# The vertex sets of subtrees of a tree


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**Abstract**

Let $\mathcal{S}$ be a set of subsets of a set $W$. When is there a tree $T$ with vertex set $W$ such that each member of $\mathcal{S}$ is the set of vertices of a subtree of $T$? It is necessary that $\mathcal{S}$ has the Helly property and the intersection graph of $\mathcal{S}$ is chordal. We will show that these two necessary conditions are together sufficient in the finite case, and more generally, they are sufficient if no element of $W$ belongs to infinitely many infinite sets in $\mathcal{S}$.

# 1 Introduction

Graphs in this paper have no loops or parallel edges. A graph is *chordal* if all its induced cycles have length three, and L. Surányi (see page 584 of [3]) and others [1, 2, 5] showed that a finite graph is chordal if and only if it is the intersection graph of a set of subtrees of a tree. (This is not always true for infinite graphs, as Halin [4] showed.)

Here is a similar but different question: let $\mathcal{S}$ be a set of subsets of a set $W$. When is there a tree $T$ with vertex set $W$ such that each member of $\mathcal{S}$ is the set of vertices of a subtree of $T$? There are two natural necessary conditions:

- **(The chordal property)** The intersection graph of $\mathcal{S}$ is chordal.
- **(The finite Helly property)** For all $S_1, \dots, S_k \in \mathcal{S}$ with $k$ finite, if $S_i \cap S_j \neq \emptyset$ for $1 \leq i < j \leq k$, then $S_1 \cap \cdots \cap S_k \neq \emptyset$.

We will prove that these two conditions are sufficient if $W$ is finite. If $W$ is infinite they are not always sufficient, but they are sufficient if no element of $W$ belongs to infinitely many infinite members of $\mathcal{S}$.

Let us see first that they are not always sufficient:

**1.1** *There is a set $\mathcal{S}$ of subsets of a set $W$, such that $\mathcal{S}$ has the chordal property and the finite Helly property, and yet there is no tree $T$ with vertex set $W$ such that each member of $\mathcal{S}$ is the vertex set of a subtree of $T$.*

**Proof.** Let $W$ be the set of non-negative integers, and let

$$\mathcal{S} = \{\{i, i+1\}; i \geq 1\} \cup \{\{0, i, i+1, i+2...\} : i \geq 1\}.$$

This satisfies the two necessary conditions, but there is no corresponding tree. To see the latter, suppose that $T$ is a tree with $V(T) = W$ and all members of $\mathcal{S}$ are vertex set of subtrees. In particular, $(i, i+1)$ is an edge of $T$ for each $i \geq 1$, so there is only one more edge in $T$, and it is incident with 0; and for each $i \geq 1$. making $(0, i)$ an edge does not work, because $\{0, i+1, i+2...\}$ is supposed to be the vertex set of a subtree. This proves 1.1. ∎

The set of 1.1 was derived from an example of Halin [4]. He gave a chordal graph $G$ that was not expressible as the intersection graph of a set of subtrees of a tree. Let $G$ be any such graph, let $W$ be the set of all maximal cliques of $G$ (a *clique* of a graph is a set of pairwise adjacent vertices, not necessarily maximal), and for each $v \in V(G)$, let $S_v$ be the set of all members of $W$ that contain $v$. Then $\mathcal{S} = \{S_v : v \in V(G)\}$ satisfies 1.1, as can easily be checked.

We observe that, in the example given above, the element 0 belongs to infinitely many infinite members of $\mathcal{S}$. Our main result is:

**1.2** *Let $\mathcal{S}$ be a set of subsets of a set $W$, such that $\mathcal{S}$ has the chordal property and the finite Helly property, and such that no member of $W$ belongs to infinitely many infinite members of $\mathcal{S}$. Then there is a tree $T$ with vertex set $W$ such that each member of $\mathcal{S}$ is the vertex set of a subtree of $T$.*

**Proof.** If $W \notin \mathcal{S}$, we could replace $\mathcal{S}$ by $\mathcal{S} \cup \{W\}$, and the chordal property and the finite Helly property would still hold; so we may assume that $W \in \mathcal{S}$. Let us say a *fleet* is a set of subsets of a set $W$, such that

- $\mathcal{S}$ has the chordal property and the finite Helly property;
- each member of $W$ belongs to only finitely many infinite members of $\mathcal{S}$; and
- $W \in \mathcal{S}$.

