Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544

Alex Scott
Oxford University, Oxford, UK

Paul Seymour[2]
Princeton University, Princeton, NJ 08544

August 15, 2015; revised August 12, 2016

[1]Supported by NSF grants DMS-1265803 and DMS-1550991.
[2]Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.

Abstract

A "string graph" is the intersection graph of a set of curves in the plane. It is known [10] that there are string graphs with clique number two and chromatic number arbitrarily large, and in this paper we study the induced subgraphs of such graphs.

Let us say a graph H is "pervasive" (in some class of graphs) if for all $\ell \geq 1$, and in every graph in the class of bounded clique number and sufficiently large chromatic number, there is an induced subgraph which is a subdivision of H, where every edge of H is replaced by a path of at least ℓ edges. In an earlier paper [4] we showed that K_3 is pervasive (in the class of all graphs).

Which graphs are pervasive in the class of string graphs? It was proved in [3] that every such graph is a "forest of chandeliers": roughly, every block is obtained from a tree by adding a vertex adjacent to its leaves, and there are rules about how the blocks fit together. In this paper we prove the converse, that every forest of chandeliers is pervasive in string graphs. Indeed, for many forests of chandeliers H, and many other graphs, every string graph with bounded clique number and sufficiently large chromatic number contains H as an induced subgraph. (This in fact implies the previous statement,)

In every string graph of very large chromatic number, some vertex has second neighbours with (quite) large chromatic number. This turns out to be a key fact: we will show that every forest of chandeliers is pervasive in every class of graphs with this property. Indeed, all that is needed is that for some fixed r, in every induced subgraph of very large chromatic number, some vertex has rth neighbours with large chromatic number.

General graphs, with no such number r, are more difficult to handle: we suspect that every forest of chandeliers H is pervasive in the class of all graphs, but so far we have only proved this for a few graphs H such as the complete bipartite graph $K_{2,n}$.

1 Introduction

All graphs in this paper are finite and simple, and if G is a graph, $\chi(G)$ denotes its chromatic number, and $\omega(G)$ denotes its clique number, that is, the cardinality of the largest clique of G. This is the fifth in a series of papers on the induced subgraphs that must be present in graphs that have bounded clique number and (sufficiently) large chromatic number. The series was originally motivated by three conjectures of Gyárfás from 1985 [7] concerning the various lengths of induced cycles in such graphs, but we have already proved two of these, in [12] and [4] respectively:

1.1 *For every integer $k \ge 0$, every graph G with $\omega(G) \le k$ and $\chi(G)$ sufficiently large contains an induced cycle of odd length at least* 5.

1.2 *For all integers $k, \ell \ge 0$, every graph G with $\omega(G) \le k$ and $\chi(G)$ sufficiently large contains an induced cycle of length at least ℓ.*

The third conjecture of Gyárfás implies the other two, but is still open, the following.

1.3 *For all integers $k, \ell \ge 0$, every graph G with $\omega(G) \le k$ and $\chi(G)$ sufficiently large contains an induced odd cycle of length at least ℓ.*

We may ask, if G has bounded clique number and very large chromatic number, which graphs must be present in G as an induced subgraph? No graph has this property except for forests, because G can have arbitrarily large girth (and it is an open conjecture of Gyárfás [6] and Sumner [13] that forests do have this property). This is an interesting question but we have nothing to say about it here.

We may ask instead for the graphs H such that every graph G with bounded clique number and sufficiently large chromatic number must contain an induced subgraph which is a subdivision of H. This certainly yields a larger class of graphs; for instance, every cycle has this property, in view of 1.2, and so does every forest, by a theorem of [11], and perhaps so do many more graphs. (For instance, Lévêque, Maffray and Trotignon [8] proved that K_4 has this property.) Solving it would considerably extend 1.2, but unfortunately this too still seems out of reach.

This paper is concerned with subdivisions of a graph, so let us clarify some definitions before we go on. Let H be a graph, and let H' be a graph obtained from H by replacing each edge uv by a path (of length at least one) joining u, v, such that these paths are vertex-disjoint except for their ends. We say that H' is a *subdivision* of H; and it is a *proper* subdivision of H if all the paths have length at least two. If each of the paths has exactly $\ell + 1$ edges we call it an *ℓ-subdivision*; if they each have at least $\ell + 1$ edges it is an *$(\ge \ell)$-subdivision*; and if they all have at most $\ell + 1$ it is an *$(\le \ell)$-subdivision*. If they all have length at least two and at most $\ell + 1$ it is a *proper* $(\le \ell)$-subdivision.

