Disjoint paths in tournaments

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Abstract

Given k pairs of vertices (s_i, t_i) $(1 \le i \le k)$ of a digraph G, how can we test whether there exist k vertex-disjoint directed paths from s_i to t_i for $1 \le i \le k$? This is NP-complete in general digraphs, even for $k = 2$ [2], but for $k = 2$ there is a polynomial-time algorithm when G is a tournament (or more generally, a semicomplete digraph), due to Bang-Jensen and Thomassen [1]. Here we prove that for all fixed k there is a polynomial-time algorithm to solve the problem when G is semicomplete.

1 Introduction

Let $s_1, t_1, \ldots, s_k, t_k$ be vertices of a graph or digraph G. The k *vertex-disjoint paths problem* is to determine whether there exist vertex-disjoint paths $P_1, \ldots, P_k$ (directed paths, in the case of a digraph) such that P_i is from s_i to t_i for $1 \le i \le k$. For undirected graphs, this problem is solvable in polynomial time for all fixed k; this was one of the highlights of the Graph Minors project of Robertson and the third author [4]. The directed version is a natural and important question, but it was shown by Fortune, Hopcroft and Wyllie [2] that, without further restrictions on the input G, this problem is NP-complete for digraphs, even for $k = 2$. This motivates the study of subclasses of digraphs for which the problem is polynomial-time solvable.

In this paper, all graphs and digraphs are finite, and without loops or parallel edges; thus if u, v are distinct vertices of a digraph then there do not exist two edges both from u to v, although there may be edges uv and vu. Also, by a "path" in a digraph we always mean a directed path. A digraph is a *tournament* if for every pair of distinct vertices u, v, exactly one of uv, vu is an edge; and a digraph is *semicomplete* if for all distinct u, v, at least one of uv, vu is an edge. It was shown by Bang-Jensen and Thomassen [1] that

- the k vertex-disjoint paths problem (for digraphs) is NP-complete if k is not fixed, even when G is a tournament;
- the two vertex-disjoint paths problem is solvable in polynomial time if G is semicomplete.

We shall show:

1.1 *For all fixed $k \ge 0$, the k vertex-disjoint paths problem is solvable in polynomial time if G is semicomplete.*

In fact we will prove a result for a wider class of digraphs, that we define next. Let P be a path of a digraph G, with vertices $v_1, \ldots, v_n$ in order. We say P is *minimal* if $j \le i + 1$ for every edge $v_i v_j$ of G with $1 \le i, j \le n$. Let $d \ge 1$; we say that a digraph G is *d-path-dominant* if for every minimal path P of G with d vertices, every vertex of G either belongs to $V(P)$ or has an out-neighbour in $V(P)$ or has an in-neighbour in $V(P)$. Thus a digraph is 1-path-dominant if and only if it is semicomplete; and 2-path-dominant if and only if its underlying simple graph is complete multipartite. We will show:

1.2 *For all fixed $d, k \ge 1$, the k vertex-disjoint paths problem is solvable in polynomial time if G is d-path-dominant.*

We stress here that we are looking for vertex-disjoint paths. One can ask the same for edge-disjoint paths, and that question has also been recently solved for tournaments, and indeed for digraphs with bounded independence number [3], but the solution is completely different. We do not know a polynomial-time algorithm for the two vertex-disjoint paths problem for digraphs with independence number two.

But we can extend 1.2 in a different way:

1.3 *For all $d, k \ge 1$, there is a polynomial-time algorithm as follows:*

- **Input:** *Vertices* $s_1, t_1, \dots, s_k, t_k$ *of a d-path-dominant digraph* G*, and integers* $x_1, \dots, x_k \geq 1$.
- **Output:** *Decides whether there exist pairwise vertex-disjoint directed paths* $P_1, \dots, P_k$ *of* G *such that for* $1 \leq i \leq k$*,* P_i *is from* s_i *to* t_i *and has at most* x_i *vertices.*

Let $s_1, t_1, \dots, s_k, t_k$ be vertices of a digraph G. We call $(G, s_1, t_1, \dots, s_k, t_k)$ a *problem instance.* A *linkage* in a digraph G is a sequence $L = (P_i \ : 1 \leq i \leq k)$ of vertex-disjoint paths, and L is a linkage *for* a problem instance $(G, s_1, t_1, \dots, s_, t_k)$ if P_i is from s_i to t_i for each i. (With a slight abuse of notation, we shall call k the "cardinality" of L, and $P_1, \dots, P_k$ its "members". Also, every subsequence of $(P_i \ : 1 \leq i \leq k)$ is a linkage L', and we say L "includes" L'.) If $x = (x_1, \dots, x_k)$ is a k-tuple of integers, we say a linkage $(P_i \ : 1 \leq i \leq k)$ is an *x-linkage* if each P_i has x_i vertices. We say a k-tuple of integers $x = (x_1, \dots, x_k)$ is a *quality* of $(G, s_1, t_1, \dots, s_k, t_k)$ if there is an x-linkage for $(G, s_1, t_1, \dots, s_, t_k)$. If $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, we say $x \leq y$ if $x_i \leq y_i$ for $1 \leq i \leq k$; and $x < y$ if $x \leq y$ and $x \neq y$. We say a quality x of $(G, s_1, t_1, \dots, s_k, t_k)$ is *key* if there is no quality y with $y < x$. Our main result is the following:

1.4 *For all* d, k*, there is an algorithm as follows:*

- **Input:** *A problem instance* $(G, s_1, t_1, \dots, s_k, t_k)$ *where* G *is d-path-dominant.*
- **Output:** *The set of all key qualities of* $(G, s_1, t_1, \dots, s_k, t_k)$.
- **Running time:** $O(n^t)$ *where* $t = 6k^2d(k + d) + 13k$.

The idea of the algorithm for 1.2 is easy described. We define an auxiliary digraph H with two special vertices s_0, t_0, and prove that there is a path in H from s_0 to t_0 if and only if there is a linkage for $(G, s_1, t_1, \dots, s_, t_k)$. Thus to solve the problem of 1.2 it suffices to construct H in polynomial time. The more general question of 1.4 is solved similarly, by assigning appropriate weights to the edges of H.

Recently we have been able to extend 1.1 to a more general class of digraphs, namely the digraphs whose vertex set can be partitioned into a bounded number of subsets such that each subset induces a semicomplete digraph. The proof is by a modification of the method of this paper, but it is considerably more difficult and not included here.