We call the members of a fleet $\mathcal{S}$ the *ships* of $\mathcal{S}$. Note that every ship $S$ is nonempty, by the finite Helly property applied to $\{S\}$ . A ship of cardinality two is called an *edge-ship*. A fleet $\mathcal{S}'$ is an *extension* of a fleet $\mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{S}'$, and $\mathcal{S}' \setminus \mathcal{S}$ contains only edge-ships. A fleet $\mathcal{S}$ is *maximal* if no extension of $\mathcal{S}$ is different from $\mathcal{S}$.

(1) *For every fleet $\mathcal{S}$, there is an extension of $\mathcal{S}$ that is maximal.*

Let $I$ be a set that is linearly ordered by some relation $<$, and for each $i \in I$ let $\mathcal{S}_i$ be a fleet, such that for all distinct $i, j \in I$, if $i < j$ then $\mathcal{S}_j$ is an extension of $\mathcal{S}_i$. Let $\mathcal{S} = \bigcup_{i \in I} \mathcal{S}_i$. We claim that $\mathcal{S}$ is a fleet. For every finite set $\mathcal{R}$ of ships of $\mathcal{S}$ that pairwise intersect, there exists $i \in I$ such that $\mathcal{R} \subseteq \mathcal{S}_i$ (since $\mathcal{R}$ is finite), and since $\mathcal{S}_i$ is a fleet, there is a vertex that belongs to every member of $\mathcal{R}$; and so $\mathcal{S}$ has the finite Helly property. Similarly it has the chordal property trivially all its ships of size different from two belong to each $\mathcal{S}_i$. This proves that $\mathcal{S}$ is a fleet extending each $\mathcal{S}_i$ ($i \in I$). From Zorn's lemma, this proves (1).

Consequently, to prove the theorem it suffices to prove it for maximal fleets, so we may assume that $\mathcal{S}$ is maximal. A nonempty subset $X \subseteq W$ is *disconnected* if the set of edge-ships included in $X$ is the edge set of a disconnected graph with vertex set $X$.

(2) *If $W$ is not disconnected then the theorem holds.*

If $W$ is not disconnected, there is a tree $T$ with vertex set $W$ such that all its edges are edge-ships of $\mathcal{S}$. Suppose that there is a ship $S$ that is not the vertex set of a tree in $T$. Hence there are at least two components of $T[S]$; choose a minimal path $P$ of $T$ that joins two vertices in different components of $T[S]$. Thus $P$ has length at least two, since the ends of $P$ are in different components of $T[S]$. Let $P$ have vertices $p_1$-$\cdots$-$p_k$ in order; thus, $k \geq 3$, and $p_1, p_k \in S$, and $p_2, \ldots, p_{k-1} \notin S$. If $k = 3$ then the three ships $\{p_1, p_2\}, \{p_2, p_3\}, S$ pairwise intersect and yet have no common vertex, contradicting that $\mathcal{S}$ has the finite Helly property; and if $k \geq 4$ then the intersection graph of the set of ships $E(P) \cup \{S\}$ is a cycle of length at least four, contradicting the chordal property. Thus every ship $S$ induces a tree in $T$, and so the theorem holds. This proves (2).

Let us say a nonempty subset of $W$ expressible as an intersection of finitely many ships is a *meeting*. (Actually, every intersection of ships is the intersection of finitely many ships, but we do not need that.) If no meeting is disconnected, then the theorem holds by (2), since $W$ is a meeting; so let us suppose that there is a disconnected meeting. Since every vertex belongs to only finitely many infinite ships, there is no infinite sequence of meetings such that each is a proper subset of its predecessor. It follows that there is a disconnected meeting $A$ such that no proper subset is a disconnected meeting. Let $F$ be the graph with vertex set $A$ and edge set all edge-ships included in $A$. Thus $F$ is not connected; let the vertex sets of its components be $\{F_d : d \in D\}$. Let $\mathcal{R}$ be the set of all ships that do not include $A$ as a subset.

