Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings

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Abstract

It is known that every graph of sufficiently large chromatic number and bounded clique number contains, as an induced subgraph, a subdivision of any fixed forest, and a subdivision of any fixed cycle. Equivalently, every forest is pervasive, and K_3 is pervasive, in the class of all graphs, where we say a graph H is "pervasive" (in some class of graphs) if for all $\ell \geq 1$, every graph in the class of bounded clique number and sufficiently large chromatic number has an induced subgraph that is a subdivision of H, in which every edge of H is replaced by a path of at least ℓ edges.

Which other graphs are pervasive? It was proved in [3] that every such graph is a "forest of chandeliers": roughly, every block is obtained from a tree by adding a vertex adjacent to its leaves, and there are rules about how the blocks fit together. It is not known whether every forest of chandeliers is pervasive in the class of all graphs; but in a later paper two of us prove that all "banana trees" are pervasive, that is, multigraphs obtained from a forest by adding parallel edges, thus generalizing the two results above. This paper contains the first half of the proof, which works for any forest of chandeliers, not just for banana trees.

Say a class of graphs is "ρ-controlled" if for every graph in the class, its chromatic number is at most some function (determined by the class) of the largest chromatic number of a ρ-ball in the graph. In this paper we prove that for every $\rho \geq 2$, and for every ρ-controlled class, every forest of chandeliers is pervasive in this class.

These results turn out particularly nicely when applied to string graphs. A "string graph" is the intersection graph of a set of curves in the plane. It is known [12] that there are string graphs with clique number two and chromatic number arbitrarily large. We prove that the class of string graphs is 2-controlled, and consequently every forest of chandeliers is pervasive in this class; but in fact something stronger is true, that every string graph of sufficiently large chromatic number and bounded clique number contains each fixed chandelier as an induced subgraph (not just as a subdivision); and the same for most forests of chandeliers (there is an extra condition on how the blocks are attached together).

1 Introduction

All graphs in this paper are finite and simple, and if G is a graph, $\chi(G)$ denotes its chromatic number, and $\omega(G)$ denotes its clique number, that is, the cardinality of the largest clique of G. This is the fifth in a series of papers on the induced subgraphs that must be present in graphs that have bounded clique number and (sufficiently) large chromatic number. The series was originally motivated by three conjectures of Gyárfás from 1985 [9] concerning the lengths of induced cycles in such graphs, but these have already been proved, in [14, 4, 5] respectively:

1.1 *For every integer $k \geq 0$, every graph G with $\omega(G) \leq k$ and $\chi(G)$ sufficiently large contains an induced cycle of odd length at least* 5.

1.2 *For all integers $k, \ell \geq 0$, every graph G with $\omega(G) \leq k$ and $\chi(G)$ sufficiently large contains an induced cycle of length at least ℓ.*

1.3 *For all integers $k, \ell \geq 0$, every graph G with $\omega(G) \leq k$ and $\chi(G)$ sufficiently large contains an induced odd cycle of length at least ℓ.*

In this paper we are particularly concerned with generalizing 1.2 to induced subgraphs different from cycles (the other two results above involve parity and the methods we use here do not work).

If G has bounded clique number and very large chromatic number, which graphs H must be present in G as induced subgraphs? No graph H has this property except for forests, because G can have arbitrarily large girth; and it is an open conjecture of Gyárfás [8] and Sumner [16] that forests do have this property. This is an interesting question but we have nothing to say about it here (except we will prove it for string graphs).

We may ask instead for the graphs H such that every graph G with bounded clique number and sufficiently large chromatic number must contain an induced subgraph which is a subdivision of H. This certainly yields a larger class of graphs; for instance, every cycle has this property, in view of 1.2, and so does every forest, by a theorem of [13], and perhaps so do many more graphs. (For instance, Lévêque, Maffray and Trotignon [10] proved that K_4 has this property.) Figuring out which graphs have this property would considerably extend 1.2, but unfortunately this too still seems out of reach.

This paper is concerned with subdivisions of a graph, so let us clarify some definitions before we go on. Let H be a graph, and let H' be a graph obtained from H by replacing each edge uv by a path (of length at least one) joining u, v, such that these paths are vertex-disjoint except for their ends. We say that H' is a *subdivision* of H; and it is a *proper* subdivision of H if all the paths have length at least two. If each of the paths has exactly $\ell + 1$ edges we call it an *ℓ-subdivision*; if they each have at least $\ell + 1$ edges it is an *($\geq \ell$)-subdivision*; and if they all have at most $\ell + 1$ it is an *($\leq \ell$)-subdivision*. If they all have length at least two and at most $\ell + 1$ it is a *proper* ($\leq \ell$)-subdivision.

Here is what seems to be a more tractable question of the same type, solving which would also extend 1.2. Let us say a graph H is *pervasive* in some class of graphs $\mathcal{C}$ if for all $\nu, \ell \geq 0$ there exists c such that for every graph $G \in \mathcal{C}$ with $\omega(G) \leq \nu$ and $\chi(G) > c$, there is an induced subgraph of G isomorphic to an ($\geq \ell$)-subdivision of H. We say H is *pervasive* if it is pervasive in the class of all graphs. Which graphs are pervasive?

If H' is a subdivision of H, then H' is pervasive if and only if H is pervasive; and 1.2 is equivalent to the statement that all cycles are pervasive (and also equivalent to the assertion that K_3 is pervasive). By the theorem of [13], all forests are pervasive; but what else?

There is a beautiful example of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter and Walczak [12]; they found a sequence of graphs SP_k for $k = 1, 2, \ldots$, each with clique number at most two and with chromatic number at least k. (The same construction was found by Burling [2], but its significance was first pointed out in [12].) Furthermore, it is a *string graph*, the intersection graph of some set of curves in the plane; and consequently for any non-planar graph H, no (≥ 1)-subdivision of H appears in any SP_k as an induced subgraph. For every pervasive graph H, some (≥ 2)-subdivision of H must appear in some SP_k as an induced subgraph, and this severely restricts the possibilities for which graphs might be pervasive. This was analyzed in a paper by Chalopin, Esperet, Li and Ossona de Mendez [3], which we explain next.

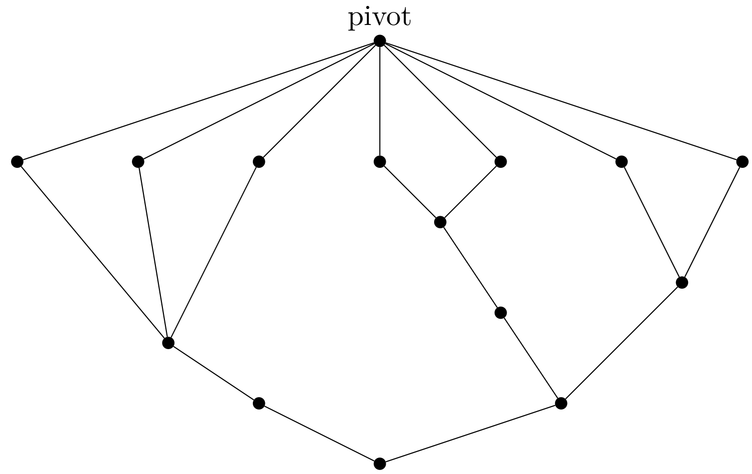

Figure 1: A chandelier

Let T be a tree with $|V(T)| \geq 2$, and let H be obtained from T by adding a new vertex v and making v adjacent to every leaf of T (and to no other vertex). Then H is called a *chandelier* with *pivot* v. (We also count the one- and two-vertex complete graphs as chandeliers, when some vertex is chosen as pivot.) More generally, if we start with a chandelier, and repeatedly take a new chandelier, and identify its pivot with some vertex of what we have already built, what results is called a *tree of chandeliers*. If every component of G is a tree of chandeliers, G is called a *forest of chandeliers*. It follows from results of Chalopin, Esperet, Li and Ossona de Mendez [3] (combine the proof of their theorem 4.5, their Theorem B.4, and the fact that every forest of chandeliers is an induced subgraph of some tree of chandeliers) that:

1.4 *For every graph H, there is a (≥ 2)-subdivision of H that appears as an induced subgraph in SP_k for some k, if and only if H is a forest of chandeliers.*

It follows that every pervasive graph is a forest of chandeliers; and perhaps the converse is true, that every forest of chandeliers is pervasive. Whether that is true or not, the goal of this paper is to begin to determine which graphs are pervasive; and we achieve this goal for a class of graphs that includes the string graphs. We only have to consider trees of chandeliers (since every forest of chandeliers is an induced subgraph of a tree of chandeliers), and they have the convenient property that every subdivision of a tree of chandeliers is another tree of chandeliers. Thus, if we could prove that for every tree of chandeliers H, every graph with bounded clique number and sufficiently large chromatic number contains a subdivision of H as an induced subgraph, then it would follow that every tree of chandeliers is pervasive. We can therefore forget about looking for $(\geq \ell)$-subdivisions, and just look for subdivisions.

If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* between two vertices u, v of G is the length of a shortest path between u, v, or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N^{\rho}_G(v)$ (or $N^{\rho}(v)$, when the graph is clear from the context) denotes the set of all vertices u with distance exactly ρ from v, and $N^{\rho}_G[v]$ or $N^{\rho}[v]$ denotes the set of all v with distance at most ρ from v. If G is a nonnull graph and $\rho \geq 1$, we define $\chi^{\rho}(G)$ to be the maximum of $\chi(N^{\rho}[v])$ taken over all vertices v of G. (For the graph G with no vertices we define $\chi^{\rho}(G) = 0$.) Let $\mathbb{N}$ denote the set of nonnegative integers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. For $\rho \geq 1$, let us say a graph G is (ρ, ϕ)-*controlled* if $\chi(H) \leq \phi(\chi^{\rho}(H))$ for every induced subgraph H of G. Roughly, this says that in every induced subgraph H of G with large chromatic number, there is a vertex v such that $\chi(N^{\rho}_H[v])$ has large chromatic number. Let us say a class of graphs $\mathcal{C}$ is ρ-*controlled* if there is a nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is (ρ, ϕ)-controlled.

Sometimes, it is helpful to know that a statement is true for all ρ-controlled classes, in order to prove that it holds for all classes. For instance, the proof of the main theorem of [13] used this approach, as did McGuinness in [11], and as we did in [4] and several other papers of this series. We hope that the same approach will be helpful for our current problem of characterizing the pervasive graphs. In this paper we will prove:

1.5 *For all $\rho \geq 2$, every tree of chandeliers is pervasive in every ρ-controlled class.*

Every ρ-controlled class is also $(\rho + 1)$-controlled, so large values of ρ give more powerful cases of 1.5; but we prove 1.5 by induction on ρ, and in fact it is the cases when ρ is small that are most challenging. The inductive proof of 1.5 is fairly easy for $\rho \geq 4$, slightly more tricky when $\rho = 3$, and most difficult by far when $\rho = 2$.

For $m \geq 0$ and $r \geq 1$, we denote the r-subdivision of $K_{m,m}$ by $K^r_{m,m}$. A "lamp" (defined later, see figure 2) is a kind of graph considerably more general than a chandelier, and we will define trees of lamps. We think that some trees of chandeliers are not trees of lamps, because the composition rule is more restrictive; but for every forest of chandeliers H there is a tree of lamps that contains a subdivision of H as an induced subgraph.

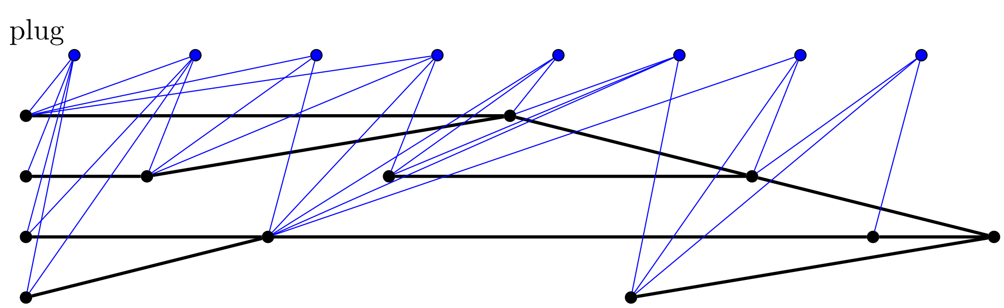

Figure 2: A lamp: each blue vertex is adjacent to the left ends of the tree edges below it

We will in fact prove something much stronger than 1.5:

1.6 *For all $\rho \geq 2$, if $\mathcal{C}$ is a ρ-controlled class of graphs, then either*

- *every graph is pervasive in $\mathcal{C}$; or*
- *for every tree of lamps Q, and for all $\nu \geq 0$, there exists c such that every graph $G \in \mathcal{C}$ with $\omega(G) \leq \nu$ and $\chi(G) > c$ contains Q as an induced subgraph.*

To prove the base case ($\rho = 2$) of 1.6, we will show:

1.7 *Let $\nu \geq 0$, let Q be a tree of lamps, and let $\mu \geq 0$. Let $\mathcal{C}$ be a 2-controlled class of graphs. Then there exists c such that every graph G in $\mathcal{C}$ with $\omega(G) \leq \nu$ and $\chi(G) > c$ contains one of $K^1_{\mu,\mu}, Q$ as an induced subgraph.*

The inductive step of the proof of 1.6 follows from:

1.8 *Let $\mu \geq 0$, and let $\rho \geq 2$. Let $\mathcal{C}$ be a ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \dots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is 2-controlled.*

We will prove 1.8 first, in sections 3–6; and then the sections 7–11 are devoted to proving 1.7.

Why work with lamps rather than chandeliers? For the application to pervasiveness we could do the whole proof using trees of chandeliers instead of trees of lamps, but there is not much gain; and 1.6 is sufficiently striking that we wanted to prove it for the most general type of graph that we could.

The class of all string graphs fits particularly well with 1.7, because:

- The graph SP_k is a string graph, so only forests of chandeliers are pervasive in the class of all string graphs.
- We will prove that the class of string graphs is 2-controlled.
- Consequently a graph is pervasive in the class of all string graphs if and only if it is a forest of chandeliers.
- Since $K^1_{3,3}$ is not a string graph, and hence not an induced subgraph of a string graph, taking $\mu = 3$ in 1.7 tells us: if $\nu \geq 0$, and Q is a tree of lamps, then there exists c such that every string graph G with $\omega(G) \leq \nu$ and $\chi(G) > c$ contains Q as an induced subgraph.
- Consequently we have inadvertently proved the Gyárfás-Sumner conjecture [8, 16] for string graphs, since every tree is a tree of lamps (and in fact proved much more.)

We handle string graphs in the final section.

What about classes that are not ρ-controlled? So far, we have not been able to prove that every tree of chandeliers is pervasive in the class of all graphs, but two of us prove in a later paper [15], using 1.5, that all "banana trees" are pervasive in this class (a *banana tree* is a multigraph obtained from a tree by adding parallel edges).

2 Defining SP_k

Before we go on, let us digress to define SP_k. We will not need it in what follows, but our work was greatly influenced by the paper [3], which is based on this construction.

First, here is a composition operation. We start with a graph A, and a stable subset S of A. Let $S = \{a_1, \ldots, a_s\}$ say, and for $1 \le i \le s$ let N_i be the set of neighbours of a_i in A.

Now take a graph consisting of $s+1$ isomorphic copies of $A \setminus S$, say $A_0, \ldots, A_s$, pairwise disjoint and with no edges between them. For $0 \le i, j \le s$, let the isomorphism from $A \setminus S$ to A_i map N_j to N_{ij}. Now add to this $3s^2$ new vertices, namely x_{ij}, y_{ij}, z_{ij} for all i, j with $1 \le i, j \le s$. Also add edges so that x_{ij}, y_{ij} are both adjacent to every vertex in $N_{0,i}$, and x_{ij}, z_{ij} are both adjacent to every vertex in N_{ij}, and $y_{ij}z_{ij}$ an edge, for $1 \le i, j \le s$. Let G be the resulting graph, and let T be the set

$$\{x_{ij}, y_{ij} \ : 1 \le i, j \le s\}.$$

We say that (G, T) is obtained by *composing* (A, S) *with itself.*

To define SP_k let SP_1 be the complete graph K_2, and let $T_1 \subseteq V(SP_1)$ with $|T_1| = 1$. Inductively let (SP_{k+1}, T_{k+1}) be obtained by composing (SP_k, T_k) with itself. It is easy to check that SP_k has no triangles, and for every colouring of SP_k with any number of colours, some vertex in T_k has neighbours of k different colours, and in particular $\chi(SP_k) \ge k + 1$. Moreover, there are graphs H such that no subdivision of H appears as an induced subgraph of any SP_k, as discussed in the previous section. SP_k is the only construction known to the authors with this property. Indeed, the following very wild statement might be true as far as we know:

2.1 Conjecture: *For all $m, i, \nu \ge 0$ there exists n such that if G has $\omega(G) \le \nu$ and $\chi(G) > n$, then either some (≥ 1)-subdivision of K_m appears in G as an induced subgraph, or SP_i appears in G as an induced subgraph.*

We have little faith in this conjecture; indeed we cannot prove it even for graphs G that are themselves induced subgraphs of some SP_k. We could make it more plausible by weakening it to: "For all $i, \nu \ge 0$ there exists n such that if G has $\omega(G) \le \nu$ and $\chi(G) > n$, then some subdivision of SP_i appears in G as an induced subgraph", and indeed then we think it might well be true; but first we should disprove the stronger form.

3 Two routing lemmas

If X, Y are subsets of the vertex set of a graph G, we say

- X is *complete* to Y if $X \cap Y = \emptyset$ and every vertex in X is adjacent to every vertex in Y;
- X is *anticomplete* to Y if $X \cap Y = \emptyset$ and every vertex in X is nonadjacent to every vertex in Y; and
- X *covers* Y if $X \cap Y = \emptyset$ and every vertex in Y has a neighbour in X.

(If $X = \{v\}$ we say v is complete to Y instead of $\{v\}$, and so on.)

Throughout the paper, we will be applying various forms of Ramsey's theorem. Here is one that contains all that we need (see theorem 5 on page 113 of [7]).

3.1 *For all integers $k, n, \alpha, \beta \geq 0$ there exists $R(k, n, \alpha, \beta) \geq n$ with the following property. Let A, B be disjoint sets, both of cardinality at least $R(k, n, \alpha, \beta)$. Let E be the set of all sets $X \subseteq A \cup B$ with $|X \cap A| = \alpha$ and $|X \cap B| = \beta$. If we partition E into k subsets, then there exist $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = n$ such that all the sets $X \in E$ with $X \subseteq A' \cup B'$ belong to the same subset.*

Before we begin the main proofs, we prove two lemmas which will be applied later. We are trying to prove that certain graphs G with bounded clique number contain a subdivision of some fixed graph H as an induced subgraph. This is true if G has an induced subgraph which is a proper subdivision of $K_{\mu,\mu}$ for appropriate μ; and so we might as well confine ourselves to graphs G that do not contain (as an induced subgraph) any proper subdivision of $K_{\mu,\mu}$, for some fixed μ. This is a little more than we actually need; we only need to exclude subdivisions in which each edge is subdivided a small number of times. For integers $\lambda \geq 2$ and $\mu, \nu \geq 0$, let us say that G is (λ, μ, ν)-*restricted* if $\omega(G) \leq \nu$, and no induced subgraph of G is a proper $(\leq \lambda)$-subdivision of $K_{\mu,\mu}$.

Let G, H be graphs. An *impression* of H in G is a map η with domain $V(H) \cup E(H)$, such that:

- $\eta(v) \in V(G)$ for each $v \in V(H)$;
- for all distinct $u, v \in V(H)$, $\eta(u) \neq \eta(v)$ and $\eta(u), \eta(v)$ are nonadjacent in G;
- for every edge $e = uv$ of H, $\eta(e)$ is a path of G with ends $\eta(u), \eta(v)$;
- if $e, f \in E(H)$ have no common end then $V(\eta(e))$ is anticomplete to $V(\eta(f))$.

The *order* of an impression η is the maximum length of the paths $\eta(e)$ ($e \in E(H)$). Our first lemma is:

3.2 *For all $\lambda \geq 1$ and $\mu, \nu \geq 0$, there exists n such that if G is (λ, μ, ν)-restricted then there is no impression of $K_{n,n}$ in G of order at most $\lambda + 1$.*

Proof. We proceed by induction on λ. If $\lambda > 1$ choose m_4 so that the theorem is satisfied with λ replaced by $\lambda - 1$ and n by m_4, and if $\lambda = 1$ let $m_4 = 0$. Let

$$\begin{aligned} m_3 &= \max(m_4 + 1, \mu, \nu + 2) \\ m_2 &= R(3^{\lambda^2}, m_3, 2, 1) \\ m_1 &= R(3^{\lambda^2}, m_2, 1, 2) \\ n &= R(\lambda, m_1, 1, 1). \end{aligned}$$

We claim that m satisfies the theorem. For let $H = K_{n,n}$, and suppose that η is an impression of H in G of order at most $\lambda + 1$.