2 A useful enumeration

If P is a path of a digraph G, its *length* is $|E(P)|$ (every path has at least one vertex); and $s(P), t(P)$ denote the first and last vertices of P, respectively. If F is a subdigraph of G, a vertex v of $G \setminus V(F)$ is *F-outward* if no vertex of F is adjacent from v in G; and *F-inward* if no vertex of F is adjacent to v in G. If F is a digraph and $v \in V(F)$, $F \setminus v$ denotes the digraph obtained from F by deleting v; if $X \subseteq V(F)$, $F|X$ denotes the subdigraph of F induced on X; and $F \setminus X$ denotes the subdigraph obtained by deleting all vertices in X.

Now let $L = (P_i \ : 1 \leq i \leq k)$ be a linkage in G. We define $V(L)$ to be $V(P_1) \cup \dots \cup V(P_k)$. A vertex v is an *internal vertex* of L if $v \in V(L)$, and v is not an end of any member of L. A linkage L is *internally disjoint* from a linkage L' if no internal vertex of L belongs to $V(L')$ (note that this does not imply that L' is internally disjoint from L); and we say that L, L' are *internally disjoint* if each of them is internally disjoint from the other (and thus all vertices in $V(L) \cap V(L')$ must be ends of paths in both L and L')

Let Q, R be vertex-disjoint paths of a digraph G. A *planar* (Q,R)*-matching* is a linkage $(M_j : 1 \le j \le n)$ for some $n \ge 0$, such that

- $M_1, \dots, M_n$ each have either two or three vertices;
- $s(M_1), \dots, s(M_n)$ are vertices of Q, in order in Q; and
- $t(M_1), \dots, t(M_n)$ are vertices of R, in order in R.

Fix $d, k \ge 1$, and let $L = (P_1, \dots, P_k)$ be a linkage in a d-path-dominant digraph G. A subset $B \subseteq V(L)$ is said to be *acceptable* (for L) if

- for $1 \le j \le k$, if uv is an edge of P_j and $v \in B$ then $u \in B$ (and so $Q_j = P_j|B$ and $R_j = P_j|(V(G) \setminus B)$ are paths if they are non-null);
- for $1 \le i, j \le k$, there is no planar (Q_i, R_j)-matching of cardinality $(k-1)d + k^2 + 2$ internally disjoint from L.

Thus $\emptyset$ and $V(L)$ are acceptable.

2.1 *Let $d \ge 1$, let $(G, s_1, t_1, \dots, s_k, t_k)$ be a problem instance, where G is d-path-dominant, let x be a key quality, and let $L = (P_1, \dots, P_k)$ be an x-linkage for $(G, s_1, t_1, \dots, s_k, t_k)$. Suppose that $B \subseteq V(L)$ is acceptable for L and $B \ne V(L)$. Then there exists $v \in V(L) \setminus B$ such that $B \cup \{v\}$ is acceptable for L.*

Proof. Let $A = V(G) \setminus B$. For $1 \le j \le k$, let $Q_j = P_j|B$ and $R_j = P_j|A$. Let q_j, r_j be the last vertex of Q_j and the first vertex of R_j, respectively (if they exist).

(1) *For $1 \le j \le k$, P_j is a minimal path of G. In particular, the only edge of G from $V(Q_j)$ to $V(R_j)$ (if there is one) is $q_j r_j$. Moreover, every three-vertex path from $V(Q_j)$ to $V(R_j)$ with internal vertex in $V(G) \setminus V(L)$ uses at least one of q_j, r_j. Consequently, there is no planar (Q_j, R_j)-matching of cardinality three internally disjoint from L.*

For suppose there is an edge uv of G such that $u, v \in V(P_j)$ and u is before v in P_j, and there is at least one vertex of P_j between u and v. If we delete from P_j the vertices of P_j strictly between u and v, and add the edge uv, we obtain a path from s_j to t_j disjoint from every member of L except P_j, and with strictly fewer vertices than P_j, contradicting that x is key. Thus P_j is induced. Similarly there is no three-vertex path from $V(Q_j)$ to $V(R_j)$ with internal vertex in $V(G) \setminus V(L)$ containing neither of q_j, r_j. The final assertion follows. This proves (1).

From (1), the theorem holds if $k = 1$, so we may assume that $k \ge 2$.

(2) *We may assume that for all $i \in \{1, \dots, k\}$, if R_i is non-null then for some $j \in \{1, \dots, k\}$ with $j \ne i$, there is a planar $(Q_i, R_j \setminus r_j)$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L.*

For suppose that some i does not satisfy the statement of (2). Thus R_i is non-null, and there is no j as in (2). Since R_i is non-null, it follows that r_i exists. We may assume that $B \cup \{r_i\}$ is not acceptable. Consequently, one of the two conditions in the definition of "acceptable" is not satisfied by $B \cup \{r_i\}$. The first is satisfied since r_i is the first vertex of R_i. Thus the second is false, and so for some $i', j \in \{1, \ldots, k\}$, there is a planar $(P_{i'}|(B \cup \{r_i\}), P_j|(A \setminus \{r_i\}))$-matching of cardinality $(k-1)d + k^2 + 2$ internally disjoint from L. Since there is no planar $(Q_{i'}, R_j)$-matching of cardinality $(k-1)d + k^2 + 2$ internally disjoint from L, and $P_j|(A \setminus \{r_i\})$ is a subpath of R_j, it follows that $P_{i'}|(B \cup \{r_i\}) \neq Q_{i'}$, and so $i' = i$. Since only one vertex of $P_i|(B \cup \{r_i\})$ does not belong to Q_i, it follows that there is a planar $(Q_i, R_j \setminus r_j)$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L. Since $(k-1)d + k^2 \geq 4$ (because $k \geq 2$), (1) implies that $j \neq i$. This proves (2).

(3) *We may assume that for some $p \geq 2$, and for all i with $1 \leq i < p$, there is a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L, and there is a planar $(Q_p, R_1 \setminus r_1)$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L.*

For by hypothesis, there exists $i \in \{1, \ldots, k\}$ such that R_i is non-null. By repeated application of (2), there exist distinct $h_1, \ldots, h_p \in \{1, \ldots, k\}$ such that for $1 \leq i \leq p$ there is a planar $(Q_{h_i}, R_{h_{i+1}} \setminus r_{h_{i+1}})$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L, where $h_{p+1} = h_1$; and $p \geq 2$ by (1). Without loss of generality, we may assume that $h_i = i$ for $1 \leq i \leq p$. This proves (3).

Let us say a planar (Q, R)-matching is *s-spaced* if no subpath of Q with at most s vertices meets more than one member of the matching, and no subpath of R with at most s vertices meets more than one member of the matching.

