Excluding a substar and an antisubstar

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Abstract

Ramsey's theorem says that for every clique H_1 and for every graph H_2 with no edges, all graphs containing neither of H_1, H_2 as induced subgraphs have bounded order. What if, instead, we exclude a graph H_1 with a vertex whose deletion gives a clique, and the complement H_2 of another such graph? This no longer implies bounded order, but it implies tightly restricted structure that we describe. There are also several related subproblems (what if we exclude a star and the complement of a star? what if we exclude a star and a clique? and so on) and we answer a selection of these.

1 Introduction

One of the nice features of minor containment is that there are theorems describing the structure of the graphs that do not contain as a minor some graph of a given type; and they are necessary and sufficient in the sense that the structure is implied by excluding a graph of this type, and implies the exclusion of another (bigger) graph of this type. We have in mind the theorem [3] that for every planar graph H, there exists k such that all graphs not containing H as a minor have tree-width at most k; and conversely, for every k there is a planar graph H such that all graphs of tree-width at most k do not contain H as a minor. There are many other theorems of the same kind for minor containment. (This paper is not about minors or tree-width, so we omit their definitions; the discussion here is just for motivation.)

In this paper we are concerned with induced subgraph containment; and we look for theorems, again necessary and sufficient for excluding a graph of a given type. Surprisingly, there are virtually no such results at all. The most obvious places to look are, presumably:

- What structure is equivalent to excluding a clique? Here nothing non-trivial is known, and perhaps nothing non-trivial can be said.
- What structure is equivalent to excluding a star? (A *star* is a complete bipartite graph $K_{s,t}$ for some s, t with $s \leq 1$.) Here at least there is a chance of some non-trivial result "explaining" the graphs without big stars in terms of graphs without big stable sets, but it is not known.
- What structure is equivalent to excluding a graph with one edge? This is equivalent to saying that every two maximal stable sets have bounded intersection, but that does not count as a "structure", and we do not see how to turn it into one.

Things go much better if we exclude a pair of graphs instead of just one. For instance, Ramsey's theorem says that for every clique H_1 and every anticlique H_2, there exists k such that every graph containing neither of H_1, H_2 as an induced subgraph has at most k vertices; and the converse is trivial, that for every k there is a clique H_1 and an anticlique H_2 such that every graph with at most k vertices does not contain H_1 or H_2. (An *anticlique* is a graph with no edges.) Thus, this is analogous to the tree-width result mentioned above, except that we exclude two graphs instead of one.

Before we go on, let us clarify what we mean by "analogous to the tree-width result". We need to make more precise the statement "necessary and sufficient for excluding a graph of a given type". The theorem says that for every planar H there exists k such that every graph not containing H as a minor has tree-width at most k, so it says something about the graphs not containing a given planar graph H. It is not necessary and sufficient for the exclusion of H, yet it is necessary and sufficient in some sense.

Let us say a class of graphs is an *ideal* if it is closed under an appropriate containment relation and isomorphism. Thus, the class of all planar graphs is a minor ideal. A minor ideal might have bounded tree-width (meaning that there exists k such that all its members have tree-width at most k) or it might not; and the property of having bounded tree-width is closed under taking subideals. We can therefore ask for the excluded subideals for this property; the minimal ideals (under subideal containment) that do not have bounded tree-width. The tree-width theorem says there is just one:

1.1 *A minor ideal has bounded tree-width if and only if it does not include the ideal of all planar graphs.*

Thus, the tree-width theorem can be regarded, perhaps most naturally, not as an excluded minor theorem, but as an excluded minor ideal theorem. Ramsey's theorem too is an excluded ideal theorem:

1.2 *An induced subgraph ideal has bounded order (meaning there is a bound on the order of its members) if and only if it includes neither of the ideal of all cliques and the ideal of all anticliques.*

Thus, here there are two excluded ideals instead of one in 1.1; but they are still uniquely determined by the property we wish to characterize by excluded subideals. Expressing results in the language of ideals sometimes helps clarify what is going on, but is a little cumbersome, so in what follows we only resort to it when it helps.

Let us look for other structure theorems for induced subgraph containment, analogous to the tree-width theorem, but now excluding pairs of graphs. Again, there are very few already known, but there are some to be discovered, and that is the topic of this paper. First, here is a result of this type that easily follows by applying 1.2 to the neighbourhoods:

1.3 *For every clique H_1 and every star H_2, there exists k such that every graph containing neither of H_1, H_2 as an induced subgraph has maximum degree at most k. Conversely, for every integer k, there is a clique H_1 and a star H_2 such that every graph with maximum degree at most k contains neither of H_1, H_2.*

Thus, an induced subgraph ideal has a bound on the maximum degree of its members if and only if it includes neither of the ideal of all cliques and the ideal of all stars.

There is another theorem of this type recently proved by two of us, in [1], as follows. Let us say a graph G is *k-split*, where $k \geq 1$, if its vertex set is the union of two sets A, B, where $G|A$ has clique number at most k and $G|B$ has stability number at most k. A *multiclique* is a graph such that each component is a clique, and a *complete multipartite graph* is the complement of a multiclique.

1.4 *For every multiclique H_1 and every complete multipartite graph H_2, there exists k such that every graph containing neither of H_1, H_2 as an induced subgraph is k-split. Conversely, for every integer k, there is a multiclique H_1 and a complete multipartite graph H_2 such that every k-split graph contains neither of H_1, H_2.*

Thus, an induced subgraph ideal has bounded "splitness" if and only if it includes neither of the ideal of all multicliques and the ideal of all complete multipartite graphs. (See [2] for some related discussion.)