(1) *$\{\eta(v) : v \in V(H)\}$ is a stable set of G, and if $e \in E(H)$ and $v \in V(H)$ is not incident with e, then $\eta(v)$ does not belong to $\eta(e)$, and has no neighbours in $V(\eta(e))$.*

The first is immediate from the definition of impression. For the second, if $e \in E(H)$ and $v \in V(H)$ not incident with e, then there is an edge f of H incident with v and with no common end with e, and since $V(\eta(e))$ is anticomplete to $V(\eta(f))$, it follows in particular that $\eta(v)$ does not belong to $\eta(e)$, and has no neighbours in $V(\eta(e))$. This proves (1).

Also we might as well assume that each path $\eta(e)$ is an induced path in G. Let (A, B) be a bipartition of $H = K_{n,n}$. There are only λ possibilities for the length of each path $\eta(e)$ ($e \in E(H)$); and so by 3.1, there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| = |B_1| = m_1$ such that the paths $\eta(ab)$ all have the same length, for all $a \in A_1$ and $b \in B_1$. Let this common length be ℓ; thus $2 \leq \ell \leq \lambda + 1$. Let us number the vertices of each path $\eta(ab)$ ($a \in A_1, b \in B_1$) as $p^0_{ab}, p^1_{ab}, \ldots, p^\ell_{ab}$ in order, where $p^0_{ab} = \eta(a)$ and $p^\ell_{ab} = \eta(b)$.

Take an ordering of B_1, denoted by $<$. For each $a \in A_1$ and all $b, b' \in B_1$ with $b < b'$, let us say the *first pattern* of (a, b, b') is the set of all pairs (i, j) with $1 \leq i, j \leq \ell - 1$ such that $p^i_{ab} = p^j_{ab'}$; and the *second pattern* of (a, b, b') is the set of all pairs (i, j) with $1 \leq i, j \leq \ell - 1$ such that $p^i_{ab}p^j_{ab'}$ are distinct and adjacent in G. There are only 3^{λ^2} possibilities for the first and second patterns; so by 3.1 there exist $A_2 \in A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2| = m_2$, such that all the triples (a, b, b') (for $a \in A_2$ and $b, b' \in B_2$ with $b < b'$) have the same first patterns and they all have the same second patterns. Let these patterns be Π_1, Π_2 say.

Similarly, by exchanging A, B, choosing an ordering $<$ of A_2 and repeating the argument, we deduce that there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ with $|A_3| = |B_3| = m_3$, and sets $\Pi_3, \Pi_4 \subseteq \{1, \ldots, \ell - 1\}^2$ such that for all $a, a' \in A_3$ with $a < a'$ and $b \in B_3$, $p^i_{ab} = p^j_{a'b}$ if and only if $(i, j) \in \Pi_3$, and $p^i_{ab}, p^j_{a'b}$ are different and adjacent if and only if $(i, j) \in \Pi_4$.

(2) $\Pi_1, \Pi_2 = \emptyset$.

For suppose that there exists $(i, j) \in \Pi_1 \cup \Pi_2$. By reversing the order on B if necessary, we may assume that $i \leq j$. Choose $b_0 \in B_3$, minimal under the ordering of B_1. For each $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, let

$$Q(ab) = \{p^j_{ab}, p^{j+1}_{ab}, , \ldots, p^\ell_{ab}\}.$$

Since $(i, j) \in \Pi_1 \cup \Pi_2$, it follows that for each $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, there is a path P_{ab} of G with ends $p^i_{ab_0}, b$ and with vertex set a subset of $\{p^i_{ab_0}\} \cup Q(ab)$. For each $b \in B_3 \setminus \{b_0\}$ let $\eta'(b) = \eta(b)$; for each $a \in A_3$, let $\eta'(a) = p^i_{ab_0}$; and for every edge ab of $H = K_{n,n}$ with $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, let $\eta'(ab) = P_{ab}$. We claim that η' is an impression of K_{m_3, m_3-1} in G. To see this, note first that the vertices $\eta'(a)$ ($a \in A_3$) are all distinct; for choose $b \in B_3 \setminus \{b_0\}$, and let $a, a' \in A_3$ be distinct. Then $p^i_{ab_0}$ is equal or adjacent to p^j_{ab}, but $p^i_{a'b_0}$ is different from and nonadjacent to p^j_{ab} since $V(\eta(a'b_0)), V(\eta(ab))$ are anticomplete, from the definition of an impression. Consequently $p^i_{ab_0}$ is different from $p^i_{a'b_0}$. If $(i, i) \in \Pi_4$, then all the vertices $p^i_{ab_0}$ ($a \in A_3$) are pairwise adjacent, contradicting that $\omega(G) \leq \nu$; so $(i, i) \notin \Pi_4$, and the vertices $\eta'(a)$ ($a \in A_3$) are pairwise nonadjacent. Also for each $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, $\eta'(a)$ is different from and nonadjacent to $\eta'(b)$ by (1). Thus the first three conditions for an impression are satisfied. For the final condition, we must check that if $a, a' \in A_3$ are distinct and $b, b' \in B_3 \setminus \{b_0\}$ are distinct, then $V(P_{ab})$ is anticomplete to $V(P_{a'b'})$. We recall that $V(P_{ab}) \subseteq \{p^i_{ab_0}\} \cup Q(ab)$, where $Q(ab)$ is a subset of the vertex set of $\eta(ab)$, and $V(P_{a'b'}) \subseteq \{p^i_{a'b_0}\} \cup Q(a'b')$. We have seen that $p^i_{ab_0}, p^i_{a'b_0}$ are distinct and nonadjacent, so, exchanging a, a' and b, b' if necessary, it suffices to show that $V(P_{ab})$ is anticomplete to $Q(a'b')$. But $V(P_{ab})$ is a subset of $V(\eta(ab_0)) \cup V(\eta(ab))$, and both the latter sets are anticomplete to $V(\eta(a'b')) \supseteq Q(a'b')$. This proves that η' is an impression as claimed.

Since $m_3 - 1 \geq m_4$, the inductive hypothesis on λ implies that the order of η' is at least $\lambda + 1$. But its order is at most $\ell - j + 1$ if $(i,j) \in \Pi_2$, and at most $\ell - j$ if $(i,j) \in \Pi_1$. Since $\ell \leq \lambda + 1$ and $j \geq 1$, we deduce that $j = 1$, and $\ell = \lambda + 1$; and so $i = 1$, since $i \leq j$, and $(1,1) \in \Pi_2$. Choose $a \in A_3$; then all the vertices p^1_{ab} $(b \in B_3 \setminus \{b_0\})$ are distinct and pairwise adjacent, contradicting that $\omega(G) \leq \nu$. This proves (2).

Similarly $\Pi_3, \Pi_4 = \emptyset$. But then G contains an ℓ-subdivision of K_{m_3,m_3}, contradicting that G is (λ, μ, ν)-restricted. This proves 3.2. ∎

The second lemma is:

3.3 *For all $\mu, \nu \geq 0$, there exists m with the following property. Let G be $(1, \mu, \nu)$-restricted, and let $X \subseteq V(G)$ with $|X| \geq m$. Then there exist distinct nonadjacent $x, x' \in X$ such that every vertex of G adjacent to both x, x' has at least one more neighbour in X.*

Proof. Choose m_4 so that 3.2 holds with n replaced by m_4. Let

$$\begin{aligned} m_3 &= \max(m_4, \nu + 1); \\ m_2 &= R(4, m_3, 2, 2); \\ m_1 &= 2m_2; \\ m &= R(2, m_1, 2, 0). \end{aligned}$$

We claim that m satisfies the theorem. For suppose that G, X are as in the theorem, and for all distinct nonadjacent $x, x' \in X$ there exists $w(x, x')$ adjacent to both x, x' and nonadjacent to all other vertices in X. Since $\omega(G) \leq \nu < m_1$, there is a stable subset X_1 of X with $|X_1| = m_1$, by 3.1. It follows that all the vertices $w(x, x')$ $(x, x' \in M_1, x \neq x')$ are distinct from one another and distinct from the vertices in M_1. Choose two disjoint subsets A_2, B_2 of X_1, both of cardinality m_2. Take an ordering of A_2 and of B_2, both denoted by $<$. Let E be the set of all quadruples (a, a', b, b') such that $a, a' \in A$, $a < a'$, and $b, b' \in B$ and $b < b'$. For all $(a, a', b, b') \in E$, we say the *first pattern* of (a, a', b, b') is 1 or 0 depending whether $w(a, b), w(a', b')$ are adjacent or not; and the *second pattern* is 1 or 0 depending whether $w(a, b'), w(a', b)$ are adjacent or not. There are four possible choices of first and second pattern; so by 3.1 there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ with $|A_3| = |B_3| = m_3$, such that, if E_3 denotes the set of $(a, a', b, b') \in E$ with $a, a' \in A_3$ and $b, b' \in B_3$, then

- either $w(a, b), w(a', b')$ are adjacent for all $(a, a', b, b') \in E_3$, or $w(a, b), w(a', b')$ are nonadjacent for all $(a, a', b, b') \in E_3$; and
- either $w(a, b'), w(a', b)$ are adjacent for all $(a, a', b, b') \in E_3$, or $w(a, b'), w(a', b)$ are nonadjacent for all $(a, a', b, b') \in E_3$.

Suppose that $w(a, b), w(a', b')$ are adjacent for all $(a, a', b, b') \in E_3$. Choose

$$\begin{aligned} a_1 < a_2 < \cdots < a_{\nu+1} \in A_3 \\ b_1 < b_2 < \cdots < b_{\nu+1} \in B_3 \end{aligned}$$

(this is possible since $m_3 \geq \nu + 1$); then the vertices $w(a_1, b_1), w(a_2, b_2), \ldots, w(a_{\nu+1}, b_{\nu+1})$ are pairwise adjacent, contradicting that $\omega(G) \leq \nu$. So the nonadjacency alternative holds in the first bullet above, and similarly nonadjacency holds in the second bullet. Let (A', B') be a bipartition of K_{m_3,m_3}, and choose η mapping A' onto A and B' onto B; and for all $a' \in A'$ and $b' \in B'$, let $\eta(a'b')$ be the path of G with vertex set $\{a, w(a,b), b\}$ where $a = \eta(a')$ and $b = \eta(b')$. Then η is an impression of K_{m_3,m_3} in G, of order 2, and the result follows from 3.2. This proves 3.3. ∎

4 Reducing control

A *levelling* in a graph G is a sequence of pairwise disjoint subsets $(L_0, L_1, \dots, L_k)$ of $V(G)$ such that

- $|L_0| = 1$;
- for $1 \le i \le k$, L_{i-1} covers L_i; and
- for $0 \le i < j \le k$, if $j > i + 1$ then L_i is anticomplete to L_j.

If $\mathcal{L} = (L_0, L_1, \dots, L_k)$ is a levelling, L_k is called the *base* of $\mathcal{L}$, and the vertex in L_0 is the *apex* of $\mathcal{L}$, and $L_0 \cup \cdots \cup L_k$ is the *union* of $\mathcal{L}$, denoted by $V(\mathcal{L})$. If $\mathcal{L} = (L_0, L_1, \dots, L_k)$ and $\mathcal{L}' = (L'_0, L'_1, \dots, L'_k)$ are levellings, we say that $\mathcal{L}'$ is *contained in* $\mathcal{L}$ if $L'_i \subseteq L_i$ for $0 \le i \le k$. For instance, one can obtain a levelling (in a connected graph) by classifying all vertices by their distance from some fixed vertex.

Let $\mathcal{L} = (L_0, L_1, \dots, L_{\rho-1})$ be a levelling in G with $\rho \ge 2$, and let $C \subseteq V(G) \setminus V(\mathcal{L})$. We say that $\mathcal{L}$ is a *ρ-cover for C* if $L_{\rho-1}$ covers C, and $L_0, \dots, L_{\rho-2}$ are anticomplete to C, that is, if $(L_1, \dots, L_{\rho-1}, C)$ is a levelling. Let $\mathcal{L} = (L_0, \dots, L_{\rho-1})$ be a ρ-cover for C, with apex x say. If $z \in C$, then z has a neighbour in $L_{\rho-1}$, and that vertex has a neighbour in $L_{\rho-2}$, and so on; and hence there is a path between z and x of length ρ, with exactly one vertex in each of $L_0, \dots, L_{\rho-1}$. Moreover, this path is induced; we call such a path an *$\mathcal{L}$-radius* for z.

If we have a ρ-controlled class that is not $(\rho-1)$-controlled, there are graphs G in the class with $\chi^{\rho-1}(G)$ bounded and $\chi^{\rho}(G)$ arbitrarily large. Choose such a graph G, with $\chi^{\rho}(G)$ very large; then there is a vertex z_1 with $\chi^{\rho}[z_1]$ very large (not quite so large). For $0 \le j \le \rho$, let $L_{1,j}$ be the set of vertices with distance j from z_0. Since $\chi^{\rho-1}(G)$ is bounded, it follows that $\chi^{\rho}(z_1) = \chi(L_{1,\rho})$ is very large. The subgraph G_2 induced on $L_{1,\rho}$ belongs to the same ρ-controlled class, and so there is a vertex z_2 in it with $\chi^{\rho}_{G_2}[z_2]$; let $L_{2,j}$ be the set of vertices in G_2 with distance j in G_2 from z_2, and then as before $\chi(L_{2,\rho})$ is very large. By continuing this process we obtain a sequence of ρ-covers, and that motivates the following definition.

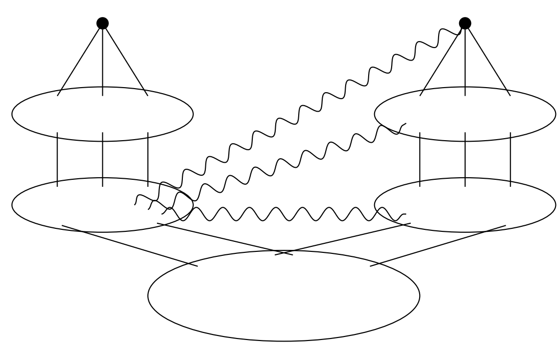

Figure 3: A 3-multicover of length two (wiggly lines indicate possible edges)

For $C \subseteq V(G)$, a *ρ-multicover for C* in G is a family $\mathcal{M} = (\mathcal{L}_i : i \in I)$, where I is a set of integers, such that

- for $1 \le i \le m$, $\mathcal{L}_i$ is a ρ-cover for C;
- for $1 \le i < j \le m$, $V(\mathcal{L}_i)$ is disjoint from $V(\mathcal{L}_j)$;
- for all $i, j \in I$ with $i < j$, every vertex in $V(\mathcal{L}_i)$ with a neighbour in $V(\mathcal{L}_j)$ belongs to the base of $\mathcal{L}_i$.

We denote the union of the sets $V(\mathcal{L}_i)$ ($i \in I$) by $V(\mathcal{M})$. We call $|I|$ the *length* of the multicover, and I is its *index set*. The next two section are devoted to proving the following:

4.1 *For all $\rho \ge 3$ and $\mu, \nu, \tau \ge 0$ there exist $m, c \ge 0$ with the following property. Let G be a $(\rho+2, \mu, \nu)$-restricted graph such that $\chi^{\rho-1}(G) \le \tau$. If $C \subseteq V(G)$ with $\chi(C) > c$, then there is no ρ-multicover of C in G with length m.*

But first, let us assume the truth of 4.1, and apply it to prove a result of great importance (for us), the following.

4.2 *Let $\mu, \nu \ge 0$ and $\rho \ge 2$. Every ρ-controlled class of $(\rho+2, \mu, \nu)$-restricted graphs is 2-controlled.*

Proof (assuming 4.1). The result is trivial for $\rho = 2$, and we proceed by induction on ρ. Let $\rho \ge 3$, and let $\mathcal{C}$ be a ρ-controlled class of $(\rho+2, \mu, \nu)$-restricted graphs. Let ϕ be nondecreasing such that every graph in $\mathcal{C}$ is (ρ, ϕ)-controlled. Let $\mathcal{C}^+$ be the class of all induced subgraphs of graphs in $\mathcal{C}$. The graphs in $\mathcal{C}^+$ are also (ρ, ϕ)-controlled and $(\rho+2, \mu, \nu)$-restricted.

Let $\tau \ge 0$, and let $\mathcal{D}$ be the set of all graphs $H \in \mathcal{C}^+$ with $\chi^{\rho-1}(H) \le \tau$. Let m, c satisfy 4.1. Define $c_0 = c$, and inductively $c_t = \phi(c_{t-1} + \tau)$ for $t > 0$. We claim:

(1) *For $0 \le t \le m$, if $H \in \mathcal{D}$ with $\chi(H) > c_t$ then there is a ρ-multicover in H with length t of some set C where $\chi(C) > c$.*

The claim is trivial if $t = 0$, and we proceed by induction on t. Let $H \in \mathcal{D}$ with $\chi(H) > c_t = \phi(c_{t-1} + \tau)$; then since H is (ρ, ϕ)-controlled, it follows that $\chi(H) \le \phi(\chi^\rho(H))$, and so $\chi^\rho(H) > c_{t-1} + \tau$. Choose $x \in V(H)$ so that $\chi^\rho[x] > c_{t-1} + \tau$. Since $\chi^{\rho-1}[x] \le \tau$, it follows that $\chi^\rho(x) > c_{t-1}$. For each $i \ge 0$, let L_i be the set of vertices in H with distance exactly i from x_1, and let $J = H[L_\rho]$. Since $\chi(J) > c_{t-1}$, from the inductive hypothesis there is a ρ-multicover in J with length $t-1$ of some set C where $\chi(C) > c$, say $(\mathcal{L}_i : 2 \le i \le t)$. Define $\mathcal{L}_1 = (L_0, L_1, \ldots, L_{\rho-1})$; then $(\mathcal{L}_i : 1 \le i \le t)$ satisfies (1). (Note that every edge between $V(\mathcal{L}_1)$ and $V(\mathcal{L}_i)$ for $i > 1$ is also between $V(\mathcal{L}_1)$ and L_ρ, and therefore has an end in $L_{\rho-1}$.) This proves (1).

From (1) and 4.1, it follows that every member of $\mathcal{D}$ has chromatic number at most c_m. At the start of the proof we made an arbitrary choice of τ, and all the subsequent variables in (1) (such as $\mathcal{D}, m$ and the sequence $c_0, c_1, \ldots$) depend on τ. In particular, c_m is a function of τ, say $\phi'(\tau)$. Thus, if $H \in \mathcal{C}^+$, then $\chi(H) \le \phi'(\chi^{\rho-1}(H))$.

We may assume that ϕ' is nondecreasing; and so every graph in $\mathcal{C}$ is $(\rho-1, \phi')$-controlled, and so $\mathcal{C}$ is $(\rho-1)$-controlled, and hence 2-controlled, from the inductive hypothesis. This proves 4.2. ∎

Next we will deduce 1.8, but before that, here is a useful lemma.

4.3 *Let $\rho \geq 2$, and let $\mathcal{C}$ be a class of graphs, such that for all $\nu \geq 0$, the class $\mathcal{C}_\nu$ of graphs $G \in \mathcal{C}$ with $\omega(G) \leq \nu$ is ρ-controlled. Then $\mathcal{C}$ is ρ-controlled.*

Proof. For each $\nu \geq 0$, let ϕ_ν be a function such that each graph G in $\mathcal{C}_\nu$ is (ρ, ϕ_ν)-controlled. For $c \geq 0$, let $\psi(c) = \max_{\nu \leq c} \phi_\nu(c)$. We claim that $\mathcal{C}$ is (ρ, ψ)-controlled. For let $G \in \mathcal{C}$, and let H be an induced subgraph of G such that $\chi(H) > \psi(c)$, for some c. Let $\nu = \omega(H)$. If $\nu > c$, choose a clique X of H with $|X| > c$, and choose $v \in X$; then X belongs to $N_H^\rho[v]$, and so $\chi^\rho(H) \geq |X| > c$ as required. Thus we may assume that $\nu \leq c$, and so $\chi(H) > \psi(c) \geq \phi_\nu(c)$. Since G is (ρ, ϕ_ν)-controlled, it follows that $\chi^\rho(H) > c$ as required. This proves 4.3. ∎

Now we prove 1.8, which we restate.

4.4 *Let $\mu \geq 0$ and $\rho \geq 2$, and let $\mathcal{C}$ be a ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is 2-controlled.*

Proof (assuming 4.1). Let $\mathcal{D}$ be the class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph. Let $\nu \geq 0$, and let $\mathcal{D}_\nu$ be the class of all graphs $G \in \mathcal{D}$ with $\omega(G) \leq \nu$. Every graph in $\mathcal{D}_\nu$ is therefore $(\rho + 2, \mu, \nu)$-restricted, and so $\mathcal{D}_\nu$ is 2-controlled by 4.2. From 4.3 it follows that $\mathcal{D}$ is 2-controlled. This proves 4.4. ∎

5 Extracting ticks from ρ-multicovers

In this section and the next we prove 4.1. Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ and $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be ρ-multicovers in G for C and for C', respectively, where $C' \subseteq C$. If $I' \subseteq I$, and $\mathcal{L}'_i$ is contained in $\mathcal{L}_i$ for each $i \in I'$, we say that $\mathcal{M}'$ is *contained in* $\mathcal{M}$.

Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ρ-multicover for C in G. Let $z \in V(G) \setminus (V(\mathcal{M}) \cup C)$, and for each $i \in I$ let S_i be an induced path of G between z and the apex x_i say of $\mathcal{L}_i$, such that

- z has no neighbours in $V(\mathcal{M}) \cup C$;
- for each $i \in I$, $V(S_i) \cap (V(\mathcal{M}) \cup C) = \{x_i\}$; and
- for each $i \in I$, every vertex in $V(\mathcal{M}) \cup C$ with a neighbour in $V(S_i)$ belongs to $V(\mathcal{L}_i)$.

(We do not require the paths S_i to be pairwise internally disjoint; they may intersect one another arbitrarily.) We say that the family $(S_i : i \in I)$ is a *tick* of G on $(\mathcal{M}, C)$, and z is its *head*, and its *order* is the maximum length of the paths S_i for $i \in I$. We will prove the following.

5.1 *For all $\rho \geq 3$ and $\mu, \nu, \tau, m', c' \geq 0$ there exist $m, c \geq 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^{\rho-1}(G) \leq \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ρ-multicover for C with length m. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a ρ-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with length m', indexed by $I' \subseteq I$, and a tick $(S_i : i \in I')$ on $(\mathcal{M}', C')$ of order at most $\rho + 3$, such that for each $i \in I'$, every vertex of S_i belongs either to $V(\mathcal{L}_i)$, or to C, or to $V(\mathcal{L}_k)$ for some $k \in I \setminus I'$.*

Before we prove 5.1, let us see that it implies 4.1, which we restate:

5.2 *For all $\rho \geq 3$ and $\mu, \nu, \tau \geq 0$ there exist $m, c \geq 0$ with the following property. Let G be a $(\rho+2, \mu, \nu)$-restricted graph such that $\chi^{\rho-1}(G) \leq \tau$. If $C \subseteq V(G)$ with $\chi(C) > c$, then there is no ρ-multicover of C in G with length m.*

Proof, assuming 5.1. First, here is a sketch. By starting with a ρ-multicover $\mathcal{M}$ with large enough length, for a set C with chromatic number large enough, and applying 5.1 repeatedly, we obtain a sequence of multicovers, each contained in its predecessor, of successively smaller (but still large) lengths, and a sequence of ticks all on the last multicover of the sequence $\mathcal{M}'$ say. The ticks are vertex-disjoint except for their vertices in $V(\mathcal{M}')$. There may be edges between them, but if say $(S_i : i \in I)$ and $(T_i : i \in I)$ are two of these ticks, and some vertex in S_i is adjacent to some vertex in T_j, then $i = j$. Consequently we have obtained an impression of $K_{n,n}$ of order at most $\rho+3$, with n large, which is impossible if G is $(\rho+2, \mu, \nu)$-restricted.

Now let us say it precisely. By 3.2, there exists an integer $n \geq 0$ such that if G is $(\rho+2, \mu, \nu)$-restricted then there is no impression of $K_{n,n}$ in G of order at most $\rho+3$. Define $m_n = n$ and $c_n = 0$; and for $j = n-1, n-2, \ldots, 0$ choose m_j, c_j so that 5.1 holds with m', c', m, c replaced by $m_{j+1}, c_{j+1}, m_j, c_j$ respectively.

Let $m = m_0$ and $c = c_0$; we claim that m, c satisfy the theorem. For let G be $(\rho+2, \mu, \nu)$-restricted with $\chi^{\rho-1}(G) \leq \tau$, let $C_0 \subseteq V(G)$ with $\chi(C_0) > c_0$, and suppose that $\mathcal{M}_0 = (\mathcal{L}_{i0} : i \in I_0)$ is a ρ-multicover for C with length m_0, indexed by I_0. Inductively, for $1 \leq j \leq n$, we define C_j, $\mathcal{M}_j$, I_j and $\mathcal{T}_j$ as follows. Since G is $(\rho+2, \mu, \nu)$-restricted and hence $(1, \mu, \nu)$-restricted, and $\mathcal{M}_{j-1}$ is a ρ-multicover for C_{j-1} with length m_{j-1}, and $\chi(C_{j-1}) > c_{j-1}$, we can apply 5.1. We deduce that there exist $C_j \subseteq C_{j-1}$ with $\chi(C_j) > c_j$, and a ρ-multicover $\mathcal{M}_j = (\mathcal{L}_{ij} : i \in I_j)$ for C_j contained in $\mathcal{M}_j$ with length m_j, and a tick $\mathcal{T}_j = (S_{ij} : i \in I_j)$ on $(\mathcal{M}_j, C_j)$ of order at most $\rho+3$, such that for each $i \in I_j$, every vertex of S_i belongs either to $V(\mathcal{L}_{i,j-1})$, or to C_{j-1}, or to $V(\mathcal{L}_{k,j-1})$ for some $k \in I_{j-1} \setminus I_j$.

For $1 \leq j \leq n$ let $\mathcal{T}_j$ have head z_j, and for $1 \leq i \leq n$ let $\mathcal{L}_{in}$ have apex x_i. Thus for $i, j \in I_n$, S_{ij} is a path joining x_i and z_j, and we claim that these paths form an impression of $K_{n,n}$. To show this, we must show:

(1) *For all $i, j, i', j' \in I_n$, if $i \neq i'$ and $j \neq j'$ then $V(S_{ij})$ is disjoint from and anticomplete to $V(S_{i'j'})$.*

We may assume that $j < j'$, from the symmetry. Suppose that $v \in V(S_{ij})$ and $v' \in V(S_{i'j'})$ are either equal or adjacent. Now $v' \in V(S_{i'j'})$ and so v' belongs either to $V(\mathcal{L}_{i',j'-1})$, or to $C_{j'-1}$, or to $V(\mathcal{L}_{k,j'-1})$ for some $k \in I_{j'-1} \setminus I_{j'}$. Hence v' belongs either to $V(\mathcal{L}_{i'j})$, or to C_j, or to $V(\mathcal{L}_{kj})$ for some $k \in I_j \setminus I_n$. But $\mathcal{T}_j$ is a tick on $(\mathcal{M}_j, C_j)$, and hence

- $V(S_{ij}) \cap (V(\mathcal{M}_j) \cup C_j) = \{x_i\}$, and so $v \neq v'$; and
- every vertex in $V(\mathcal{M}_j) \cup C_j$ with a neighbour in $V(S_{ij})$ belongs to $V(\mathcal{L}_{ij})$.

It follows in particular that $v' \in V(\mathcal{L}_{ij})$; but we already showed that v' belongs either to $V(\mathcal{L}_{i'j})$, or to C_j, or to $V(\mathcal{L}_{kj})$ for some $k \in I_j \setminus I_n$, a contradiction. This proves (1).

Since each S_{ij} has length at most $\rho+3$, it follows that G contains an impression of $K_{n,n}$ of order at most $\rho+3$, a contradiction. This proves 5.2. ∎

The proof of 5.1 breaks into two cases, depending whether $\rho = 3$ or not. In this section we handle the easier case $\rho \ge 4$, and postpone $\rho = 3$ until the next section. When $\rho \ge 4$, a stronger statement holds, the following:

5.3 *For all $\rho \ge 4$ and $\tau, m, c' \ge 0$ there exists $c \ge 0$ with the following property. Let G be a graph such that $\chi^{\rho-1}(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ρ-multicover for C, with $|I| = m$. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a ρ-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with length m, and a tick $(S_i : i \in I)$ on $(\mathcal{M}', C')$ with head $z \in C \setminus C'$, such that for each $i \in I$, S_i has length ρ, and $V(S_i) \subseteq V(\mathcal{L}_i) \cup \{z\}$ (and so the paths S_i $(i \in I)$ are pairwise disjoint except for z).*

Proof. Let $c = c' + (m(\rho-1)+1)\tau$, and let G, C and $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be as in the theorem. Let x_i be the apex of $\mathcal{L}_i$ for each $i \in I$, and let $X = \{x_i : i \in I\}$. For each $i \in I$, let C_i be the set of vertices in C with distance at most $\rho - 1$ from x_i in G. Then by hypothesis, $\chi(C_i) \le \tau$; let D be the set of vertices in C that do not belong to the union of the sets C_i $(i \in I)$. It follows that $\chi(D) > c - m\tau$. Since $c \ge m\tau$, there exists $z \in D$; choose some such z. For each $i \in I$ let S_i be some $\mathcal{L}_i$-radius for z.

(1) *For all distinct $i, i' \in I$, $x_{i'}$ has no neighbours in $V(S_i)$.*

Suppose that some $x_{i'}$ is adjacent to a vertex in S_i. Since S_i has length ρ, and the distance from $x_{i'}$ to z is at least ρ (because $z \notin C_{i'}$), it follows that $x_{i'}$ is adjacent to x_i or to the neighbour of x_i in S_i; but this contradicts that $\mathcal{M}$ is a multicover, since $\rho \ge 4$. This proves (1).

Let S be the union of the sets $V(S_i)$ $(i \in I)$. Thus $|S| = m(\rho-1)+1$. Let C' be the set of vertices in C with distance at least ρ in G from every vertex in S. Since $X \subseteq S$ it follows that $C' \subseteq D$, and $z \in D \setminus C'$, and $\chi(C') > c - (m(\rho-1)+1)\tau = c'$. For each $j \in I$, let $\mathcal{L}_j = (L_{0,j}, \ldots, L_{\rho-1,j})$ say, and for $0 \le i \le \rho - 1$ let $L'_{i,j}$ be the set of vertices $v \in L_{i,j}$ such that some $\mathcal{L}_j$-radius contains both v and a vertex in C'; and let $\mathcal{L}'_j = (L'_{0,j}, \ldots, L'_{\rho-1,j})$. Then $\mathcal{L}'_j$ is a ρ-cover for C'; let $\mathcal{M}' = (\mathcal{L}'_j : j \in I)$, and then $\mathcal{M}'$ is a ρ-multicover for C' contained in $\mathcal{M}$. We claim that it satisfies the theorem. Certainly $z \in C \setminus C'$.

(2) *$V(S_i) \cap V(\mathcal{M}') = \{x_i\}$ for each $i \in I$.*

For suppose that $u \in V(S_j) \cap V(\mathcal{M}')$, and choose $j' \in I$ so that $u \in V(\mathcal{L}'_{j'})$. Since $V(S_j) \subseteq V(\mathcal{L}_j)$ and $V(\mathcal{L}'_{j'}) \subseteq V(\mathcal{L}_{j'})$, it follows that $V(\mathcal{L}_j)$ is not disjoint from $V(\mathcal{L}_{j'})$, and so $j' = j$. Since $u \in V(\mathcal{L}'_j)$, there exists i with $0 \le i \le \rho - 1$ such that $u \in L'_{i,j}$; and so the distance in G between u and some vertex in C' is at most $\rho - i$. But from the definition of C', since $u \in S$ it follows that this distance is at least ρ, and so $i = 0$, that is, $u = x_j$. This proves (2).

(3) *For each $j \in I$, if some $u \in V(S_j)$ is adjacent to some $v \in V(\mathcal{M}') \cup C'$ then $v \in V(\mathcal{L}'_j)$.*

Assume that $u \in V(S_j)$ and $v \in V(\mathcal{M}') \cup C'$ are adjacent. Since $u \in S$ and so has distance at least ρ from every vertex in C', it follows that $v \notin C'$, and so $v \in V(\mathcal{L}'_{j'})$ for some $j' \in I'$. Choose i so that $v \in L'_{i,j'}$; then the distance in G between v and some vertex in C' is at most $\rho - i$, and so the distance between u and some vertex in C' is at most $\rho + 1 - i$. Since this distance is at least ρ, it follows that $i \leq 1$, and so v is equal to or adjacent to $x_{j'}$, and in either case v does not belong to the base of $\mathcal{L}_{j'}$. If u belongs to the base of $\mathcal{L}_j$, then u is adjacent to z (because only one vertex in S_j belongs to the base of $\mathcal{L}_j$, namely the neighbour of z); and since $i \leq 1$, and therefore the distance between u and $x_{j'}$ in G is at most 2, it follows that the distance between z and $x_{j'}$ is at most 3, contrary to the definition of D (since $\rho \geq 4$). Thus u does not belong to the base of $\mathcal{L}_j$; and since $\mathcal{M}$ is a multicover, it follows that $j = j'$. This proves (3).

From (1), (2) and (3) it follows that $(S_i : i \in I)$ is a tick on $(\mathcal{M}', C')$. This proves 5.3. ∎

6 Extracting ticks from 3-multicovers

In this section we prove 5.1 when $\rho = 3$. We will need the following lemma, proved in [6]:

6.1 *Let $\mathcal{A}$ be a set of nonempty subsets of a finite set V, and let $k \geq 0$ be an integer. Then either:*

- *there exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$;*
- *there are k distinct members $A_1, \dots, A_k \in \mathcal{A}$, and for all i, j with $1 \leq i < j \leq k$ an element $v_{ij} \in V$, such that for all $h, i, j \in \{1, \dots, k\}$ with $i < j$, $v_{ij} \in A_h$ if and only if $h \in \{i, j\}$; or*
- *there exists $X \subseteq V$ with $|X| \leq 11(k+4)^5$ such that $X \cap A \neq \emptyset$ for all $A \in \mathcal{A}$.*

The idea of using 6.1 in this context is due to Bousquet and Thomassé [1]. We use it to prove the following.

6.2 *For all $\mu, \nu \geq 0$, there exists $m \geq 0$ with the following property. Let G be $(1, \mu, \nu)$-restricted, and let $X \subseteq V(G)$, such that every two vertices in X have distance at most two in G. Then there exists $Y \subseteq V(G)$ with $|Y| \leq m$ such that every vertex in $X \setminus Y$ has a neighbour in Y.*

Proof. Choose k so that 3.3 holds with m replaced by k, and let $m = 11(k+4)^5$. We claim that m satisfies the theorem; for let G, X be as in the theorem. For each $x \in X$, let $N[x]$ be the set of all vertices equal to or adjacent in G to x, and let $\mathcal{A}$ be the set $\{N[x] \ : x \in X\}$. By hypothesis, no two members of $\mathcal{A}$ are disjoint. Let $A_1, \dots, A_k \in \mathcal{A}$ be distinct, where $A_i = N[x_i]$ for $1 \leq i \leq k$; then by 3.3 and the choice of k, there exist i, j with $1 \leq i < j \leq k$ such that x_i, x_j are nonadjacent, and every vertex of G adjacent to both x_i, x_j has a third neighbour in $\{x_1, \dots, x_k\}$. Consequently there is no vertex v_{ij} in $V(G)$ such that for all $h \in \{1, \dots, k\}$ with $i < j$, $v_{ij} \in A_h$ if and only if $h \in \{i, j\}$.

From 6.1 we deduce that there exists $Y \subseteq V$ with $|Y| \leq 11(k+4)^5 = m$ such that $Y \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. But then every vertex in X either belongs to Y or has a neighbour in Y. This proves 6.2. ∎

If $\mathcal{M} = (\mathcal{L}_i : i \in I)$ is a 3-multicover of C, and $i, j \in I$ are distinct, and $z \in C$, let P, Q be $\mathcal{L}_i$- and $\mathcal{L}_j$-radii for z respectively; then $P \cup Q$ is a path of G (not necessarily induced), and we call such a path an *$(\mathcal{L}_i, \mathcal{L}_j)$-diameter*. We need another lemma.

6.3 *For all $\mu, \nu, \tau, c' \geq 0$ and $m > 0$ there exist $c \geq 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^2(G) \leq \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a 3-multicover for C with $|I| = m$. Let x_i be the apex of $\mathcal{L}_i$ for $i \in I$. Let $k \in I$ be maximum. For each $g \in I \setminus \{k\}$, there exist*

- *a subset $I' \subseteq I \setminus \{k\}$ with $|I'| \ge m/2$ and with $\{i \in I : i \le g\} \subseteq I'$;*
- *a subset $C' \subseteq C$ with $\chi(C') > c'$;*
- *for each $i \in I'$, a 3-cover $\mathcal{L}_i'$ for C' contained in $\mathcal{L}_i$, such that for all distinct $i, i' \in I'$, x_i has no neighbour in $V(\mathcal{L}_{i'})$; and*
- *an $(\mathcal{L}_g, \mathcal{L}_k)$-diameter S, such that $V(S)$ is anticomplete to C', and $V(S)$ is anticomplete to $V(\mathcal{L}_i')$ for each $i \in I' \setminus \{g\}$, and $V(S) \cap V(\mathcal{L}_g') = \{x_g\}$, and $V(S) \subseteq V(\mathcal{L}_g) \cup V(\mathcal{L}_k) \cup C$.*

Proof. Choose m_0 so that 6.2 holds with m replaced by m_0. Let

$$c = \max((m + m_0)\tau, (12 + m)\tau + c'2^{m+1}).$$

We claim that c satisfies the theorem. For let G, C, $\mathcal{M} = (\mathcal{L}_i : i \in I), k, g$ be as in the theorem, where $\mathcal{L}_i = (\{x_i\}, A_i, B_i)$ for each $i \in I$, say. Since the set of vertices in C with distance at most two from one of the vertices x_i $(i \in I)$ has chromatic number at most $m\tau$, there exists $C_0 \subseteq C$ with $\chi(C_0) > c - m\tau$ such that every vertex in C_0 has distance at least three from each x_i. Let D be the set of vertices in B_g with a neighbour in C_0.

(1) *There exist $y_1, y_2 \in D$ with distance at least three in G.*

For if not, then by 6.2 applied with $X = D$, there exists $Y \subseteq V(G)$ with $|Y| \le m_0$ such that every vertex in $D \setminus Y$ has a neighbour in Y. Then every vertex in C_0 has distance at most two from a vertex in Y, and so $\chi(C_0) \le |Y|\tau$; and since $\chi(C_0) > c - m\tau$, it follows that $|Y| > c\tau^{-1} - m \ge m_0$, a contradiction. This proves (1).

Choose $z_1, z_2 \in C_0$ adjacent to y_1, y_2 respectively. Let S_1 be an $(\mathcal{L}_g, \mathcal{L}_k)$-diameter containing y_1 and z_1, and choose S_2 for y_2, z_2 similarly. The union of S_1 and S_2 has at most 12 vertices, and so the set of vertices in C_0 with distance at most two from a vertex in $S_1 \cup S_2$ has chromatic number at most 12τ. Consequently there exists $C_1 \subseteq C_0$ with $\chi(C_1) > c - m\tau - 12\tau$ such that every vertex in C_1 has distance at least three from every vertex in $S_1 \cup S_2$. For $1 \le i \le g$, let $\mathcal{L}_i'$ be the levelling $(\{x_i\}, A_i', B_i')$, where B_i' is the set of vertices in B_i with a neighbour in C_1, and A_i' is the set of vertices in A_i with a neighbour in B_i'. Then $V(S_1 \cup S_2) \cap V(\mathcal{L}_g') = \{x_g\}$, because every vertex in C_1 has distance at least three from $S_1 \cup S_2$. Also $V(S_1 \cup S_2)$ is anticomplete to $V(\mathcal{L}_i')$ if $i < g$, since every vertex in $V(\mathcal{L}_i)$ with a neighbour in $S_1 \cup S_2$ belongs to B_i (from the definition of a 3-multicover) and hence does not belong to B_i' (because vertices in B_i' have neighbours in C_1 and therefore have no neighbours in $S_1 \cup S_2$). Also, for $j \in I$ with $j \ne g, k$, x_j has no neighbour in $S_1 \cup S_2$ (from the definition of a multicover, and since $z_1, z_2 \in C_0$ and therefore have distance at least three from x_j). Moreover,

$$V(S_1 \cup S_2) \subseteq V(\mathcal{L}_g) \cup V(\mathcal{L}_k) \cup C.$$

Now we shall choose one of S_1, S_2 to satisfy the other requirements of the theorem. For each $j \in I \setminus \{k\}$ with $j > g$ and each $v \in C_1$, let P_{jv} be an $\mathcal{L}_j$-radius for v. Fix $v \in C_1$ for the moment. Now P_{jv} has length three; let its vertices be x_j-a_{jv}-b_{jv}-v in order. We have seen that x_j has no neighbours in $S_1 \cup S_2$. Since $v \in C_1$ and therefore has distance at least three from every vertex in $S_1 \cup S_2$, it follows that v, b_{jv} have no neighbours in $S_1 \cup S_2$; but a_{jv} might have neighbours in $S_1 \cup S_2$. From the definition of a multicover, every neighbour of a_{jv} in $S_1 \cup S_2$ is one of y_1, y_2; and since y_1, y_2 have distance at least three in G, a_{jv} is not adjacent to them both. Consequently $V(P_{jv})$ is anticomplete to at least one of S_1, S_2. Choose $I_v \subseteq I \setminus \{k\}$ including $\{i \in I : i \le g\}$, with $|I_v| \ge m/2$, such that for one of S_1, S_2 (say S_v), each of the paths P_{jv} ($j \in I_v, j > g$) is anticomplete to S_v. There are only 2^{m+1} possibilities for the pair (S_v, I_v); and so there exists $C' \subseteq C_1$ with $\chi(C') \ge \chi(C_1)2^{-m-1} > c'$, and one of S_1, S_2, say S, and a set I', such that $S_v = S$ and $I_v = I'$ for all $v \in C'$. For each $j \in I \setminus \{k\}$ with $j > g$, let $\mathcal{L}'_j$ be the levelling $(\{x_j\}, A'_j, B'_j)$, where $A'_j = \{a_{jv} : v \in C'\}$ and $B'_j = \{b_{jv} : v \in C'\}$.