(4) *We may assume that for some $p \geq 2$, and for all i with $1 \leq i < p$, there is a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching L_i, and there is a planar $(Q_p, R_1 \setminus r_1)$-matching L_p, such that*

- *$L_1, \ldots, L_p$ all have cardinality k;*
- *they are pairwise internally disjoint;*
- *each of $L_1, \ldots, L_p$ is internally disjoint from L; and*
- *each of $L_1, \ldots, L_p$ is $(d+1)$-spaced.*

For let L'_i be a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L, for $1 \leq i < p$, and let L'_p be a planar $(Q_p, R_1 \setminus r_1)$-matching of cardinality $(k-1)d + k^2$ internally disjoint from L. We choose $L_i \subseteq L'_i$ inductively. Suppose that for some $h < p$, we have chosen $L_1, \ldots, L_h$, such that

- $L_1, \ldots, L_h$ all have cardinality k;
- they are pairwise internally disjoint;
- each of $L_1, \ldots, L_h$ is internally disjoint from L; and
- each of $L_1, \ldots, L_h$ is $(d+1)$-spaced.

We define L_{h+1} as follows. The union of the sets of internal vertices of $L_1, \ldots, L_h$ has cardinality at most $hk \leq k(k-1)$, and so L'_{h+1} includes a planar $(Q_{h+1}, R_{h+2} \setminus r_{h+2})$-matching (or $(Q_p, R_1 \setminus r_1)$-matching, if $h = p-1$) of cardinality $(k-1)d + k^2 - k(k-1) = 1 + (k-1)(d+1)$, internally disjoint from each of $L_1, \ldots, L_h$. By ordering the members of this matching in their natural order, and taking only the ith terms, where $i = 1, 1+(d+1), 1+2(d+1)\ldots$, we obtain a $(d+1)$-spaced matching of cardinality k. Let this be L_{h+1}. This completes the inductive definition of $L_1, \ldots, L_p$, and so proves (4).

For $1 \leq i \leq p$, let $L_i = \{M_i^1, \ldots, M_i^k\}$, numbered in order; thus, if q_i^h and r_{i+1}^h denote the first and last vertices of M_i^h, then $q_i^1, \ldots, q_i^k$ are distinct and in order in Q_i, and $r_{i+1}, r_{i+1}^1, \ldots, r_{i+1}^k$ are distinct and in order in R_{i+1} (or in R_1 if $i = p$). For $1 \leq i \leq p$ and $2 \leq h \leq k$, let Q_i^h be the subpath of P_i with d vertices and with last vertex q_i^h. (Thus q_i^{h-1} does not belong to Q_i^h since L_i is d-spaced, and indeed $(d+1)$-spaced.) Since P_i and hence Q_i^h is a minimal path of G, and G is d-path-dominant, it follows that for $1 \leq i \leq p$ and $2 \leq h \leq k$, r_i^{h-1} is adjacent to or from some vertex v of Q_i^h. Since $r_i^{h-1} \neq r_i$, (1) implies that r_i^{h-1} is not adjacent from any vertex of Q_i^h; and so there is a path R_i^{h-1} from r_i^{h-1} to q_i^h of length at most d, such that all its internal vertices belong to Q_i^h. For $1 \leq i \leq p$, and $1 \leq h < k$, let S_i^h be the path

$$q_i^h\text{-}M_i^h\text{-}r_{i+1}^h\text{-}R_{i+1}^h\text{-}q_{i+1}^{h+1},$$

or

$$q_p^h\text{-}M_p^h\text{-}r_1^h\text{-}R_1^h\text{-}q_1^{h+1}$$

if $i = p$; then S_i^h is a path from q_i^h to q_{i+1}^{h+1} (or to q_1^{h+1} if $i = p$), of length at most $d+2$. Thus (reading subscripts modulo p) concatenating $S_i^1, S_{i+1}^2, \ldots, S_{i+p-2}^{p-1}$ and M_{i-1}^p gives a path T'_i from q_i^1 to r_i^p of length at most $(p-1)(d+2)+2$. The subpath T_i of P_i from q_i^1 to r_i^p has length at least $(p+k-2)(d+1)+2$, since L_{i-1}, L_i are $(d+1)$-spaced and r_i is different from r_i^1; and since $p+k-2 \geq 2(p-1)$ and $d+1 > (d+2)/2$, it follows that T_i has length strictly greater than that of T'_i. Let P'_i be obtained from P_i by replacing the subpath T_i by T'_i, for $1 \leq i \leq p$, and let $P_{i'} = P_i$ for $p+1 \leq i \leq k$. Then $\{P'_1, \ldots, P'_k\}$ is a linkage for $(G, s_1, t_1, \ldots, s, t_k)$, contradicting that x is key. This proves 2.1. ∎

We deduce:

2.2 *Let $d \geq 1$, let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance where G is d-path-dominant, let x be a key quality, and let $L = (P_1, \ldots, P_k)$ be an x-linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$. Let $c = (k-1)d + k^2 + 2$. Then there is an enumeration $(v_1, \ldots, v_n)$ of $V(L)$, such that*

- *for $1 \leq h \leq k$ and $1 \leq p, q \leq n$, if $v_p v_q$ is an edge of P_h then $p < q$;*
- *for $1 \leq h, i \leq k$ and $1 \leq p \leq n-1$, and every cd-vertex subpath Q of $P_h|\{v_1, \ldots, v_p\}$, and every cd-vertex subpath R of $P_i|\{v_{p+1}, \ldots, v_n\}$, there are at most $c(2k+1)$ vertices of G that are both Q-outward and R-inward.*

Proof. Since $\emptyset$ is acceptable for L, by repeated application of 2.1 implies that there is an enumeration $(v_1, \ldots, v_n)$ of $V(L)$, such that $\{v_1, \ldots, v_p\}$ is acceptable for $0 \leq p \leq n$. We claim that this enumeration satisfies the theorem. For certainly the first bullet holds; we must check the second.

Thus, let $1 \leq p \leq n$, and let $B = \{v_1, \ldots, v_p\}$ and $A = \{v_{p+1}, \ldots, v_n\}$. For $1 \leq h \leq k$, let $Q_h = P_h|B$ and $R_h = P_h|A$. Now let $1 \leq h, i \leq k$, and let Q, R be cd-vertex subpaths of Q_h, R_i respectively. Let X be the set of all vertices of G that are both Q-outward and R-inward. We must show that $|X| \leq c(2k+1)$.

