its backedge graph under the numbering $v_1, \dots, v_n$. From 4.1, it follows that $\chi(T) \geq d/t = d_2$. By 1.2, there exist disjoint $A', B' \subseteq V(T)$ with A' complete to B', such that $T[A'], T[B']$ both have chromatic number at least c_2. Choose i minimum such that one of

$$\{v_1, \dots, v_i\} \cap A', \{v_1, \dots, v_i\} \cap B'$$

induces a tournament with chromatic number at least $c_2/2$. Now there are two cases.

Suppose first that $\{v_1, \dots, v_i\} \cap A'$ induces a tournament with chromatic number at least $c_2/2$. Let $A = \{v_1, \dots, v_i\} \cap A'$ and $B = \{v_{i+1}, \dots, v_k\} \cap B'$. Thus $A \neq \emptyset$ and from the minimality of i, $\chi(T \setminus B) \leq c_2/2$, and so $\chi(T[B]) \geq c_2/2 \geq d_1$. Since A is complete to B in T, and $h \leq i < j$ for all h, j with $v_h \in A$ and $v_j \in B$, it follows that every vertex of A is adjacent in G to every vertex in B. But every subset that is stable in G is acyclic in T, and so $\chi(G[B]) \geq c_2/2 \geq d_1$, contradicting that for every vertex $v \in V(G)$, the subgraph induced on its neighbour set has chromatic number less than d_1.

Thus $\{v_1, \dots, v_i\} \cap B'$ induces a tournament with chromatic number at least $c_2/2$. Let $B = \{v_1, \dots, v_i\} \cap B'$ and $A = \{v_{i+1}, \dots, v_k\} \cap A'$. As before, $\chi(T[A]), \chi(T[B]) \geq c_2/2 \geq c$. Since A is complete to B in T, and $h \leq i < j$ for all h, j with $v_j \in A$ and $v_h \in B$, it follows that every vertex of A is nonadjacent in G to every vertex in B. Moreover, every subset that is stable in G is acyclic in T, and so $\chi(G[A]) \geq \chi(T[A]) \geq c$ and similarly $\chi(G[B]) \geq c$. This proves 1.1. ∎

In the proof above that 1.2 implies 1.1, the tournament T constructed has domination number less than t, and so do all its subtournaments. (To see this, choose a clique X of G that is optimal in the following sense: it contains v_n, and it contains v_j where $j < n$ is maximum such that v_j, v_n are adjacent, and it contains v_i where $i < j$ is maximum such that v_i is adjacent to both v_j, v_n, and so on. This clique is dominating in T.)

So one might consider restricting 1.2 to tournaments such that all subtournaments have bounded domination number, since that would still be strong enough to imply 1.1, as we just saw.

But here is a surprise: that conjecture, apparently much weaker, is the *hard* part of 1.2: the latter is true for all tournaments with sufficiently large domination number, because of 2.3.

5 The equivalence of 1.1 and 1.2: the hard half

As we mentioned earlier, F. Klingelhoefer and A. Newman [10] proved (since an earlier version of this paper was submitted for publication) that the truth of 1.1 implies 1.2. They used a result about the chromatic number of digraphs (that is, the smallest k such that the vertex set of the digraph can be partitioned into k sets each inducing an acyclic digraph), the following:

5.1 *For all integers $a, c \geq 1$ there exists d such that if D is a digraph such that:*

- *all directed cycles of D have length at least three;*
- *the undirected graph underlying D has no stable set of size a; and*
- *$D[X]$ has chromatic number at most c for every edge uv of D, where X denotes the set of vertices that are adjacent to u and from v,*

then D has chromatic number at most d.

Their derivation of the equivalence of 1.1 and 1.2 from 5.1 was quick (about 1.5 pages), but the proof of 5.1 itself was quite lengthy (about 7.5 pages). We have now found a different way to do it, using some ideas from [10] but bypassing 5.1, that we present here.

If T is a tournament, let J_k be the graph with vertex set $V(T)$ and edge set all pairs $\{u, v\}$ such that one of uv, vu is k-heavy in T. We call J_k the *graph of k-heavy pairs* of T. We begin with:

5.2 *Let T be a tournament, and let $d, k \geq 1$ be integers. If there exists $X \subseteq V(T)$ such that $T[X]$ is transitive and $\chi(J_{2k}[X]) \geq 5^d$, then $J_k[X]$ has a d-clique.*

Proof. The proof is by induction on d, and trivial for $d = 1$, so we assume $d \geq 2$ and the result holds for $d - 1$. Number $X = \{x_1, \dots, x_n\}$ such that $x_i x_j \in E(T)$ for $1 \leq i < j \leq n$. (Now the proof is like that of 3.2.) Choose $i_1 \leq n$ maximum such that $\chi(J_{2k}[X_1]) \leq 2(5^{d-1})$, where $X_1 = \{x_1, \dots, x_{i_1}\}$; so either $i_1 = n$ or $\chi(J_{2k}[X_1]) = 2(5^{d-1})$. If $i_1 < n$, choose i_2 with $i_1 \leq i_2 \leq n$ maximum such that $\chi(J_{2k}[X_2]) \leq 2(5^{d-1})$, where $X_2 = \{x_{i_1+1}, \dots, x_{i_2}\}$, and so on. This partitions X into some number of intervals, say $X_1, \dots, X_t$, such that $\chi(J_{2k}[X_s]) = 2(5^{d-1})$ for $1 \leq s < t$, and $\chi(J_{2k}[X_t]) \leq 2(5^{d-1})$. Let us choose two disjoint sets of $2(5^{d-1})$ colours, say P, Q, and colour the graphs $J_{2k}[X_s]$ with colours from P for $1 \leq s \leq t$ with s odd, and use colours from Q to colour the graphs $J_{2k}[X_s]$ with s even. Since $\chi(J_{2k}[X]) \geq 5^d > |P| + |Q|$, this does not make a valid colouring of $J_{2k}[X]$, and so there exist $1 \leq r < s \leq t$ with $s \geq r + 2$ such that there is an edge of $J_{2k}[X]$ between X_r and

X_s. In particular, there exist $1 \le i \le j \le n$ such that x_ix_j is $2k$-heavy, and $\chi(J_{2k}[X']) \ge 2(5^{d-1})$, where $X' = \{x_{i+1}, \dots, x_{j-1}\}$. Let Y be the set of all vertices adjacent from x_j and to x_i in T; thus $\chi(Y) \ge 2k$. Let $x \in X'$. Since $\chi(Y) \ge 2k$, either the set of out-neighbours, or the set of in-neighbours, of x in Y has chromatic number at least k. In the first case x_ix is k-heavy, and in the second xx_j is k-heavy. Thus there exist $u \in \{x_i, x_j\}$ and $X'' \subseteq X'$ with $\chi(J_{2k}[X'']) \ge 5^{d-1}$ such that u is adjacent in J_k to each vertex in X''; and the result follows from the inductive hypothesis applied to X''. This proves 5.2. ∎

In what follows, we will often have a graph (J_k for instance) and a tournament T with the same vertex set, and for $X \subseteq V(T)$, we need to talk about the chromatic number of the subgraph of J_k induced on X, and the chromatic number of the subtournament of T induced on X. We denote the first by $\chi(J_k[X])$, and the second by $\chi(T[X])$ or just $\chi(X)$.

Let us say a tournament T is *c-bad* if there do not exist disjoint $A, B \subseteq V(T)$, such that $A \Rightarrow B$ and $\chi(A), \chi(B) \ge c$.

5.3 *Let $c, d \ge 1$ be integers. If 1.1 is true, then for all $k \ge c$ there exists $L \ge 0$ such that if T is a c-bad tournament with $\chi(T) \ge L$, then its graph J_k of k-heavy pairs has a d-clique.*

Proof. The proof is by induction on d, and is trivial for $d = 1$, so we assume that $d \ge 2$ and the result holds for $d-1$. Choose L' such that for every c-bad tournament T with $\chi(T) \ge L'$, J_k has a $(d-1)$-clique. Let $K = \max(2k, 2L')$. By 3.1 there exists p such that every tournament with no diamond of chromatic number at least K admits a numbering with local chromatic number at most p. By 3.2 there exists $q \ge 0$ such that every tournament with chromatic number at least q has a k-heavy edge.

We are assuming the truth of 1.1, and so there exists M such that if a graph G has chromatic number at least M and has no clique of size d, then there are subsets $A, B \subseteq V(G)$ with $\chi(G[A]), \chi(G[B]) \ge 5^d(2c + 2k + 4p)$, such that there are no edges between A and B. Let $L = qM$, and let T be a c-bad tournament with $\chi(T) \ge L$. Suppose first that T contains a diamond (a, b, P, Q) with chromatic number at least K. For each $v \in P$, either the set of out-neighbours, or the set of in-neighbours, of v in Q has chromatic number at least $K/2 \ge k$; and so one of the edges av, vb is k-heavy. Consequently, for one of a, b (say u), the set of vertices in P adjacent to u in J_k induces a tournament with chromatic number at least $K/2 \ge L'$, and so includes a $(d-1)$-clique of J_k, from the inductive hypothesis. Adding u to this clique gives a d-clique of J_k as required.

So we assume that there is no such diamond, and consequently T admits a numbering with local chromatic number at most p, say $v_1, \dots, v_n$. We may assume that J_k has no d-clique; and so by 5.2, for every subset $X \subseteq V(T)$, $\chi(J_k[X]) \le 5^d\chi(X)$. Conversely, every subset $X \subseteq V(T)$ satisfies $\chi(X) \le q\chi(J_k[X])$, since every set that is stable in J_k induces a tournament with no k-heavy edge and which therefore has chromatic number at most q. In particular, $\chi(J_k) \ge \chi(T)/q \ge L/q = M$. By 1.1, there are subsets $A', B' \subseteq V(T)$ with $\chi(J_k[A']), \chi(J_k[B']) \ge 5^d(2c + 2k + 4p)$, such that there are no edges of J_k between A' and B'. It follows that $\chi(A'), \chi(B') \ge 2c + 2k + 4p$. Choose $i \le n$ maximum such that $\chi(A_1), \chi(B_1) \le c + k + 2p$, where $A_1 = A' \cap \{v_1, \dots, v_i\}$ and B_1 is defined similarly. Let $A_2 = A' \setminus A_1$ and $B_2 = B' \setminus B_1$. Thus one of $\chi(A_1), \chi(B_1) = c + k + 2p$, and both $\chi(A_2), \chi(B_2) \ge c + k + 2p$. We assume $\chi(A_1) = c + k + 2p$ (the other case is similar, and indeed the same if we reverse all edges). Choose $h \le i$ maximum such that $\chi(A) \le c$, where $A = A_1 \cap \{v_1, \dots, v_h\}$; thus $\chi(A) = c$, and $\chi(\{v_{h+1}, \dots, v_i\}) \ge k + 2p$. Let $B = B_2$. As in the proof of 3.2, every edge of T from B to A is k-heavy, and so there are no such edges, since there are no edges of J_k between A' and B'. Consequently $A \Rightarrow B$, which is impossible since T is c-bad. This proves 5.3. ∎

5.4 *Let $c \geq 1$. If 1.1 is true, then for all integers $d \geq 1$ there exist L_d, n_d such that every c-bad tournament with chromatic number at least L_d has a subtournament with chromatic number at least d and with at most n_d vertices.*