We claim that for all distinct $i, i' \in I'$, x_i has no neighbour in $V(\mathcal{L}'_{i'})$. Suppose it does; then $i > i'$ and x_i has a neighbour in $B'_{j'}$. But every vertex in $B'_{j'}$ has a neighbour in $C_1 \subseteq C_0$, and the distance between x_i and every vertex in C_0 is at least three, a contradiction. This proves the claim, and so proves 6.3. ∎

We deduce:

6.4 *For all $\mu, \nu, \tau, c' \ge 0$, and $t > 0$, and $m \ge t2^t$, there exist $c \ge 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^2(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a 3-multicover for C with $|I| = m$. Let $k \in I$ be maximum. Then there exist*

- *a subset $I' \subseteq I \setminus \{k\}$ with $|I'| \ge m2^{-t} \ge t$; $I' = \{i_1, \dots, i_n\}$ say, where $i_1 < i_2 < \dots < i_n$;*
- *a subset $C' \subseteq C$ with $\chi(C') > c'$;*
- *for each $i \in I'$, a 3-cover $\mathcal{L}'_i$ for C', contained in $\mathcal{L}_i$;*
- *for each $i \in \{i_1, \dots, i_t\}$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C', and $V(S_i)$ is anticomplete to $V(\mathcal{L}'_j)$ for all $j \in I' \setminus \{i\}$, and $V(S_i) \cap V(\mathcal{L}'_i) = \{x_i\}$, and $V(S_i) \subseteq V(\mathcal{L}_i) \cup V(\mathcal{L}_k) \cup C$.*

Proof. We assume first that $t = 1$. Choose c so that 6.3 is satisfied. Choose $g \in I$, minimum; then the result follows from 6.3. Thus the result holds if $t = 1$.

We fix μ, ν, τ, m, and proceed by induction on t (assuming $m \ge t2^t$). Thus we assume that $t > 1$ and the result holds with t replaced by $t - 1$. Choose c'' so that 6.3 is satisfied with c replaced by c'' (and the given value of m). Let c have the value that satisfies the theorem with t, c' replaced by $t - 1, c''$; we claim that c satisfies the theorem.

For let G, C and $\mathcal{M} = (\mathcal{L}_i : i \in I), k$ be as in the theorem, where $|I| = m \ge t2^t$. From the inductive hypothesis, there exist

- a subset $I'' \subseteq I \setminus \{k\}$ with $|I''| \ge m2^{1-t}$; $I'' = \{i_1, \dots, i_n\}$ say, where $i_1 < i_2 < \dots < i_n$;
- a subset $C'' \subseteq C$ with $\chi(C'') > c''$;
- for each $i \in I''$, a 3-cover $\mathcal{L}''_i$ for C'', contained in $\mathcal{L}_i$;
- for each $i \in \{i_1, \dots, i_{t-1}\}$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C'', and $V(S_i)$ is anticomplete to $V(\mathcal{L}''_j)$ for all $j \in I'' \setminus \{i\}$, and $V(S_i) \cap V(\mathcal{L}''_i) = \{x_i\}$.

Let $\mathcal{L}''_k = \mathcal{L}_k$. Thus $\mathcal{M}'' = (\mathcal{L}''_i : i \in I'' \cup \{k\})$ is a 3-multicover of C'', contained in $\mathcal{M}$. Also $n \ge 2t$, since $n \ge m2^{1-t}$ and $m \ge t2^t$. From 6.3 applied to $\mathcal{M}''$ taking $g = i_t$, we deduce that there exist

- a subset $I' \subseteq I''$ with $|I'| \geq (|I''|+1)/2 \geq m2^{-t}$ and with $\{i_1,\dots,i_t\} \subseteq I'$;
- a subset $C' \subseteq C''$ with $\chi(C') > c'$;
- for each $i \in I'$, a 3-cover $\mathcal{L}'_i$ for C' contained in $\mathcal{L}''_i$;
- an $(\mathcal{L}''_{i_t}, \mathcal{L}''_k)$-diameter S_{i_t} (which is therefore also an $(\mathcal{L}_{i_t}, \mathcal{L}_k)$-diameter), such that $V(S_{i_t})$ is anticomplete to C', and $V(S_{i_t})$ is anticomplete to $V(\mathcal{L}'_i)$ for all $i \in I' \setminus \{i_t\}$, and $V(S_{i_t}) \cap V(\mathcal{L}'_{i_t}) = \{x_{i_t}\}$, and $V(S_{i_t}) \subseteq V(\mathcal{L}_i) \cup V(\mathcal{L}_k) \cup C$.

But then I', C', $\mathcal{L}'_i$ $(i \in I')$, and the paths S_i $(i \in \{i_1,\dots,i_t\})$ satisfy the theorem. This proves 6.4. ∎

Now we prove the main result of this section, the case of 5.1 for 3-multicovers:

6.5 *For all $\mu,\nu,\tau,m',c' \geq 0$ there exist $m,c \geq 0$ with the following property. Let G be a $(1,\mu,\nu)$-restricted graph such that $\chi^2(G) \leq \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a 3-multicover for C, with length m. Let $k \in I$ be maximum. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a 3-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with length m', with index set some $I' \subseteq I \setminus \{k\}$, and a tick $(S_i : i \in I)$ on $(\mathcal{M}', C')$ of order at most 6, such that $V(S_i) \subseteq V(\mathcal{L}_i) \cup V(\mathcal{L}_k) \cup C$ for each $i \in I$.*

Proof. Let $m = m'2^{m'}$ and let c satisfy 6.4 with this choice of m, taking $t = m'$. We claim that m, c satisfy the theorem. For let G, C, $\mathcal{M} = (\mathcal{L}_i : i \in I)$ and k be as in the theorem. For each $i \in I$, let $\mathcal{L}_i = (\{x_i\}, A_i, B_i)$ say.

By 6.4 applied to $\mathcal{M}$, there exist

- a subset $I' \subseteq I \setminus \{k\}$ with $|I'| = |I|2^{-m'} = m'$ (we only take the first m' elements of the set I' claimed by 6.4);
- a subset $C' \subseteq C$ with $\chi(C') > c'$;
- for each $i \in I'$, a 3-cover $\mathcal{L}'_i$ for C', contained in $\mathcal{L}_i$;
- for each $i \in I'$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C', and $V(S_i)$ is anticomplete to $V(\mathcal{L}'_j)$ for all $j \in I' \setminus \{i\}$, and and $V(S_i) \cap V(\mathcal{L}'_i) = \{x_i\}$, and $V(S_i) \subseteq V(\mathcal{L}_i) \cup V(\mathcal{L}_k) \cup C$.

Let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$. Then $\mathcal{M}'$ is a 3-multicover of C', and $(S_i : i \in I')$ is a tick on $(\mathcal{M}', C')$ of order at most six, with head x_k. This proves 6.5. ∎

Together 6.5 and 5.3 imply 5.1, so we have completed the proof of 5.1, and hence of 4.1, 4.2 and 1.8. Henceforth we need only consider 2-controlled class of graphs.

7 Clique control

Now we come to the second part of the paper, in which we handle 2-controlled graphs. We will follow the approach taken in [4]; and in particular, it will be helpful to introduce a refinement of control, called "clique-control". If X is a clique with $|X| = \xi$ we call X a *ξ-clique.* We denote by $N_G^1(X)$ the set of all vertices in $V(G) \setminus X$ that are complete to X; and by $N_G^2(X)$ the set of all vertices in $V(G) \setminus X$ with a neighbour in $N^1(X)$ and with no neighbour in X. When $X = \{v\}$ we write $N_G^i(v)$ for $N_G^i(X)$ $(i = 1, 2)$. (We omit the subscript G when the graph is clear from context.) We are assuming that in every induced subgraph H of large χ, there is a vertex v such that $N_H^2(v)$ also has large χ; and perhaps the same is true for cliques larger than singletons. For instance, it may or may not be true that in every induced subgraph H of large χ, there is a 2-clique X such that $N_H^2(X)$ also has large χ. If this is false, we can find induced subgraphs H (of graphs in the class) with arbitrarily large χ such that $N_H^2(X)$ has bounded χ for all 2-cliques X, and we focus on these subgraphs. If it is true, then we ask the same question for triples, and so on; we must soon hit a clique-size for which the answer is "false", because none of our graphs have a clique larger than ν. Let us say this more precisely.

If $\mathcal{C}$ is a class of graphs, we denote by $\mathcal{C}^+$ the class of all induced subgraphs of members of $\mathcal{C}$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, and let $\xi \geq 1$ be an integer. We say a graph G is *(ξ, ϕ)-clique-controlled* if for every induced subgraph H of G and every integer $n \geq 0$, if $\chi(H) > \phi(n)$ then there is a ξ-clique X of H such that $\chi(N^2(X)) > n$. Roughly, this means that in every induced subgraph H of large chromatic number, there is a ξ-clique X with $N_H^2(X)$ of large chromatic number. We say a class of graphs $\mathcal{C}$ is *ξ-clique-controlled* if there is a nondecreasing function ϕ such that every graph in $\mathcal{C}$ is (ξ, ϕ)-clique-controlled.

7.1 *Let $\nu \geq 1$ and $\tau_1 \geq 0$, and let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is 2-controlled;*
- *$\omega(G) \leq \nu$ for each $G \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$; and*
- *there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number.*

Then there exist ξ with $1 \leq \xi \leq \nu$ and $\tau_2 \geq 0$ with the following properties:

- *$\mathcal{C}$ is ξ-clique-controlled; and*
- *for all $c \geq 0$ there is a graph $H \in \mathcal{C}^+$ with $\chi(H) > c$, such that $\chi(N_H^2(X)) \leq \tau_2$ for every $(\xi + 1)$-clique X of H.*

Proof. Suppose that $\mathcal{C}$ is ν-clique-controlled, and choose a function ϕ such that every graph in $\mathcal{C}$ is (ν, ϕ)-clique-controlled. Let $c = \phi(0)$; then by hypothesis, there exists $G \in \mathcal{C}$ with $\chi(G) > c$. From the definition of (ν, ϕ)-clique-controlled, there is a ν-clique X in G with $\chi(N^2(X)) > 0$, which is impossible since $N^1(X) = \emptyset$ (because $\omega(G) \leq \nu$).

This proves that $\mathcal{C}$ is not ν-clique-controlled. We claim that $\mathcal{C}$ is 1-clique-controlled. Choose ϕ such that every graph in $\mathcal{C}$ is $(2, \phi)$-controlled, and let $\phi'(c) = \phi(c + \tau_1 + 1)$ for each $c \geq 0$. We claim that every $G \in \mathcal{C}$ is $(1, \phi)$-clique-controlled. For let $c \geq 0$, and let H be an induced subgraph of $G \in \mathcal{C}$, with $\chi(H) > \phi'(c)$. Then $\chi(H) > \phi(c + \tau_1 + 1)$, and since G is $(2, \phi)$-controlled, it follows that $\chi^2(H) > c + \tau_1 + 1$. Hence there is a vertex v of H such that $\chi(N_H^2[v]) > c + \tau_1 + 1$. Now $\chi(N_H^1[v]) \leq \tau_1 + 1$, since the subgraph of H induced on $N_H^1(v)$ has clique number at most $\nu - 1$. Consequently $\chi(N_H^2(v)) > c$. This proves that $\mathcal{C}$ is 1-clique-controlled.

Choose ξ maximum such that $\mathcal{C}$ is ξ-clique-controlled; then $1 \leq \xi < \nu$. Suppose that for all $\kappa \geq 0$, there exists m_κ such that for every $G \in \mathcal{C}$ and every induced subgraph H of G with $\chi(H) > m_\kappa$, there is a $(\xi+1)$-clique X of H with $\chi(N_H^2(X)) > \kappa$. Then G is $(\xi+1, \phi')$-clique-controlled, where we define $\phi'(\kappa) = m_\kappa$ for each $\kappa \geq 0$ (having arranged that $m_0 \leq m_1 \leq \ldots$). Consequently $\mathcal{C}$ is $(\xi+1)$-clique-controlled, a contradiction.

Thus there exists $\kappa \geq 0$ such that for all c, there are graphs $H \in \mathcal{C}^+$ such that $\chi(H) > c$ and $\chi(N_H^2(X)) \leq \kappa$ for every $(\xi+1)$-clique X of H. Let $\tau_2 = \kappa$. This proves 7.1. ∎

The advantage of looking at a class of graphs that is ξ-clique-controlled is the following. Start with a graph in the class with huge chromatic number. Consequently it contains a ξ-clique X_1 with $\chi_G(N^2(X_1))$ (not quite so) huge; let C_1 be the set of vertices with a neighbour in $N(X_1)$ and with none in X_1. Since $\chi(C_1)$ is huge, there is a ξ-clique X_2 of $G_1 = G[C_1]$ such that $\chi_{G_1}(N^2(X_2))$ fairly huge; and so on. We generate a sequence of "ξ-clique-covers" of some ultimate set C, of any desired length, and this gives us some structured thing to explore in the hope of finding the induced subgraph we want. We call this a "ξ-clique-multicover" of C.

Formally: let G be a graph, and $X, N, C \subseteq V(G)$, such that

- X, N, C are pairwise disjoint;
- X is a ξ-clique;
- X is complete to N;
- X is anticomplete to C; and
- N covers C.

We say that the triple $\mathcal{L} = (X, N, W)$ is a *ξ-clique-cover* of C. We write $X(\mathcal{L}) = X$, $N(\mathcal{L}) = N$, and $V(N(\mathcal{L}) = X \cup N$. Thus (X, N) is a 1-clique-cover of C if and only if (X, N) is a 2-cover for C.

A *ξ-clique-multicover* of C of *length* $|I|$ is a family $(\mathcal{L}_i : i \in I)$ of ξ-clique-covers of C, where I is a set of integers, such that for all $i, j \in I$ with $i < j$:

- the sets $V(\mathcal{L}_i)(i \in I)$ are pairwise disjoint; and
- $X(\mathcal{L}_i)$ is anticomplete to $N(\mathcal{L}_j)$.

For $i, j \in I$ with $i < j$, we say that the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is *independent (with respect to C)* if there exists $x_j \in X(\mathcal{L}_j)$ such that no vertex in $N(\mathcal{L}_i)$ with a neighbour in C is adjacent to x_j. A ξ-clique-multicover $\mathcal{M} = (\mathcal{L}_i : i \in I)$ of C is *independent* if all its pairs $(\mathcal{L}_i, \mathcal{L}_j)$ (where $j > i$) are independent (with respect to C). For brevity, let us say a graph G is *(ξ, ζ, c)-free* if for each $C \subseteq V(G)$ with $\chi(C) > c$, there is no independent ξ-clique-multicover of C with length ζ.

In [4] we proved something like 4.1 for $\rho = 2$, but it only applies to "strongly-independent" 2-multicovers. Let us say a 2-multicover $\mathcal{M} = (\mathcal{L}_i : i \in I)$ is *strongly-independent* if for all $i, j \in I$ with $i < j$, the apex of $\mathcal{L}_j$ has no neighbour in the base of $\mathcal{L}_i$. (Thus, any edge between $V(\mathcal{L}_i)$ and $V(\mathcal{L}_j)$

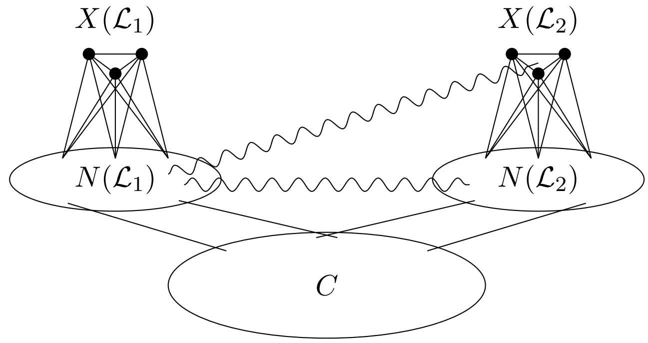

Figure 4: A 3-clique-multicover of length two (wiggly lines indicate possible edges).

is between the two bases, so this is stronger than just independence as 1-clique-covers.) A warning: in [4] we used the term "multicover" to mean what in this paper is called a strongly-independent 2-multicover. The result of [4] that we need is the following, theorem 2.3 of that paper.

7.2 *For all $n, \nu, \tau_1 \geq 0$ there exist $m, d \geq 0$ with the following property. Let G be a graph, such that there is no impression of $K_{n,n}$ in G of order two, and $\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \nu$. If $C \subseteq V(G)$ with $\chi(C) > d$, then there is no strongly-independent 2-multicover of C in G with length m.*

In view of 3.2, we can strengthen this to:

7.3 *For all $\mu, \nu, \tau_1 \geq 0$ there exist $m, d \geq 0$ with the following property. Let G be $(1, \mu, nu)$-restricted, and such that $\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \nu$. If $C \subseteq V(G)$ with $\chi(C) > d$, then there is no strongly-independent 2-multicover of C in G with length m.*

Proof. Choose n to satisfy 3.2 taking $\lambda = 1$; and choose $m, d \geq 0$ to satisfy 7.2. Now let G be as in the theorem; then G is $(1, \mu, \nu)$-restricted, and so by 3.2, there is no impression of $K_{n,n}$ in G of order at most 2. The result follows from 7.2. This proves 7.3. ∎

Because of 7.3, for our pervasiveness problem, we win if we can find a strongly-independent 2-multicover in G of sufficient length and covering a set C with large enough chromatic number; and so several theorems to come will have as a hypothesis that there is no such 2-multicover. For brevity, let us say G is *(m, c)-limited* if for every subset $C \subseteq V(G)$ with $\chi(G) > c$, there is no strongly-independent 2-multicover of C of length m in G.

The next result is closely related to theorem 3.1 of [4].

7.4 *For all $m \geq 0$ and $\xi \geq 1$, there exist $\zeta_0 \geq 0$ such that for all $c \geq 0$, every (m, c)-limited graph is (ξ, ζ_0, c)-free.*

Proof. Choose an integer $\zeta_0 \geq 0$ such that for every partition of the edges of K_{ζ_0} into ξ classes, some K_m subgraph has all its edges in the same class. We claim that ζ_0 satisfies the theorem. For let G be a graph that is not (ξ, ζ_0, c)-free. Consequently for some $C \subseteq V(G)$ with $\chi(C) > c$, there is an independent ξ-clique-multicover of C with length ζ_0, say $(\mathcal{L}_i : i \in I)$ where $|I| = \zeta_0$. For each $i \in I$, let $\mathcal{L}_i = (X_i, N_i)$, and take an enumeration of X_i. Thus we may speak of the pth vertex of X_i

for $1 \le p \le \xi$. For each i, let $N_i' \subseteq N_i$ be the set of vertices in N_i with a neighbour in C. For each pair $i, j \in I$ with $i < j$, choose $p \in \{1, \dots, \xi\}$ such that the pth vertex of X_j has no neighbours in N_i' (this is possible since $(\mathcal{L}_i : i \in I)$ is independent); we call p the *colour* of the pair (i, j). From the choice of ζ_0, there exists $I' \subseteq I$ with $I' = m$ such that all pairs (i, j) with $i, j \in I'$ and $i < j$ have the same colour, say p. For each $i \in I'$ let x_i be the pth vertex of X_i; and let $\mathcal{L}_i' = (\{x_i\}, N_i')$. Then $(\mathcal{L}_i' : i \in I')$ is a strongly-independent 2-multicover of C in G with length m; and so G is not (m, c)-limited. This proves 7.4. ∎

8 Where are we going?

It might be helpful at this stage if we try to sketch the difficulties that lie ahead and our route around them. We have seen that we can assume we have a ξ-clique-multicover of huge length, covering some set C with huge chromatic number. Any subsequence is also a ξ-clique-multicover, and because of 7.4, there is no long independent subsequence. This is asking for us to apply Ramsey's theorem, and obtain a long sequence where each pair of terms are the "opposite" of independent, but what does that mean? Just "not independent" does not tell us anything worthwhile. Before we apply Ramsey's theorem, it is better to tidy up each pair of terms first, shrinking them as necessary, to make them either independent or "very" non-independent; what can we arrange?

If (X_1, N_1) and (X_2, N_2) are terms (in this order) of the ξ-clique-multicover of C, we would like to arrange that some vertex in X_2 has no neighbour in the set of vertices in N_1 that have neighbours in C; and it would be enough to arrange that no vertex in N_1 is complete to X_2 (because then, since $|X_2|$ has bounded size, some vertex in X_2 would be nonadjacent to a big subset of N_1, big enough to cover a large chromatic number part of C, and we could throw away the rest). So the problem is, vertices in N_1 that are complete to X_2. If the set of vertices in N_1 that are not complete to X_2 covers a big-χ part of C, we could just take that, and delete the remainder of N_1; and if not then the vertices in N_1 that are complete to X_2 cover a big-χ part of C, so we could just take that. That would be one way to tidy up the pair; we would obtain a pair that is either independent, or has the property that every vertex in N_1 is complete to X_2. We tidy up every pair in this way, and then we apply Ramsey; one outcome is a long sequence of ξ-clique-covers, pairwise independent, which is impossible; and the other is a long sequence of ξ-clique-covers where the base of each is complete to the clique of every later term. This unfortunately does not work; the second outcome is not rich enough to be useful. We have to tidy up the pairs more carefully.