(1) *If $x_1, \ldots, x_c \in X$ are distinct, then there exist $y_1, \ldots, y_c \in V(Q)$, distinct and in order in Q, such that $y_j x_j$ is an edge for $1 \leq j \leq c$.*

For Q has cd vertices; let its vertices be $q_1, \ldots, q_{cd}$ in order. Let $1 \leq j \leq c$. The subpath of Q induced on $\{q_s : (j-1)d < s \leq jd\}$ has d vertices, and since Q is a minimal path of G and G is d-path-dominant, and $X \cap V(Q) = \emptyset$, it follows that x_j is in- or out-adjacent to a vertex of this subpath, say y_j. Since $x_j \in X$ and hence is Q-outwards, it follows that $x_j y_j$ is not an edge, and so $y_j x_j$ is an edge. But then $y_1, \ldots, y_c$ satisfy (1). This proves (1).

(2) *The sets $X \setminus V(L)$, $X \cap V(Q_g)$ $(1 \leq g \leq k)$ and $X \cap V(R_g)$ $(1 \leq g \leq k)$ all have cardinality at most $c-1$, and hence $|X| \leq (2k+1)(c-1)$.*

For suppose that there exist distinct $x_1, \ldots, x_c \in X \setminus V(L)$. By (1) there exist distinct $y_1, \ldots, y_c \in V(Q)$, in order in Q, such that $y_j x_j$ is an edge for $1 \leq j \leq c$; and similarly there exist $z_1, \ldots, z_c \in V(R)$, in order in R, such that $x_j z_j$ is an edge for $1 \leq j \leq c$. But then the c paths y_j-x_j-z_j $(1 \leq j \leq c)$ form a planar (Q_h, R_i)-matching of cardinality c, internally disjoint from L, contradicting that $\{v_1, \ldots, v_p\}$ is acceptable. Thus $|X \setminus V(L)| \leq c-1$. Now suppose that for some $g \in \{1, \ldots, k\}$, there exist distinct $x_1, \ldots, x_c$ in $X \cap V(R_g)$, numbered in order in R_g. Choose $y_1, \ldots, y_c$ as in (1); then the paths $y_j x_j$ $(1 \leq j \leq c)$ form a planar (Q_h, R_g)-matching of cardinality c, internally disjoint from L, contradicting that $\{v_1, \ldots, v_p\}$ is acceptable. Thus $|X \cap V(R_g)| \leq c-1$, and similarly $|X \cap V(Q_g)| \leq c-1$, for $1 \leq g \leq k$. This proves (2).

From (2), the theorem follows. ∎

3 Confusion and the auxiliary digraph

Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance, and let $L = (M_1, \ldots, M_k)$ be a linkage in G (not necessarily a linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$). Let $A(L)$ be the set of all vertices in $V(G) \setminus V(L)$ that are $M_j \setminus t(M_j)$-inward for some $j \in \{1, \ldots, k\}$ such that $t(M_j) \neq t_j$ and let $B(L)$ be the set of all vertices in $V(G) \setminus V(L)$ that are $M_j \setminus s(M_j)$-outward for some $j \in \{1, \ldots, k\}$ such that $s(M_j) \neq s_j$. We call $|A(L) \cap B(L)|$ the *confusion* of L ; and it is helpful to keep the confusion small, as we shall see.

A (k, m, c)-*rail* in a problem instance $(G, s_1, t_1, \ldots, s_k, t_k)$ is a triple (L, X, Y), where

- L is a linkage in G consisting of k paths $(M_1, \ldots, M_k)$ (but not necessarily a linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$);
- for $1 \leq j \leq k$, M_j has at most $2m$ vertices, and if it has fewer than $2m$ vertices then M_j either has first vertex s_j or last vertex t_j;

- L has confusion at most c;
- X, Y are disjoint subsets of $V(G) \setminus V(L)$; and
- $X \subseteq A(L)$, $Y \subseteq B(L)$, and $X \cup Y = A(L) \cup B(L)$.

3.1 *For all $k, m, c \geq 0$, if $(G, s_1, t_1, \dots, s_k, t_k)$ is a problem instance and G has n vertices then there are at most $2^c n^{2km}(2km)^k$ (k, m, c)-rails in $(G, s_1, t_1, \dots, s_k, t_k)$. Moreover, for all fixed $k, m, c \geq 0$, there is an algorithm which, with input a problem instance $(G, s_1, t_1, \dots, s_k, t_k)$, finds all its (k, m, c)-rails in time $O(n^{2km+1})$, where $n = |V(G)|$.*

Proof. First, if L is a linkage with k paths each with at most $2m$ vertices, then $|V(L)| \leq 2km$, and so the number of such linkages is at most $n^{2km}(2km)^k$, as is easily seen. Now fix a linkage L satisfying the first two bullets in the definition of (k, m, c)-rail; let us count the number of pairs (X, Y) such that (L, X, Y) is a (k, m, c)-rail. There are none unless $|A(L) \cap B(L)| \leq c$; and in that case, there are at most 2^c possibilities for the pair (X, Y), since X consists of $A(L) \setminus B(L)$ together with some subset of $A(L) \cap B(L)$, and $Y = (A(L) \cup B(L)) \setminus X$.

For the algorithm, we first find all linkages L with k paths each with at most $2m$ vertices, by examining all ordered $2km$-tuples of distinct vertices of G. For each such L, we check whether it satisfies the first three bullets in the definition of (k, m, c)-rail (this takes time $O(n)$); if not we discard it and otherwise we partition $A(L) \cap B(L)$ into two subsets in all possible ways, and output the corresponding (k, m, c)-rails. The result follows. ∎

Let (L, X, Y) and (L', X', Y') be distinct (k, m, c)-rails in G, and let $L = (P_1, \dots, P_k)$ and $L' = (P'_1, \dots, P'_k)$. We write $(L, X, Y) \to (L', X', Y')$ if the following hold:

- for $1 \leq i \leq k$, $P_i \cup P'_i$ is a path from the first vertex of P_i to the last vertex of P'_i;
- for $1 \leq i \leq k$, $V(P'_i) \subseteq V(P_i) \cup X$, and $V(P_i) \subseteq V(P'_i) \cup Y'$; and
- $X' \subseteq X$, and $Y \subseteq Y'$.