Here is what seems to be a more tractable question of the same type, solving which would also extend 1.2. Let us say a graph H is *pervasive* in some class of graphs $\mathcal{C}$ if for all $\nu, \ell \ge 0$ there exists c such that for every graph $G \in \mathcal{C}$ with $\omega(G) \le \nu$ and $\chi(G) > c$, there is an induced subgraph of G isomorphic to a $(\ge \ell)$-subdivision of H. We say H is *pervasive* if it is pervasive in the class of all graphs. Which graphs are pervasive?

If H' is a subdivision of H, then H' is pervasive if and only if H is pervasive; and 1.2 is equivalent to the statement that all cycles are pervasive (and also equivalent to the assertion that K_3 is pervasive). By the theorem of [11], all forests are pervasive; but what else?

There is a beautiful example of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter and Walczak [10]; they found a sequence of graphs SP_k for $k = 1, 2, \ldots$, each with clique number at most two and with chromatic number at least k. (The same construction was found by Burling [2], but its significance was first pointed out in [10].) Furthermore, it is a *string graph*, the intersection graph of some set of curves in the plane; and consequently there are many graphs H such that no (≥ 2)-subdivision of H appears in any SP_k as an induced subgraph. For every pervasive graph H, some (≥ 2)-subdivision of H must appear in some SP_k as an induced subgraph, and this severely restricts the possibilities for which graphs might be pervasive. This was analyzed in a paper by Chalopin, Esperet, Li and Ossona de Mendez [3], which we explain next.

Let T be a tree with $|V(T)| \geq 2$, and let H be obtained from T by adding a new vertex v and making v adjacent to every leaf of T (and to no other vertex). Then H is called a *chandelier* with *pivot* v. (We also count the one- and two-vertex complete graphs as chandeliers, when some vertex is chosen as pivot.) More generally, if we start with a chandelier, and repeatedly take a new chandelier, and identify its pivot with some vertex of what we have already built, what results is called a *tree of chandeliers*. If every component of G is a tree of chandeliers, G is called a *forest of chandeliers*. Chalopin, Esperet, Li and Ossona de Mendez [3] proved:

1.4 *For every graph H, there is a (≥ 2)-subdivision of H that appears as an induced subgraph in SP_k for some k, if and only if H is a forest of chandeliers.*

It follows that every pervasive graph is a forest of chandeliers; and perhaps the converse is true, that every forest of chandeliers is pervasive. Whether that is true or not, the goal of this paper is to begin to determine which graphs are pervasive; and we achieve this goal for a class of graphs that includes the string graphs. We only have to consider forests of chandeliers, and they have the convenient property that every subdivision of a forest of chandeliers is another forest of chandeliers. Thus, if we could prove that for every forest of chandeliers H, every graph with bounded clique number and sufficiently large chromatic number contains a subdivision of H as an induced subgraph, then it would follow that every forest of chandeliers is pervasive. We can therefore forget about looking for $(\geq \ell)$-subdivisions, and just look for subdivisions.

It is helpful to break this problem down into two steps, which we describe next. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* between two vertices u, v of G is the length of a shortest path between u, v, or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N^\rho_G(v)$ or $N^\rho(v)$ denotes the set of all vertices u with distance exactly ρ from v, and $N^\rho_G[v]$ or $N^\rho[v]$ denotes the set of all v with distance at most ρ from v. If G is a nonnull graph and $\rho \geq 1$, we define $\chi^\rho(G)$ to be the maximum of $\chi(N^\rho[v])$ taken over all vertices v of G. (For the null graph G we define $\chi^\rho(G) = 0$.) Let $\mathbb{N}$ denote the set of nonnegative integers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. For $\rho \geq 1$, let us say a graph G is (ρ, ϕ)-*controlled* if $\chi(H) \leq \phi(\chi^\rho(H))$ for every induced subgraph H of G. Roughly, this says that in every induced subgraph H of G with large chromatic number, there is a vertex v such that $\chi(N^\rho_H[v])$ has large chromatic number. Let us say a class of graphs $\mathcal{C}$ is ρ-*controlled* if there is a nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is (ρ, ϕ)-controlled.

Suppose that we are trying to prove that every graph with bounded clique number and sufficiently large chromatic number has some induced subgraph with property X, whatever that may be. In other words, we wish to show that for all $\nu \geq 0$ there exists n such that for every graph G, if no induced subgraph has property X, and $\omega(G) \leq \nu$, then $\chi(G) \leq n$. We proceed by induction on ν; so fix some

ν, such that the result holds for all smaller ν. Let $\mathcal{C}$ be the class of all graphs G with $\omega(G) \le \nu$ such that no induced subgraph of G has property X. We have to prove an upper bound on the chromatic number of the members of $\mathcal{C}$. One way to do so would be to complete two steps:

- prove that $\mathcal{C}$ is ρ-controlled for some $\rho \ge 1$; and
- use this fact to prove an upper bound on the chromatic number of the members of $\mathcal{C}$.