Proof. We use induction on d, and can assume that $d > 1$, and L_{d-1}, n_{d-1} exist. By 5.3, there exists $L_d \geq 0$ such that if T is a c-bad tournament with $\chi(T) \geq L_d$, then its graph of L_{d-1}-heavy pairs has a d-clique. Define $n_d = d + d(d-1)n_{d-1}/2$. We claim that L_d, n_d satisfy the theorem. Let T be c-bad with $\chi(T) \geq L_d$. By 5.3, there is a d-clique X of the graph of L_{d-1}-heavy pairs. For each edge uv of $T[X]$, there exists $Y_{uv} \subseteq V(T)$ that is complete to u and from v, with $\chi(Y_{uv}) \geq L_{d-1}$; and hence from the inductive hypothesis, there exists $Z_{uv} \subseteq Y_{uv}$ with $\chi(Z_{uv}) \geq d-1$ and $|Z_{uv}| \leq n_{d-1}$. The union of X and all the Z_{uv} has chromatic number at least d (because no two members of X can receive the same colour in a $(d-1)$-colouring), and has at most n_d vertices. This proves 5.4. ∎

Finally, we deduce that 1.1 implies 1.2, because:

5.5 *If 1.1 is true, then for all $c \geq 1$, every c-bad tournament has chromatic number at most $\max(c2^{n_{2c}} + n_{2c}, L_{2c})$.*

Proof. Suppose that T is a c-bad tournament with $\chi(T) \geq L_{2c}$). By 5.4, T has a subtournament X with $\chi(X) \geq 2c$ and with at most n_{2c} vertices. Partition $V(T) \setminus V(X)$ into at most $2^{n_{2c}}$ sets Y_i $(i \in I)$ say, such that for each $i \in I$, all members of Y_i have the same out-neighbours in $V(X)$. Consequently each set Y_i is complete to or from some subset of $V(X)$ with chromatic number at least c, since $\chi(X) \geq 2c$; and so $\chi(Y_i) < c$, since T is c-bad. It follows that $\chi(T) \leq |X| + 2^{n_{2c}}c$. This proves 5.5. ∎

One can recast the idea of 5.4 above so that it does not assume the truth of 1.1, as follows:

5.6 *Let $d \geq 1$ be an integer, and let T be a tournament such that no subtournament has chromatic number at least d and has at most $(d!)^2$ vertices. Then for every choice of integers $L_1, \ldots, L_d \geq 1$, if $\chi(T) \geq L_d$ then there exists $i \in \{2, \ldots, d\}$ and a subset $Y \subseteq V(T)$ such that $\chi(Y) \geq L_i$ and the graph of L_{i-1}-heavy edges of $T[Y]$ has no clique of size i.*

Proof. We proceed by induction on d. The statement is true when $d = 1$, so we assume that $d > 1$ and it holds for $d-1$. We may assume that $\chi(T) \geq L_d$, and the graph of L_{d-1}-heavy edges of T has a clique of size d. So there is a d-vertex subtournament X of T such that all its edges are L_{d-1}-heavy. For each edge uv of X, let $N(uv)$ be the set of all vertices adjacent to v and from u. If for some edge uv of X, there exists $i \in \{2, \ldots, d-1\}$ and a subset $Y \subseteq N_{uv}$ such that $\chi(Y) \geq L_i$ and the graph of L_{i-1}-heavy edges of $T[Y]$ has no clique of size i, then the theorem is satisfied. Since $\chi(N(uv)) \leq L_{d-1}$ for each uv, we may therefore assume that for each uv, $T[N(uv)]$ has a subtournament Y_{uv} with chromatic number at least $d-1$ and at most $((d-1)!)^2$ vertices. Then the union of all the $V(Y_[uv])$ together with $V(X)$ induces a tournament with chromatic number at least d and with at most $(d!)^2$ vertices. This proves 5.6. ∎

It is perhaps of interest to compare this with 13.3, which tells us something similar when we exclude one particular tournament $\mathcal{S}_t$.

6 Crossing tournaments

Let us say the *clique number* of a numbering of a tournament is the size of the largest clique in its backedge graph. Can we test whether a tournament admits a numbering with small clique number (even in an approximate sense) in polynomial time? Is it in co-NP?

In an attempt to show that 1.1 implies 1.2, before this was proved in [10], we reduced proving 1.2 to proving it for tournaments with small local chromatic number (by applying 13.3; we omit the details, since the result is superceded). On the other hand, 1.1 tells us something about tournaments that admit numberings with small clique number. Could we use it to deduce something about tournaments with small local chromatic number?

Not all numberings with small local chromatic number have small clique number; for instance, the backedge graph could be a complete graph. But in that case, we could reverse the numbering and get a numbering with small clique number. That suggests the question, is it true that if a tournament T admits a numbering with small local chromatic number, then it also admits a numbering with small clique number? The answer is no: we will give a class of tournaments that admit a numbering with local chromatic number two, such that every numbering has arbitrarily large clique number.

Take a set P of pairs of integers (a, b), such that all the integers used are distinct (we call this an *integer matching*). The graph H with vertex set P, in which (a, b) and (c, d) are adjacent if either $a < c < b < d$ or $c < a < d < b$, is called a *circle graph*, and these are very interesting graphs; but we can also derive an interesting tournament from P, as follows.

Let T be the tournament with vertex set P, in which (c, d) is adjacent from (a, b) if either $a < b < c < d$, or $c < a < b < d$, or $c < a < d < b$. (In other words, if we arrange a, b, c, d in increasing order, then the second term is one of a, b.) Let us call such a tournament a *crossing* tournament. In fact, if we number P by second terms (so (a, b) is earlier than (c, d) in the numbering if $b < d$), then the circle graph H is the backedge graph of T under this numbering, as we shall see. We begin with:

6.1 *Every crossing tournament admits a numbering such that its backedge graph is a circle graph, and for every vertex, its right in-neighbours are transitive, and its left out-neighbours are transitive; and consequently the numbering has local chromatic number at most two.*

Proof. Let T be a crossing tournament defined by an integer matching P. Let

$$P = \{(a_1, b_1), \dots, (a_n, b_n)\}$$

where $b_1 < b_2 < \cdots < b_n$. This defines a numbering of T. Let $1 \leq i, j \leq n$. Then under this numbering, (a_j, b_j) is a right in-neighbour of (a_i, b_i) if and only if $a_i < a_j < b_i < b_j$; and so the backedge graph is a circle graph. If $(a_j, b_j), (a_k, b_k)$ are both right in-neighbours of (a_i, b_i), then the adjacency between them is determined by whichever of a_j, a_k is larger; (a_j, b_j) is adjacent to (a_k, b_k) if and only if $a_k < a_j$. Consequently the set of all right in-neighbours of (a_i, b_i) is transitive. Also, (a_j, b_j) is a left out-neighbour of (a_i, b_i) if and only if $a_j < a_i < b_j < b_i$; and so similarly the set of left out-neighbours of (a_i, b_i) is transitive. This proves 6.1. ∎

We do not need the next result; it is included just because we find crossing tournaments interesting. If H_1, H_2, H_3 are tournaments, we denote by $\Delta(H_1, H_2, H_3)$ the tournament T with vertex set the disjoint union of three sets A_1, A_2, A_3, where $T[A_i]$ is isomorphic to H_i for $i = 1, 2, 3$, and $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow A_1$. When H_3 is a one-vertex tournament we write $\Delta(H_1, H_2, 1)$ for $\Delta(H_1, H_2, H_3)$. Let $\mathcal{S}_1$ be the tournament with one vertex, and for $t \geq 2$, let $\mathcal{S}_t = \Delta(\mathcal{S}_{t-1}, \mathcal{S}_{t-1}, 1)$.

6.2 *Crossing tournaments do not contain $\mathcal{S}_3$.*

Proof. Suppose that $\mathcal{S}_3$ is a crossing tournament, defined by the integer matching P say. Let $P = A \cup B \cup \{c\}$ where A, B are cyclic triangles, and $\{c\} \Rightarrow A \Rightarrow B \Rightarrow \{c\}$. Let $c = (a, b)$ say. Since A is not transitive, there exists $(p, q) \in A$ such that either $p > a$ or $q < a$; and since c is adjacent to (p, q), it follows that $p > b$. Similarly there exists $(r, s) \in B$ with $s < a$. But then (r, s) is adjacent to (p, q), contradicting that $A \Rightarrow B$. This proves 6.2. ∎

If P is an integer matching, $V(P)$ denotes the set of $2|P|$ ends of its members. If P, Q are integer matchings, we say that Q is a *copy* of P if $|V(P)| = |V(Q)|$ and the (unique) order-preserving bijection from $V(P)$ to $V(Q)$ maps P to Q. We need the following lemma:

6.3 *Let P be an integer matching. Then there is an integer matching Q, with the property that for all Q_1, Q_2 with $Q_1 \cup Q_2 = Q$, there is a copy of P that is a subset of one of Q_1, Q_2.*

Proof. Let $|P| = p$ say. We may assume that $V(P) = \{1, \ldots, 2p\}$. Let N be an integer such that for every partition of the dge set of the complete graph K_N into two classes, there is a K_{2p} subgraph with all edges in the same class. For $1 \leq i \leq N$ let

$$V_i = \{(i-1)(N-1)+1, \ldots, i(N-1)\};$$

thus, $V_1, \ldots, V_N$ form a partition of $\{1, \ldots, (N-1)N\}$ into sets of cardinality $N-1$. For each pair (a, b) of integers with $1 \leq a < b \leq N$, let

$$f(a, b) = ((a-1)(N-1)+b, (b-1)(N-1)+a).$$

Thus the set, Q say, of all these pairs $f(a, b)$ is an integer matching, and for every choice of a, b with $1 \leq a < b \leq N$, there is a pair $(c, d) \in Q$ with $c \in V_a$ and $d \in V_b$.

We claim that Q satisfies the theorem. Let $Q_1, Q_2 \subseteq Q$ with union Q. From the choice of N, there exists $I \subseteq \{1, \ldots, N\}$ with $|I| = 2p$, such that all the members of Q with both ends in $\bigcup_{i \in I} V_i$ belong to the same one of Q_1, Q_2, say to Q_1. But then Q_1 contains a copy of P. This proves 6.3. ∎

We deduce:

6.4 *For each integer $k \geq 1$, there is a crossing tournament such that every numbering has clique number at least k. Consequently its chromatic number is at least k.*

Proof. Let us define a crossing tournament $\mathcal{U}_k$ inductively for $k \geq 1$ as follows. $\mathcal{U}_k$ has one vertex. For $k \geq 2$, we may assume inductively that $\mathcal{U}_{k-1}$ is defined, by the integer matching P say. By 6.3, there is an integer matching Q such that for all Q_1, Q_2 with union Q, one of Q_1, Q_2 contains a copy of P.

We may assume that Q is a set of pairs of integers in $\{1, \ldots, 2q\}$. For $i \geq 0$ let Q^{+i} be the integer matching $\{(a+i, b+i) : (a, b) \in Q\}$. Let $b_i = k + 1 + (2q+1)i$ for $1 \leq i \leq k$, and let $R = \{(i, b_i) : 1 \leq i \leq k\}$. Let S be the union of R and all the sets Q^{+b_i} for $1 \leq i \leq k-1$. Thus S is an integer matching. Let $\mathcal{U}_k$ be the crossing tournament defined by S. This completes the inductive definition.

We claim that every numbering of $\mathcal{U}_k$ has clique number at least k. The claim is true for $k = 1$, so inductively we assume that $k \geq 2$ and the claim holds for $\mathcal{U}_{k-1}$. Let τ be a numbering of $\mathcal{U}_k$, and let G be the corresponding backedge graph. We need to prove that T has a clique of size k. Let P, Q, R, S and so on be as in the inductive definition of $\mathcal{U}_k$.