In this paper we present some relatives of 1.3. We study the structure that results from excluding a "substar" (a graph such that, if it has a vertex, then it has a vertex incident with all edges) and the complement of one; and also the structure equivalent to excluding a substar of various types, and the complement of one. (There are many possible combinations here, and we just did those that seemed to us most interesting.)

All the proofs of this paper could easily be turned into polynomial-time algorithms, to find either one of the two excluded subgraphs or the structural decomposition, but we omit those details.

2 Layouts

All graphs in this paper are finite, and have no loops or parallel edges. Let us say a graph H is a *dash* if it has at most one edge. H is an *antidash* or *antistar* or *antisubstar* if its complement $\overline{H}$ is a dash or star or substar respectively. Thus a graph with a vertex is a substar if deleting some vertex gives an anticlique. $\Delta(G)$ denotes the maximum degree of the vertices of G. (In what follows, if X is a clique or anticlique, we shall sometimes regard it as a graph, and sometimes as a set of vertices, and hope this causes no confusion.)

Let G be a graph, and let $c_1, c_2, c_3 \geq 0$ be integers or ∞. Let $\mathcal{F}$ be a set of subsets of $V(G)$, with the following properties:

- $|A \cap B| \leq c_1$ for all distinct $A, B \in \mathcal{F}$,
- $\Delta(G|A) \leq c_2$, for each $A \in \mathcal{F}$, and
- for each $v \in V(G)$, there are at most c_3 vertices u in G such that u, v are non-adjacent and there is no $A \in \mathcal{F}$ containing both u, v.

We call such a set $\mathcal{F}$ a (c_1, c_2, c_3)-*layout*, and we say G *admits* $\mathcal{F}$. Thus G admits a $(0,0,0)$-layout if and only if it is complete multipartite. The graphs that admit a (c_1, c_2, c_3)-layout for fixed c_1, c_2, c_3 are thus "close" to being complete multipartite, and it may be helpful to view our results in that light. In the theorems that follow, the structures will involve a (c_1, c_2, c_3)-layout for some choice of c_1, c_2, c_3, sometimes with additional restrictions, such as:

- $V(G) = \cup_{A \in \mathcal{F}} A$; we call this a *covering layout*
- each $v \in V(G)$ belongs to at most c_4 members of $\mathcal{F}$; we call this a *layout with degree at most* c_4
- $|\mathcal{F}| \leq c_5$.

We begin by proving that certain kinds of substars and antisubstars do not admit certain kinds of layouts. (Note that if G admits a (c_1, c_2, c_3)-layout then so does every induced subgraph of G.)

2.1 *For all integers $c \geq 0$, if H is a clique with at least $c^2 + c + 1$ vertices, then H does not admit a covering (∞, c, ∞)-layout with cardinality at most c.*

Proof. If H admits a covering (∞, c, ∞)-layout $\mathcal{F}$ of cardinality at most c, then since $|V(H)| \geq c^2 + c + 1$ and $|\mathcal{F}| \leq c$, some $A \in \mathcal{F}$ contains at least $c + 2$ vertices of H, which is impossible since $\Delta(H|A) \leq c$. Thus H admits no such layout. This proves 2.1. ∎

2.2 *For all integers $c \geq 0$, if H is a dash with at least $c^3 + 2c + 3$ vertices and with $|E(H)| = 1$, then H does not admit a $(c, 0, c)$-layout with degree at most c.*

Proof. Suppose that H admits a $(c, 0, c)$-layout $\mathcal{F}$ with degree at most c. Let uv be the unique edge of H, and let $X = V(H) \setminus \{u, v\}$. Now for each $A \in \mathcal{F}$ containing u, and each $B \in \mathcal{F}$ containing v, it follows that $v \notin A$ since A is an anticlique, and so $A \neq B$, and therefore $|A \cap B| \leq c$. In particular $|X \cap A \cap B| \leq c$. Since this holds for all choices of A and B, and since there are at most c choices of A containing u and at most c choices of B containing v, it follows that at most c^3 members of

X belong to a member of $\mathcal{F}$ containing u and to a member of $\mathcal{F}$ containing v. But by hypothesis, since each vertex in X is non-adjacent to u, at most c such vertices are not contained in members of $\mathcal{F}$ containing u, and the same for v. We deduce that $|X| \le c^3 + 2c$, a contradiction. This proves 2.2. ∎

2.3 *For all integers $c \ge 0$, if H is an antistar with at least $(c+1)^2+1$ vertices that is not a clique, then H does not admit a (c,c,c)-layout with degree at most c.*

Proof. Suppose that H admits a (c,c,c)-layout $\mathcal{F}$ with degree at most c. Let v be the vertex of H with degree zero, and let X be the set of vertices u different from v such that some member of $\mathcal{F}$ contains both u, v. If $A \in \mathcal{F}$ with $v \in A$, then since $A \setminus \{v\}$ is a clique and $\Delta(H|A) \le c$, it follows that $|A \setminus \{v\}| \le c+1$; and since v is contained in at most c members of $\mathcal{F}$, it follows that $|X| \le c(c+1)$. But since v is non-adjacent to every other vertex of H, there are at most c vertices not in $X \cup \{v\}$; and so $|V(H)| \le (c+1)^2$, a contradiction. This proves 2.3. ∎