Let $(G, s_1, t_1, \dots, s_k, t_k)$ be a problem instance, and let $\mathcal{T}$ be the set of all (k, m, c)-rails in $(G, s_1, t_1, \dots, s_k, t_k)$. Take two new vertices s_0, t_0, and let us define a digraph H with vertex set $\mathcal{T} \cup \{s_0, t_0\}$ as follows. Let $u, v \in V(H)$. If $u, v \in \mathcal{T}$ are distinct, then $uv \in E(H)$ if and only if $u \to v$. If $u = s_0$ and $v \in \mathcal{T}$, let $v = (L, X, Y)$ where $L = (M_1, \dots, M_k)$; then $uv \in E(H)$ if and only if M_j has first vertex s_j for all $j \in \{1, \dots, k\}$. Similarly, if $u \in \mathcal{T}$ and $v = t_0$, let $u = (L, X, Y)$ where $L = (M_1, \dots, M_k)$; then $uv \in E(H)$ if and only if M_j has last vertex t_j for all $j \in \{1, \dots, k\}$. This defines H. We call H the *(k, m, c)-tracker* of $(G, s_1, t_1, \dots, s_k, t_k)$.

We shall show that with an appropriate choice of m, c, when G is d-path-dominant we can reduce our problems about linkages for $(G, s_1, t_1, \dots, s_k, t_k)$ to problems about paths from s_0 to t_0 in the (k, m, c)-tracker. Let $(G, s_1, t_1, \dots, s_k, t_k)$ be a problem instance, let $(P_1, \dots, P_k)$ be a linkage for $(G, s_1, t_1, \dots, s_k, t_k)$, and let P be a path from s_0 to t_0 in the (k, m, c)-tracker. Let P have vertices

$$s_0, (L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n), t_0$$

in order, and let $L_p = (M_{p,1}, \dots, M_{p,k})$ for $1 \leq p \leq n$. We say that P *traces* $(P_1, \dots, P_k)$ if P_j is the union of $M_{1,j}, \dots, M_{n,j}$ for all $j \in \{1, \dots, k\}$.

3.2 *Let $k, m, c \geq 0$ be integers, and let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance, with (k, m, c)-tracker H. Every path in H from s_0 to t_0 traces some linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$.*

Proof. Let P be a path of H, with vertices

$$s_0, (L_1, X_1, Y_1), \ldots, (L_n, X_n, Y_n), t_0$$

in order, and let $L_p = (M_{p,1}, \ldots, M_{p,k})$ for $1 \leq p \leq n$. For $1 \leq p \leq n$ and $1 \leq j \leq k$, let $P_{p,j}$ be the union of $M_{1,j}, \ldots, M_{p,j}$.

(1) *For $1 \leq p \leq n$ and $1 \leq j \leq k$, every vertex of $P_{p,j}$ belongs to $Y_p \cup V(M_{p,j})$.*

We prove this by induction on p. If $p = 1$ the claim is true, since then $P_{1,j} = M_{1,j}$. We assume then that $p > 1$ and the result holds for $p - 1$. Let $v \in V(P_{p,j})$. If $v \in V(M_{p,j})$ then the claim is true, so we assume not. Since $v \in V(P_{p,j})$, and $P_{p,j} = P_{p-1,j} \cup M_{p,j}$, it follows that $v \in V(P_{p-1,j})$, and so from the inductive hypothesis, $v \in Y_{p-1} \cup V(M_{p-1,j})$. But since $(L_{p-1}, X_{p-1}, Y_{p-1}) \to (L_p, X_p, Y_p)$, we deduce that $Y_{p-1} \subseteq Y_p$, and $V(M_{p-1,j}) \subseteq V(M_{p,j}) \cup Y_p$, and so $v \in V(M_{p,j}) \cup Y_p$. This proves (1).

(2) *For $1 \leq p \leq n$ and $1 \leq j \leq k$, $P_{p,j}$ is a path from s_j to the last vertex of $M_{p,j}$.*

The claim holds if $p = 1$; so we assume that $p > 1$ and the claim holds for $p - 1$. Thus $P_{p-1,j}$ is a path from s_j to the last vertex of $M_{p-1,j}$; and also, $M_{p-1,j} \cup M_{p,j}$ is a path, from the first vertex of $M_{p-1,j}$ to the last vertex of $M_{p,j}$, since $(L_{p-1}, X_{p-1}, Y_{p-1}) \to (L_p, X_p, Y_p)$. We claim that every vertex v that belongs to both of $P_{p-1,j}, M_{p,j}$ also belongs to $M_{p-1,j}$. For suppose not; then by (1), $v \in Y_{p-1}$ since $v \in V(P_{p-1,j}) \setminus V(M_{p-1,j})$, and $v \in X_{p-1}$, since $v \in V(M_{p,j}) \setminus V(M_{p-1,j})$. This is impossible since $X_{p-1} \cap Y_{p-1} = \emptyset$. This proves that every vertex that belongs to both of $P_{p-1,j}, M_{p,j}$ also belongs to $M_{p-1,j}$. Since $M_{p-1,j}$ is non-null, we deduce that $P_{p-1,j} \cup M_{p,j}$ is a path from s_j to the last vertex of $M_{p,j}$. This proves (2).

(3) *For $1 \leq p \leq n$, the paths $P_{p,1}, \ldots, P_{p,k}$ are pairwise vertex-disjoint.*

For again we proceed by induction on p, and may assume that $p > 1$ and the result holds for $p - 1$. Suppose that v belongs to two of the paths $P_{p,1}, \ldots, P_{p,k}$, say to $P_{p,1}$ and $P_{p,2}$. From the inductive hypothesis, v does not belong to both of $P_{p-1,1}$ and $P_{p-1,2}$, so we may assume that $v \in V(M_{p,1})$. Now $v \notin V(M_{p,2})$, because L_p is a linkage, and so $v \in V(P_{p-1,2})$. From (1) we deduce that $v \in Y_{p-1} \cup V(M_{p-1,2})$. But $Y_{p-1} \subseteq Y_p$, and $V(M_{p-1,2}) \setminus V(M_{p,2}) \subseteq Y_p$, and so $v \in Y_p$; but $Y_p \cap V(L_p) = \emptyset$ since (L_p, X_p, Y_p) is a (k, m, c)-rail, a contradiction. This proves (3).

From (2) and (3) we deduce that $(P_{n,1}, \ldots, P_{n,k})$ is a linkage L for $(G, s_1, t_1, \ldots, s_k, t_k)$. Thus P traces L. This proves 3.2. ∎

The next result is a kind of partial converse; but we have to choose m, c carefully, and we need G to be d-path-dominant, and the proof only works for linkages that realize a key quality.