Since (i, b_i) is adjacent from (j, b_j) for $1 \leq i < j \leq k$, we may assume that there exists i with $1 \leq i < k$ such that $(i + 1, b_{i+1})$ precedes (i, b_i) in the numbering τ, since otherwise G has a clique of size k as desired. Every member of Q^{+b_i} is a vertex of $\mathcal{U}_k$, and is a pair (a, b) with $b_i < a < b < b_{i+1}$. Consequently every member of Q^{+b_i} is either a left out-neighbour of (i, b_i) or a right in-neighbour of $(i + 1, b_{i+1})$. Let Q_1 be the set of left out-neighbours of (i, b_i), and Q_2 the set of right in-neighbours of $(i + 1, b_{i+1})$. From the choice of Q, one of Q_1, Q_2, say Q_j, contains a copy of P; and so there is a clique of $G[Q_j]$ of cardinality $k - 1$. If $j = 1$ then (i, b_i) is adjacent in G to every vertex of this clique, and if $j = 2$ then $(i + 1, b_{i+1})$ has the same property. Consequently G contains a clique of size k. This proves that every numbering of $\mathcal{U}_k$ has clique number at least k.

If $\mathcal{U}_k$ has chromatic number t say, take a partition into t transitive sets, and take the numbering τ where we first list the members of the first of the transitive sets, in their natural order, and then list the members of the second set, and so on. Then the backedge graph is t-colourable, and so has no clique of size larger than t. Hence $t \leq k$. This proves 6.4. ∎

So admitting a numbering with small local chromatic number does not imply that there is one with small clique number.

7 Rebels and posets

A tournament H is a *rebel* if for some $c > 0$, every H-free tournament has domination number less than c. Which tournaments are rebels? This section and the next give several new results towards answering this question.

If we arrange the members of a set in a circular order, then every ordered triple of elements rotates clockwise or counterclockwise in the natural sense. In particular, if we arrange the vertices of a digraph in a circular order, then every cyclic triangle is directed clockwise or counterclockwise. Let us say a tournament H is a *poset tournament* if its vertex set can be arranged in a circular order, such that there is no clockwise cyclic triangle; or equivalently, if its vertex set can be numbered $v_1, \ldots, v_n$ such that for all i, j, k with $1 \leq i < j < k \leq n$, if v_iv_j and v_jv_k are edges, then v_iv_k is an edge. Chudnovsky, Kim, Liu, Seymour and Thomassé [4] proved that not all tournaments are rebels, and indeed:

7.1 *Every rebel is a poset tournament.*

They proposed the conjecture, still open, that the converse also holds:

7.2 Conjecture: *H is a rebel if and only if H is a poset tournament.*

The conjecture 7.2 is very strong, and here is an entertaining way that one might try to disprove it. Take a class of graphs $\mathcal{F}$, that is closed under taking induced subgraphs, and let $\mathcal{F}'$ be the class of all tournaments T that admit a numbering with backedge graph in $\mathcal{F}$. Since $\mathcal{F}'$ is closed under taking subtournaments, if some rebel is not contained in $\mathcal{F}'$ then all members of $\mathcal{F}'$ have bounded domination number; and if all rebels are contained in $\mathcal{F}'$ then 7.2 would imply that every poset tournament is in $\mathcal{F}'$. Consequently 7.2 would imply that either every poset tournament is in $\mathcal{F}'$, or all members of $\mathcal{F}'$ have bounded domination number. We tested this on a few familiar classes $\mathcal{F}$, and showed the following:

- If $\mathcal{F}$ is the class of all split graphs (graphs with vertex set the union of a clique and a stable set) then every member of $\mathcal{F}'$ has domination number at most two.
- If $\mathcal{F}$ is the class of all line graphs, then every member of $\mathcal{F}'$ has domination number at most three.
- If $\mathcal{F}$ is the class of all cographs (graphs that do not contain a four-vertex path as an induced subgraph) then there exists k such that such that all members of $\mathcal{F}'$ have domination number at most k.
- More generally, if $\mathcal{F}$ is the class of all circle graphs (the intersection graph of a set of chords of a circle), there exists k such that all members of $\mathcal{F}'$ have domination number at most k.
- If $\mathcal{F}$ is the class of all graphs with no four-vertex induced cycle, then every member of $\mathcal{F}'$ has domination number at most four.
- If H is a permutation graph (that is, there are two numberings σ, τ of $V(H)$, such that u, v are adjacent if u is earlier than v in one of these numberings and later in the other), and $\mathcal{F}$ is the class of all graphs that do not contain H as an induced subgraph, then there exists k such that all members of $\mathcal{F}'$ have domination number at most k.

Here are sketches of the proofs. The first bullet is easy, because tournaments that have a split graph as a backedge graph are two-colourable (because the subtournaments induced on the clique and on the stable set are both one-colourable). For the second bullet, the set of in-neighbours of the first vertex in the numbering is the union of two cliques. The third is a special case of the fourth, because cographs are permutations graphs (easily proved by induction) and hence circle graphs. For the fourth, we observe that circle graphs have bounded VC-dimension, and hence yield tournaments with bounded VC-dimension, and such tournaments have bounded domination number. For the fifth, let $v_1, \dots, v_n$ be a numbering of a tournament T, such that its backedge graph G does not contain the four-vertex cycle C_4 as an induced subgraph. Choose $1 \le i < j \le n$ with j minimum such that v_j is adjacent from v_i. Thus $T[\{v_1, \dots, v_{j-1}\}]$ is transitive and so has domination number at most one; the set of vertices in $\{v_{j+1}, \dots, v_n\}$ that are adjacent to both v_i, v_j is transitive, since G does not contain C_4; and all other vertices in $\{v_{j+1}, \dots, v_n\}$ are dominated by one of v_i, v_j. Hence $\operatorname{dom}(T) \le 4$. For the sixth, observe that if H is a permutation graph, then it admits a numbering such that the corresponding tournament is transitive, and so the claim follows from 7.4 below.

Let us say a graph H is *good* if there exists k such that all members of $\mathcal{F}'$ have domination number at most k, where $\mathcal{F}$ is the class of all graphs that do not contain H as an induced subgraph. The last bullet above says that permutation graphs are good, but which other graphs are good? The complement of any good graph is good, and the disjoint union of two good graphs is also good (the latter can be shown by an easy modification of the proof above that C_4 is good). In the other direction, if we set $\mathcal{F}$ to be the class of all comparability graphs, there is no bound on the domination number of the members of $\mathcal{F}'$; and so if H is good, then $\mathcal{F}$ contains H, and therefore H is a comparability graph. So the class of all good graphs is a superclass of the class of permutation graphs, and a subclass of the class of all comparability graphs; and as far as we know, it might be equal to either one. In particular, here is a cousin of 7.2 (as far as we know, it neither implies nor is implied by 7.2), the following (the "only if" part is true):

7.3 Conjecture: *For every graph H, there exists k such that for every tournament T with domination number at least k and every numbering of T, the backedge graph contains H as an induced subgraph, if and only if H is a comparability graph.*

Here is another variation, even prettier. We recall that an *ordered tournament* is a pair (T, τ), where T is a tournament and τ is a numbering of its vertex set; and its *domination number* is the domination number of T. We say an ordered tournament (H, σ) is a *legend* if there exists k such that every ordered tournament (T, τ) with domination number at least k contains (H, σ) in the natural sense. (Thus a legend is the analogue of a rebel for ordered tournaments.) Which ordered tournaments are legends? We can answer this.

7.4 *An ordered tournament (H, σ) is a legend if and only if H is transitive.*

Proof. Let us show first that for every legend (H, σ), H is transitive. The *reverse* of an ordered tournament is obtained by reversing the direction of all edges, *without* reversing the numbering. We say a ordered tournament (H, σ) is an *ordered poset tournament*, where σ is the numbering $\sigma_1, \ldots, \sigma_n$, if $\sigma_k\sigma_i$ is an edge of H for all $1 \le i < j < k \le n$, such that $\sigma_j\sigma_i$ and $\sigma_k\sigma_j$ are edges of H. There are ordered poset tournaments with arbitrarily large domination number, as was shown in [4], so every legend must be an ordered poset tournament. If we take an ordered poset tournament, and reverse all its edges, and reverse the numbering, we obtain another ordered poset tournament; and therefore there are ordered tournaments with arbitrarily large domination number, such that their reverses are ordered poset tournaments. Hence, the reverse of every legend must also be an ordered poset tournament. But for an ordered tournament (H, σ), if both (H, σ) and its reverse are ordered poset tournaments, then H is transitive (as can be seen by checking that no three vertices make a cyclic triangle). This proves the "only if" part of the theorem.

If A is a subset of $V(T)$, where T is a tournament, we define the *external domination number* $\mathrm{edom}(A)$ of A to be the size of the smallest subset $X \subseteq V(T)$ such that every vertex in $A \setminus X$ is adjacent from a vertex in X. (This differs from the domination number of the subtournament induced on A; and in particular, edom is subadditive.) Now let (H, σ) be an ordered tournament where H is transitive, and let σ be the numbering $\sigma_1, \ldots, \sigma_h$ where $h = |H|$. We need to show that (H, σ) is a legend, but for purposes of induction, we will prove a stronger statement, the following:

(1) *If an ordered tournament (T, τ) has domination number at least $d2^h$ where $d \ge h$ is an integer, and τ is $\tau_1, \ldots, \tau_n$, then there exist $t_1 < t_2 < \cdots t_h$ such that the restriction of (T, τ) to $\{\tau_{t_1}, \ldots, \tau_{t_h}\}$ is a copy of (H, σ), and each of the $h + 1$ sets*

$$\{\tau_1, \ldots, \tau_{t_1 - 1}\}, \{\tau_{t_1+1}, \ldots, \tau_{t_2-1}\}, \ldots, \{\tau_{t_h+1}, \ldots, \tau_n\}$$

has external domination number at least $d - h$.

The claim is true if $h = 0$, so we assume that $h \geq 1$ and the result holds for $h-1$. Choose f with $1 \leq f \leq h$ such that σ_f is the vertex of H with in-degree zero; let H' be the transitive tournament obtained from H by deleting σ_f; and let σ' be the sequence obtained from σ by removing the term σ_f. In other words, $\sigma'_i = \sigma_i$ for $1 \leq i < f$, and $\sigma'_i = \sigma_{i+1}$ for $f \leq i \leq h-1$. Thus (H', σ') is an ordered tournament and H' is transitive. Let (T, τ) be an ordered tournament with domination number at least $d2^h$, where $d \geq h$ is an integer and τ is $\tau_1, \dots, \tau_n$. Temporarily, define $t_0 = 0$ and $t_h = n+1$. From the inductive hypothesis (with d replaced by $2d$) there exist $1 \leq t_1 < t_2 < \cdots t_{h-1} \leq n$ such that the restriction of (T, τ) to $\{\tau_{t_1}, \dots, \tau_{t_{h-1}}\}$ is a copy of (H', σ'), and each of the h sets $\{\tau_{t_{i-1}+1}, \dots, \tau_{t_i - 1}\}$ $(1 \leq i \leq h)$ has external domination number at least $2d - (h-1)$. Let $V = \{\tau_{t_{f-1}+1}, \dots, \tau_{t_f - 1}\}$; thus, $\text{edom}(V) \geq 2d - (h-1)$. Let Y be the set of vertices in V that are adjacent from one of the vertices $\tau_{t_1}, \dots, \tau_{t_{h-1}}$. Since $\text{edom}(Y) \leq h-1$, it follows that $\text{edom}(V \setminus Y) \geq 2d - 2(h-1)$. Each vertex $v \in V \setminus Y$ is adjacent to all of $\tau_{t_1}, \dots, \tau_{t_{h-1}}$, and so the restriction of (T, τ) to $\{\tau_{t_1}, \dots, \tau_{t_h}, v\}$ is a copy of (H, σ). But we need to choose v more carefully, to arrange the final requirement about external domination number. Since $\text{edom}(V \setminus Y) \geq 2d - 2(h-1)$, we may choose k minimum such that

$$\text{edom}(\{\tau_1, \dots, \tau_k\} \cap (V \setminus Y)) > d - h.$$

The minimality of k implies that $\tau_k \in V \setminus H$, and that

$$\text{edom}(\{\tau_1, \dots, \tau_{k-1}\} \cap (V \setminus Y)) \leq d - h.$$

Since adding a vertex to a set changes its external domination number by at most one, it follows that

$$\text{edom}(\{\tau_1, \dots, \tau_k\} \cap (V \setminus Y)) = d - h + 1,$$

and so

$$\text{edom}(\{\tau_{k+1}, \dots, \tau_n\} \cap (V \setminus Y)) \geq (2d - 2(h-1)) - (d - h + 1) = d - h + 1.$$

Thus if we insert k into the sequence $t_1, \dots, t_{h-1}$, so that the new sequence remains increasing, then our requirements are met. This proves (1).