2.4 *For all integers $c \ge 1$, if H is a substar in which some vertex v has at least c^2+1 neighbours and at least c^3+2c+1 non-neighbours, then H does not admit a (c,c,c)-layout with degree at most c.*

Proof. Since v has degree at least two and H is a substar, it follows that v is incident with all edges of H. Suppose that H admits a (c,c,c)-layout $\mathcal{F}$ with degree at most c. Let N be the set of vertices of $H \setminus v$ adjacent to v, and let M be the set that are non-adjacent to v. Let $X(v)$ be the set of vertices u different from v such that some member of $\mathcal{F}$ contains both u, v. If $A \in \mathcal{F}$ with $v \in A$, then since $\Delta(H|A) \le c$, it follows that $|N \cap A| \le c$, and since v belongs to at most c members of $\mathcal{F}$, we deduce that $|N \cap X(v)| \le c^2$. Hence there exists $w \in N \setminus X(v)$. Let $X(w)$ be the set of vertices u different from w such that some member of $\mathcal{F}$ contains both u, w. Now for each $A \in \mathcal{F}$ containing v, and each $B \in \mathcal{F}$ containing w, since $A \ne B$ (because $w \notin X(v)$) it follows that $|A \cap B| \le c$; and since v, w are each contained in at most c members of $\mathcal{F}$, we deduce that $|X(v) \cap X(w)| \le c^3$. But since v, w are both non-adjacent to all members of M, it follows that $|M \setminus X(v)| \le c$, and $|M \setminus X(w)| \le c$; and so $|M| \le c^3 + 2c$, a contradiction. This proves 2.4. ∎

3 The main result

The main result of this paper is a theorem describing the structure implied by excluding a general antisubstar and a general substar; but before that we prove several other results describing the structure implied by excluding some special kinds of antisubstar and some special kinds of substar. Some of these results are needed for lemmas towards the main theorem, and others are included for their own sake. We will study the following combinations, in theorems 3.1, 3.3, 3.4, 3.5, 3.7, 3.8, and 3.9 respectively:

- antistar and star
- clique and dash

- antidash and dash
- antistar and dash
- clique and substar
- antistar and substar
- antisubstar and substar.

The first of these is easy, and does not need layouts:

3.1 *For every antistar H_1 and star H_2 there exists $c \geq 0$ such that for every graph G containing neither of H_1, H_2 as an induced subgraph, either $\Delta(G) \leq c$ or $\Delta(\overline{G}) \leq c$. Conversely, for every integer $c \geq 0$, there is an antistar H_1 and a star H_2 such that every graph G with either $\Delta(G) \leq c$ or $\Delta(\overline{G}) \leq c$ contains neither of H_1, H_2.*

Proof. For the first assertion, by enlarging H_1, H_2, we may assume that $\overline{H_1}, H_2$ are both isomorphic to some star H with $h \geq 1$ vertices, not an anticlique. Suppose that G contains a clique A with $|A| = 2h$, and an anticlique B with $|B| = 2h$, with $A \cap B = \emptyset$. If there are at least $2h^2$ edges between A and B, then some vertex in A has at least h neighbours in B, and G contains H, a contradiction. Otherwise some vertex in B has at least h non-neighbours in A, and G contains $\overline{H}$, a contradiction. It follows that either G contains no clique of cardinality $2h + 1$, or G contains no anticlique of cardinality $2h + 1$, and in either case the result follows from 1.3. For the second assertion, observe that for all c, every sufficiently large star H satisfies $\Delta(H), \Delta(\overline{H}) > c$. This proves 3.1. ∎

We need the following lemma.

3.2 *Let $d, t \geq 0$ be integers, let G be a graph, and let $A_1, \ldots, A_t \subseteq V(G)$, such that*

- *$|A_i| > d(t-1)$ for $1 \leq i \leq t$,*
- *$|A_i \cap A_j| \leq d$ for $1 \leq i < j \leq t$, and*
- *each vertex in $A_i \setminus A_j$ has at most d non-neighbours in A_j for $1 \leq i < j \leq t$.*

Then there is a clique with cardinality t included in $A_1 \cup \cdots \cup A_t$.

Proof. Choose $j \leq t$ maximal such that there exist $v_i \in A_i$ for $1 \leq i \leq j$, distinct and pairwise adjacent, where each v_i belongs to none of $A_{i+1}, \ldots, A_t$. Suppose that $j < t$. For $1 \leq i \leq j$ there are at most d vertices in A_{j+1} that are non-adjacent to v_i, since $v_i \notin A_{j+1}$; and for $j + 1 < k \leq t$ there are at most d vertices of A_{j+1} that belong to A_k; and so there are at least

$$|A_{j+1}| - jd - (t - j - 1)d > 0$$

vertices in A_{j+1} that are adjacent to all of $v_1, \ldots, v_j$ and belong to none of $A_{j+2}, \ldots, A_t$, contrary to the maximality of j. Consequently $j = t$. This proves 3.2. ∎

The *Ramsey number* $R(s,t)$ is the least integer n such that every graph with at least n vertices has either a clique of cardinality s or an anticlique of cardinality t.

3.3 *For every clique H_1 and dash H_2, there is an integer $c \geq 0$ such that every graph containing neither of H_1, H_2 as an induced subgraph admits a covering $(c,0,c)$-layout of cardinality at most c. Conversely, for all integers $c \geq 0$, there is a clique H_1 and a dash H_2 such that every graph admitting a covering $(c,0,c)$-layout of cardinality at most c contains neither of H_1, H_2.*

Proof. For the first statement, we may assume that H_1, H_2 both have $h \geq 1$ vertices, and $|E(H_2)| = 1$. Let $m = R(h, h^2)$, and let $c = h + m$. We claim c satisfies the theorem.