When our sequence of ξ-clique-covers was created in the first place, we first chose one, say (X_1, N_1), covering C_1; then we chose (X_2, N_2) covering C_2 in $G[C_1]$, and so on. In particular, every vertex of every later $X_j \cup N_j$ has a neighbour in every N_i. So far we have used the fact that every vertex in the ultimate set C has a neighbour in each N_i, and have been resigned to the fact that vertices in $X_j \cup N_j$ might have neighbours in earlier N_i's; but in fact they do have such neighbours, and these edges are useful and need to be carefully guarded, particularly in the case when we fail to get a long independent subsequence. Here is a better way to tidy up the pairs, that is not so cavalier about the edges between N_i and N_j. (But it doesn't seem to work if we start with a sequence and try to tidy it; it only works if we grow the sequence term-by-term and tidy as we go.)

Again, start with (X_1, N_1), covering C_1 say. For a vertex $v \in C_1$, look at the set of vertices of C_1 that have distance two from v, where the intermediate vertex belongs to N_1. And actually, we only care about the vertices that can be reached in two steps starting from some ξ-clique that contains v. So, let us say the "up-down-χ" of v is the maximum, over all ξ-cliques in C_1 containing v, of the chromatic number of the set of vertices in C_1 that can be reached in two steps from v, where the intermediate vertex is complete to the clique. We can show that the set of vertices in C_1 with small up-down-χ has small chromatic number; let us delete it, and just work with the set of all v with big up-down-χ.

Here there is a problem; when we remove some of C_1, the up-down-χ of the vertices we keep might drop. So, we have a subset of C_1 with big χ, such that each of its vertices used to have big up-down-χ. To make use of this property, we need to keep track of the old C_1. As we grow more terms in the clique-multicover there will be more "old" sets that we need to keep track of, and we assemble them in a sequence called a "world". Anyway, let us ignore the world for this sketch.

Choose a ξ-clique-cover (X_2, N_2) of C_2 say, all in $G[C_1]$, and let Y be the set of vertices in N_1 complete to X_2. The vertices in C_2 all have neighbours in N_1. If many (in the big-χ sense) have a neighbour in $N_1 \setminus Y$, we can tidy to make an independent pair of ξ-clique-covers by deleting the other part of C_2, and we rejoice; so either that, or by throwing away a small part of C_2, we can arrange that C_2 is anticomplete to $N_1 \setminus Y$. But each vertex v in N_2 used to have big up-down-χ; and it only had small up-down-χ via Y, because any vertex that v could reach in two steps via Y belongs to $N^2(X_2 \cup \{v\})$, and the clique $X_2 \cup \{v\}$ is too large to have second neighbours with big χ. (This step is the primary reason why we are looking at ξ-clique-covers with ξ maximum instead of 1-clique-covers.) So v had a neighbour in $N_1 \setminus Y$, and therefore still has such a neighbour (we discarded part of C_1 but did not change N_1). This is still the argument we used in [4], but now comes a refinement; v has *many* neighbours in $N_1 \setminus Y$, enough that it used to have big up-down-χ via these neighbours. This is a key observation. The two possible outcomes are, therefore, that either we obtain an independent pair, or we obtain a pair $(X_1, N_1), (X_2, N_2)$ where every vertex in N_2 has big up-down-χ via $N_1 \setminus Y$ (with notation as before) and some extra set W_2 (that was the old C_1 before we discarded some of it), and C_2 is anticomplete to $N_2 \setminus Y$. We call this a "skew" pair.

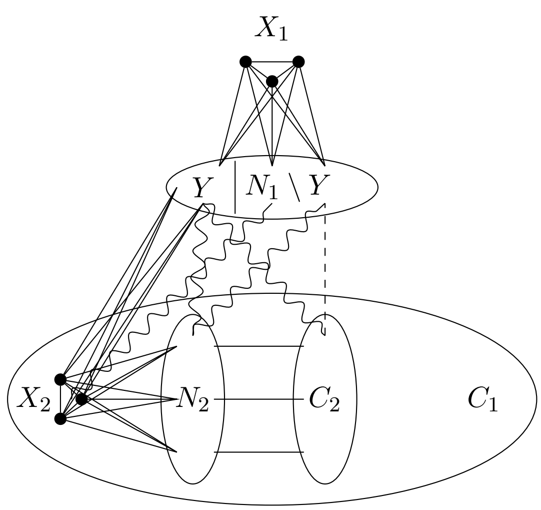

Figure 5: Birth of a skew pair (dashed = anticomplete).

Now we go on to the birth of the third pair (X_3, N_3), chosen within $G[C_2]$. We have to tidy up both the pairs $(X_1,N_1),(X_3,N_3)$ and $(X_2,N_2),(X_3,N_3)$, in the same way. One problem is, this might mess up what we already did. For instance, perhaps we have arranged the pair $(X_1,N_1),(X_2,N_2)$ to be skew, and the pair $(X_1,N_1),(X_3,N_3)$ wants to be independent, and we therefore have to shrink N_1 to make this so. There is a danger that shrinking N_1 will mess up the fact that vertices in N_2 have big up-down-χ via $N_1 \setminus Y$ (with notation as before). But we will be careful that the vertices we remove from N_1 all have neighbours in C_3, and the vertices in $N_1 \setminus Y$ do not.

So the third pair can be tidied, and so on; eventually we get a long sequence of ξ-clique covers of some set C, such that each pair is either independent or skew. Now we apply Ramsey; and get a long subsequence such that all pairs are independent, or all pairs are skew. The first is impossible, as always, so we have built a long sequence of ξ-clique-covers, all pairwise skew.

This is an interesting object. We can show it contains any chandelier, and indeed any lamp, as an induced subgraph; it is much richer than the thing we had before. One can greedily embed a tree into it; first embed the root at some vertex v_k of some N_k with k large. Next we embed the neighbours of the root. There are vertices in each earlier N_j that are adjacent to v_k; so choose one such vertex from N_{k-1}, one from N_{k-2} and so on until we have enough. We have to make these pairwise nonadjacent; and this is where we use the key observation from above, that v_k has many neighbours in N_j, enough that it used to have big second neighbours via these neighbours, and we can argue that there is always one nonadjacent to all the vertices we have already chosen (except v_k). Now start filling in the second neighbours of v_k in the tree, and so on. To get a chandelier, arrange that each leaf of the tree is chosen from N_1; and then we can use a vertex from X_1 as the pivot. Lamps can be embedded the same way.

Unfortunately, this is not yet good enough: we don't want lamps, we want trees of lamps. How can we modify this to get a tree of lamps? (Or tree of chandeliers, say, for this sketch – though it is not quite true that we can get every tree of chandeliers.) Notice that the pivot in the chandelier we just built could be chosen to be any vertex of X_1; so whenever we find a ξ-clique-cover (X_1,N_1) of some set C and we can extend it to a long sequence of pairwise skew ξ-clique-covers, we can get a chandelier with pivot in X_1. And the definition of "big up-down-χ" ensures that when we embed the chandelier, all the vertices we use belong to cliques X such that there is a ξ-clique-cover (X,N) of some "semi-private" big-χ set in which we can try to grow any desired pendant tree of lamps without too much interruption from other vertices (again, this is a place where the world intrudes; and not true for the leaves of the tree, embedded in N_1, which explains the curious composition rule for trees of lamps, and explains why we cannot get every tree of chandeliers).

So our problem is, we have a ξ-clique-cover (X,N) covering a set C with big χ, and we would be happy if we could prove that it can be extended to a long sequence of pairwise skew ξ-clique-covers. Certainly it can be extended to a long sequence of ξ-clique-covers, and we can tidy them and then apply Ramsey; but the skew subsequence we get might no longer include the first term. We have to do something so that we can get the long skew sequence without discarding the first term.

Can we always get a skew sequence of length two with specified first term? If we could, then look at the set they cover in common, and do it again, tidying up all the pairs as we go; we would generate a long sequence of ξ-clique-covers, still including the given first term, such that the first term and ith term are skew, for all i. Then apply Ramsey to the sequence with first term removed, get a long skew subsequence, and put the first term back, and we have won. So, the problem is just getting a skew sequence of length two with a specific first term.

Suppose the first term is (X_1,N_1), covering C_1, and in $G[C_1]$ we cannot find a suitable second term. Again, we can assume (throwing away part of C_1) that every vertex in C_1 has (or used to have, at least) big up-down-χ. Now look at the longest independent sequence of ξ-clique-covers in $G[C_1]$, say $(X_2, N_2), \dots, (X_k, N_k)$. As before, we tidy up all the pairs $(X_1, N_1), (X_i, N_i)$, and if one of them comes out skew we are happy. If they all come out independent, then, including (X_1, N_1), we have a sequence of k pairwise independent ξ-clique-covers in G, *strictly longer than the longest in* $G[C_1]$. So here comes the last trick; we do induction on the size of the longest independent sequence of ξ-clique-covers. If we can move to a subgraph with large χ in which this number is smaller, we do, and start over again; so we can assume that every subgraph with large χ has an independent sequence of ξ-clique-covers of the same length as the longest in G, and so the problem set C_1 cannot occur. More exactly, we have to figure in the chromatic number of the set being covered; and this is the reason for the idea of "ξ-multiclique control", which we explore next.

9 Multiclique control

Let ϕ be nondecreasing, and let $\xi, \zeta \geq 0$. We say that G is (ξ, ζ, ϕ)*-multiclique-controlled* if for every induced subgraph H of G and all $c \geq 0$, if $\chi(H) > \phi(c)$ then H is not (ξ, ζ, c)-free. We say a class of graphs is (ξ, ζ)*-multiclique-controlled* if there is a function ϕ such that all graphs in the class are (ξ, ζ, ϕ)-multiclique-controlled.

It follows that a class of graphs is ξ-clique-controlled if and only if it is $(\xi, 1)$-multiclique-controlled. To see this, note that a graph G is (ξ, ϕ)-clique-controlled if and only if for every induced subgraph H of G with $\chi(H) > \phi(c)$, there is a ξ-clique-cover $(X, N, V(H))$ of a set $C \subseteq V(H)$ with $\chi(C) > c$ (where $N = N_H^1(X)$ and $C = N_H^2(X)$), that is, if and only if every induced subgraph H of G with $\chi(H) > \phi(c)$ is not $(\xi, 1, c)$-free.

9.1 *For all $\tau_1, m, \nu, d \geq 0$ and $\xi \geq 1$, there exists $\zeta_0 \geq 1$ with the following property. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(G) \leq \nu$ for every graph $G \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\mathcal{C}$ is (ξ, ζ_0)-multiclique-controlled; and*
- *every graph in $\mathcal{C}$ is (m, d)-limited.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof. Let ζ_0 satisfy 7.4; and suppose that $\mathcal{C}$ is a class of graphs that is (ξ, ζ_0)-multiclique-controlled, and all graphs in $\mathcal{C}$ have clique number at most ν, and are (m, d)-limited, and $\chi(H) \leq \tau_1$ for all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$. Choose a function ϕ such that all graphs in $\mathcal{C}$ are (ξ, ζ_0, ϕ)-multiclique-controlled. We claim that $c = \phi(d)$ satisfies the theorem. If there exists $G \in \mathcal{C}$ with $\chi(G) > \phi(d)$, then from the definition of "(ξ, ζ_0, ϕ)-multiclique-controlled", G is not (ξ, ζ_0, d)-free, contrary to 7.4. Consequently every graph in $\mathcal{C}$ has chromatic number at most $\phi(d) = c$. This proves 9.1. ∎

9.1 tells us that we can choose ζ maximum such that our class is (ξ, ζ)-multiclique-controlled. That motivates the following.

9.2 *For all $\xi, \zeta \geq 1$, let $\mathcal{C}$ be a class of graphs that is (ξ, ζ)-multiclique-controlled and not $(\xi, \zeta+1)$-multiclique-controlled. Then there exists τ_3 such that for all c, some graph in $\mathcal{C}^+$ has chromatic number more than c, and is $(\xi, \zeta+1, \tau_3)$-free.*

Proof. Choose ϕ such that every graph in $\mathcal{C}$ is (ξ, ζ, ϕ)-multiclique-controlled. If for all $\sigma \geq 0$ there exists m_σ such that no $H \in \mathcal{C}^+$ with $\chi(H) > m_\sigma$ is $(\xi, \zeta+1, \sigma)$-free, then, defining $\phi'(\sigma) = m_\sigma$ (and having arranged that $m_0 \leq m_1 \leq m_2 \leq \ldots$), it follows that every graph in $\mathcal{C}$ is $(\xi, \zeta+1, \phi')$-multiclique-controlled, and hence $\mathcal{C}$ is $(\xi, \zeta+1)$-multiclique-controlled, a contradiction. Consequently, for some σ there is no such m_σ; that is, there exists τ_3 as in the theorem. This proves 9.2. ∎

In our search for the graphs in our class that contain trees of chandeliers, we will focus on the induced subgraphs mentioned in 9.2. We will show the following, in later sections. (A "tree of lamps" is defined later, and is closely related to a tree of chandeliers).

9.3 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2, \tau_3, \nu \geq 0$. Let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(H) \leq \nu$ for every $H \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\chi(N_G^2(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi+1)$-clique X in G;*
- *every member of $\mathcal{C}$ is $(\xi, \zeta+1, \tau_3)$-free;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled; and*
- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c.

Before we begin the proof of 9.3, let us assume its truth and unravel the various inductions implicit in 9.2, 9.1 and 7.1.

9.4 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2, \nu \geq 0$, and let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(H) \leq \nu$ for each $H \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi+1)$-clique X in G;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled; and*
- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then $\mathcal{C}$ is $(\xi, \zeta+1)$-multiclique-controlled.

Proof (assuming 9.3). Suppose that $\mathcal{C}$ is not $(\xi, \zeta+1)$-multiclique-controlled, and let τ_3 be as in 9.2. Let $\mathcal{D}$ be the class of all $(\xi, \zeta+1, \tau_3)$-free graphs in $\mathcal{C}^+$. By 9.2 applied to $\mathcal{C}$, there are graphs in $\mathcal{D}$ with arbitrarily large chromatic number. But by 9.3 applied to $\mathcal{D}$, with $\nu = \omega(G)$, there exists c such that every graph in $\mathcal{D}$ has chromatic number at most c, a contradiction. Thus $\mathcal{C}$ is $(\xi, \zeta+1)$-multiclique-controlled. This proves 9.4. ∎

9.5 *Let $\tau_1, \tau_2, m, \nu, d \geq 0$ and $\xi \geq 1$, and let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(G) \leq \nu$ for all $G \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\mathcal{C}$ is ξ-clique-controlled;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi+1)$-clique X in G;*
- *all graphs in $\mathcal{C}$ are (m, d)-limited; and*
- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 9.3). Let ζ_0 be as in 9.1. Now $\mathcal{C}$ is $(\xi, 1)$-multiclique-controlled, and so for all ζ with $1 \leq \zeta < \zeta_0$, it follows from 9.4 that $\mathcal{C}$ is $(\xi, \zeta+1)$-multiclique-controlled, and hence (ξ, ζ_0)-multiclique-controlled. By 9.1, there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c. This proves 9.5. ∎

9.6 *Let $\tau_1, \nu, m, d \geq 0$, and let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(G) \leq \nu$ for all $G \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\mathcal{C}$ is 2-controlled;*
- *all graphs in $\mathcal{C}$ are (m, d)-limited;*
- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 9.3). Suppose that there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number, and let ξ, τ_2 be as in 7.1. Let $\mathcal{D}$ be the class of all graphs $H \in \mathcal{C}^+$ such that $\chi(N^2_H(X)) \leq \tau_2$ for every $(\xi+1)$-clique X of H. Then from 7.1, $\mathcal{D}$ is ξ-clique-controlled, and for all $c \geq 0$ there is a graph $H \in \mathcal{D}$ with $\chi(H) > c$, contrary to 9.5 applied to $\mathcal{D}$. This proves 9.6. ∎

We deduce:

9.7 *Let $m, \nu, d \geq 0$, and let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(G) \leq \nu$ for all $G \in \mathcal{C}$;*
- *$\mathcal{C}$ is 2-controlled;*
- *all graphs in $\mathcal{C}$ are (m, d)-limited; and*

- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 9.3). We proceed by induction on ν. We may assume that $\nu \geq 1$ and the result holds for $\nu - 1$. Let $\mathcal{D}$ be the class of all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$. Thus by the inductive hypothesis, there exists τ_1 such that all graphs in $\mathcal{D}$ have chromatic number at most τ_1. By 9.6, there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c. This proves 9.7. ∎

Because of 7.3, we have the corollary:

9.8 *Let $\mu, \nu \geq 0$, and let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is 2-controlled;*
- *all graphs in $\mathcal{C}$ are $(1, \mu, \nu)$-restricted; and*
- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof. Choose m, d as in 7.3; then since every graph in $\mathcal{C}$ is $(1, \mu, \nu)$-restricted, they are all (m, d)-limited by 7.3, and the result follows from 9.7. ∎

We see that 1.7 is an immediate consequence of 9.8. Let us prove 1.5, which we restate:

9.9 *For all $\rho \geq 2$, every forest of chandeliers is pervasive in every ρ-controlled class.*

Proof (assuming 9.3). Let T be a forest of chandeliers, and let $\nu, \ell \geq 0$. We must show that there exists c such that for every graph $G \in \mathcal{C}$ with $\omega(G) \leq \nu$ and $\chi(G) > c$, there is an induced subgraph of G isomorphic to an $(\geq \ell)$-subdivision of T. Let T_1 be the ℓ-subdivision of T; then T_1 is also a forest of chandeliers. Choose a tree of lamps Q such that some subdivision of T_1 is an induced subgraph of Q (that this is always possible is discussed after the definition of "tree of lamps", later), and choose $\mu \geq 0$ such that some subdivision of T_1 is an induced subgraph of $K^1_{\mu,\mu}$ (and hence each of $K^1_{\mu,\mu}, \ldots, , K^{\rho+2}_{\mu,\mu}$ contains some $(\geq \ell)$-subdivision of T as an induced subgraph). Let $\mathcal{C}$ be a ρ-controlled class, and let $\mathcal{D}$ be the class of graphs $G \in \mathcal{C}$ with clique number at most ν such that no induced subgraph of G is an $(\geq \ell)$-subdivision of T. It follows that every graph in $\mathcal{D}$ is $(\rho + 2, \mu, \nu)$-restricted, and hence $\mathcal{D}$ is 2-controlled by 4.2. By 9.8 applied to $\mathcal{D}$ and Q, the members of $\mathcal{D}$ have bounded chromatic number. This proves 9.9. ∎

10 Skew pairs

If $Z, W \subseteq V(G)$ are disjoint and $\beta \geq 0$ and $\xi > 0$, we say that a vertex $v \in W$ is (β, ξ)-*earthed via* (Z, W) if there is a ξ-clique $X \subseteq W$ with $v \in X$, such that $\chi(M) > \beta$, where M is the set of all vertices in W that are anticomplete to X and have a neighbour in Z that is complete to X. (This is the concept we called "big up-down-χ" in section 8.) We observe that if $Z \subseteq Z' \subseteq V(G) \setminus W$, then every vertex of W that is (β, ξ)-earthed via (Z, W) is also (β, ξ)-earthed via (Z', W).