It follows that if (T, τ) is a tournament with domination number at least $h2^h$, then it contains (H, σ); and so (H, σ) is a legend. This proves 7.4. ∎

We remark that if H is transitive and (H, σ) is an ordered tournament, then its backedge graph is a permutation graph, or equivalently the comparability graph of a poset of dimension two. This makes some connection with our two "poset" conjectures, 7.2 and 7.3.

8 Making rebels

But, despite all the attacks on it described in the previous section, 7.2 remains open. In support of it, Chudnovsky et al. [4] proved the following two results:

8.1 *Every tournament with chromatic number two is a rebel.*

8.2 $\mathcal{S}_3$ *is a rebel.*

The proof of the second was complicated, using a randomized version of a theorem about VC-dimension, and we will give a much simpler proof. But first, let us observe:

8.3 *Let H be obtained from the disjoint union of rebels H_1, H_2 by making $V(H_1)$ complete to $V(H_2)$. Then H is a rebel.*

Proof. Choose d such that every tournament with domination number at least d contains both H_1 and H_2, and let $c = d + |H_2|$. Let T be a tournament with $\text{dom}(T) \geq c$; we claim that T contains H and hence H is a rebel. Since $c \geq d$, there is a copy S of H_2 in T. Let X be the set of vertices of $T \setminus V(S)$ that are adjacent to every vertex in $V(S)$. It follows that $\text{dom}(T[X]) \geq c - |H_2|$, since every vertex of T not in X either belongs to $V(S)$ or is adjacent from a vertex of S. Since $c - |H_2| = d$, we deduce that $T[X]$ contains H_1, and so T contains H. This proves 8.3. ∎

A tournament H is a *hero* if for some $c > 0$, every H-free tournament has chromatic number less than c. (Thus, "hero" is the concept analogous to "rebel" for chromatic number instead of domination number.) All heroes are rebels, but not all rebels are heroes, and indeed we know which tournaments are heroes: Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé [3] showed that:

8.4 *A tournament is a hero if and only if all its strongly-connected components are heroes. A strongly-connected tournament with more than one vertex is a hero if and only if it equals $\Delta(H, K, 1)$ or $\Delta(K, H, 1)$ for some hero H and some acyclic tournament K.*

The methods of this paper will give a simpler proof of the main part of 8.4, as we will explain later.

Until now, $\mathcal{S}_3$ was the only non-two-colourable tournament that was known to be a rebel. 8.3 trivially gives more, but they are not strongly connected. The next result gives more that are strongly connected, and also gives a much simpler proof that $\mathcal{S}_3$ is a rebel. Let us say a *ring* in a tournament T is a sequence $X_1, X_2, \ldots, X_n$ of subsets of $V(T)$ with $n \geq 3$, such that

$$X_1 \Rightarrow X_2 \Rightarrow \cdots \Rightarrow X_n \Rightarrow X_1.$$

(Thus consecutive terms in this sequence are disjoint, but non-consecutive terms may intersect.) Using rings, we will show that if H_1, H_2, H_3 are heroes then $\Delta(H_1, H_2, H_3)$ is a rebel. (This implies that $\mathcal{S}_3$ is a rebel, by setting H_1, H_2 to be cyclic triangles and H_3 to have one vertex.) The claim follows from the following:

8.5 *For all integers $c \geq 0$ there exist K, k such that if a tournament T with $\text{dom}(T) \geq K$, then there are three disjoint sets $A, B, C \subseteq V(T)$ such that $|A|, |B|, |C| \leq k$, and $\chi(A), \chi(B), \chi(C) \geq c$, and $A \Rightarrow B \Rightarrow C \Rightarrow A$.*

Proof. By 2.1 there exist integers $K, k \geq 1$ such that every tournament T with $\text{dom}(T) \geq K$ contains a subtournament on at most k vertices having chromatic number at least $3c$. Also by 2.1 there exist integers $K', k' \geq 1$ such that every tournament T with $\text{dom}(T) \geq K'$ contains a subtournament on at most k' vertices having chromatic number at least $2^{2k}c + 3c$. We will show that every tournament with domination number at least $\max(K, K') + k + k'$ contains sets A, B, C as in the theorem, each of cardinality at most $k + k'$.

Let T be a tournament with $\text{dom}(T) \geq \max(K', K) + k + k'$. Let $\mathcal{S}$ be the set of all subsets $X \subseteq V(T)$ with $|X| \leq k$ such that $T[X]$ has chromatic number exactly $3c$. Let $\mathcal{S}'$ be the set of all subsets $X \subseteq V(T)$ with $|X| \le k'$ such that $T[X]$ has chromatic number exactly $2^{2k}c + 3c$. Let $\mathcal{R}$ be the set of all subsets that are the union of a member of $\mathcal{S}$ and a member of $\mathcal{S}'$. For every $X \subseteq V(T)$, if $\mathrm{dom}(T[X]) \ge \max(K, K')$, then X includes a member of $\mathcal{S}$ and a member of $\mathcal{S}'$, and hence includes a member of $\mathcal{R}$. In particular, since $\mathrm{dom}(T) \ge \max(K, K') + k + k'$, it follows that $\mathcal{R} \ne \emptyset$.

(1) *For each $X \in \mathcal{R}$ there exists $Y \in \mathcal{R}$ with $X \cap Y = \emptyset$ such that $Y \Rightarrow X$.*

Let $X \in \mathcal{R}$. Since $|X| \le k + k'$ and $\mathrm{dom}(T) \ge \max(K', K) + k + k'$, it follows that the set of vertices in $V(T) \setminus X$ that are complete to X induces a tournament with domination number at least $\max(K, K')$, and so contains a member of $\mathcal{R}$. This proves (1).

From (1) and since T is finite, there is a ring of members of $\mathcal{R}$, and consequently there is a ring $X_1, X_2, \ldots, X_n$ such that

- $|X_1| \le k + k'$ and $\chi(X_1) \ge c$;
- $|X_2| \le k$ and $\chi(X_2) \ge 2c$;
- $|X_3| \le k$ and $\chi(X_3) = 3c$ (that is, $X_3 \in \mathcal{S}$); and
- $X_i \in \mathcal{R}$ for $4 \le i \le n$.

Let us call such a sequence a *special* ring. Choose a special ring $X_1, X_2, \ldots, X_n$ with n minimum. We may assume that $n \ge 4$, since if $n = 3$ then the theorem holds.

Since X_n includes a member of $\mathcal{S}'$ and hence has chromatic number at least $2^{2k}c + 3c$, it follows that $\chi(X_n \setminus X_3) \ge 2^{2k}c$. Moreover, $X_n \cap X_2 = \emptyset$, since $X_n \Rightarrow X_1 \Rightarrow X_2$. For each $Z \subseteq X_2 \cup X_3$, let P_Z be the set of vertices in $X_n \setminus X_3$ that are complete to Z and complete from $(X_2 \cup X_3) \setminus Z$. Since there are at most 2^{2k} choices of Z, and each vertex of $X_n \setminus X_3$ belongs to one of the sets P_Z, it follows that $\chi(P_Z) \ge c$ for some choice of Z. If $\chi(X_2 \setminus Z) \ge c$, then the theorem holds, since

$$P_Z \Rightarrow X_1 \Rightarrow X_2 \setminus Z \Rightarrow P_Z;$$

and so we may assume that $\chi(X_2 \setminus Z) < c$. Consequently $\chi(X_2 \cap Z) \ge c$. If $\chi(X_3 \setminus Z) \ge c$ then the theorem holds, since

$$P_Z \Rightarrow X_2 \cap Z \Rightarrow X_3 \setminus Z \Rightarrow P_Z;$$

and so we may assume (for a contradiction) that $\chi(X_3 \setminus Z) < c$. Hence $\chi(X_3 \cap Z) \ge 2c$ (and consequently $n \ge 5$, since X_3 is not complete to X_n). Choose $Y \subseteq X_4$ with $Y \in \mathcal{S}$; then

$$P_Z, X_3 \cap Z, Y, X_5, \ldots, X_{n-1}$$

is a special ring, contrary to the minimality of n. This proves 8.5. ∎

We deduce:

8.6 *If H_1, H_2, H_3 are heroes then $\Delta(H_1, H_2, H_3)$ is a rebel.*

Proof. Choose c sufficiently large that every tournament with chromatic number at least c contains each of H_1, H_2, H_3. Choose K, k as in 8.5. We claim that every tournament T with $\mathrm{dom}(T) \ge K$ contains $\Delta(H_1, H_2, H_3)$. Let T be a tournament with $\mathrm{dom}(T) \ge K$. By 8.5 there exist A, B, C as in 8.5. But $T[A]$ contains H_1, and $T[B]$ contains H_2, and $T[C]$ contains H_3, and so T contains $\Delta(H_1, H_2, H_3)$. This proves 8.6. ∎

In view of 8.3 and 8.6, one might hope for the following:

8.7 Conjecture: *If H_1, H_2, H_3 are rebels then $\Delta(H_1, H_2, H_3)$ is a rebel.*

This is consistent with 7.2, but we cannot yet prove it. But here is a special case of 8.7 that we can prove:

8.8 *Let H be a rebel, and let K be a transitive tournament. Then $\Delta(H, K, K)$ is a rebel.*

Proof. For each integer $r \geq 1$, let H^r be obtained from r disjoint copies of H (say with vertex sets $S_1, \ldots, S_r$), by making $V(S_i)$ complete to $V(S_j)$ for $1 \leq i < j \leq r$. If $T[X]$ is isomorphic to H^r, we call the subsets of X corresponding to $S_1, \ldots, S_r$ the *parts* of X. By 8.3, H^r is a rebel. Let $k = |K|$, let $p = k^2$, $q = p + k|H|$, and $r = k(p^k + q^k)|H|^k + 1$. Choose c such that every tournament with domination number at least c contains H^r. Now let $d = c + r|H|$, and let T be a tournament with domination number at least d; we will show that T contains $\Delta(H, K, K)$. Suppose not (for a contradiction). Let $\mathcal{R}$ be the set of all subsets of $V(T)$ that induce a copy of H^r. Since $d \geq c$, T contains a copy of H^r, so $\mathcal{R} \neq \emptyset$; and for each $X \in \mathcal{R}$, the set of vertices not dominated by X (that is, the set of $v \in V(T) \setminus X$ that are adjacent to every vertex of X) induces a subtournament with domination number at least $d - |X| = c$, and consequently also includes a member of $\mathcal{R}$. Hence there is a ring $X_1, \ldots, X_n$ of members of $\mathcal{R}$.