3.3 *Let $d, k \geq 1$ be integers, and let*

$$\begin{aligned} c &= ((k-1)d + k^2 + 2)(2k+1)k^2 \\ m &= ((k-1)d + k^2 + 2)d + 1. \end{aligned}$$

Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance where G is d-path-dominant, let x be a key quality, and let $(P_1, \ldots, P_k)$ be an x-linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$. Let H be the (k, m, c)-tracker of $(G, s_1, t_1, \ldots, s_k, t_k)$. Then there is a path in H from s_0 to t_0 tracing $(P_1, \ldots, P_k)$.

Proof. Let $L = (P_1, \ldots, P_k)$. By 2.2, there is an enumeration $(v_1, \ldots, v_n)$ of $V(L)$, such that

- for $1 \leq j \leq k$ and $1 \leq p, q \leq n$, if $v_p v_q$ is an edge of P_j then $p < q$;
- for $1 \leq i, j \leq k$ and $1 \leq p \leq n-1$, and every $(m-1)$-vertex subpath Q of $P_i|\{v_1, \ldots, v_p\}$, and every $(m-1)$-vertex subpath R of $P_j|\{v_{p+1}, \ldots, v_n\}$, there are at most $((k-1)d+k^2+2)(2k+1)$ vertices of G that are both Q-outward and R-inward.

For each $v \in V(L)$, let $\phi(v) = i$ where $v = v_i$; thus ϕ is a bijection from $V(L)$ onto $\{1, \ldots, n\}$.

For all $p \in \{0, \ldots, n\}$ and all $j \in \{1, \ldots, k\}$, if $\phi(s_j) \leq p$, let $Q_{p,j}$ be the maximal subpath of P_j with at most m vertices and with last vertex v_q, where $q \leq p$ is maximum such that $v_q \in V(P_j)$. If $\phi(s_j) > p$, let $Q_{p,j}$ be the null digraph. Similarly, if $\phi(t_j) > p$, let $R_{p,j}$ be the maximal subpath of P_j with at most m vertices and with first vertex v_r, where $r > p$ is minimum such that $v_r \in V(P_j)$. If $\phi(t_j) \leq p$, let $R_{p,j}$ be the null digraph. Thus, if $Q_{p,j}, R_{p,j}$ are both non-null, then $t(Q_{p,j})$ and $s(R_{p,j})$ are consecutive in P_j.

For all $p \in \{0, \ldots, n\}$ and all $j \in \{1, \ldots, k\}$, let $M_{p,j}$ be the subpath of P_j defined as follows: if both $Q_{p,j}, R_{p,j}$ are non-null, $M_{p,j}$ consists of $Q_{p,j} \cup R_{p,j}$ together with the edge of P_j from $t(Q_{p,j})$ to $s(R_{p,j})$, while if one of $Q_{p,j}, R_{p,j}$ is null, $M_{p,j}$ equals the other (not both can be null). We see that, for all p, j, $M_{p,j}$ has at most $2m$ vertices; and either it has exactly $2m$, or its first vertex is s_j, or its last vertex is t_j. For all $p \in \{0, \ldots, n\}$, let L_p be the linkage $(M_{p,1}, \ldots, M_{p,k})$.

(1) *For all $p \in \{0, \ldots, n\}$, L_p has confusion at most c.*

Let $v \in A(L_p) \cap B(L_p)$, where $A(L_p), B(L_p)$ are as in the definition of confusion. Thus there exists $j \in \{1, \ldots, k\}$ such that v is $M_{p,j} \setminus t(M_{p,j})$-inward and $t(M_{p,j}) \neq t_j$. Since $t(M_{p,j}) \neq t_j$, it follows from the choice of $R_{p,j}$ that $R_{p,j}$ has exactly m vertices. Moreover, v is $R_{p,j} \setminus t(R_{p,j})$-inward, since v is $M_{p,j} \setminus t(M_{p,j})$-inward. Similarly, there exists $i \in \{1, \ldots, k\}$ such that v is $Q_{p,i} \setminus s(Q_{p,i})$-outward and $Q_{p,i}$ has m vertices. For each choice of $i, j \in \{1, \ldots, k\}$, there are at most $((k-1)d + k^2 + 2)(2k+1)$ vertices that are both $Q_{p,i} \setminus s(Q_{p,i})$-outward and $R_{p,j} \setminus t(R_{p,j})$-inward, from the choice of the enumeration $(v_1, \ldots, v_n)$. Consequently in total there are only c possibilities for v, and so $|A(L_p) \cap B(L_p)| \leq c$. This proves (1).

(2) *For $0 \leq p \leq n$ and each $v \in V(L) \setminus V(L_p)$, if $\phi(v) > p$ then $v \in A(L_p)$, and if $\phi(v) \leq p$ then $v \in B(L_p)$.*

For let $v \in V(P_j)$ say. Assume first that $\phi(v) > p$. Since $v \notin V(L_p)$, it follows that $M_{p,j}$ does not have last vertex t_j; and since x is key, v is not adjacent from any vertex in $M_{p,j}$ except possibly

$t(M_{p,j})$. Consequently v is $M_{p,j} \setminus t(M_{p,j})$-inward, and hence belongs to $A(L_p)$. Similarly, if $\phi(v) \leq p$ then $v \in B(L_p)$. This proves (2).

For all $p \in \{0, \ldots, n\}$, define X_p, Y_p as follows:

$$\begin{aligned} X_p &= \{v \in V(L) \setminus V(L_p) \, : \phi(v) > p\} \cup (A(L_p) \setminus B(L_p)) \\ Y_p &= (A(L_p) \cup B(L_p)) \setminus X_p. \end{aligned}$$

(3) *For all $p \in \{0, \ldots, n\}$, (L_p, X_p, Y_p) is a (k, m, c)-rail.*

From (1), it suffices to check that

- X_p, Y_p are disjoint subsets of $V(G) \setminus V(L_p)$;
- $X_p \subseteq A(L_p)$, $Y_p \subseteq B(L_p)$; and
- $X_p \cup Y_p = A(L_p) \cup B(L_p)$.

Certainly they are disjoint, and have union $A(L_p) \cup B(L_p)$. Moreover, from (2), $X_p \subseteq A(L_p)$. It remains to show that $Y_p \subseteq B(L_p)$. Let $v \in Y_p$. Thus $v \in A(L_p) \cup B(L_p)$; and $v \notin A(L_p) \setminus B(L_p)$, since $v \notin X_p$. Consequently $v \in B(L_p)$ as required. This proves (3).