10.1 *Let $\xi > 0$ and $\tau_3 \geq 0$, and ϕ a nondecreasing function. Let G be (ξ, ζ, ϕ)-multiclique-controlled and $(\xi, \zeta+1, \tau_3)$-free. Let $\mathcal{L} = (X, N)$ be a ξ-clique-cover of C in G. For all $\beta \geq 0$, the set of vertices in C that are not (β, ξ)-earthed via (N, C) has chromatic number at most $\phi(\zeta\beta + \xi^\zeta\tau_3)$.*

Proof. Let C' be the set of vertices in C that are not (β, ξ)-earthed via (N, C), and suppose that $\chi(C') > \phi(\zeta\beta + \xi^\zeta\tau_3)$. Since G be (ξ, ζ, ϕ)-multiclique-controlled, there is an independent ξ-clique-multicover $(\mathcal{L}_i : 1 \leq i \leq \zeta)$ of some $D \subseteq C'$, with $V(\mathcal{L}_i) \subseteq C'$ for $1 \leq i \leq \zeta$, and with $\chi(D) > \zeta\beta + \xi^\zeta\tau_3$. Let $\mathcal{L}_i = (X_i, N_i)$ for each i, and let N' be the set of vertices in N that are not complete to any X_i $(1 \leq i \leq \zeta)$. For $i \in I$, since the vertices in X_i are not (β, ξ)-earthed via (N, C), the set of vertices in D that have a neighbour in N complete to X_i has chromatic number at most β. Consequently the set of vertices in D that have a neighbour in N complete to X_i for some $i \in I$ has chromatic number at most $\zeta\beta$; and since every vertex in D has a neighbour in N, the set C_0 of vertices in D that have a neighbour in N' has chromatic number at least $\chi(D) - \zeta\beta > \xi^\zeta\tau_3$. But there is a partition of N' into ξ^ζ parts, such that for each part P and each $i \in I$, some vertex in X_i is anticomplete to P; choose such a part P with $\chi(P) > \tau_3$, and let $\mathcal{L}_0 = (X, P)$. Then $(\mathcal{L}_i : 0 \leq i \leq \zeta)$ is a ξ-clique-multicover of C_0 of length $\zeta + 1$, which is impossible since $\chi(C_0) > \tau_3$ and G is $(\xi, \zeta+1, \tau_3)$-free. This proves 10.1. ∎

Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G. A *world* for $\mathcal{M}, C$ is a family $\mathcal{W} = (W_i : i \in I)$ of subsets of $V(G)$ such that for all $i, j \in I$:

- if $i \leq j$ then $W_i \supseteq W_j \supseteq C$;
- if $i < j$ then $V(\mathcal{L}_i) \cap W_j = \emptyset$, and if $i \geq j$ then $V(\mathcal{L}_i) \subseteq W_j$;
- if $i < j$ then $X(\mathcal{L}_i)$ is anticomplete to W_j.

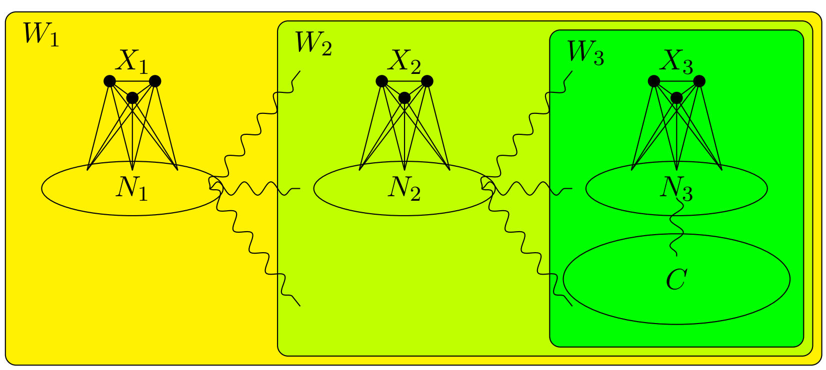

Figure 6: A world for a clique-multicovering

For instance, earlier we mentioned that a way to obtain a ξ-clique-multicover is to start with a ξ-clique cover $\mathcal{L}_1 = (X_1, N_1)$ of some set C_1 with huge chromatic number; then choose $\mathcal{L}_2$ covering C_2, all within C_1; and so on. This generates a ξ-clique-multicover $(\mathcal{L}_1, \mathcal{L}_2, \ldots)$; and $(V(G), C_1, C_2, \ldots\ldots)$ is a world for it. If instead we choose $C_1' \subseteq C_1$ to be the set of vertices in C_1 that are (β, ξ)-earthed via (N_1, C_1) (and 10.1 will tell us that this set still has large chromatic number), and choose $\mathcal{L}_2 = (X_2, N_2)$ covering C_2, all within C_1'; then let C_2' be the vertices in C_2 that are (β, ξ)-earthed via both (N_1, C_2) and (N_2, C_2), and so on, then $(V(G), C_1, C_2, \ldots)$ will again be a world, and now for all $i < j$, every $v \in N_j$ is (β, ξ)-earthed via (N_i, C_{j-1}).

Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G, where $\mathcal{L}_i = (X_i, N_i)$ for each $i \in I$, and let $\mathcal{W} = (W_i : i \in I)$ be a world for $\mathcal{M}, C$. Let $i, j \in I$ with $i < j$,and let Z be the set of vertices in N_i that are not complete to X_j; we say that the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is

- *skew with respect to* $\mathcal{M}, C, \mathcal{W}$ if Z is anticomplete to C and to W_k for all $k \in I$ with $k > j$;
- β*-skew with respect to* $\mathcal{M}, C, \mathcal{W}$ if it is skew with respect to $\mathcal{M}, C, \mathcal{W}$, and every vertex in N_j is (β, ξ)-earthed via (Z, W_j).

We say that $\mathcal{M}$ is *skew with respect to* $C, \mathcal{W}$ if all its pairs are skew with respect to $\mathcal{M}, C, \mathcal{W}$; and similarly define β*-skew with respect to* $C, \mathcal{W}$ if all its pairs have the corresponding property.

Let (X, N) be a ξ-clique-cover of C, and let $N' \subseteq N$. If every vertex in $N \setminus N'$ has a neighbour in C, we say that (X, N') is a C*-residue* of (X, N) (*covering* C' if $C' \subseteq C$ and N' covers C'). Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C, and let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be a ξ-clique-multicover of C'. We say that $\mathcal{M}'$ is an $(\mathcal{M}, C)$*-residue covering* C' if $I' \subseteq I$, $C' \subseteq C$, and $\mathcal{L}'_i$ is a C-residue of $\mathcal{L}_i$ for each $i \in I'$.

We need:

10.2 *Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G, and let $\mathcal{W} = (W_i : i \in I)$ be a world for $\mathcal{M}, C$. Let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be an $(\mathcal{M}, C)$-residue covering $C' \subseteq C$, and let $\mathcal{W}' = (W_i : i \in I')$ (and so $\mathcal{W}'$ is a world for $\mathcal{M}', C'$). For all $i, j \in I'$ with $i < j$:*

- *if the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is independent with respect to C then $(\mathcal{L}'_i, \mathcal{L}'_j)$ is independent with respect to C';*
- *if $(\mathcal{L}_i, \mathcal{L}_j)$ is skew with respect to $\mathcal{M}, C, \mathcal{W}$ then $(\mathcal{L}'_i, \mathcal{L}'_j)$ is skew with respect to $\mathcal{M}', C', \mathcal{W}'$; and*
- *if $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C, \mathcal{W}$ then $(\mathcal{L}'_i, \mathcal{L}'_j)$ is β-skew with respect to $\mathcal{M}', C', \mathcal{W}'$.*

Proof. Let $\mathcal{L}_i = (X_i, N_i)$ for each $i \in I$, and $\mathcal{L}'_i = (X_i, N'_i)$ for each $i \in I'$. Let $i, j \in I'$ with $i < j$, and assume first that $(\mathcal{L}_i, \mathcal{L}_j)$ is independent with respect to C. Consequently there exists $x_j \in X_j$ such that no vertex in N_i has a neighbour in C and is adjacent to x_j; and so no vertex in N'_i has a neighbour in C' and is adjacent to x_j. Thus $(\mathcal{L}'_i, \mathcal{L}'_j)$ is independent with respect to C'.

Now assume that $(\mathcal{L}_i, \mathcal{L}_j)$ is skew with respect to $\mathcal{M}, C, \mathcal{W}$. Thus every vertex in N_i is either complete to X_j or anticomplete to C and to W_k for all $k \in I$ with $k > j$. Consequently every vertex in N'_i is either complete to X_j or anticomplete to C' and to W_k for all $k \in I'$ with $k > j$, and so $(\mathcal{L}'_i, \mathcal{L}'_j)$ is skew with respect to $\mathcal{M}', C', \mathcal{W}'$.

Finally assume that $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C, \mathcal{W}$. Let $v \in N_j$. Thus v is (β, ξ)-earthed via (Z, W_j), where Z is the set of vertices in N_i that are not complete to X_j. Since $(\mathcal{L}_i, \mathcal{L}_j)$ is skew with respect to $\mathcal{M}, C, \mathcal{W}$, it follows that Z is anticomplete to C; and so $Z \subseteq N'_i$, since every vertex in $N_i \setminus N'_i$ has a neighbour in C. Consequently $(\mathcal{L}'_i, \mathcal{L}'_j)$ is β-skew with respect to $\mathcal{M}', C', \mathcal{W}'$. This proves 10.2. ∎

Note that if $C'' \subseteq C' \subseteq C$ and $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are ξ-clique-multicovers of C, C', C'' respectively, and $\mathcal{M}'$ is an $(\mathcal{M}, C)$-residue, and $\mathcal{M}''$ is an $(\mathcal{M}', C')$-residue, then $\mathcal{M}''$ is an $(\mathcal{M}, C)$-residue (we leave the proof to the reader). We call this *transitivity of residues.*

If $\mathcal{M} = (\mathcal{L}_i : i \in I)$ is a ξ-clique-multicover of C in G, and $\mathcal{W}$ is a world, a pair $(\mathcal{L}_i, \mathcal{L}_j)$ is *β-tidy* with respect to $\mathcal{M}, C, \mathcal{W}$ if it is either independent with respect to C or β-skew with respect to $\mathcal{M}, C, \mathcal{W}$. If every pair in $\mathcal{M}$ is β-tidy with respect to $\mathcal{M}, C, \mathcal{W}$, we say that $\mathcal{M}$ is *β-tidy* with respect to $C, \mathcal{W}$. Our next goal is to get rid of the untidy pairs. Pairs involving the last term of the multicover can be handled as follows.

10.3 *Let $\xi > 0$ and $\tau_1, \tau_2, \beta \geq 0$; and let G be such that $\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$, and $\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G. Define $\gamma = \beta+\tau_2+\xi\tau_1+\xi$. Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G, where $\chi(C) > (\xi+1)\gamma$, and let $\mathcal{W} = (W_i : i \in I)$ be a world for $\mathcal{M}, C$. Let $\mathcal{L}_i = (X_i, N_i)$ for each $i \in I$. Let $k \in I$ be the largest member of I, and let $i \in I$ with $i < k$. Assume that every vertex in N_k is (γ, ξ)-earthed via (N_i, W_k). Then there exist $C' \subseteq C$ with $\chi(C') \geq \chi(C)/(\xi+1)$, and a C-residue $\mathcal{L}_i'$ of $\mathcal{L}_i$ covering C', such that $(\mathcal{L}_i', \mathcal{L}_k)$ is β-tidy with respect to $\mathcal{M}', C', \mathcal{W}$, where $\mathcal{M}'$ denotes the ξ-clique-multicover obtained from $\mathcal{M}$ by replacing the term L_i by L_i'.*

Proof. For each $x \in X_k$, let Y_x be the set of vertices in N_i that are adjacent to x and have a neighbour in C, and let C_x be the set of vertices in C with a neighbour in $N_i \setminus Y_x$. Suppose that there exists $x \in X_k$ with $\chi(C_x) \geq \chi(C)/(\xi + 1)$. Let $\mathcal{L}_i' = (X_i, N_i \setminus Y_x)$; then $(\mathcal{L}_i', \mathcal{L}_k)$ is an $(\mathcal{M}, C)$-residue covering C_x, and is independent with respect to C_x, and therefore the pair $(\mathcal{L}_i', \mathcal{L}_k)$ is β-tidy with respect to $\mathcal{M}', C_x, \mathcal{W}$, and the theorem is satisfied.

Thus we may assume that $\chi(C_x) < \chi(C)/(\xi+1)$ for each $x \in X_k$. Let C' be the set of all vertices in C that are not in any of the sets C_x $(x \in X_k)$. It follows that $\chi(C') \geq \chi(C) - \chi(C)\xi/(\xi + 1) = \chi(C)/(\xi + 1)$. Let U be the set of vertices in N_i that are complete to X_k. Thus every vertex in C' has no neighbour in any of the sets $N_i \setminus Y_x$ $(x \in X_k)$, and therefore all its neighbours in N_i belong to U.

We claim that $\mathcal{M}$ itself, with C', satisfy the theorem in this case. Thus we need to show that the pair $(\mathcal{L}_i, \mathcal{L}_k)$ is β-skew with respect to $\mathcal{M}, C', \mathcal{W}$. Since every vertex in N_i is either complete to X_k or anticomplete to C', the pair $(\mathcal{L}_i, \mathcal{L}_k)$ is skew with respect to $\mathcal{M}, C', \mathcal{W}$. It remains to show that every $v \in N_k$ is (β, ξ)-earthed via $(N_i \setminus U, W_k)$.

Let $v \in N_k$. By hypothesis, v is (γ, ξ)-earthed via (N_i, W_k), and so there is a ξ-clique $X \subseteq W_k$ with $v \in X$, such that $\chi(M) > \gamma$, where M is the set of all vertices in $W_k \setminus X$ that are anticomplete to X and have a neighbour in N_i that is complete to X. Let U' be the set of vertices in U adjacent to v. We need to show that v is (β, ξ)-earthed via $(N_i \setminus U, W_k)$ (using the same clique X); and to show this it suffices to prove that the set of vertices in M with a neighbour in U' has chromatic number at most $\gamma - \beta$. To see the latter, let $m \in M$ with a neighbour $u \in U'$. If $m \notin X_k$ and has no neighbour in X_k, then $m \in N^2(X_k \cup \{v\})$, and since $X_k \cup \{v\}$ is a $\xi + 1$-clique, the set of all such m has chromatic number at most τ_2. On the other hand, the set of all vertices that either belong to X_k or have a neighbour in X_k has chromatic number at most $\xi\tau_1 + \xi$; and so, adding, the set of vertices in M with a neighbour in U' has chromatic number at most $\tau_2 + \xi\tau_1 + \xi = \gamma - \beta$. This proves that v is (β, ξ)-earthed via $(N_i \setminus U, W_k)$, and so proves 10.3. ∎

From 10.3 we deduce the following result, that given any ξ-clique cover (and suitable conditions), we can extend it (or at least some residue of it) to a β-skew ξ-clique-multicover of length two. It was in order to prove this result and its consequence 10.6 that we introduced the concept of (ξ, ζ)-multiclique-control.

10.4 *Let $\xi > 0$ and $\tau_1, \tau_2, \tau_3, \beta \geq 0$, and ϕ a nondecreasing function. For all $c' \geq 0$ there exists $c \geq 0$ with the following property. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L} = (X, N)$ be a ξ-clique-cover of C in G, where $\chi(C) > c$. Then there is a C-residue $\mathcal{L}' = (X, N')$ of $\mathcal{L}$ covering $C' \subseteq C$ with $\chi(C') > c'$, and a ξ-clique-cover $\mathcal{L}^* = (X^*, N^*)$ of C' with $X^*, N^* \subseteq C$, such that the ξ-clique-multicover $(\mathcal{L}', \mathcal{L}^*)$ is β-skew with respect to C' and the world $(V(G), C)$.

Proof. Let $\gamma = \beta + \tau_2 + \xi\tau_1 + \xi$, and let

$$c = \phi((\xi+1)^\zeta \max(c', \tau_3)) + \phi(\zeta\gamma + \xi^\zeta \tau_3).$$

We claim that c satisfies the theorem. For let G, C and $\mathcal{L} = (X, N)$ be as in the theorem, with $\chi(C) > c$. Let D be the set of vertices in C that are (γ, ξ)-earthed via (N, C). By 10.1, $\chi(C \setminus D) \leq \phi(\zeta\gamma + \xi^\zeta\tau_3)$, and so $\chi(D) > \phi(\max(c', \tau_3)(\xi+1)^\zeta)$. Since G is (ξ, ζ, ϕ)-multiclique-controlled, it follows that there is an independent ξ-clique-multicover $(\mathcal{L}_i : 1 \leq i \leq \zeta)$ of some $C_0 \subseteq D$, with $\chi(C_0) > (\xi+1)^\zeta \max(c', \tau_3)$, and with $V(\mathcal{L}_i) \subseteq D$ for $1 \leq i \leq \zeta$. Now for every C-residue $\mathcal{L}'$ of $\mathcal{L}$ covering $C' \subseteq C_0$, the pair $(\mathcal{L}', \mathcal{L}_i)$ is a ξ-clique-multicover of C' of length two, and $\mathcal{W} = (V(G), C)$ is a world for it. Let $\mathcal{L}'_0 = \mathcal{L}$. By ζ applications of 10.3, to the ξ-clique-multicovers $(\mathcal{L}'_{i-1}, \mathcal{L}_i)$ for $i = 1, \dots, \zeta$ in turn, and successive subsets of C_1, we deduce that for $i = 1, \dots, \zeta$ there exist $C_i \subseteq C_{i-1}$ with $\chi(C_i) > \chi(C_{i-1})/(\xi+1)$, and a C_{i-1}-residue $\mathcal{L}'_i$ of $\mathcal{L}'_{i-1}$ (and hence of $\mathcal{L}$) covering C_i, such that the pair $(\mathcal{L}'_i, \mathcal{L}_i)$ is β-tidy with respect to $C_i, \mathcal{W}$. In particular, this is true when $i = \zeta$; let $C' = C_\zeta$ and $\mathcal{L}' = \mathcal{L}'_\zeta$. Thus $\chi(C') > \max(c', \tau_3)$, and $\mathcal{L}'$ is a C-residue of $\mathcal{L}$, covering C'. Moreover, by 10.2, each of the pairs $(\mathcal{L}', \mathcal{L}_i)$ is β-tidy with respect to $C', \mathcal{W}$. Suppose that each of the pairs $(\mathcal{L}', \mathcal{L}_i)$ is independent with respect to C', for $i = 1, \dots, \zeta$; then since each of the pairs $(\mathcal{L}_i, \mathcal{L}_j)$ for $1 \leq i < j \leq \zeta$ is independent with respect to C', by 10.2, it follows that $(\mathcal{L}', \mathcal{L}_1, \dots, \mathcal{L}_\zeta)$ is an independent ξ-clique-multicover of C', which is impossible since $\chi(C') > \tau_3$. Thus there exists $i \in \{1, \dots, \zeta\}$ such that $(\mathcal{L}', \mathcal{L}_i)$ is not independent with respect to C'; and since it is β-tidy with respect to $C', \mathcal{W}$, it follows that it is β-skew with respect to $C', \mathcal{W}$. This proves 10.4. ∎

This implies:

10.5 *Let $\xi, t > 0$ and $\tau_1, \tau_2, \tau_3, \beta \geq 0$, and let ϕ be a nondecreasing function. For all $c' \geq 0$ there exists $c \geq 0$ with the following property. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*

- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta + 1, \tau_3)$-free.*

Let $\mathcal{L}$ be a ξ-clique-cover of C in G, where $\chi(C) > c$. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a C-residue $\mathcal{L}_1$ of $\mathcal{L}$ covering C', and ξ-clique-covers $\mathcal{L}_2, \dots, \mathcal{L}_t$ of C', and $\mathcal{W}$, such that

- *$V(\mathcal{L}_i) \subseteq C$ for $2 \le i \le t$;*
- *$\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_t)$ is a ξ-clique-multicover of C', and $\mathcal{W}$ is a world for $\mathcal{M}, C'$;*
- *$\mathcal{M}$ is β-tidy with respect to $C', \mathcal{W}$; and*
- *for $2 \le i \le t$, the pair $(\mathcal{L}_1, \mathcal{L}_i)$ is β-skew with respect to $\mathcal{M}, C', \mathcal{W}$.*

Proof. The result is true when $t = 1$, taking $c' = c$; so we assume that $t > 1$ and the result holds for $t - 1$. Define $\gamma = \beta + \tau_2 + \xi\tau_1 + \xi$. Choose c_0 such that setting $c = c_0$ satisfies 10.4 when c' is replaced by $(\xi + 1)^{t-2} \max(\gamma, c')$. Choose a value of c that satisfies the result with t, c' replaced by $t - 1, c_0 + (t - 1)\phi(\zeta\gamma + \xi^\zeta\tau_3)$ respectively. We claim that c satisfies the theorem. For let G, C and $\mathcal{L} = (X, N)$ be as in the theorem, with $\chi(C) > c$. From the choice of c, there exist $D' \subseteq C$ with $\chi(D') > c_0 + (t - 1)\phi(\zeta\gamma + \xi^\zeta\tau_3)$, and a C-residue $\mathcal{L}'_1$ of $\mathcal{L}$ covering D', and ξ-clique-covers $\mathcal{L}'_2, \dots, \mathcal{L}'_{t-1}$ of D', and $\mathcal{W}'$, such that

- $V(\mathcal{L}'_i) \subseteq C$ for $2 \le i \le t - 1$;
- $\mathcal{M}' = (\mathcal{L}'_1, \mathcal{L}'_2, \dots, \mathcal{L}'_{t-1})$ is a ξ-clique-multicover of D', and $\mathcal{W}'$ is a world for $\mathcal{M}', D'$;
- $\mathcal{M}'$ is β-tidy with respect to $D', \mathcal{W}'$; and
- for $2 \le i \le t - 1$, the pair $(\mathcal{L}'_1, \mathcal{L}'_i)$ is β-skew with respect to $\mathcal{M}', D', \mathcal{W}'$.

For $1 \le i \le t - 1$, let $\mathcal{L}'_i = (X_i, N_i)$. By 10.1, for $1 \le i \le t - 1$, the set of vertices in D' that are not (γ, ξ)-earthed via (N_i, D') has chromatic number at most $\phi(\zeta\gamma + \xi^\zeta\tau_3)$. So the set D_2 of vertices in D' that are (γ, ξ)-earthed via (N_i, D') for all $i \in \{1, \dots, t - 1\}$ has chromatic number more than $\chi(D') - (t - 1)\phi(\zeta\gamma + \xi^\zeta\tau_3) \ge c_0$.