For suppose that G contains neither of H_1, H_2. Let $\mathcal{F}_0$ be the set of all maximal anticliques of cardinality at least h^2.

(1) *If $A \in \mathcal{F}_0$, and $v \in V(G) \setminus A$, then v has at most $h-3$ non-neighbours in A. Consequently, every two members of $\mathcal{F}_0$ have at most $h-3$ vertices in common.*

Since A is maximal, v has a neighbour in A, and if it also has $h-2$ non-neighbours, then G contains H_2, a contradiction. This proves (1).

(2) $|\mathcal{F}_0| \leq h - 1$.

Suppose that $A_1, \dots, A_h \in \mathcal{F}_0$ are distinct. By 3.2 (with d, t replaced by $h-3, h$), since each A_i has cardinality more than $(h-3)(h-1)$, it follows that G contains H_1, a contradiction. This proves (2).

Let Z be the set of vertices of G not in any member of $\mathcal{F}_0$.

(3) $|Z| \leq m$; *and for every vertex v, there are at most m vertices u in G such that u, v are non-adjacent and there is no $A \in \mathcal{F}_0$ containing both u, v.*

For $G|Z$ has no clique of cardinality h and no anticlique of cardinality h^2 (since the latter would be a subset of a member of $\mathcal{F}_0$), so $|Z| < m$. This proves the first assertion. For the second, let $v \in V(G)$, and let X be the set of vertices u such that u, v are non-adjacent and there is no $A \in \mathcal{F}$, containing both u, v. It follows that X has no anticlique of cardinality $h^2 - 1$ (because then adding v would give an anticlique of cardinality h^2 containing v, and that could be extended to a member of $\mathcal{F}_0$), and no clique of cardinality h, and so $|X| < m$. This proves (3).

Let $\mathcal{F}$ be the union of $\mathcal{F}_0$ and the set $\{\{v\} : v \in Z\}$. Then by (2) and (3), $|\mathcal{F}| \leq h + m$. Thus $\mathcal{F}$ is a covering $(c,0,c)$-layout of cardinality at most c. This proves the first assertion of the theorem. The second follows from 2.1 and 2.2. This proves 3.3. ∎

3.4 *For every antidash H_1 and dash H_2, there is an integer $c \geq 0$ such that for every graph G containing neither of H_1, H_2 as an induced subgraph, one of $G, \overline{G}$ admits a covering $(c,0,c)$-layout of cardinality at most c. Conversely, for all integers $c \geq 0$, there is an antidash H_1 and a dash H_2 with the following property: every graph G such that G or $\overline{G}$ admits a covering $(c,0,c)$-layout of cardinality at most c contains neither of H_1, H_2.*

Proof. We may assume that $\overline{H_1}, H_2$ both have $h \geq 3$ vertices, and H_1 has a non-edge, and H_2 has an edge. Suppose that G has a clique A and an anticlique B, disjoint and both with $2h$ vertices. There is at most one vertex in A with no neighbour in B (because it there were two, they would be adjacent, and together with an appropriate subset of B would form a copy of H_2); and every vertex in A with a neighbour in B has at most $h-3$ non-neighbours in B (because G does not contain H_2). It follows that there are at least $(2h-1)(h+3) > 2h^2$ edges between A and B. But similarly there are more than $2h^2$ edges of $\overline{G}$ between A and B, a contradiction. It follows that either G contains no clique of cardinality $2h+1$, or no anticlique of cardinality $2h+1$, so the first assertion of the theorem follows from 3.3. The second follows from 2.1 and 2.2 (since a large antidash contains a large clique). ▮

3.5 *For every antistar H_1 and every dash H_2, there is an integer $c \geq 0$ such that every graph containing neither of H_1, H_2 as an induced subgraph admits a $(c, 0, c)$-layout with degree at most c. Conversely, for all integers $c \geq 0$, there is an antistar H_1 and a dash H_2 such that every graph admitting a $(c, 0, c)$-layout of degree at most c contains neither of H_1, H_2.*

Proof. We may assume that H_1, H_2 both have $h \geq 1$ vertices, and H_1 is not a clique (by 3.3), and H_2 has an edge (by 1.3 applied in the complement). Let $c = R(h, h^2)$. Now let G be a graph containing neither of H_1, H_2. Let $\mathcal{F}$ be the set of all maximal anticliques of G with cardinality at least h^2. As in the proof of 3.3 we have

(1) *If $A \in \mathcal{F}$, and $v \in V(G) \setminus A$, then v has at most $h-3$ non-neighbours in A. Consequently, every two members of $\mathcal{F}$ have at most $h-3$ vertices in common.*

(2) *Every vertex belongs to at most $h-2$ members of $\mathcal{F}$.*

If $A_1, \ldots, A_{h-1}$ are distinct members of $\mathcal{F}$, all containing some vertex v, then by 3.2 (with t, d replaced by $h-1, h-3$), the union of the sets $A_1 \setminus \{v\}, \ldots, A_{h-1} \setminus \{v\}$ includes a clique with $h-1$ vertices, and so G contains H_1, a contradiction. This proves (2).