Let us say a *special ring* is a ring $X_1, \ldots, X_n$, where $T[X_1]$ is a copy of H^p, $T[X_2]$ is a copy of H, and $X_3, \ldots, X_n \in \mathcal{R}$. It follows that T contains a special ring; let us choose a special ring $X_1, \ldots, X_n$ with n minimum. Thus $n \geq 3$.

Let P be the set of vertices in X_3 that have an out-neighbour in at least k parts of X_1.

(1) *There are fewer than $k(p|H|)^k$ parts of X_3 that have a vertex in P.*

Suppose that there is a set $Y \subseteq X_3 \cap P$ with $|Y| \geq k(p|H|)^k$ such that all vertices in Y belong to different parts of X_3 (and hence Y is transitive). Each $y \in Y$ has k out-neighbours in X_1 that all belong to different parts of X_1, and hence form a transitive set of cardinality k; and there are at most $(p|H|)^k$ choices of such a set, since $|X_1| = p|H|$. So there is a subset $Y' \subseteq Y$ with $|Y'| \geq |Y|/(p|H|)^k \geq k$, and a transitive subset $Z \subseteq X_1$ with cardinality k, such that $Y' \Rightarrow Z$. But then $T[X_2 \cup Y' \cup Z]$ is a copy of $\Delta(H, K, K)$, a contradiction. This proves (1).

The set X_n has r parts; let us fix some q of them, called the *primary parts* of X_n. Let Q be the set of vertices in X_3 that have an out-neighbour in at least k primary parts of X_n.

(2) *There are fewer than $k(p^k + q^k)|H|^k$ parts of X_3 that have a vertex in $P \cup Q$.*

Suppose not; then, by (1), there is a set $Y \subseteq X_3 \cap (Q \setminus P)$ with $|Y| \geq k(q|H|)^k$ such that all vertices in Y belong to different parts of X_3 (and hence Y is transitive). As in the proof of (1), there exists $Y' \subseteq Y$ with $|Y'| = k$, and a transitive subset $Z \subseteq X_n$ with $|Z| = k$, such that $Y' \Rightarrow Z$. Since $Y' \cap P = \emptyset$, each vertex in Y' has an out-neighbour in fewer than k parts of X_1, and since $|Y'| = k$ and X_1 has $p = k^2$ parts, there is a part S of X_1 such that $S \Rightarrow Y'$. But then $T[S \cup Y' \cup Z]$ is a copy of $\Delta(H, K, K)$, a contradiction. This proves (2).

Since $r > k(p^k + q^k)|H|^k$, it is not the case that $X_3 \Rightarrow X_n$, and so $n \geq 5$. For the same reason, there is a part S of X_3 that is disjoint from Q. Each vertex in S has out-neighbours in fewer than k primary parts of X_n, and, since there are $q = k|H| + p$ primary parts of X_n, there are p of them that are all complete to S. Hence there is a copy T of H^p in $T[X_n]$ that is complete to S. But then $S, X_4, \ldots, X_{n-1}, T$ is a special ring, contrary to the minimality of n. This proves 8.8. ∎

9 Two conjectures of Harutyunyan et al.

Several of our results are based on the breakthrough result 2.1 by Harutyunyan et al. [9]. In the same paper they also proposed two strengthenings, that are both still open. First, they proposed:

9.1 Conjecture: *For every integer $c \geq 1$, there exist integers $K, k \geq 1$ such that every tournament with domination number at least K contains a subtournament on at most k vertices having domination number at least c.*

For the second conjecture, we need some definitions. We recall that $\mathcal{S}_1$ is the tournament with one vertex, and $\mathcal{S}_t = \Delta(\mathcal{S}_{t-1}, \mathcal{S}_{t-1}, 1)$ for $t \geq 2$. For tournaments T, H, we say T *contains* H if T has a subtournament isomorphic to H.

Harutyunyan et al. [9] proposed:

9.2 Conjecture: *For every integer $t \geq 1$, there exist $K \geq 1$ such that every tournament with domination number at least K contains $\mathcal{S}_t$.*

The proof of 8.5 can be adjusted to show that conjecture 9.1 implies conjecture 9.2. Indeed, if 9.1 is true then something stronger than 9.2 holds. Let $\mathcal{T}_1$ be a tournament with one vertex, and for $t > 1$ let $\mathcal{T}_t = \Delta(\mathcal{T}_{t-1}, \mathcal{T}_{t-1}, \mathcal{T}_{t-1})$. We will show that:

9.3 *Suppose that for every integer $c \geq 0$, there exists integers $K, k \geq 1$ such that every tournament with domination number at least K contains a subtournament on at most k vertices having domination number at least c. Then for every integer $t \geq 1$, there exists K such that every $\mathcal{T}_t$-free tournament has domination number at most K.*

Proof. We may assume that $t \geq 2$, and we proceed by induction on t. Thus we may assume that there exists c such that every $\mathcal{T}_{t-1}$-free tournament has domination number less than c. From the hypothesis, there exist integers $K, k \geq 1$ such that every tournament T with $\text{dom}(T) \geq K$ contains a subtournament on at most k vertices having domination number at least $3c$; and there exist integers $K', k' \geq 1$ such that every tournament T with $\text{dom}(T) \geq K'$ contains a subtournament on at most k' vertices having domination number at least $2^{2k}c + 3c$. We will show that every tournament with domination number at least $\max(K, K') + k + k'$ contains $\mathcal{T}_t$.

Let T be a tournament with $\text{dom}(T) \geq \max(K', K) + k + k'$. Let $\mathcal{S}$ be the set of all subsets $X \subseteq V(T)$ with $|X| \leq k$ such that $\text{dom}(T[X]) = 3c$; and let $\mathcal{S}'$ be the set of all subsets $X \subseteq V(T)$ with $|X| \leq k'$ such that $\text{dom}(T[X]) = 2^{2k}c + 3c$.

Now the proof proceeds exactly as the proof of 8.5: claim (1) holds, and we get a special ring, which now means a ring $X_1, X_2, \ldots, X_n$ such that

- $|X_1| \leq k + k'$ and $\text{dom}(X_1) \geq c$;

- $|X_2| \leq k$ and $\text{dom}(X_2) \geq 2c$;
- $|X_3| \leq k$ and $\text{dom}(X_3) = 3c$ (that is, $X_3 \in \mathcal{S}$); and
- $X_i \in \mathcal{R}$ for $4 \leq i \leq n$.

We choose such a ring with n minimum, and as before, we prove that $n = 3$. But each of $T[X_1], T[X_2], T[X_3]$ has domination number at least c and so contains $\mathcal{T}_{t-1}$, and hence T contains $\mathcal{T}_t$. This proves 9.3. ∎

It is not easy to think of tournaments with arbitrarily large domination number such that there is some tournament not contained in them. For instance:

- A uniformly random tournament probably has large domination number, but it also probably contains every small tournament.
- Let $q \geq 3$ be a prime congruent to 3 modulo 4. The *Paley tournament* has vertex set the element set of the field $\mathcal{F}_q$, in which y is adjacent from x if $x - y$ is a square. It was shown by Graham and Spencer [8] that the Paley tournament has domination number at least $\Omega(\log q)$. But Chung and Graham [5] showed that for every tournament H, if q is large enough then the Paley tournament contains H.
- Take $2k-1$ linear orderings of the same set V, and make a tournament with vertex set V where v is adjacent from u if v is later than u in at least k of the orderings. This is called a *k-majority tournament*. Alon, Brightwell, Kierstead, Kostochka and Winkler [2] showed that there are k-majority tournaments with domination number at least $\Omega(k/\log k)$. But every tournament is a k-majority tournament if k is large enough.

Unlike chromatic number, domination number is not monotone; let us define the *subdomination number* of a tournament to be the maximum of the domination number of all subtournaments. (This is called *hereditary domination number* in [1].) The first conjecture 9.1 would imply that we can show that subdomination number is big (when it is) with a constant-time non-deterministic algorithm. More exactly, for all k there exists K, such that if the subdomination number of T is at least K, one can demonstrate with a non-deterministic algorithm (with constant running time if k is fixed) that its subdomination number is at least k. This would be a very nice thing to have, not necessarily via 9.1.

One can show that domination number is small (when it is), just by exhibiting a small dominating set. But can we show that subdomination number is small (when it is)? Is it true that for all integers $k \geq 0$ there exists K, and a poly-time algorithm when k is fixed, that would decide either that T has subdomination number at most K or that T has subdomination number at least k? More plausibly, is there a non-deterministic poly-time algorithm that would do this?

There are many other basic questions about domination number and rebels that we cannot answer; for instance the following three:

9.4 Conjecture: *For all $c \geq 0$, there exists $d \geq 0$ such that if a tournament T has $\text{dom}(T) \geq d$, then it has a subtournament whose reverse has domination number at least c.*

This is trivially true when $c = 2$, but we cannot even prove it when $c = 3$. It would imply the following, an analogue of 2.2 for subdomination number (the analogue for domination number is false, as we saw in 2.11):

9.5 Conjecture: *For all integers $c \geq 0$ there exists $d \geq 0$ such that for every tournament T with subdomination number at least d, there exists $v \in V(T)$ such that $T[N^+(v)]$ has subdomination number at least c.*

(This conjecture also appears in [1].) 9.4 would also imply:

9.6 Conjecture: *If H is a rebel, then the reverse of H is a rebel.*

10 The density property.

In this section we prove a result that will be used to deduce 1.3 (and some extensions of it), and to prove 12.2. It will be applied to variants of chromatic number, but it holds for general submeasures, and so we have written it in terms of submeasures (we recall that a submeasure on a tournament T is a function $\mu : 2^{V(T)} \to \mathbb{R}^+$, such that $\mu(\emptyset) = 0$, and μ is increasing and subadditive). We call $\mu(X)$ the *μ-value* of X.

Let T be a tournament, and let μ be a submeasure on T. If $P, Q \subseteq V(T)$ are disjoint and $c \geq 0$, we denote by $\langle P \xrightarrow{c,\mu} Q \rangle$ the set of all $v \in P$ such that $\mu(N^+(v) \cap Q) \leq c$. Similarly, $\langle P \xleftarrow{c,\mu} Q \rangle$ denotes the set of all $v \in P$ such that $\mu(N^-(v) \cap Q) \leq c$ (not to be confused with $\langle Q \xrightarrow{c,\mu} P \rangle$). When μ is the chromatic number χ, we omit the reference to it and write $\langle P \xrightarrow{c} Q \rangle$, and so on.

Let $g : \mathbb{R}^+ \to \mathbb{R}$ be some function, and let $k \in \mathbb{R}^+$. A pair (P, Q) of disjoint subsets of $V(T)$ has the *(g, k, μ)-out-density property* if

$$\mu(\langle P \xrightarrow{c,\mu} Q' \rangle) < k$$

for all $c \geq 0$ and for every $Q' \subseteq Q$ with $\mu(Q') \geq g(c)$.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be some function. With g, k, μ as before, we say a tournament T has the *(f, g, k, μ)-branching property* if for all $c \geq 0$, and for every $X \subseteq V(T)$ with $\mu(X) \geq f(c)$, there is a vertex $a \in X$ and two subsets P, Q of X, with the following properties:

- $P, Q, \{a\}$ are pairwise disjoint, and $Q \Rightarrow \{a\} \Rightarrow P$;
- $\mu(P), \mu(Q) \geq c$; and
- (P, Q) has the (g, k, μ)-out-density property.