Let $\mathcal{W}_1 = (W_1, \dots, W_{t-1})$, and define $W_t = D_2$ and $\mathcal{W} = (W_1, \dots, W_t)$. Now $\mathcal{L}'$ is a ξ-clique-cover of D_2, so by 10.4 and the choice of c_0, there exist $D_3 \subseteq D_2$ with $\chi(D_3) > (\xi + 1)^{t-1} \max(\gamma, c')$, and a D_2-residue $\mathcal{L}_1$ of $\mathcal{L}'_1$ covering D_3, and a ξ-clique-cover $\mathcal{L}_t = (X_t, N_t)$ of D_3, such that $X_t, N_t \subseteq D_2$, and the ξ-clique multicover $(\mathcal{L}_1, \mathcal{L}_t)$ is β-skew with respect to D_3 and the world $(V(G), D_2)$. Let

$$\mathcal{M}_2 = (\mathcal{L}_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_{t-1})$$

and

$$\mathcal{M}_3 = (\mathcal{L}_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_{t-1}, \mathcal{L}_t);$$

these are both ξ-clique-multicovers of D_3. Also, $\mathcal{W}_1$ is a world for $\mathcal{M}_2, D_3$, and $\mathcal{W}$ is a world for $\mathcal{M}_3, D_3$. Moreover, $\mathcal{L}_1$ is a C-residue of $\mathcal{L}$, by the transitivity of residues.

(1) *Every pair of $\mathcal{M}_3$ is β-tidy with respect to $\mathcal{M}_3, D_3, \mathcal{W}$ except possibly the pairs $(\mathcal{L}'_i, \mathcal{L}_t)$ where $2 \le i \le t - 1$; and in particular, for $2 \le i \le t$, the pair $(\mathcal{L}_1, \mathcal{L}'_i)$ is β-skew with respect to $\mathcal{M}_3, D_3, \mathcal{W}$.*

To see this, there are three kinds of pairs to consider:

- The pair $(\mathcal{L}_1, \mathcal{L}'_i)$ where $2 \le i \le t-1$: the pair $(\mathcal{L}', \mathcal{L}'_i)$ is β-skew with respect to $\mathcal{M}_1, D_1, \mathcal{W}_1$, and therefore $(\mathcal{L}_1, \mathcal{L}'_i)$ is β-skew with respect to $\mathcal{M}_2, D_3, \mathcal{W}_1$, by 10.2. Since $W(\mathcal{L}_t) \subseteq D_1$, it is also β-skew with respect to $\mathcal{M}_3, D_3, \mathcal{W}$.
- The pair $(\mathcal{L}_1, \mathcal{L}_t)$: this is β-skew with respect to $\mathcal{M}_3, D_3, \mathcal{W}$, since as a ξ-clique-multicover, it is β-skew with respect to D_3 and the world $(V(G), D_2)$.
- The pair $(\mathcal{L}'_i, \mathcal{L}'_j)$ where $2 \le i < j \le t-1$: this is β-tidy with respect to $\mathcal{M}_1, D_1, \mathcal{W}_1$, and therefore with respect to $\mathcal{M}_2, D_3, \mathcal{W}_1$, by 10.2; and hence also with respect to $\mathcal{M}_3, D_3, \mathcal{W}$ since $W_t \subseteq D_1$.

This proves (1).

Let $C_1 = D_3$. By $t-2$ applications of 10.3, applied to the pairs $(\mathcal{L}'_i, \mathcal{L}_t)$ and $C_{i-1}, \mathcal{W}$ for $2 \le i \le t-1$ in turn, we deduce that for $1 \le i \le t-1$ there exist $C_i \subseteq C_{i-1}$ with $\chi(C_i) \ge \chi(C_{i-1})/(\xi+1)$, and a C_{i-1}-residue $\mathcal{L}_i$ of $\mathcal{L}'_i$ covering C_i, such that $(\mathcal{L}_i, \mathcal{L}_t)$ is β-tidy with respect to the ξ-clique-multicovering $(\mathcal{L}_1, \ldots, , \mathcal{L}_i, \mathcal{L}'_{i+1}, \ldots, \mathcal{L}'_{t-1}, \mathcal{L}_t)$ and $C_i, \mathcal{W}$. It follows from 10.2 and (1) that

$$\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_{t-1}, \mathcal{L}_t)$$

(setting $C' = C_{t-1}$) satisfies the theorem. This proves 10.5. ∎

By choosing t large enough in 10.5, and applying Ramsey's theorem to the sequence $(\mathcal{L}_2, \ldots, \mathcal{L}_t)$, we deduce since G is $(\xi, \zeta+1, \tau_3)$-free that the same result as 10.5 is true with "β-tidy" replaced by "β-skew". This result is important enough that it deserves to be said explicitly:

10.6 *Let $\xi, t > 0$ and $\tau_1, \tau_2, \tau_3, \beta \ge 0$, and ϕ a nondecreasing function. For all $c' \ge 0$ there exists $c \ge 0$ with the following property. Let G be such that*

- *$\chi(H) \le \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \le \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}$ be a ξ-clique-cover of $C \subseteq V(G)$, where $\chi(C) > c$. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a C-residue $\mathcal{L}_1$ of $\mathcal{L}$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_t$ of C', and $\mathcal{W}$, such that

- *$V(\mathcal{L}_i) \subseteq C$ for $1 \le i \le t$;*
- *$\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_t)$ is a ξ-clique-multicover of C', and $\mathcal{W}$ is a world for $\mathcal{M}, C'$; and*
- *$\mathcal{M}$ is β-skew with respect to $C', \mathcal{W}$.*

Proof. Choose an integer $s \ge 0$ such that for every partition of the edges of K_{s-1} into two classes, either some K_{t-1} subgraph has all its edges in the first class, or some $K_{\zeta+1}$ subgraph has all its edges in the second. Let c satisfy 10.5 with t replaced by s, and c' replaced by $\max(c', \tau_3)$. We claim t satisfies the theorem; for let $G, \mathcal{L}$ and C be as in the theorem. By 10.5 there exist $C' \subseteq C$ with $\chi(C') > \max(c', \tau_3)$, and a C-residue $\mathcal{L}_1$ of $\mathcal{L}$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_s$ of C', and $\mathcal{W}$, such that

- $V(\mathcal{L}_i) \subseteq C$ for $2 \leq i \leq s$;
- $\mathcal{M}' = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s)$ is a ξ-clique-multicover of C';
- $\mathcal{M}'$ is β-tidy with respect to $C', \mathcal{W}$; and
- for $2 \leq i \leq s$, the pair $(\mathcal{L}_1, \mathcal{L}_i)$ is β-skew with respect to $\mathcal{M}', C'$.

For each pair (i,j) with $2 \leq i < j \leq s$, the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is β-tidy with respect to $\mathcal{M}', C'$, and so is either independent with respect to C', or β-skew with respect to $\mathcal{M}', C', \mathcal{W}$. From the choice of s, either

- there exists $I \subseteq \{2, \dots, s\}$ with $|I| = t-1$ such that $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C', \mathcal{W}$ for all $i < j$ with $i, j \in I$, or
- there exists $J \subseteq \{2, \dots, s\}$ with $|J| = \zeta + 1$ such that $(\mathcal{L}_i, \mathcal{L}_j)$ is independent with respect to C, for all $i < j$ with $i, j \in J$.

The second is impossible, since G is $(\xi, \zeta+1, \tau_3)$-free and $\chi(C') > \tau_3$, and so the first holds. But then by 10.2, every pair of terms in $\mathcal{M} = (\mathcal{L}_i : i \in \{I \cup \{1\})$ is β-skew with respect to $\mathcal{M}, C', \mathcal{W}'$, where $\mathcal{W} = (W_r, \dots, W_t)$ and $\mathcal{W}' = (W_i : i \in I \cup \{1\})$; and so $\mathcal{M}$ is β-skew with respect to $C', \mathcal{W}'$. This proves 10.6. ∎

The next two results are lemmas for use in the next section. Let $\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_t)$ be a ξ-clique-multicover of $C \subseteq V(G)$, that is β-skew with respect to $C, \mathcal{W}$. For $1 \leq i \leq t$, let $\mathcal{L}_i = (X_i, N_i)$, and let $\mathcal{W} = (W_1, \dots, W_t)$. Define $W_{t+1} = C$ (thus, $C \cup W_{j+1} \cup \dots \cup W_t = W_{j+1}$ for all $j \in \{1, \dots, t\}$). For $1 \leq i < j \leq t$, let $Z_{i,j}$ be the set of vertices in N_i that have a neighbour in W_j and are anticomplete to W_{j+1}. We call the family of sets $Z_{i,j} (1 \leq i < j \leq t)$ the *standard refinement* of $\mathcal{M}, C$.

10.7 *In the notation just given:*

- *the sets $Z_{i,i+1}, \dots, Z_{i,t}$ are pairwise disjoint subsets of N_i;*
- *X_j is complete to $Z_{i,k}$ for $1 \leq i \leq j < k \leq t$, and to every vertex in N_i with a neighbour in C, for $1 \leq i \leq j$;*
- *X_j is anticomplete to $Z_{i,k}$ for all $i, j, k \in \{1, \dots, t\}$ with $i < k$ if $j < i$ or $k < j$; and*
- *every vertex in N_j is (β, ξ)-earthed via $(Z_{i,j}, W_j)$ for $1 \leq i < j \leq t$.*

Proof. The first statement is clear from the definition. Let $1 \leq i < j \leq t$, and let Z be the set of all vertices in N_i anticomplete to W_{j+1}. Thus $Z = Z_{i,i+1} \cup \dots \cup Z_{i,j} \cup U_i$, where U_i is the set of vertices in N_i anticomplete to W_{i+1}. From the definition of "β-skew", every vertex in $N_i \setminus Z$ is complete to X_j, so the second statement follows if $i < j$; and if $i = j$ then it follows since X_i is complete to N_i. Now X_j is anticomplete to $Z_{i,k}$ if $j < i$ from the definition of a ξ-clique-multicover; and X_j is anticomplete to $Z_{i,k}$ if $k < j$, since $Z_{i,k}$ is anticomplete to $W_{k+1} \supseteq X_j$, so the third statement follows. From the definition of "β-skew", every vertex in N_j is (β, ξ)-earthed via (Z, W_j), and since $Z_{i,j}$ is the set of all vertices in Z that have a neighbour in N_j, the fourth statement follows. This proves 10.7. ∎

10.8 *Let $\xi, \zeta > 0$ and $\tau_1, \tau_2, \beta \geq 0$. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$; and*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*

Let $\mathcal{W} = (W_1, \dots, W_t)$, define $W_{t+1} = C \subseteq V(G)$, let $\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_t)$ be a ξ-clique-multicover of C that is β-skew with respect to $C, \mathcal{W}$, and let $Z_{i,j} (1 \leq i < j \leq t)$ be its standard refinement. Let $1 \leq i < j \leq t$, and let

$$r \in \left(\bigcup_{1 \leq h < i} X_h \cup (N_h \setminus Z_{h,i}) \right) \cup \left(\bigcup_{i \leq h < j} N_h \right) \cup W_{j+1}.$$

Let A be the set of vertices in $V(G)$ that are equal or adjacent to r, or have a neighbour in $Z_{i,j}$ adjacent to r. Then $\chi(A) \leq \tau_2 + (\xi+1)(\tau_1+1)$.

Proof. If r has no neighbour in $Z_{i,j}$ then every vertex in A is equal to or adjacent to r and hence $\chi(A) \leq \tau_1 + 1$ and the result holds. So we may assume that r has a neighbour in $Z_{i,j}$, and so $r \notin W_{j+1}$; choose $h \in \{1, \dots, j-1\}$ with $r \in X_h \cup N_h$.

(1) *One of X_h, X_i is complete to $Z_{i,j} \cup \{r\}$.*

For $r \notin X_i$ by hypothesis, and if $r \in N_i$ then the claim holds, so we may assume that $h \neq i$. If $i < h < j$, then $r \in N_h$ by hypothesis; and then X_h is complete to r and to $Z_{i,j}$ by 10.7. Finally, if $h < i$, then since r has a neighbour in $Z_{i,j}$, it follows that $r \in N_h$. If r is complete to X_i then the claim holds, so we assume not. Consequently 10.7 implies that r has no neighbour in C; and therefore $r \in Z_{h,k}$ for some k. Again, since r is not complete to X_i, 10.7 implies that $k \leq i$. Since r has a neighbour in N_i, it follows that $k = i$, contrary to the hypothesis. This proves (1).

Let X be a ξ-clique that is complete to $Z_{i,j} \cup \{r\}$. Since $N^2(X \cup \{r\}) \leq \tau_2$ (because $X \cup \{r\}$ is a $(\xi+1)$-clique), and X is complete to $Z_{i,j}$, it follows that the set of vertices in A that are adjacent to a neighbour of r in $Z_{i,j}$ and anticomplete to $X \cup \{r\}$ has chromatic number at most τ_2. But the chromatic number of the set of vertices in A that belong to or have a neighbour in $X \cup \{r\}$ is at most $(\xi+1)(\tau_1+1)$; and so $\chi(A) \leq \tau_2 + (\xi+1)(\tau_1+1)$. This proves 10.8. ∎

11 Finding a tree of lamps

Now we come to reap the benefit of all the complications of 10.6; we show that any graph satisfying the conditions of 10.6 contains any given tree of lamps as an induced subgraph, if the number t and the chromatic number are large enough.

Here at last is a definition of a tree of lamps. (See figure 2.) Start with a tree T, and select a vertex of T called the *root*; then every vertex different from the root has a unique *parent*, its neighbour on the path towards the root. Take a map w from $V(T)$ into the set of positive integers, such that

- for all $u, v \in V(T)$, if v is the parent of u then $w(v) > w(u)$ (and consequently the w-value of the root is strictly larger than all the other values);
- there is a vertex v with $w(v) = 1$ (necessarily, either v is the root and $|V(T)| = 1$, or v is a leaf of T);
- for all vertices u, v with $u \neq v$, if $w(u) = w(v)$ then $w(u) = 1$.

We call such a function w a *height function* for T. Let $w(V(T))$ denote the set $\{w(v) : v \in V(T)\}$.

Now choose a set J of integers, each at least 1 and at most the w-value of the root, with $J \cap w(V(T)) = \{1\}$. For each $j \in J$, take a new vertex x_j; and make x_j adjacent to v for every edge uv of T such that $w(v) \leq j$ and $w(u) > j$. (If $|V(T)| = 1$, make x_1 adjacent to the root.) A graph constructed this way is called a *lamp*, and x_1 is its *plug*. Thus every chandelier is a lamp, but many lamps are not chandeliers.

Analogously to trees of chandeliers, we can make trees of lamps, by taking a new lamp, and attaching trees of lamps already constructed to this new lamp by their plugs. However, we are not permitted to attach anything to neighbours of the plug of the new lamp. Let us say this more precisely. A *spotlight* is a one-vertex graph, with plug its vertex. No tree of lamps has negative height; and the spotlight is the only tree of lamps of height zero. Inductively for $r > 0$, having defined trees of lamps of height $\leq r - 1$ and their plugs, we proceed as follows. Let L be a lamp with plug ℓ. For each $v \in V(L)$, let Q_v be a tree of lamps of height at most $r - 1$, such that all the graphs L and Q_v ($v \in V(L)$) are pairwise anticomplete, and such that if v is equal to or adjacent to ℓ, then Q_v is a spotlight. Now identify v with the plug of Q_v, for each $v \in V(L)$. (More precisely, add new edges joining v to every neighbour of the plug of Q_v, and then delete the plug of Q_v, for each $v \in V$.) Let the result be Q. Any such graph Q, with plug ℓ, is said to be a tree of lamps of height $\leq r$ (and so is the spotlight).

We mentioned earlier that we think that not every tree of chandeliers is a tree of lamps; the reason for this (if true) is the more restrictive composition rule. In fact, there is a third class: we have

- trees of lamps (call this $\mathcal{A}$)
- connected induced subgraphs of trees of lamps ($\mathcal{B}$)
- trees of chandeliers ($\mathcal{C}$).

Evidently $\mathcal{A} \subseteq \mathcal{B}$, but we are not sure whether equality holds, or whether $\mathcal{C}$ is a subclass of either of the other two, although we expect the answer is "no" in each case.

We used earlier the fact that for every tree of chandeliers H, there is a tree of lamps Q such that some subdivision of H is an induced subgraph of Q. We leave it to the reader to verify this. (When growing a tree of chandeliers, there is no need to attached new chandeliers to the pivot of what we have already built, because a graph formed by two chandeliers with their pivots identified is an induced subgraph of one bigger chandelier with the same pivot. So, grow it adding one chandelier at a time, and identifying the pivot of the new chandelier with a non-pivot vertex of what we have already built. Now change this; for each new chandelier that we want to attach, first subdivide all the edges incident with its pivot and attach that instead. What we construct is a tree of lamps that is a subdivision of our original tree of chandeliers.)

We will show the following.

11.1 *Let $\xi, \zeta > 0$ and $\tau_1, \tau_2, \tau_3 \geq 0$, and ϕ a nondecreasing function. Let Q be a tree of lamps. Then there exists $c \geq 0$ with the following property. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}_0$ be a ξ-clique-cover of $C \subseteq V(G)$, where $\chi(C) > c$, and let $a \in X(\mathcal{L}_0)$. Then there is an isomorphism from Q to an induced subgraph of G, mapping the plug of Q to a.

Proof. We proceed by induction on $|V(Q)|$. Certainly it is true if $|V(Q)| = 1$, so we assume that $|V(Q)| > 1$ and the result holds for all smaller trees of lamps. Since, up to isomorphism, there are only finitely many smaller trees of lamps, we can choose $c_0 \geq 0$ such that the theorem is true with c replaced by c_0 for every tree of lamps with at most $|V(Q)| - 1$ vertices. Let $\beta = c_0 + |V(Q)|(\tau_2 + (\xi+1)(\tau_1+1))$.

There is a lamp L with plug ℓ say, and trees of lamps Q_v ($v \in V(L)$) such that Q is obtained from L and the graphs Q_v ($v \in V(L)$) as in the definition above.

There is a tree T, a height function w, a set J of integers, and vertices x_j ($j \in J$) in L, as in the definition of a lamp. Choose w such that $w(v)$ is congruent to 1 modulo 3 for all v, and every member of J is also congruent to 1 modulo 3. Let q_0 be the root of T, and let $t = w(q_0)$. Choose c such that 10.6 holds with $c' = 0$. We claim that c satisfies the theorem.

Let $G, \mathcal{L}_0$ and C be as in the theorem. By 10.6, there exist $C' \subseteq C$ with $\chi(C') > 0$, and a C-residue $\mathcal{L}_1$ of $\mathcal{L}_0$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_t$ of C', and $\mathcal{W} = (W_1, \ldots, W_t)$, such that

- $V(\mathcal{L}_i) \subseteq C$ for $2 \leq i \leq t$;
- $\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_t)$ is a ξ-clique-multicover of C', and $\mathcal{W}$ is a world for $\mathcal{W}, C'$; and
- $\mathcal{M}$ is β-skew with respect to $C', \mathcal{W}$.

For $1 \leq i \leq t$ let $\mathcal{L}_i = (X_i, N_i)$, and let $Z_{i,j} (1 \leq i < j \leq t)$ be the standard refinement of $\mathcal{M}, C'$.

Now we begin to construct the isomorphism η from Q to an induced subgraph of G. We recall that q_0 is the root of T; choose some vertex in N_t, and call it $\eta(q_0)$. At a general stage of the process, we will have defined $\eta(p)$ only for the vertices p in a subset $\mathrm{dom}(\eta)$ of $V(Q)$. We will ensure that η is injective, and for all $u, v \in \mathrm{dom}(\eta)$, u, v are adjacent in Q if and only if $\eta(u), \eta(v)$ are adjacent in G. If $|V(T)| = 1$, then $|J| = 1$, and (since no pendant lamp can be attached at the plug or at one of its neighbours) it follows that $|V(Q)| \leq 2$ and the claim is trivial; so we may assume that $|V(T)| \geq 2$.

First we extend $\mathrm{dom}(\eta)$ to equal $V(T)$, in such a way that $\eta(p) \in N_{w(p)}$ for each $p \in V(T)$, by repeating the following process.

- Choose an integer n maximum such that $w(v) = n$ for some $v \in V(T) \setminus \mathrm{dom}(\eta)$. (When $\mathrm{dom}(\eta) = V(T)$, stop).
- Let u be the neighbour of v in $\mathrm{dom}(\eta)$ (necessarily unique). Note that $w(v) < w(u)$.