(3) *For every vertex v, there are at most c vertices u in G such that u, v are non-adjacent and there is no $A \in \mathcal{F}$ containing both u, v.*

Let $v \in V(G)$, and let X be the set of vertices u such that u, v are non-adjacent and there is no $A \in \mathcal{F}$ containing both u, v. It follows that X has no anticlique of cardinality $h^2 - 1$ (because then adding v would give an anticlique of cardinality h^2 containing v, and that could be extended to a member of $\mathcal{F}$), and no clique of cardinality h, and so $|X| < c$. This proves (3).

From (1), (2) and (3) it follows that $\mathcal{F}$ is a $(c, 0, c)$-layout of degree at most c. This proves the first assertion of the theorem, and the second follows from 2.3 and 2.2. This proves 3.5. ▮

The proofs of 3.7 and 3.8 have a great deal in common, and we have extracted the common part in the following lemma.

3.6 *Let H_1 be an antistar with h vertices, and let H_2 be a substar with h vertices. Let $m = R(h,h)$ and $k = h(h+2)m$. Let G be a graph containing neither of H_1, H_2, and let $\mathcal{F}$ be the set of all maximal subsets $A \subseteq V(G)$ such that $|A| \geq k$ and $\Delta(G|A) \leq m$. Then the following hold:*

- *If $A \in \mathcal{F}$ then $\Delta(G|A) < m$.*
- *If $A \in \mathcal{F}$ and $v \in V(G) \setminus A$ then v has at most $(h+1)m$ non-neighbours in A.*
- *If $A, B \in \mathcal{F}$ are distinct, then $|A \cap B| \leq (h+2)m$.*
- *For every vertex v, there are fewer than $R(h,k)$ vertices u such that u, v are non-adjacent and no member of $\mathcal{F}$ contains u, v.*

Proof. For the first assertion, let $A \in \mathcal{F}$, and suppose that some $v \in A$ has exactly m neighbours in A. This set of neighbours includes a set X of cardinality h which is either a clique or an anticlique. Let Z be the set of all vertices in $A \setminus (X \cup \{v\})$ with no neighbour in $X \cup \{v\}$. At most $(h+1)m$ vertices in $A \setminus X$ have neighbours in $X \cup \{v\}$, and so $|Z| \geq m$. Consequently X is not a clique since G does not contain H_1. Now Z includes a set Y of cardinality h that is either a clique or an anticlique. If Y is a clique, then $G|(Y \cup \{v\})$ contains H_1, while if Y is an anticlique then $G|(X \cup Y \cup \{v\})$ contains H_2, in either case a contradiction. This proves the first assertion.

For the second, let $A \in \mathcal{F}$ and $v \in V(G) \setminus A$. From the maximality of A it follows that $\Delta(A \cup \{v\}) > m$, and from the first assertion, v has more than m neighbours in A. Consequently there is a clique or anticlique $X \subseteq A$ of cardinality h such that v is adjacent to every vertex in X. Let M be the set of non-neighbours of v in A, and suppose that $|M| \geq (h+1)m$. Each vertex in X has at most m neighbours in M, and so there is a subset $Z \subseteq M$ with $|Z| = m$ with no neighbours in $X \cup \{v\}$. Consequently X is not a clique, since G does not contain H_1. Now Z includes a clique or anticlique Y of cardinality h, and that gives a copy of H_1 (if Y is a clique) or of H_2 (if Y is an anticlique), a contradiction. This proves the second assertion.

For the third assertion, let $A, B \in \mathcal{F}$ be distinct. Let $v \in B \setminus A$. Since $\Delta(G|B) \leq m$, it follows that v has at most m neighbours in $A \cap B$; and by the second assertion, v has at most $(h+1)m$ non-neighbours in $A \cap B$. Consequently, $|A \cap B| \leq (h+2)m$. This proves the third assertion.

For the fourth, let $v \in V(G)$, and let Z be the set of all vertices u such that u, v are non-adjacent and no member of $\mathcal{F}$ contains u, v. Suppose that $|Z| \geq R(h,k)$; then Z includes either a clique of cardinality h or an anticlique of cardinality k. The first, with v, gives a copy of H_1; and the second, with v, extends to a member of $\mathcal{F}$ containing v and a member of Z, in either case a contradiction. This proves the fourth assertion, and hence completes the proof of 3.6. ∎

3.7 *For every clique H_1 and every substar H_2, there is an integer $c \geq 0$ such that every graph that contains neither of H_1, H_2 as an induced subgraph admits a covering (c,c,c)-layout of cardinality at most c. Conversely, for all integers $c \geq 0$, there is a clique H_1 and a substar H_2 such that every graph admitting a covering (c,c,c)-layout of cardinality at most c contains neither of H_1, H_2.*

Proof. We may assume that H_1, H_2 both have $h \geq 1$ vertices. Let $m = R(h,h)$, $k = h(h+2)m$, and $c = R(h,k) + h$. Now let G be a graph containing neither of H_1, H_2, and let $\mathcal{F}_0$ be the set of all maximal subsets $A \subseteq V(G)$ such that $|A| \geq k$ and $\Delta(G|A) \leq m$. Since H_1 is a clique and hence an antistar, we may apply 3.6; and consequently the four assertions of 3.6 hold (with $\mathcal{F}$ replaced by

$\mathcal{F}_0$.)

(1) $|\mathcal{F}_0| < h$.

If $A_1, \dots, A_h \in \mathcal{F}_0$ are distinct, then by 3.2 (with t, d replaced by $h, (h+2)m$) it follows that G contains H_1, a contradiction. This proves (1).