Roughly, this says that in every subset with large μ, we can find a vertex a and two sets P, Q of out-neighbours and in-neighbours of a respectively, both with large μ, such that for all $Q' \subseteq Q$ with large μ, there are fewer than k vertices in P with out-neighbour set in Q' of small μ.

How to arrange the (f, g, k, μ)-branching property is a separate issue, and there are combinations of hypotheses that will give this, that we discuss later. But now we need to prove the following.

10.1 *Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be functions, and let $k \geq 0$. For all $c \geq 0$ and integers $s, t \geq 1$ there exists $d_{c,s,t}$ with the following property. Let T be a tournament, and let μ be a submeasure in T, such that T has the (f, g, k, μ)-branching property. Let $P \subseteq V(T)$, and let $Q_1, \dots, Q_s$ be subsets of $V(T) \setminus P$ (not necessarily disjoint from each other), with the following properties:*

- *$\mu(P), \mu(Q_1), \dots, \mu(Q_s) \geq d_{c,s,t}$;*
- *for $1 \leq i \leq s$, (P, Q_i) has the (g,k,μ)-out-density property.*

Then there is a copy S of $\mathcal{S}_t$ in $T[P]$, and for $1 \leq i \leq s$ there is a subset $C_i \subseteq B_i$ complete from $V(S)$ with $\mu(C_i) \geq c$.

Proof. We prove by induction on t that the statement holds for all choices of s, c and the given value of t. Suppose first that $t = 1$. We claim that we may set $d_{c,s,1} = \max(k, g(c))$. Let $T, P, Q_1, \dots, Q_s$ be as in the theorem. Since (P, Q_i) has the (g,k,μ)-out-density property, and $\mu(Q_i) \geq d_{c,s,1} \geq g(c)$, it follows that $\mu(\langle P \xrightarrow{c,\mu} Q_i \rangle) < k$ for $1 \leq s$, and so the union of those sets has μ-value less than ks. Since $\mu(P) \geq d_{c,s,1} \geq ks$, there exists $a \in P$ in none of the sets $\langle P \xrightarrow{c,\mu} Q_i \rangle$, and consequently satisfying the theorem.

Thus the theorem holds when $t = 1$; so we assume that $t > 1$, and the theorem holds for $t-1$ and all choices of s, c. Let $c' = d_{c,s,t-1}$, and $d' = d_{c',s+1,t-1}$. We claim that setting $d_{c,s,t} = \max(g(d'), sk + f(d'))$ satisfies the theorem. To see this, let $T, \mu, P, Q_1, \dots, Q_s$ be as in the theorem. For $1 \leq i \leq s$, since (P, Q_i) has the (g,k,μ)-out-density property, and $\mu(Q_i) \geq d_{c,s,t} \geq g(d')$, it follows that

$$\mu(\langle P \xrightarrow{d',\mu} Q_i \rangle) < k.$$

Consequently the set P_0 of vertices in P that belongs to none of the sets $\langle P \xrightarrow{d',\mu} Q_i \rangle$ has μ-value at least $\mu(P) - sk \geq d_{c,s,t} - sk \geq f(d')$.

Since T has the (f,g,k,μ)-branching property, and $\mu(P_0) \geq f(d')$, there is a vertex $a \in P_0$ and two subsets P', Q' of P_0, with the following properties:

- $P', Q', \{a\}$ are pairwise disjoint, and $Q' \Rightarrow \{a\} \Rightarrow P'$;
- $\mu(P'), \chi_\mu(Q') \geq d'$; and
- (P', Q') has the (g,k,μ)-out-density property.

For $1 \leq i \leq s$, let Q'_i be the set of vertices in Q_i that are adjacent from a. So $\mu(Q'_i) \geq d'$ for $1 \leq i \leq s$, since $a \in P_0$. Then Q' and $Q'_1, \dots, Q'_s$ are all disjoint from P'; (P', Q') and all the pairs (P', Q'_i) have the (g,k,μ)-out-density property; and P', Q' and all the sets Q'_i have μ-value at least $d' = d_{c',s+1,t-1}$. From the inductive hypothesis, there is a copy R of $\mathcal{S}_{t-1}$ in $T[P']$, and for $1 \leq i \leq s$ there is a subset $C_i \subseteq Q'_i$ complete from $V(R)$ with $\mu(C_i) \geq c'$, and there is a subset $C \subseteq Q'$ complete from $V(R)$ with $\mu(C) \geq c'$. Now $C_1, \dots, C_s$ are all disjoint from C; all the pairs (C, C_i) have the (g,k,μ)-out-density property; and C and all the sets C_i have μ-value at least $c' = d_{c,s,t-1}$. From the inductive hypothesis, there is a copy R' of $\mathcal{S}_{t-1}$ in $T[C]$, and for $1 \leq i \leq s$ there is a subset $C'_i \subseteq C_i$ complete from $V(R')$ with $\mu(C'_i) \geq c$. But then the subtournament S with vertex set $V(R) \cup V(R') \cup \{a\}$ is a copy of $\mathcal{S}_t$ satisfying the theorem. This proves 10.1. ∎

11 $\mathcal{L}$-colouring

Colouring a tournament means partitioning its vertex set into subsets none of which includes a cyclic triangle; and this can usefully be generalized, as follows. Let T be a tournament, and let $\mathcal{L}$ be a set of subsets of $V(T)$, each including the vertex set of a cyclic triangle of T. We call $\mathcal{L}$ a *law* for T. The *order* of a law is the maximum cardinality of its members (or $|T|$ if $\mathcal{L} = \emptyset$). We define the *$\mathcal{L}$-chromatic number* $\chi_{\mathcal{L}}(T)$ to be the minimum k such that $V(T)$ can be partitioned into k sets, each including no member of $\mathcal{L}$. The function $\chi_{\mathcal{L}}$ is a submeasure. Colouring in this sense is the same as colouring the hypergraph $\mathcal{L}$ in the usual sense of hypergraph colouring; and $\chi(T) = \chi_{\mathcal{L}}(T)$ when $\mathcal{L}$ is the set of all vertex sets of cyclic triangles of T.

Now we give the first application of 10.1, to prove 2.14 (and therefore its corollaries 2.13 and 1.3), which we restate:

11.1 *Let H be a tournament and $d \geq 0$. Then there exists an integer w such that for every tournament T, either:*

- *there exists a copy S of H in T, and two subsets $P, Q \subseteq V(T) \setminus V(S)$ with $Q \Rightarrow V(S) \Rightarrow P$, such that $\chi(P), \chi(Q) \geq d$; or*
- *$V(T)$ can be partitioned into w subsets each inducing an H-free subtournament.*

This is implied by the following, by taking $\mathcal{L}$ to be the law of all $X \subseteq V(T)$ such that $T[X]$ is isomorphic to H, and $\ell = |H|$.

11.2 *Let $d, \ell \geq 0$. Then there exists w such that for every tournament T, and every law $\mathcal{L}$ in T of order at most ℓ, if $\chi_{\mathcal{L}}(T) > w$ then there exists $A \in \mathcal{L}$ and two subsets $P, Q \subseteq V(T) \setminus A$ with $Q \Rightarrow A \Rightarrow P$, such that $\chi(P), \chi(Q) \geq d$.*

(Note that the condition $\chi(P), \chi(Q) \geq d$ refers to chromatic number, and not $\mathcal{L}$-chromatic number. We do not know if the same holds using $\mathcal{L}$-chromatic number.)

We first prove the following weaker statement:

11.3 *Let $d, \ell \geq 0$. Then there exists w such that for every tournament T, and every law $\mathcal{L}$ in T of order at most ℓ, if $\chi_{\mathcal{L}}(T) > w$ then there exist $A \in \mathcal{L}$ and a subset $Q \subseteq V(T) \setminus A$ with $Q \Rightarrow A$, such that $\chi(Q) \geq d$.*

Proof. Let g be the function defined by $g(c) = c\ell + d$ for $c \geq 0$. By 2.1, there exist K, k such that every tournament with domination number at least K has a subtournament with chromatic number at least d and with at most k vertices. Let f be the function defined by

$$f(c) = (c+1)\ell + d + \max(K(c+1), k(c+1)+1)$$

for $c \geq 0$. With this choice of f, g let $d_{c,s,t}$ be as in 10.1, for all $c, s, t \geq 0$. Define $w = 2d_{2,1,d}$, let T be a tournament, and let $\mathcal{L}$ be a law in T such that $\chi_{\mathcal{L}}(T) > w$. Let $\mu(X) = \chi_{\mathcal{L}}(X)$ for each $X \subseteq V(T)$; thus μ is a submeasure. We suppose for a contradiction that there do not exist $A \in \mathcal{L}$ and a subset $Q \subseteq V(T) \setminus A$ with $Q \Rightarrow A$, such that $\chi(Q) \geq d$.

(1) *If $P, Q \subseteq V(T)$ are disjoint then (P, Q) has the $(g, 2, \mu)$-out-density property.*

Let $c \geq 0$ and let $Q' \subseteq Q$ with $\mu(Q') \geq g(c)$. Suppose that there exists $S \in \mathcal{L}$ with $S \subseteq \langle P \xrightarrow{c,\mu} Q' \rangle$. For each $v \in S$, the set of out-neighbours of v in Q' has μ-value at most c, and since $\mathcal{L}$ has order at most ℓ, the set of vertices in Q' that have an in-neighbour in S has μ-value at most $c\ell$. Consequently the set of vertices in Q' that are complete to S has μ-value at least $\mu(Q') - c\ell \geq d$, a contradiction. Thus there is no such S; and so $\mu(\langle P \xrightarrow{c,\mu} Q' \rangle) < 2$. This proves (1).

(2) *For every $c \geq 0$ and every $P \subseteq V(T)$ with $\mu(P) \geq (c+1)\ell + d$, there exists $v \in P$ such that $\mu(P \cap N^+(v)) \geq c$.*

Choose $S \in \mathcal{L}$ with $S \subseteq P$. The set of vertices in $P \setminus S$ that are complete to S has chromatic number less than d, and hence has $\mathcal{L}$-chromatic number less than d (here we use that every member of the law includes a cyclic triangle). So the union over $v \in S$ of the sets $P \cap N^+[v]$ has μ-value at least $\mu(P) - d \geq (c+1)\ell$, and since $|S| \leq \ell$, there exists $v \in S$ such that $\mu(P \cap N^+[v]) \geq c+1$, and so $\mu(P \cap N^+(v)) \geq c$ (because μ is subadditive and $\mu(\{v\}) \leq 1$). This proves (2).

(3) *For every $c \geq 0$ and every $P \subseteq V(T)$ with $\mu(P) \geq \max(K(c+1), k(c+1)+1)$, there exists $v \in P$ such that $\mu(P \cap N^-(v)) \geq c$.*

From the choice of K, k, applied to the reverse of $T[P]$, we deduce that either there is a subset $X \subseteq P$ with $|X| < K$ such that every vertex in $P \setminus X$ has an out-neighbour in X, or there exists $C \subseteq P$ with $\chi(C) \geq d$ and $|C| \leq k$. In the first case, since $\mu(P) \geq K(c+1)$, there exists $v \in X$ such that $\mu(P \cap N^-[v]) \geq c+1$, and hence $\mu(P \cap N^-(v)) \geq c$ as required. In the second case, the set of vertices in $P \setminus C$ that are complete from C contains no member of $\mathcal{L}$, by hypothesis, and so has $\mathcal{L}$-chromatic number at most one; and so for some $v \in C$, $\mu(P \cap N^-[v]) \geq (\mu(P)-1)/k \geq c+1$, and hence $\mu(P \cap N^-(v)) \geq c$. This proves (3).