(4) *There is no sequence of ships $R_1, \dots, R_k \in \mathcal{R}$ such that for some distinct $d, d' \in D$, $R_1 \cap F_d \neq \emptyset$ and $R_k \cap F_{d'} \neq \emptyset$, and $R_i \cap R_{i+1} \neq \emptyset$ for $1 \leq i < k$.*

Suppose there is such a sequence $R_1, \dots, R_k$, and choose one with $k$ minimum. Choose finitely many ships $S_1, \dots, S_\ell$ such that $S_1 \cap \dots \cap S_\ell = A$. If $k = 1$, then $R_1 \cap A$ is a disconnected meeting, and is a proper subset of $A$, a contradiction. If $k = 2$, then from the finite Helly property, since every two of the ships $S_1, \dots, S_\ell, R_1, R_2$ have nonempty intersection, they have a common member $c$ say. So $c \in A \cap R_1 \cap R_2$, and so one of $R_1, R_2$ meets two of the sets $F_d$ ($d \in D$), contrary to the minimality of $k$. Thus $k \geq 3$, and $R_i \cap R_j = \emptyset$ for $1 \leq i, j \leq k$ with $j > i + 1$; and $R_2, \dots, R_{k-1}$ are all disjoint from $A$ (because otherwise we could reduce $k$). If each of $S_1, \dots, S_\ell$ has nonempty intersection with $R_2$, then $S_1 \cap \dots \cap S_\ell \cap R_2$ is nonempty by the finite Helly property, contradicting that $A \cap R_2 = \emptyset$. So we assume that $S_1 \cap R_2 = \emptyset$. Since $S_1 \cap R_k \supseteq A \cap R_k \neq \emptyset$, we may choose $j \in \{3, \dots, k\}$ minimum such that $S_1 \cap R_j \neq \emptyset$. Thus $S_1$ has nonempty intersection with $R_1, R_j$ and is disjoint from $R_2, \dots, R_{j-1}$. Since $j \geq 3$, the intersection graph of the set of ships $\{R_1, \dots, R_j, S_1\}$ is a cycle of length at least four, contradicting that $\mathcal{S}$ has the chordal property. This proves (3).

Choose distinct $d, d' \in D$ and $a \in F_d$ and $b \in F_{d'}$. Since $\{a, b\}$ is not an edge-ship, the maximality of $\mathcal{S}$ tells us that $\mathcal{S} \cup \{\{a, b\}\}$ is not a fleet, and therefore violates either the chordal property or the finite Helly property.

Suppose that $\mathcal{S} \cup \{\{a, b\}\}$ does not have the finite Helly property. Thus there are finitely many ships $P_1, \dots, P_m$, pairwise with nonempty intersection, and each containing one or both of $a, b$, such that neither of $a, b$ belongs to all of $P_1, \dots, P_m$. We may assume that $a \notin P_1$ and $b \notin P_2$, and so $P_1, P_2 \in \mathcal{R}$. Since $P_1 \cap P_2 \neq \emptyset$, this contradicts (3).

Thus $\mathcal{S} \cup \{\{a, b\}\}$ does not have the chordal property; and so there are finitely many ships $P_1, \dots, P_m$ such that the intersection graph of $\{P_1, \dots, P_m, \{a, b\}\}$ is a cycle of length at least four. Consequently each of $P_1, \dots, P_m$ contains at most one of $a, b$, and so $P_1, \dots, P_m \in \mathcal{R}$, contrary to (3). This contradiction show that there is no disconnected meeting, and so the theorem holds by (2). This proves 1.2. ∎

Finally, we obtain a slight strengthening of the theorem of Halin [4]. He proved that if $G$ is a chordal graph with no infinite clique, then $G$ is the intersection graph of a set of subtrees of a tree. We will prove:

**1.3** *Let $G$ be a chordal graph, such that no clique has infinitely many vertices that each belong to infinitely many maximal cliques. Then $G$ is the intersection graph of a set of subtrees of a tree.*

**Proof.** Let $W$ be the set of all maximal cliques of $G$, and for each $v \in V(G)$, let $S_v$ be the set of all maximal cliques that contain $v$. From the hypothesis, $\mathcal{S} = \{S_v : (v \in V(G))\}$ satisfies the hypotheses of 1.2, and so there is a tree $T$ with vertex set $W$ such that each member of $\mathcal{S}$ is the vertex set of a subtree of $T$. But then $G$ is the intersection graph of this set of trees. This proves 1.3. ∎

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