Let Z be the set of all vertices of G that are not in any member of $\mathcal{F}_0$.

(2) $|Z| < R(h, k)$.

For Z includes no clique with cardinality h, because G does not contain H_1, and Z includes no anticlique with cardinality k, because such an anticlique could be extended to a member of $\mathcal{F}_0$ containing a vertex of Z. This proves (2).

Let $\mathcal{F}$ be the union of $\mathcal{F}_0$ and $\{\{v\} : v \in Z\}$. From (1), (2) and 3.6 we deduce that $\mathcal{F}$ is a covering (c, c, c)-layout of cardinality at most c. This proves the first assertion of the theorem, and the second follows from 2.1 and 2.4. This proves 3.7. ∎

3.8 *For every antistar H_1 and substar H_2 there is an integer $c \geq 0$ such that every graph containing neither of H_1, H_2 admits a (c, c, c)-layout with degree at most c. Conversely, for all integers $c \geq 0$, there is an antistar H_1 and a substar H_2 such that every graph admitting a (c, c, c)-layout of degree at most c contains neither of H_1, H_2.*

Proof. We may assume that H_1, H_2 both have $h \geq 1$ vertices. Let $m = R(h, h)$, $k = h(h+2)m$, and $c = R(h, k)$. Now let G be a graph containing neither of H_1, H_2. Let $\mathcal{F}$ be the set of all maximal subsets $A \subseteq V(G)$ such that $|A| \geq k$ and $\Delta(G|A) \leq m$. By 3.6, the four assertions of 3.6 hold.

(1) *Every vertex v is contained in at most $h - 1$ members of $\mathcal{F}$.*

Suppose that $A_1, \dots, A_h$ are distinct members of $\mathcal{F}$, all containing some vertex v. For $1 \leq i \leq h$ let B_i be the set of vertices in $A_i \setminus \{v\}$ that are non-adjacent to v. Thus each B_i has cardinality at least $h(h+2)m - m > (h-1)(h+2)m$. By 3.2 (with t, d replaced by $h, (h+2)m$), it follows that $B_1 \cup \cdots \cup B_h$ includes a clique with h vertices, and hence G contains H_1, a contradiction. This proves (1).

From (1) and 3.6, $\mathcal{F}$ is a (c, c, c)-layout with degree at most c. This proves the first assertion of the theorem, and the second follows from 2.3 and 2.4. This proves 3.8. ∎

In the next result, the layout structure proved is not invariant under taking complements, which is unexpected since the hypothesis is invariant under taking complements. But we prove that one of $G, \overline{G}$ admits this layout structure, and so the conclusion of the theorem is invariant under complements, as it should be. This unexpected behaviour is also exhibited by 3.1.

3.9 *For every antisubstar H_1 and every substar H_2 there is an integer c such that for every graph G containing neither of H_1, H_2 as an induced subgraph, one of $G, \overline{G}$ (say G') admits a partition (P_1, P_2) of $V(G')$, and a covering (c,c,c)-layout $\mathcal{F}_1$ of $G'|P_1$ of cardinality at most c, and a (c,c,c)-layout $\mathcal{F}_2$ of $\overline{G'}|P_2$ with degree at most c, with the following properties:*

- *for each $v \in P_1$ and each $A \in \mathcal{F}_2$, either v has at most c neighbours in A or it has at most c non-neighbours in A;*
- *for each $v \in P_2$ and each $A \in \mathcal{F}_1$, either v has at most c neighbours in A or it has at most c non-neighbours in A; and*
- *for each $v \in P_1$, there exists $\mathcal{F}' \subseteq \mathcal{F}_2$ with $|\mathcal{F}'| \leq c$, such that every non-neighbour of v in P_2 belongs to $\cup_{A \in \mathcal{F}'} A$.*

Conversely, for all integers $c \geq 0$ there is an antisubstar H_1 and a substar H_2 such that every graph G for which one of $G, \overline{G}$ admits a partition (P_1, P_2) as above contains neither of H_1, H_2.

Proof. We may assume that $\overline{H_1}, H_2$ are equal, say both equal to a substar H with $h \geq 1$ vertices. Let $t = h(h+2)$, and let J be the graph consisting of the disjoint union of t cliques each of cardinality h. By 1.4, there exists k such that every graph containing neither of $J, \overline{J}$ as an induced subgraph is k-split, and we may choose $k \geq h$.

Let H_0 be the h-vertex antistar with a vertex of degree zero. Choose an integer $c' \geq 2$ such that

- 3.7 holds (with c replaced by c') for a $(k+1)$-vertex clique and the substar H, and
- 3.8 holds (with c replaced by c') for H_0 and H,

and let $c = c'(c'+1)h$. We claim that c satisfies the theorem. Let G be a graph containing neither of $H, \overline{H}$. We observe first:

(1) *If $A, B \subseteq V(G)$ are disjoint, and $\Delta(G|A) \leq d$, and $\Delta(\overline{G}|B) \leq d$, then every vertex in B has either at most $(2d+1)h$ neighbours in A, or at most $(2d+1)h$ non-neighbours in A. Also, every vertex in A has either at most $(2d+1)h$ neighbours in B, or at most $(2d+1)h$ non-neighbours in B.*

Let $v \in B$, and let X, Y be the sets of neighbours and non-neighbours of v in A, respectively. We assume that $|X|, |Y| \geq (2d+1)h$. Since each vertex in A has at most d neighbours in A, X is $(d+1)$-colourable, and so includes an anticlique X' with h vertices. Each vertex in X' has at most d neighbours in Y, and so there are at least $|Y| - dh \geq (d+1)h$ vertices in Y with no neighbours in X'; and by the same argument, this set includes an anticlique of cardinality h. But then G contains H, a contradiction. The second statement follows by taking complements. This proves (1).