(4) *For every $c \geq 0$ and every $P \subseteq V(T)$ with $\mu(P) \geq f(c)$, there exists $v \in P$ such that $\mu(P \cap N^+(v)) \geq c$ and $\mu(P \cap N^-(v)) \geq c$.*

Let X be the set of vertices $v \in P$ such that $\mu(P \cap N^+(v)) \geq c$. By (2), $\mu(P \setminus X) < (c+1)\ell + d$, and so

$$\mu(X) \geq \mu(P) - ((c+1)\ell + d) \geq \max(K(c+1), k(c+1)+1).$$

Hence the claim follows from (3). This proves (4).

From (4) and (1), it follows that T has the $(f, g, 2, \mu)$-branching property. Since $\mu(T) \geq 2d_{2,1,d}$, there exist disjoint $P, Q \subseteq V(T)$ both with $\mathcal{L}$-chromatic number at least $d_{2,1,d}$. By 10.1, with $k = 2$, $c = 2$, $s = 1$ and $t = d$, we deduce that there is a copy S of $\mathcal{S}_d$ in $T[P]$, and there is a subset $C \subseteq Q$ complete from $V(S)$ with $\mu(C) \geq 2$. Choose $A \in \mathcal{L}$ with $A \subseteq C$; then A is complete from a copy of $\mathcal{S}_d$, and the latter has chromatic number at least d, a contradiction. This proves 11.3. ∎

We remark that the proof of 11.3 almost never refers to (normal) chromatic number; the only place we need it is when we apply 2.1. It would be good to have a version of 11.3 with $\chi(Q)$ replaced by $\chi_{\mathcal{L}}(Q)$, but we have not been able to prove this.

Now let us deduce 11.2, which we restate:

11.4 *Let $d, \ell \geq 0$. Then there exists w such that for every tournament T, and every law $\mathcal{L}$ in T of order at most ℓ, if $\chi_{\mathcal{L}}(T) > w$ then there exists $A \in \mathcal{L}$ and two subsets $P, Q \subseteq V(T) \setminus A$ with $Q \Rightarrow A \Rightarrow P$, such that $\chi(P), \chi(Q) \geq d$.*

Proof. Choose m such that for every tournament T, and every law $\mathcal{L}$ in T of order at most ℓ, if $\chi_{\mathcal{L}}(T) > m$ then there exists $A \in \mathcal{L}$ and a subset $P \subseteq V(T) \setminus A$ with $A \Rightarrow P$, such that $\chi(P) \geq d$. Define $w = m^2$. We will show that w satisfies the theorem.

(1) *For every tournament T, and every law $\mathcal{L}$ in T of order at most ℓ, if $\chi_{\mathcal{L}}(T) > m$ then there exists $A \in \mathcal{L}$ and a subset $P \subseteq V(T) \setminus A$ with $A \Rightarrow P$, such that $\chi(P) \geq d$.*

This is immediate from 11.3, by reversing the direction of all edges.

Now let T be a tournament, and let $\mathcal{L}$ be a law in T of order at most ℓ, such that $\chi_{\mathcal{L}}(T) > w$. Let $\mathcal{L}'$ be the set of all $S \in \mathcal{L}$ such that there exists $P \subseteq V(T) \setminus S$ with $S \Rightarrow P$, and $\chi(P) \geq d$. By (1), if $X \subseteq V(T)$ and $\chi_{\mathcal{L}'}(X) \leq 1$, then $\chi_{\mathcal{L}}(X) \leq m$; and so, for each $X \subseteq V(T)$, $\chi_{\mathcal{L}}(X) \leq m\chi_{\mathcal{L}'}(X)$. It follows that $\chi_{\mathcal{L}'}(T) \geq w/m = m$; and hence by 11.3 applied to $\mathcal{L}'$, there exists $S \in \mathcal{L}'$ and a subset $Q \subseteq V(T) \setminus S$ with $Q \Rightarrow S$, such that $\chi(Q) \geq d$. But since $S \in \mathcal{L}'$ there exists $P \subseteq V(T) \setminus S$ with $S \Rightarrow P$, and $\chi(P) \geq d$. This proves 11.4. ∎

12 Excluding $\Delta(H_1, H_2, 1)$.

We showed in 8.6 that if H_1, H_2, H_3 are heroes then every $\Delta(H_1, H_2, H_3)$-free tournament has bounded domination number. But all tournaments that admit numberings with bounded local chromatic number have bounded domination number, as we show below, so one might hope to strengthen 8.6: is it true that if H_1, H_2, H_3 are heroes then all $\Delta(H_1, H_2, H_3)$-free tournaments admit numberings with bounded local chromatic number? We shall show below that this is true if one of H_1, H_2, H_3 has only one vertex: that is, if H_1, H_2 are heroes then all $\Delta(H_1, H_2, 1)$-free tournaments admit numberings with bounded local chromatic number. But otherwise it is false. To see this, observe that if H_1, H_2, H_3 each have at least two vertices, then $\mathcal{S}_t$ is $\Delta(H_1, H_2, H_3)$-free, and so by 13.1 of the next section, not all $\Delta(H_1, H_2, H_3)$-free tournaments admit numberings with small local chromatic number.

First let us observe that:

12.1 *If a tournament T admits a numbering with local chromatic number at most c, then* $\mathrm{dom}(T) \leq c + 1$.

Proof. Let $v_1, \ldots, v_n$ be a numbering with local chromatic number at most c. Thus $\chi(N^-(v_1)) \leq c$, and hence $\mathrm{dom}(T[N^-(v_1)]) \leq c$. But v_1 dominates all other vertices of T, and so $\mathrm{dom}(T) \leq c + 1$. This proves 12.1. ∎

A *hereditary class* $\mathcal{C}$ is a class of tournaments such that for all $T \in \mathcal{C}$, if a tournament H is isomorphic to a subtournament of T, then $H \in \mathcal{C}$. We can define heroes relative to a hereditary class, as follows. Let $\mathcal{C}$ be a hereditary class, and let H be a tournament. We say that H is a *hero relative to* $\mathcal{C}$ if there exists $c \geq 0$ such that every H-free tournament in $\mathcal{C}$ has chromatic number less than c. (Consequently, tournaments not in $\mathcal{C}$ are heroes relative to $\mathcal{C}$ if all members of $\mathcal{C}$ have bounded chromatic number; this is for later technical convenience.) A tournament H is a hero in the earlier sense if and only if it is a hero relative to the class of all tournaments. The greater generality of the current definition will be useful when we explore $\mathcal{S}_t$-free tournaments in section 13.

We already defined the (g,k,μ)-out-density property when μ is a submeasure; and we say a pair (P,Q) of disjoint subsets of $V(T)$ has the *(g,k,μ)-in-density property* if $\mu(\langle P \overset{c,\mu}{\longleftarrow} Q'\rangle) < k$ for all $c \geq 0$ and for every $Q' \subseteq Q$ with $\mu(Q') \geq g(c)$. When $\mu = \chi$ we omit reference to it, and speak of the *(g,k)-out-density property* and *(g,k)-in-density property*, and the *(f,g,k)-branching property*. We will use 10.1 to show the next result, which is needed to prove 13.3:

12.2 *Let $\mathcal{C}$ be a hereditary class of tournaments, and let H_1, H_2 be heroes relative to $\mathcal{C}$. There exists d such that for every $\Delta(H_1,H_2,1)$-free tournament $T \in \mathcal{C}$, every diamond in T has chromatic number at most d. Consequently there exists c such that every $\Delta(H_1,H_2,1)$-free member of $\mathcal{C}$ admits a numbering with local chromatic number at most c.*

Proof. The first statement implies the second, by 3.1, and so it suffices to prove the first. Choose c_0 such that every tournament in $\mathcal{C}$ with chromatic number at least c_0 contains both H_1 and H_2. Choose $t = c_0 + 1$.

Let ϕ be as in 2.12. Let g be the function defined by $g(c) = c\max(|H_1|,|H_2|) + c_0$ for $c \geq 0$. Let f be the function defined by $f(c) = \phi(g(c))$ for $c \geq 0$. By 10.1 (with $\mu = \chi$, and $c = s = 0$, and $k = c_0$, and taking $P = V(T)$), there exists d such that every tournament T with $\chi(T) \geq d$ and with the (f,g,c_0)-branching property contains $\mathcal{S}_t$.

Let T be a $\Delta(H_1,H_2,1)$-free member of $\mathcal{C}$. We will show that every diamond in T has chromatic number less than $\max(d, 2c_0)$, and so the theorem holds.

(1) *For every diamond (a,b,P,Q) in T, the pair (P,Q) has the (g,c_0)-out-density property and the (g,c_0)-in-density property.*

Let $c > 0$ and let $Q' \subseteq Q$ with $\chi(Q') \geq g(c)$. We must show that $\chi(\langle P \overset{c}{\rightarrow} Q'\rangle) < c_0$ and $\chi(\langle P \overset{c}{\leftarrow} Q'\rangle) < c_0$. First we show that $\chi(\langle P \overset{c}{\rightarrow} Q'\rangle) < c_0$.

Let $X = \langle P \overset{c}{\rightarrow} Q'\rangle$, and suppose that $\chi(X) \geq c_0$. Consequently $T[X]$ includes a copy S of H_2. For each vertex $v \in V(S)$, the set of vertices in Q' adjacent from v has chromatic number at most c, and so the set of vertices in Q' that have an in-neighbour in $V(S)$ has chromatic number at most $c|H_2|$. Since $\chi(Q') \geq c|H_2| + c_0$, the set Y of vertices in Q' that are adjacent to each vertex of S has chromatic number at least c_0, and so contains a copy S' of H_1. But then the subtournament induced on $V(S) \cup V(S') \cup \{b\}$ is isomorphic to $\Delta(H_1,H_2,1)$, a contradiction. This proves that $\chi(\langle P \overset{c}{\rightarrow} Q'\rangle) < c_0$.

Now we show that $\chi(\langle P \overset{c}{\leftarrow} Q'\rangle) < c_0$. (This proof is almost identical to what we just did, using a instead of b.) Let $X = \langle P \overset{c}{\leftarrow} Q'\rangle$, and suppose that $\chi(X) \geq c_0$. Consequently $T[X]$ includes a copy S of H_1. For each vertex $v \in V(S)$, the set of vertices in Q' adjacent to v has chromatic number at most c, and so the set of vertices in Q' that have an out-neighbour in $V(S)$ has chromatic number at most $c|H_1|$. Since $\chi(Q') \geq c|H_1| + c_0$, the set Y of vertices in Q' that are adjacent from each vertex of S has chromatic number at least c_0, and so contains a copy S' of H_1. But then the subtournament induced on $V(S) \cup V(S') \cup \{a\}$ is isomorphic to $\Delta(H_1,H_2,1)$, a contradiction. This proves that $\chi(\langle P \overset{c}{\leftarrow} Q'\rangle) < c_0$, and so proves (1).