(4) *For all $p \in \{0, \ldots, n-1\}$, and all $j \in \{1, \ldots, k\}$, $M_{p,j} \cup M_{p+1,j}$ is a path from the first vertex of $M_{p,j}$ to the last vertex of $M_{p+1,j}$.*

For $M_{p,j}, M_{p+1,j}$ are both subpaths of P_j, and we may assume they are distinct, and so $v_{p+1} \in V(P_j)$. Hence, since $m > 0$, v_{p+1} is the first vertex of $R_{p,j}$, and the last vertex of $Q_{p+1,j}$; and so $M_{p,j} \cup M_{p+1,j}$ is a path. Moreover, it follows from the definition of the paths $M_{p,j}$ that $M_{p,j} \cup M_{p+1,j}$ is a path from the first vertex of $M_{p,j}$ to the last vertex of $M_{p+1,j}$. This proves (4).

(5) *For all $p \in \{0, \ldots, n-1\}$, and all $j \in \{1, \ldots, k\}$, $A(L_{p+1}) \subseteq A(L_p) \cup V(L)$ and $B(L_p) \subseteq B(L_{p+1}) \cup V(L)$.*

For let $v \in A(L_{p+1})$. We need to prove that $v \in A(L_p) \cup V(L)$, and so we may assume that $v \notin V(L)$. Choose j with $1 \leq j \leq k$ such that v is $M_{p+1,j} \setminus t(M_{p+1,j})$-inward and $t(M_{p+1,j}) \neq t_j$. Consequently $t(M_{p,j}) \neq t_j$, and so if v is $M_{p,j} \setminus t(M_{p,j})$-inward then $v \in A(L_p)$ as required, so we may assume that v is adjacent from some vertex of $M_{p,j}$. In particular, $M_{p,j} \neq M_{p+1,j}$ and so $v_{p+1} \in V(P_j)$, and $v_{p+1} = s(R_{p,j}) = t(Q_{p+1,j})$. Moreover, since $s(M_{p,j})$ is the only vertex of $M_{p,j}$ that may not belong to $M_{p+1,j}$, we deduce that $s(M_{p,j})$ is adjacent to v, and $s(M_{p,j})$ does not belong to $M_{p+1,j}$. Consequently $s(M_{p+1,j}) \neq s_j$, and so $Q_{p+1,j}$ has m vertices. Since v is $M_{p+1,j} \setminus t(M_{p+1,j})$-inward, and G is d-path-dominant, and $M_{p+1,j} \setminus t(M_{p+1,j})$ is a minimal path of G, and it has $m - 1 \geq d + 2$ vertices, there is a subpath of $M_{p+1,j} \setminus t(M_{p+1,j})$ with d vertices, not containing the first or second vertex of $M_{p+1,j} \setminus t(M_{p+1,j})$; and so v is adjacent to some vertex w of $M_{p+1,j} \setminus t(M_{p+1,j})$ different from its first and second vertices. But v is adjacent from u, so by replacing the subpath of P_j between u and w by the path u-v-w, we contradict that x is key. This proves that $v \in A(L_p)$, and so $A(L_{p+1}) \subseteq A(L_p) \cup V(L)$. Similarly $B(L_p) \subseteq B(L_{p+1}) \cup V(L)$. This proves (5).

(6) *For all* $p \in \{0, \dots, n-1\}$, $X_{p+1} \subseteq X_p$, *and* $Y_p \subseteq Y_{p+1}$.

Let $v \in X_{p+1}$. Suppose first that $v \notin V(L)$. Then $v \in A(L_{p+1}) \setminus B(L_{p+1})$. By (5), $v \in A(L_p) \setminus B(L_p)$, and so $v \in X_p$ as required. Thus we may assume that $v \in V(L)$. Since $v \in X_{p+1}$, it follows that either $\phi(v) > p+1$, or $v \notin B(L_{p+1})$. If $\phi(v) > p+1$, then since $v \notin V(L_{p+1})$, it follows that $v \notin V(L_p)$, and hence $v \in X_p$ from the definition of X_p. Thus we may assume that $\phi(v) \leq p+1$ and $v \notin B(L_{p+1})$, contrary to (2). This proves that $X_{p+1} \subseteq X_p$.

For the second inclusion, let $v \in Y_p$. Suppose first that $v \notin V(L)$. Then $v \in B(L_p)$; and so $v \in B(L_{p+1})$ by (5), and hence $v \in Y_{p+1}$ as required. Thus we may assume that $v \in V(L)$. Since $v \in Y_p$, it follows that $\phi(v) \leq p$. Now $v \notin V(L_p)$, and therefore $v \notin V(L_{p+1})$. But $\phi(v) \leq p+1$, and so by (2), $v \in B(L_{p+1})$, and consequently $v \notin X_{p+1}$. Thus $v \in Y_{p+1}$, as required. This proves that $Y_p \subseteq Y_{p+1}$, and so proves (6).

(7) *For all* $p \in \{0, \dots, n-1\}$, *and all* $j \in \{1, \dots, k\}$, $V(P_{p+1,j}) \subseteq V(P_{p,j}) \cup X_p$, *and* $V(P_{p,j}) \subseteq V(P_{p+1,j}) \cup Y_{p+1}$.

To prove the first assertion, let $v \in V(P_{p+1,j}) \setminus V(P_{p,j})$. It follows that $\phi(v) > p$; but then $v \in X_p$ from the definition of X_p. For the second assertion, let $v \in V(P_{p,j}) \setminus V(P_{p+1,j})$; then $\phi(v) \leq p+1$, and so $v \in B(L_{p+1})$ by (2). Consequently $v \notin X_{p+1}$, and so $v \in Y_{p+1}$ as required. This proves (7).

(8) *For all* $p \in \{0, \dots, n-1\}$, $(L_p, X_p, Y_p) \to (L_{p+1}, X_{p+1}, Y_{p+1})$.

This is immediate from (4), (6) and (7).

Now $(L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n)$ are not necessarily all distinct. But we have:

(9) *For all* p, r *with* $0 \leq p \leq r \leq n$, *if* $(L_p, X_p, Y_p) = (L_r, X_r, Y_r)$, *then* $(L_p, X_p, Y_p) = (L_q, X_q, Y_q)$ *for all* q *with* $p \leq q \leq r$.

For (6) implies that $X_q \subseteq X_p$, and $X_r \subseteq X_q$, and so $X_p = X_q$, and similarly $Y_p = Y_q$. If some vertex v belongs to $V(L_q) \setminus V(L_p)$, then by (7) and (6), $v \in X_p = X_q$, a contradiction. Similarly, if $v \in V(L_p) \setminus V(L_q)$ then $v \in Y_q = Y_p$, a contradiction. This proves (9).