If G contains neither of $J, \overline{J}$, then it is k-split by 1.4; let (P_1, P_2) be a partition of $V(G)$ where P_1 includes no clique of cardinality $k+1$ and P_2 includes no anticlique of cardinality $k+1$. By 3.7 applied to $G|P_1$, $G|P_1$ admits a covering (c',c',c')-layout of cardinality at most c'; and by 3.7 applied to $\overline{G}|P_2$, $\overline{G}|P_2$ admits a covering (c',c',c')-layout of cardinality at most c'. From (1), since $c \geq (2c'+1)h$, the result holds in this case.

Thus we may assume that G contains one of $J, \overline{J}$; and since the result we are proving is symmetric between $G, \overline{G}$, we may assume that G contains J, by replacing G by $\overline{G}$ if necessary. Thus, we assume that there are t disjoint cliques $Z_1, \ldots, Z_t$, each of cardinality h, with no edges between them. Let $Z = Z_1 \cup \cdots \cup Z_t$. Let P_1 be the set of vertices $v \in V(G) \setminus Z$ such that v has a non-neighbour in at most h of $Z_1, \ldots, Z_t$, and let $P_2 = V(G) \setminus P_1$.

(2) *For every vertex $v \in P_2$, v has a neighbour in at most h of $Z_1, \ldots, Z_t$.*

If $v \in Z$ this is true, so we assume that $v \notin Z$. Let K be the set of $i \in \{1, \ldots, t\}$ such that v has a neighbour in Z_i, and L the set of i where v has a non-neighbour in Z_i. Since $v \notin P_1$, it follows that $|L| \geq h$. Suppose also that $|K| \geq h$. Now $|K \cup L| = t \geq 2h$, and hence there exists $K' \subseteq K$ and $L' \subseteq L$, with $|K'| = |L'| = h$, and with $K' \cap L' = \emptyset$. But then G contains H, a contradiction. This proves (2).

(3) *$G|P_1$ has no clique with cardinality h.*

Suppose that $X \subseteq P_1$ is a clique with cardinality h. For each $v \in X$, there are at most h of $Z_1, \ldots, Z_t$ in which v has a non-neighbour; and since $t \geq h^2 + 2$, there exist distinct $i, j \in \{1, \ldots, t\}$ such that every vertex in X is adjacent to every vertex in $Z_i \cup Z_j$. Choose $v \in Z_j$; then v has h neighbours and h non-neighbours in the clique $X \cup Z_i$, and so G contains $\overline{H}$, a contradiction. This proves (3).

(4) *$G|P_2$ does not contain $\overline{H_0}$.*

Suppose that $X \subseteq P_2$ with $|X| = h$, and $G|X$ is isomorphic to $\overline{H_0}$. For each $v \in X$, v has neighbours in at most h of $Z_1, \ldots, Z_t$; and since $t \geq h^2 + h$, there are h values of $i \in \{1, \ldots, t\}$ such that no vertex in X has a neighbour in Z_i. By choosing one vertex from each such Z_i, we obtain an anticlique $Y \subseteq Z$ with $|Y| = h$ such that no vertex in X has a neighbour in Y; but then G contains H, a contradiction. This proves (4).

From (3) and 3.7, $G|P_1$ admits a covering (c', c', c')-layout $\mathcal{F}_1$ of cardinality at most c'. By (4) and 3.8 applied to $\overline{G}|P_2$, $\overline{G}|P_2$ admits a (c', c', c')-layout $\mathcal{F}_2$ with degree at most c'. We claim they satisfy the theorem, and so must check they satisfy the three bulletted statements of the theorem. The first two follow from (1). For the third, we observe

(5) *If $v \in P_1$, there exists $\mathcal{F}' \subseteq \mathcal{F}_2$ with $|\mathcal{F}'| \leq c'(c'+1)h$, such that every non-neighbour of v in P_2 belongs to $\cup_{A \in \mathcal{F}'} A$.*

Choose $X \subseteq P_2$ maximal such that v has no neighbours in X and no member of $\mathcal{F}_2$ contains more than one member of X. Suppose that $|X| \geq (c'+1)h$. Since $\mathcal{F}_2$ is a (c', c', c')-layout of $\overline{G}|P_2$, it follows that each vertex in X is adjacent to at most c' other members of X, and so X includes an anticlique X' of cardinality h. Now v has non-neighbours in at most h of $Z_1, \ldots, Z_t$, and each $x \in X'$ has neighbours in at most h of $Z_1, \ldots, Z_t$, and since $t \geq (h+2)h$, there are h values of $i \in \{1, \ldots, t\}$ such that every vertex of Z_i is adjacent to v and has no neighbour in X'. But then G contains H, a contradiction. This proves that $|X| < (c'+1)h$. Since $\mathcal{F}_2$ is a (c', c', c')-layout with degree at most

c', it follows that there are at most $c'|X|$ members of $\mathcal{F}_2$ that contain a vertex of X; but from the maximality of X, every non-neighbour of v in P_2 belongs to a member of $\mathcal{F}_2$ meeting X. This proves (5).