(2) *For every diamond (a,b,P,Q) in T, if $\chi(Q) \geq 2c_0$ then $T[P]$ has the (f,g,c_0)-branching property.*

We must show that for all $c \geq 0$, and for every $X \subseteq P$ with $\chi(X) \geq f(c)$, there is a vertex $a' \in X$ and two subsets P', Q' of X, with the following properties:

- $P', Q', \{a'\}$ are pairwise disjoint, and $Q' \Rightarrow \{a'\} \Rightarrow P'$;
- $\chi(P'), \chi(Q') \geq c$; and
- (P', Q') has the (g, c_0)-out-density property.

By the definition of ϕ, and since $\chi(X) \geq f(c) = \phi(g(c))$, there exists $a' \in X$ such that $\chi(X \cap N^+(a')), \chi(X \cap N^-(a')) \geq g(c)$. Let $P_1 = X \cap N^+(a')$ and $Q_1 = X \cap N^-(a')$. By (1) applied to the diamond (b, a, Q, P), since $\chi(P_1) \geq g(c)$ and $\chi(Q_1) \geq g(c)$, it follows that $\chi(\langle Q \xleftarrow{c} P_1\rangle) < c_0$ and $\chi(\langle Q \xrightarrow{c} Q_1\rangle) < c_0$. Since $\chi(Q) \geq 2c_0$, there exists $b' \in Q$ that belongs to neither of the sets $\langle Q \xleftarrow{c} P_1\rangle, \langle Q \xrightarrow{c} Q_1\rangle$. Let P' be the set of in-neighbours of b' in P_1, and let Q' be the set of out-neighbours of b' in Q_1; thus $\chi(P'), \chi(Q') \geq c$, and (a', b', P', Q') is a diamond. Hence by (1), (P', Q') has the (g, c_0)-out-density property, and so a', P', Q' satisfy the definition of the (f, g, c_0)-branching property. This proves (2).

If $\mathcal{S}_{t-1} \in \mathcal{C}$ then it contains H_1 and H_2, since its chromatic number is at least $t - 1 = c_0$; and so if $\mathcal{S}_t \in \mathcal{C}$ then it contains $\Delta(H_1, H_2, 1)$. Hence T does not contain $\mathcal{S}_t$; and it follows from (2) and the choice of d that there is no diamond (a, b, P, Q) in T such that $\chi(P) \geq d$ and $\chi(Q) \geq 2c_0$. From 3.1, this proves 12.2. ∎

13 Excluding $\mathcal{S}_t$.

There is no converse to 12.1, because the tournament $\mathcal{S}_t$ and all its subtournaments have domination number at most three, and yet:

13.1 *Every numbering of $\mathcal{S}_t$ has local chromatic number at least $(t-1)/2$.*

Proof. We may assume that $t \geq 3$. Let $V(\mathcal{S}_t)$ be the disjoint union of $A, B, \{c\}$, where $c \Rightarrow A \Rightarrow B \Rightarrow c$ and A, B induce subtournaments isomorphic to $\mathcal{S}_{t-1}$. Suppose that $v_1, \ldots, v_n$ is a numbering with local chromatic number less than $(t-1)/2$, and choose i minimum such that one of $A \cap \{v_1, \ldots, v_i\}$, $B \cap \{v_1, \ldots, v_i\}$ has chromatic number at least $(t-1)/2$. Let $I = \{v_1, \ldots, v_i\}$ and $J = \{v_{i+1}, \ldots, v_n\}$. Suppose that $\chi(B \cap I) \geq (t-1)/2$. For each $v \in J$, the set of out-neighbours of v in I has chromatic number less than $(t-1)/2$ (by assumption, as they are left out-neighbours), and so $A \cap J = \emptyset$. Hence $\chi(A \cap I) = t - 1$, and since $t - 2 \geq (t-1)/2$, this contradicts the minimality of i. It follows that $\chi(B \cap I) < (t-1)/2$, and so $\chi(A \cap I) \geq (t-1)/2$. For the same reason it follows that $c \notin J$, and so $c \in I$. Hence the set of in-neighbours of c in J has chromatic number less than $(t-1)/2$, and so $\chi(B \cap J) < (t-1)/2$. Since $\chi(B \cap I) < (t-1)/2$, we deduce that $\chi(B) < t - 1$, a contradiction. This proves 13.1. ∎

$\mathcal{S}_t$ is one of the simplest tournaments with large chromatic number, and it would be nice to understand better the tournaments that do not contain it, particularly since if a tournament contains $\mathcal{S}_t$ with t sufficiently large then it satisfies 1.2. Conjecture 9.2 says that all tournaments not containing

$\mathcal{S}_t$ have bounded domination number, and for $t = 3$, theorem 12.2 says they all have bounded local chromatic number. Does 12.1 have a converse if we exclude $\mathcal{S}_t$? Yes if $t = 3$, by 12.2, but for $t \geq 5$, the simplest hope for a converse is false: for all $t \geq 5$, one can make tournaments that do not contain $\mathcal{S}_t$ that also do not admit numberings with small local chromatic number. To see this, take a tournament H with large chromatic number that does not contain $\mathcal{S}_3$ (for instance, a tournament with backedge graph that has large girth and chromatic number: this has the desired properties); partition its vertex set into two sets P, Q both with large chromatic number, and add two new vertices a, b, where $a \Rightarrow P \Rightarrow b \Rightarrow Q \Rightarrow a$, forming T. The presence of the diamond guarantees that all numberings of T have large local chromatic number, by 3.1, and yet T does not contain $\mathcal{S}_5$, since H does not contain $\mathcal{S}_3$.

Here is a more plausible conjecture:

13.2 Conjecture: *For all integers $t \geq 1$ there exist K, k such that for every $\mathcal{S}_t$-free tournament T, there is a partition of $V(T)$ into K sets each inducing a subtournament that admits a numbering with local chromatic number at most k.*

We are far from proving this, because by 3.1 it would imply 9.2, but if such K, k exist, then in every $\mathcal{S}_t$-free tournament T, there is a subtournament with chromatic number at least $\chi(T)/K$ in which every diamond has chromatic number at most $2k$. Here is a step in this direction. The quantifiers are complicated, but it says, roughly, that if T has large chromatic number and does not contain $\mathcal{S}_t$, then there is a subtournament with large chromatic number, in which every diamond has small chromatic number. (We remark that excluding $\mathcal{S}_t$ is necessary: if $T = \mathcal{S}_t$ then every subtournament with large chromatic number has a diamond with large chromatic number.)

13.3 *Let $t \geq 2$ be an integer, and let $c_2 = 1$. Then*

$$\forall d_2 \exists c_3 \forall d_3 \exists c_4 \cdots \forall d_{t-1} \exists c_t$$

(where $d_2, c_3, d_3, \ldots, c_t$ are all non-negative integers) such that the following holds. If T is an $\mathcal{S}_t$-free tournament, then either every diamond in T has chromatic number less than c_t, or for some $i \in \{2, \ldots, t-1\}$, there is a subtournament T' with $\chi(T') \geq d_i$, such that every diamond of T' has chromatic number less than c_i.

Proof. If $t = 2$, the statement is trivial, since every tournament with a diamond of positive chromatic number contains $\mathcal{S}_2$. Thus, inductively, we may assume that $t \geq 3$ and the result holds for $t - 1$. Hence

$$\forall d_2 \exists c_3 \forall d_3 \exists c_4 \cdots \exists c_{t-1},$$

such that if T is a tournament with a diamond of chromatic number at least c_{t-1}, and for each $i \in \{2, \ldots, t-2\}$, every subtournament T' with $\chi(T') \geq d_i$ has a diamond of T' with chromatic number at least c_i, then T contains $\mathcal{S}_{t-1}$. Let $c_2, d_2, \ldots, c_{t-1}$ be given; then for $d_{t-1} \geq 0$, we need to show that there exists c_t such that the last sentence of the theorem holds.

Let $\mathcal{C}$ be the hereditary class of all tournaments T such that for each $i \in \{2, \ldots, t-1\}$, every subtournament T' with $\chi(T') \geq d_i$ has a diamond of T' with chromatic number at least c_i. Thus if $T \in \mathcal{C}$ with $\chi(T) \geq d_{t-1}$, then T has a diamond with chromatic number at least c_{t-1}, from the definition of $\mathcal{C}$; and from the inductive hypothesis T contains $\mathcal{S}_{t-1}$. Consequently, $\mathcal{S}_{t-1}$ is a hero relative to $\mathcal{C}_{t-1}$.

From 12.2, there exists c_t such that for every $\Delta(\mathcal{S}_{t-1}, \mathcal{S}_{t-1}, 1)$-free tournament $T \in \mathcal{C}$, every diamond in T has chromatic number at most c_t. Since $\Delta(\mathcal{S}_{t-1}, \mathcal{S}_{t-1}, 1) = \mathcal{S}_t$, this proves 13.3. ∎

14 Constructing heroes

Finally, we apply these methods to give a different proof of an old result. The result of 8.4 gives a construction for all heroes. One half of it, that every hero can be constructed by the construction given, is easy; the hard half is to show 14.1 and 14.2 below, and took about eight pages in [3]. We shall see that, given the machinery developed in this paper, we can easily deduce those two results.

14.1 *If H is obtained from the disjoint union of heroes H_1, H_2 by making $V(H_1)$ complete to $V(H_2)$, then H is a hero.*

Proof. Choose c_0 such that every tournament with chromatic number at least c_0 contains both H_1 and H_2. Choose n as in 11.1, taking $d = c_0$ and $H = H_1$, and let $c = nc_0$. Now let T be a tournament with chromatic number at least c. If $V(T)$ can be partitioned into n subsets each inducing an H_1-free subtournament, then each of these subtournaments has chromatic number less than c_0, and so $\chi(T) < nc_0$, a contradiction. So, from the choice of n, there exists a copy S of H_1 in T, and a subset $P \subseteq V(T) \setminus V(S)$ with $V(S) \Rightarrow P$, such that $\chi(P) \geq c_0$. But then $T[P]$ contains H_2, and so T contains H. This proves 14.1. ∎

14.2 *Let H be a hero and $k \geq 0$ an integer: then $\Delta(H, K, 1)$ is a hero, where K is a transitive tournament with k vertices.*

This is evidently implied by the following extension of 3.2 (since $T[P]$ contains H if c is large enough, and $T[Q]$ contains K if k in 14.3 is large enough):

14.3 *For all integers $k, c \geq 0$ there exists $d \geq 0$, such that if T is a tournament with $\chi(T) \geq d$, then there exist $v \in V(T)$ and subsets $P \subseteq N^+(v)$ and $Q \subseteq N^-(v)$, where $\chi(P) \geq c$, and $|Q| \geq k$, and $P \Rightarrow Q$.*

Proof. (Sketch.) Choose $t \geq 2$ such that $\mathcal{S}_t$ contains v, P, Q as specified. From 3.1 and an easy modification of the proof of theorem 4.4 of [3], taking all the sets X_i of that theorem to be singletons, we deduce:

(1) *For all c, there exists $f(c)$ such that every tournament with chromatic number at least $f(c)$ and with no diamond of chromatic number at least c contains v, P, Q as specified.*

Now let T be a tournament that does not contain the desired v, P, Q. Consequently it does not contain $\mathcal{S}_t$, and we must show that its chromatic number is bounded. By (1), for all $c \geq 0$, every subtournament of T with chromatic number at least $f(c)$ has a diamond of chromatic number at least c. From 13.3, taking $c_2 = 1$ and $d_i = f(c_i)$ for $2 \leq i \leq t-1$, we deduce that, with c_t as in that theorem, every diamond in T has chromatic number less than c_t. Consequently $\chi(T) \leq f(c_t)$. This proves 14.3. ∎

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