(10) *For all* $j \in \{1, \dots, k\}$, $M_{0,j}$ *has first vertex* s_j, *and* $M_{n,j}$ *has last vertex* t_j.

This follows from the definitions of $M_{0,j}$ and $M_{n,j}$.

We recall that H is the (k, m, c)-tracker, with two special vertices s_0, t_0. Now (10) implies that s_0 is adjacent to (L_1, X_1, Y_1) in H, and (L_n, X_n, Y_n) is adjacent to t_0. From (8) and (9), there is a subsequence of the sequence

$$s_0, (L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n), t_0,$$

which lists the vertex set in order of a path of H from s_0 to t_0. By 3.2, this path traces some linkage L' for $(G, s_1, t_1, \dots, s_k, t_k)$. But for all $j \in \{1, \dots, k\}$, $M_{0,j}, M_{1,j}, \dots, M_{n,j}$ are all subpaths of P_j;

and since their union is a path from s_j to t_j, it follows that their union is P_j. Hence $L' = L$. This proves 3.3. ∎

4 The algorithm

Next, we need a polynomial algorithm to solve a kind of vector-valued shortest path problem. If $n \geq 0$ is an integer, K_n denotes the set of all k-tuples $(x_1, \ldots, x_k)$ of nonnegative integers such that $x_1 + \cdots + x_k \leq n$.

4.1 *There is an algorithm as follows:*

- **Input:** *A digraph H, and distinct vertices $s_0, t_0 \in V(H)$; an integer $n \geq 0$; and for each edge e of H, a member $l(e)$ of K_n.*
- **Output:** *The set of all minimal (under component-wise domination) vectors $l(P)$, over all paths P of H from s_0 to t_0; where for a path P with edge set $\{e_1, \ldots, e_p\}$, $l(P) = l(e_1) + \cdots + l(e_p)$.*
- **Running time:** *$O(n^k|V(H)||E(H)|)$.*

Proof. Let $Q_0(s_0) = \{(0, \ldots, 0)\}$, and let $Q_0(v) = \emptyset$ for every other vertex v of D. Inductively, for $1 \leq i \leq |V(H)|$, let $Q_i(v)$ be the set of minimal vectors in K_n that either belong to $Q_{i-1}(v)$ or are expressible in the form $l(e) + x$ for some edge $e = uv$ of H and some $x \in Q_{i-1}(u)$.

Now here is an algorithm for the problem:

- For $i = 1, \ldots, |V(H)|$ in turn, compute $Q_i(v)$ for every $v \in V(H)$.
- Output $Q_{|V(H)|}(t_0)$.

It is easy to check that this output is correct, and we leave it to the reader. To compute $Q_i(v)$ at the ith step takes time $O(n^k)d^-(v)$, where $d^-(v)$ is the in-degree of v in H (since K_n has at most $(n+1)^k$ members), and so the ith step in total takes time $O(n^k|E(H)|)$. Thus the running time is $O(n^k|V(H)||E(H)|)$. ∎

Finally, we can give the main algorithm, 1.4, which we restate.

4.2 *For all $d, k \geq 1$, there is an algorithm as follows:*

- **Input:** *A problem instance $(G, s_1, t_1, \ldots, s_k, t_k)$ where G is d-path-dominant.*
- **Output:** *The set of all key qualities of $(G, s_1, t_1, \ldots, s_k, t_k)$.*
- **Running time:** *$O(n^t)$ where $t = 6k^2d(k + d) + 13k$.*

Proof. Here is the algorithm.

- Compute the (k,m,c)-tracker H, where

$$\begin{aligned} c &= ((k-1)d + k^2 + 2)(2k+1)k^2 \\ m &= ((k-1)d + k^2 + 2)d + 1. \end{aligned}$$

- For each edge $e = uv$ of H, define $l(e)$ as follows:
  - if $u = s_0$ and $v = (L, X, Y)$ where $L = (M_1, \ldots, M_k)$, let $l(e) = (|V(M_1)|, \ldots, |V(M_k)|)$;
  - if $u = (L, X, Y)$ where $L = (M_1, \ldots, M_k)$, and $v = (L', X', Y')$ where $L' = (M'_1, \ldots, M'_k)$, let $l(e) = (|V(M'_1) \setminus V(M_1)|, \ldots, |V(M'_k) \setminus V(M_k)|)$;
  - if $v = t_0$ let $l(e) = (0, \ldots, 0)$.
- Run the algorithm of 4.1 with input H, s_0, t_0, l.
- Output its output.

To see its correctness, we must check that every key quality is in the output, and everything in the output is a key quality. We show first that every vector in the output is a quality. For let x be in the output, and let P be a path in H from s_0 to t_0 with $l(P) = x$. By 3.2, P traces some linkage $L = (P_1, \ldots, P_k)$ for $(G, s_1, t_1, \ldots, s_k, t_k)$; and so $(|V(P_1)|, |V(P_2)|, \ldots, |V(P_k)|) = l(P) = x$. Hence x is a quality.

Next, we show that every key quality is in the output. For let x be a key quality. Let L be an x-linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$. By 3.3, there is a path P of H from s_0 to t_0 tracing L; and hence $l(P) = x$ (where $l(P)$ is defined as in the statement of 4.1). Thus the output of 4.1 contains a vector dominated by x. But x does not dominate any other quality, since it is key; and since every member of the output is a quality, it follows that x belongs to the output.

Third, we show that every member of the output is key. For let x be in the output, and suppose it is not key. Hence x dominates some other quality, and hence dominates some other key quality y say. Consequently y is in the output. But no two members of the output dominate one another, a contradiction. This proves that every member of the output is key, and so completes the proof that the output of the algorithm is as claimed.

Finally, for the running time: by 3.1, we can find all (k,m,c)-rails in time $O(n^{2km+1})$; and since there are at most $O(n^{2km})$ of them (by 3.1), we can compute H and the function l in time $O(n^{4km})$. Then running 4.1 takes time $O(n^k |V(H)|^3)$, and hence time at most $O(n^{6km+k})$. Thus the total running time is $O(n^{6km+k})$. Since $m = ((k-1)d + k^2 + 2)d + 1$, the running time is $O(n^t)$ where

$$t = 6k(k-1)d^2 + 6k(k^2+2)d + 7k = 6k^2d^2 + 6k^3d + 12kd + 7k - 6kd^2 \leq 6k^2d(k+d) + 13k$$

as claimed. This proves 4.2. ∎

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