From (5), this completes the proof of the first assertion of 3.9. For the second assertion, let $c \geq 0$, let $h = (c+1)^3$, and let H be a substar with $2h+1$ vertices, in which some vertex r has h neighbours and h non-neighbours. Let N be the set of neighbours of r in H, and M the set of vertices in $V(H) \setminus \{r\}$ non-adjacent to r. Let G be a graph, satisfying the following:

- (P_1, P_2) is a partition of $V(G)$;
- $\mathcal{F}_1$ is a covering (c, c, c)-layout of $G|P_1$ of cardinality at most c;
- $\mathcal{F}_2$ is a (c, c, c)-layout of $\overline{G}|P_2$ with degree at most c;
- for each $v \in P_1$ and each $A \in \mathcal{F}_2$, either v has at most c neighbours in A or it has at most c non-neighbours in A;
- for each $v \in P_2$ and each $A \in \mathcal{F}_1$, either v has at most c neighbours in A or it has at most c non-neighbours in A; and
- for each $v \in P_1$, there exists $\mathcal{F}' \subseteq \mathcal{F}_2$ with $|\mathcal{F}'| \leq c$, such that every non-neighbour of v in P_2 belongs to $\cup_{A \in \mathcal{F}'} A$.

By replacing G by its complement if necessary, it suffices to show that G contains neither of $H, \overline{H}$ as an induced subgraph. We observe first:

(6) *For every anticlique Z of G, if $Z \cap P_1 \neq \emptyset$ then $|Z \cap P_2| \leq c^2 + c$.*

Let $z \in Z \cap P_1$. Now for each $B \in \mathcal{F}_2$, since every vertex in B is non-adjacent to at most c other members of B, it follows that $|B \cap Z| \leq c+1$. From the final statement above, there exists $\mathcal{F}' \subseteq \mathcal{F}_2$ with $|\mathcal{F}'| \leq c$, such that every non-neighbour of z in P_2 belongs to a member of $\mathcal{F}'$. Consequently $|Z \cap P_2| \leq c(c+1)$. This proves (6).

Suppose that H is an induced subgraph of G and $r \in P_1$. Since $\Delta(G|A) \leq c$ for each $A \in \mathcal{F}_1$, and $|\mathcal{F}_1| \leq c$, it follows that $|N \cap P_1| \leq c^2$. By (6), since $M \cup \{r\}$ is an anticlique, $|M \cap P_2| \leq c(c+1)$. Since $|M| > c(c+1)$, there exists $w \in M \cap P_1$. Since $N \cup \{w\}$ is an anticlique, (6) implies that $|N \cap P_2| \leq c(c+1)$; and so $|N| \leq 2c^2 + c$, a contradiction.

Next suppose that H is an induced subgraph of G and $r \in P_2$. By 2.3 (applied to $\overline{G}|P_2$), it follows that $|N \cap P_2| < (c+1)^2$. Consequently $|N \cap P_1| > h - (c+1)^2 \geq c(c+1)^2$, and since $|\mathcal{F}_1| \leq c$ and $\mathcal{F}_1$ is covering, there exists $A \in \mathcal{F}_1$ with $|N \cap A| > c(c+1)$. Consequently $|A \cap M| \leq c$, by the fifth bullet above. Choose $w \in A \cap N$. Now $M \cup \{w\}$ is an anticlique, so by (6), $|M \cap P_2| \leq c(c+1)$. Since $|A \cap M| \leq c$ and $|M| = h > c(c+1) + c$, there exists $u \in M \cap P_1$ with $u \notin A$. Now u is non-adjacent to all members of N, and in particular to all the members of $N \cap A$. Let X be the set of vertices $x \in A$ such that some member of $\mathcal{F}_1$ contains u, x. Since $u \notin A$, it follows that each $A' \in \mathcal{F}_1$ containing u satisfies $|A \cap A'| \leq c$; and since $|\mathcal{F}_1| \leq c$, we deduce that $|X| \leq c^2$. But there are at most c vertices in $A \setminus X$ non-adjacent to u, and so $|N \cap A| \leq c^2 + c$, a contradiction.

Next suppose that $\overline{H}$ is an induced subgraph of G. By 2.1, $|(M \cup N) \cap P_1| \leq c^2 + c$. If $r \in P_2$, then by 2.4 applied to $\overline{G}|P_2$, it follows that either $|N \cap P_2| \leq c^2$ or $|M \cap P_2| \leq c^3 + 2c$, in either case a contradiction, since $|M|, |N| \geq h$ and $|(M \cup N) \cap P_1| \leq c^2 + c$. Thus $r \in P_1$. Since $|N \cap P_2| \geq h - c^2 - c > c^3 + c^2$, and there exists $\mathcal{F}' \subseteq \mathcal{F}_2$ with $|\mathcal{F}'| \leq c$ such that $N \cap P_2 \subseteq \cup_{A \in \mathcal{F}'} A$, it follows that there exists $A \in \mathcal{F}_2$ such that $|N \cap A| > c^2 + c$. Consequently $|M \cap A| \leq c$ by the fourth bullet above, and so there exists $u \in (M \cap P_2) \setminus A$. Let X be the set of vertices $x \in A$ such that some member of $\mathcal{F}_2$ contains both x, u. Since there are at most c members of $\mathcal{F}_2$ that contain u, and each of them contains at most c vertices of A (since $u \notin A$), it follows that $|X| \leq c^2$. But at most c members of $N \cap A$ do not belong to X, since they are all adjacent to u; and so $|N \cap A| \leq c^2 + c$, a contradiction. This completes the proof of 3.9. ∎

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