- Choose a vertex $y \in Z_{w(v),w(u)}$ adjacent to $\eta(u)$ and nonadjacent to all the vertices $\eta(p)(p \in \text{dom}(\eta) \setminus \{u\})$. To see that this is possible, let $p \in \text{dom}(\eta) \setminus \{u\}$. Since $w(u) > w(v) \geq 1$, and therefore $w(p) \neq w(u)$, it follows from 10.8, and from the fact that $\eta(p) \in N(w(p))$, that the set of vertices in $V(G)$ that have a neighbour in $Z_{w(v),w(u)}$ adjacent to $\eta(p)$ has chromatic number at most $\tau_2 + (\xi + 1)(\tau_1 + 1)$. Consequently the set of vertices in $W_{w(u)}$ that have a neighbour in $Z_{w(v),w(u)}$ with a neighbour in $\{\eta(p) : p \in \text{dom}(\eta) \setminus \{u\}\}$ has chromatic number at most $|V(Q)|(\tau_2 + (\xi + 1)(\tau_1 + 1))$. Since $\eta(u)$ is (β, ξ)-earthed via $(Z_{w(v),w(u)}, W_{w(u)})$ by 10.7, and $\beta > |V(Q)|(\tau_2 + (\xi+1)(\tau_1+1))$, there is at least one vertex $x \in W_{w(u)}$ that has a neighbour $y \in Z_{w(v),w(u)}$ adjacent to $\eta(u)$, and has no neighbour in $Z_{w(v),w(u)}$ that is adjacent to any of $\eta(p)(p \in \text{dom}(\eta) \setminus \{u\})$. In particular, y is nonadjacent to all of $\eta(p)(p \in \text{dom}(\eta) \setminus \{u\})$. This shows the existence of the vertex y as claimed.

- Define $\eta(v) = y$, and add v to $\text{dom}(\eta)$.

Note that for all i, j with $1 \leq i < j \leq t$, if some vertex of T is mapped into $Z_{i,j}$ by η, then both i, j are equal to 1 modulo 3.

Next we add all the vertices $x_j (j \in J)$ to $\text{dom}(\eta)$, defining $\eta(x_j)$ to be some vertex in X_j for each $j \in J$, and in particular choosing $\eta(x_1) = a$. We claim that η still defines an isomorphism from $\text{dom}(\eta)$ into $V(G)$. To see this, let $j \in J$ and $v \in V(T)$. We must check that $\eta(x_j), \eta(v)$ are adjacent if and only if either v has a parent u in T and $w(u) > w(x_j) \geq w(v)$, or $i = j = 1$ and $|V(T)| = 1$. Let $v \in Z_{i,k}$ say. If $i > j$ then $\eta(x_j), \eta(v)$ are nonadjacent since X_j is anticomplete to N_i; so we may assume that $i \leq j$. Consequently, if v has no parent, then $i = 1$ and $|V(T)| = 1$, a contradiction; so we may assume that v has a parent u. From the construction, $\eta(u) \in N_k$. Now $Z_{i,k}$ is anticomplete to X_j if $k < j$, from 10.7, so we may assume that $j \leq k$; and so $j < k$ since $k \neq 1$. Thus $i \leq j < k$; and so $\eta(x_j), \eta(v)$ are adjacent since X_j is complete to $Z_{i,k}$ by 10.7. This proves that we can add all the vertices $x_j (j \in J)$ to $\text{dom}(\eta)$ so that η still defines an isomorphism. At this stage, then, $\text{dom}(\eta) = V(L)$.

Now we turn to adding the "pendant" trees of lamps $Q_v (v \in V(L))$. The plug of each Q_v, namely v, already belongs to $\text{dom}(\eta)$, and we must add the other vertices of Q_v; and we shall do so mapping $V(Q_v) \setminus \{v\}$ into $W_{w(v)-1}$. We do them in order: for $n = t, t-3, t-6, \ldots, 1$ in turn, if there is a vertex $v \in \text{dom}(\eta)$, we shall extend $\text{dom}(\eta)$ to include $V(Q_v) \setminus \{v\}$. At the start of a general step of the process, let $R = \{\eta(v) : v \in \text{dom}(\eta)\}$; then $|R| \leq |Q|$, and every $r \in R$ belongs either to W_{n+2}, or to some $X_i \cup N_i$ where $i < n$ and $i = 1$ modulo 3. Moreover, if $R \cap Z_{h,i} \neq \emptyset$ where $h \leq n+1$, then both h, i equal 1 modulo 3.

If $n = 1$, then since all the Q_v are spotlights when $w(v) = 1$, the process stops. So we assume that $n \geq 2$. If there is no $u \in L$ with $w(u) = n$, go on to the next value of n. So now, there is such a vertex u, unique since $n > 1$, and $\eta(u) \in X_n \cup N_n$. Either $u \in V(T)$ or $u = x_n$; the arguments in the two cases are almost identical, but slightly different (this is why we need two values of m in (1)).

(1) *For each $r \in R \setminus \eta(u)\}$, and for $m = n, n+1$, the set of vertices in $V(G)$ that have a neighbour in $Z_{n-1,m}$ adjacent to r has chromatic number at most $\tau_2 + (\xi + 1)(\tau_1 + 1)$.*

Let $r \in R \setminus \{\eta(u)\}$. Then r belongs either to W_{n+2}, or to some $X_i \cup N_i$ where $i < n$ and $i = 1$ modulo 3. Moreover, if $R \cap Z_{h,i} \neq \emptyset$ where $h \leq n+1$, then both h, i equal 1 modulo 3. Since $W_{n+2} \subseteq W_{m+1}$, and $n-1$ does not equal 1 modulo 3, it follows that

$$r \in \left(\bigcup_{1 \leq h < n-1} X_h \cup (N_h \setminus Z_{h,n-1}) \right) \cup \left(\bigcup_{i \leq h < m} N_h \right) \cup W_{m+1}.$$

Hence the claim follows from 10.8. This proves (1).

Now there are two cases, depending whether $u \in V(T)$ or $u = x_n$.

- Assume that $u \in V(T)$. Let Z be the set of vertices in $Z_{n-1,n}$ with no neighbour in $R \setminus \{\eta(u)\}$, and let W be the set of vertices in W_n with no neighbour in $R \setminus \{\eta(u)\}$. By (1), the set of vertices in $V(G)$ that either belong to $W_n \setminus W$ or have a neighbour in $Z_{n-1,n} \setminus Z$ has chromatic number at most $|Q|(\tau_2 + (\xi+1)(\tau_1+1))$; and since $\eta(u)$ is (β, ξ)-earthed via $(Z_{n-1,n}, W_n)$, by 10.7, it follows that $\eta(u)$ is (c_0, ξ)-earthed via (Z, W). From the inductive hypothesis, there is an isomorphism from Q_u to an induced subgraph of $G[Z \cup W \cup \{\eta(u)\}]$, mapping the plug of Q_u to $\eta(u)$. This provides the desired extension of η and $\text{dom}(\eta)$ to include $V(Q_u)$. Then go to the next value of n.

- Assume that $u = x_n$, and so $n < t$ and there are vertices in N_{n+1}; choose one. Since it is (β, ξ)-earthed via $(Z_{n-1,n+1}, W_{n+1})$, by 10.7, it follows that the set of vertices in W_{n+1} that have a neighbour in $Z_{n-1,n+1}$ has chromatic number more than β. Since X_n is anticomplete to W_{n+1} and complete to $Z_{n-1,n+1}$, it follows that $\eta(u)$ is (β, ξ)-earthed via $(Z_{n-1,n+1}, W_{n+1})$.

  Let Z be the set of vertices in $Z_{n-1,n+1}$ with no neighbour in $R \setminus \{\eta(u)\}$, and let W be be the set of vertices in W_{n+1} with no neighbour in $R \setminus \{\eta(u)\}$. By (1), the set of vertices in $V(G)$ that either belong to $W_{n+1} \setminus W$ or have a neighbour in $Z_{n-1,n+1} \setminus Z$ has chromatic number at most $|Q|(\tau_2 + (\xi+1)(\tau_1+1))$; and since $\eta(u)$ is (β, ξ)-earthed via $(Z_{n-1,n+1}, W_{n+1})$, it follows that $\eta(u)$ is (c_0, ξ)-earthed via (Z, W). From the inductive hypothesis, there is an isomorphism from Q_u to an induced subgraph of $G[Z \cup W \cup \{\eta(u)\}]$, mapping the plug of Q_u to $\eta(u)$. This provides the desired extension of η and $\text{dom}(\eta)$ to include $V(Q_u)$. Then go to the next value of n.

This completes the construction of the isomorphism, and so completes the proof of 11.1. ∎

We deduce 9.3, which we restate:

11.2 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2, \tau_3, \nu \geq 0$. Let Q be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\omega(H) \leq \nu$ for each $H \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\chi(N_G^2(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi+1)$-clique X in G;*
- *every member of $\mathcal{C}$ is $(\xi, \zeta+1, \tau_3)$-free;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled; and*

- *no graph in $\mathcal{C}$ contains Q as an induced subgraph.*

Then there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c.

Proof. Choose ϕ such that every graph in $\mathcal{C}$ is (ξ, ζ, ϕ)-multiclique-controlled. Choose c' such that 11.1 is satisfied with c replaced by c', and let $c = \phi(c')$. We claim that c satisfies the theorem. For let $\mathcal{C}$ be as in the theorem, let $G \in \mathcal{C}$, and suppose that $\chi(G) > c$. Since $\chi(G) > \phi(c')$, there is a ξ-clique X_1 of G with $\chi(N^2(X_1)) > c'$. By 11.1, G contains Q as an induced subgraph, a contradiction. This proves that $\chi(G) \leq c$, and so proves 9.3. ∎

12 String graphs

A *curve* means a subset of the plane which is homeomorphic to the interval $[0, 1]$. Given a finite set C of curves in the plane, its *intersection graph* is the graph with vertex set C in which distinct $S, T \in C$ are adjacent if $S \cap T \neq \emptyset$; and the intersection graphs of sets of curves are called *string graphs*. Every string graph can be realized by a set of piecewise linear curves, and in this paper, a *string* means a piecewise linear curve. In this section we prove that the class of string graphs is 3-controlled, and consequently the theorems of this paper can be applied to the class. The proof that they are 3-controlled is a modification and simplification of an argument of McGuinness [11], who showed that a similar statement holds for a triangle-free subclass of string graphs satisfying another condition that we omit.

Let $(v_1, \ldots, v_n)$ be a sequence of distinct vertices of a graph G. We say that $(v_1, \ldots, v_n)$ has the *cross property* if for all h, i, j, k with $1 \leq h < i < j < k \leq n$, if P, Q are paths of G between v_h, v_j and between v_i, v_k respectively, then $V(P)$ is not anticomplete to $V(Q)$. We need the following.

12.1 *Let Δ be a closed disc in the plane, and let C be a finite set of strings all within Δ. Let C_1 be the set of members of C with nonempty intersection with the boundary of Δ. Then C_1 can be ordered as $\{v_1, \ldots, v_n\}$ such that $(v_1, \ldots, v_n)$ has the cross property in the string graph of C.*

Proof. Let G be the string graph of C. Choose a point $d \in bd(\Delta)$ such that every member of C_1 contains a point of $bd(\Delta) \setminus \{d\}$, and for each $x \in C_1$ choose a point $f(x) \in x \cap (bd(\Delta) \setminus \{d\})$. Number C_1 so that the points $f(x)$ ($x \in C_1$) are in clockwise order, starting from d and breaking ties arbitrarily. Let the numbering of C_1 be $\{v_1, \ldots, v_n\}$. If $1 \leq h < i < j < k \leq n$, and P is a path of G between v_h and v_j, then the union of the strings in $V(P)$ is an arcwise connected subset of Δ, containing $f(v_h)$ and $f(v_j)$; and therefore includes a string s with ends $f(v_h)$ and $f(v_j)$ (not necessarily in C) with $s \subseteq \Delta$. Similarly if Q is between v_i, v_k, there is a string t between $f(v_i)$ and $f(v_k)$. The strings s, t intersect, and so one of the strings in $V(P)$ has nonempty intersection with one of the strings in $V(Q)$. This proves 12.1. ∎

A *homomorphism* from a graph H to a graph G is a map $\eta : V(H) \to V(G)$, such that for all adjacent $u, v \in V(H)$, $\eta(u), \eta(v)$ are distinct and adjacent in G.

12.2 *Let G be a non-null string graph. Then there is a graph H and $V = \{v_1, \ldots, v_n\} \subseteq V(H)$, such that*

- *$(v_1, \dots, v_n)$ has the cross property in H;*
- *every vertex in $V(H) \setminus V$ has a neighbour in V;*
- *there is a homomorphism from H to G; and*
- *$\chi(H \setminus V) \geq \chi(G)/2$.*

Proof. We may assume that $\chi(G) \geq 3$ for otherwise the result is trivial. Choose a component D of G with maximum chromatic number, and let $z \in D$. For $i \geq 0$ let L_i be the set of vertices of D with distance i from z. Choose k such that $\chi(L_k) \geq \chi(G)/2$. Thus $k \neq 0$, and if $k = 1$ then let H be the subgraph induced on $L_0 \cup L_1$, and let $n = 1$ and $v_1 = z$, and the theorem holds. So we may assume that $k \geq 2$. Let D' be a component of $G[L_k]$ with maximum chromatic number. The union of the set of strings in D' is a closed arcwise connected subset of the plane, say S_1; and also the union of the strings in $L_0 \cup \dots \cup L_{k-2}$ is nonnull, closed and arcwise connected, say S_2; and $S_1 \cap S_2 = \emptyset$. Consequently there is a closed disc Δ in the plane disjoint from S_2 and with S_1 in its interior. Moreover, we can choose Δ such that for each string in L_{k-1}, its intersection with Δ is the disjoint union of a finite set of strings. Let V be the set of all strings s such that s is a component of the intersection with Δ of a string in L_{k-1}, and let H be the intersection graph of the set of strings $V \cup L_k$. For each $s \in V$, we claim that $s \cap bd(\Delta) \neq \emptyset$. For there exists $t \in L_{k-1}$ such that s is a component of $t \cap \Delta$; then since t is adjacent in G to a vertex in S_2, and consequently $t \cap S_2 \neq \emptyset$, it follows that every component of $t \cap \Delta$ has nonempty intersection with $bd(\Delta)$, and in particular, $s \cap bd(\Delta) \neq \emptyset$ as claimed. The map $\eta : V(H) \to V(G)$ mapping each string in $V(H)$ to the string in $V(G)$ of which it is a component, is a homomorphism. Moreover, let $r \in V(H) \setminus V = L_k$; we claim that r is adjacent in H to a vertex in V. For let $t \in L_{k-1}$ be adjacent to r in G; then $r \cap t \neq \emptyset$, and since $r \subseteq S_1$, it follows that $r \cap s \neq \emptyset$ for some $s \in V$. Consequently r is adjacent in H to a vertex in V. The result follows from 12.1. This proves 12.2. ∎

Finally we need:

12.3 *Let H be a graph, let $V \subseteq V(H)$, and let $V = \{v_1, \dots, v_n\}$ where $(v_1, \dots, v_n)$ has the cross property in H. Assume also that every vertex in $V(H) \setminus V$ has a neighbour in V. Then*

$$\chi^3(H) \geq \chi(H \setminus V)/20.$$

Proof. Let $\kappa = \chi^3(H)$, and suppose that $\chi(H \setminus V) > 20\kappa$. We may assume that H is connected (by choosing a component of H with maximum chromatic number, and working inside that). For each $i \geq 0$, let L_i be the set of vertices of H with distance exactly i from v_1. Choose k such that $\chi(L_k \setminus V) \geq \chi(H \setminus V)/2$. Thus $\chi(L_k \setminus V) > 10\kappa$. Since every vertex in L_k has a neighbour in V, there are disjoint subsets $X_1, \dots, X_n$ of $L_k \setminus V$ with union $L_k \setminus V$, such that every vertex in X_i is adjacent to v_i for $1 \leq i \leq n$. Consequently $\chi(X_i) \leq \kappa$ for $1 \leq i \leq n$.

(1) *There exist a, b, c, d with $1 \leq a < b < c < d \leq n$, such that there is a path of length three between v_a, v_d, and both its internal vertices belong to $L_k \setminus V$, and the subgraph of H induced on $\bigcup_{b \leq i \leq c} X_i$ has chromatic number more than 4κ.*

For $0 \leq h \leq j \leq n$, let $Y(h, j) = \bigcup_{h < i \leq j} X_i$. Let $i_0 = 0$. Inductively, having defined i_{j-1}, choose i_j with $i_{j-1} \le i_j \le n$ minimal such that $\chi(Y(i_{j-1}, i_j)) > 4\kappa$, if such a choice is possible; and otherwise let $i_j = n$ and stop. Let this process stop with $j = t$ and $i_t = n$ say. For $1 \le j < t$, the minimality of i_j implies that $\chi(Y(i_{j-1}, i_j)) \le 5\kappa$, since $\chi(X_{i_j}) \le \kappa$. Also $\chi(Y(i_{t-1}, i_t)) \le 4\kappa$ since the sequence stopped. Since each of $Y(i_0, i_1), Y(i_1, i_2), \dots, Y(i_{t-1}, i_t)$ has chromatic number at most 5κ, and $\chi(L_k \setminus V) > 10\kappa$, there exist h, k with $1 \le h \le k \le t$ and $h + 2 \le k$ such that there is an edge between Y_{i_{h-1}, i_h} and Y_{i_{k-1}, i_k}. Choose j with $h < j < k$; then, taking $b = i_{j-1} + 1$ and $c = i_j$, and choosing $a \le i_{j-1}$ and $d > i_j$ such that there is an edge between X_a and X_d, this proves (1).

Choose a, b, c, d as in (1), and let Q be a path between v_a, v_d of length three.

(2) *For each $v \in \bigcup_{b \le i \le c} X_i$, there is a vertex q of Q such that the distance between v, q is at most three.*

Since $v \in L_k$, there is a path P between v_1, v of length k. Let its vertices be p_0-p_1-$\cdots$-p_k in order, where $p_0 = v_1$ and $p_k = v$. Choose e with $b \le e \le c$ such that v is adjacent to v_e. Then there is a path of H between v_e, v_1 with interior included in $V(P)$. By the cross property, there is a vertex $q \in V(Q)$ that either belongs to $V(P) \cup \{v_e\}$ or has a neighbour in $V(P) \cup \{v_e\}$. Now since the interior vertices of Q belong to L_k, it follows that for $0 \le i \le k-3$, $p_i \notin V(Q)$ and has no neighbour in $V(Q)$. So q equals or is adjacent to one of $p_{k-2}, p_{k-1}, p_k = v, v_e$. In each case the distance between v, q is at most three. This proves (2).

Since the subgraph of H induced on $\bigcup_{b \le i \le c} X_i$ has chromatic number more than 4κ, (2) implies that for one of the four vertices of Q, say q, $\chi(N^3[q]) > \kappa$, a contradiction. Thus $\chi(H \setminus V) \le 20\kappa$. This proves 12.3. ∎

From 12.2 and 12.3, we deduce:

12.4 *For every string graph G, $\chi(G) \le 40\chi^3(G)$.*

Proof. Let G be a string graph, and choose H and V as in 12.2. Thus $\chi(H \setminus V) \ge \chi(G)/2$. By 12.3, $\chi^3(H) \ge \chi(H \setminus V)/20$, and so $\chi^3(H) \ge \chi(G)/40$. But $\chi^3(G) \ge \chi^3(H)$ since there is a homomorphism from H to G. This proves 12.4. ∎

In particular, the class of string graphs is 3-controlled. Since no string graph has an induced subgraph which is a proper subdivision of $K_{3,3}$, 4.2 and 4.3 imply a result mentioned in section 1, which we restate:

12.5 *The class of string graphs is 2-controlled.*

Consequently the theorems of this paper apply to string graphs, and in particular, 9.8 implies a result mentioned in section 1, which we restate:

12.6 *Let $\nu \ge 0$, and let H be a tree of lamps. Then there exists c such that every string graph with clique number at most ν and chromatic number greater than c contains H as an induced subgraph.*

13 Linearity

In this paper we proved many theorems of the form "For all integers $c' \geq 0$ there exists $c \geq 0$ with the following property...", and the reader may have noticed that in each case, we were able to give an explicit formula for c in terms of c' (and other fixed parameters), and the dependence of c on c' is linear. While it seemed not worth the trouble to mention this linearity at each step, it also seems a pity just to ignore it, so let us see what adjustments we need to retain it. First, let us say a class of graphs $\mathcal{C}$ is *linearly ρ-controlled* if there is a linear nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is (ρ, ϕ)-controlled. Then we check that all the claims in this paper about ρ-controlled classes are also true for linearly ρ-controlled classes. For instance, 1.8 becomes

13.1 *Let $\mu \geq 0$ and $\rho \geq 2$, and let $\mathcal{C}$ be a linearly ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is linearly 2-controlled.*

If we wished, we could make the analogous modifications for clique-control and multiclique-control, and then all the results of the paper would have linear analogues. Note that some of these linear analogues are not strengthenings of the original, because for instance, 13.1 needs the stronger hypothesis that $\mathcal{C}$ is linearly ρ-controlled.

Conveniently, 12.4 implies that the class of string graphs is indeed linearly 3-controlled, and so by 13.1, it is also linearly 2-controlled. This answers a question of Bartosz Walczak (private communication).

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