Let us see in more detail what is involved in these two steps.

- For the first step, we need to find ρ such that for all integers $c \ge 0$, there exists n_c such that $\chi(G) \le n_c$ for all $G \in \mathcal{C}$ with $\chi^{\rho}(G) \le c$. If we can do this, then for each $c \ge 0$ we set $\phi(c) = \max_{0 \le c' \le c} n_{c'}$ (or just arrange that the sequence $n_0, n_1, \ldots$ is nondecreasing, and set $\phi(c) = n_c$ for each c); then every $G \in \mathcal{C}$ is (ρ, ϕ)-controlled, and so $\mathcal{C}$ is ρ-controlled.
- For the second step, we are given that $\mathcal{C}$ is ρ-controlled. We only need to prove an upper bound on $\chi(G)$ ($G \in \mathcal{C}$) for this given choice of ρ; but to do so we prove something more general. We prove that for *all* choices of ρ, and every ρ-controlled subclass $\mathcal{D} \subseteq \mathcal{C}$, there is an upper bound on the chromatic number of the members of $\mathcal{D}$. This approach has the advantage that we can use induction on ρ. The base case of this induction, when $\rho = 1$, is easy because of the induction on ν. Consequently, we can assume that we are looking at a class $\mathcal{D}$ that is not $(\rho - 1)$-controlled; and that means that for some $c \ge 0$, there are induced subgraphs H of members of $\mathcal{D}$, with $\chi^{\rho-1}(H) \le c$ and with $\chi(H)$ arbitrarily large. (In our case, the arguments for $\rho = 2$ and for $\rho = 3$ are different from and more difficult than the argument for larger values of ρ.)

This is quite a powerful proof strategy, and we already use forms of it in several earlier papers; and we try to use it again, this time taking X to be the property of being a subdivision of some fixed graph H.

In this paper we only carry out the step described in the second bullet above; the first bullet is giving us trouble, and so far we have managed it only for very simple-structured graphs H, such as the complete bipartite graph $K_{2,n}$. That being so, it seems that our success with the second bullet is wasted, unless we can complete the corresponding first bullet program. Fortunately not; for there are interesting graphs that automatically fall under the second bullet, and so our results will apply to them, whether we can complete the first bullet program or not. For instance, string graphs have this property; we will prove that the class of string graphs is 2-controlled.

For $m \ge 0$ and $r \ge 1$, we denote the r-subdivision of $K_{m,m}$ by $K^r_{m,m}$. A "lamp" (defined later) is a kind of graph more general than a chandelier, and we will define trees and forests of lamps; but some trees of chandeliers are not trees of lamps, because the composition rule is more restrictive. For every tree of chandeliers H there is a tree of lamps J such that some subdivision of H is an induced subgraph of J, so the two classes are closely related.

Here are our results:

1.5 *Let $\nu \ge 0$, let H be a forest of lamps, and let $\mu \ge 0$. Let $\mathcal{C}$ be a 2-controlled class of graphs. Then there exists c such that every graph G in $\mathcal{C}$ with $\omega(G) \le \nu$ and $\chi(G) > c$ contains one of $K^1_{\mu,\mu}, H$ as an induced subgraph.*

Note the anomaly; for our purposes it would be enough to show that G contains a subdivision of H above, but in fact it will contain H itself. Also, we remark that since $K^1_{\mu,\mu}$ (for μ sufficiently large) has an induced subgraph which is a 3-subdivision of H, for any graph H, it follows that under the same hypotheses, every graph G in $\mathcal{C}$ with $\omega(G) \le \nu$ and $\chi(G) > c$ contains one of H, H^3 as an induced subgraph.

1.6 *For all $\rho \ge 2$, every forest of chandeliers is pervasive in every ρ-controlled class.*

1.7 *Let $\mu \ge 0$, and let $\rho \ge 2$. Let $\mathcal{C}$ be a ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is 2-controlled.*

We will deduce that

1.8 *The class of all string graphs is 2-controlled.*

For string graphs, we have a strengthening of 1.6:

1.9 *Let $\nu \ge 0$, and let H be a forest of lamps. Then there exists c such that every string graph with clique number at most ν and chromatic number greater than c contains H as an induced subgraph.*

This implies:

1.10 *A graph H is pervasive for the class of string graphs if and only if H is a forest of chandeliers.*

Finally, we have a result about general graphs extending 1.2:

1.11 *For all $n \ge 1$, the graph $K_{2,n}$ is pervasive.*

2 Defining SP_k

Before we go on, let us digress to define SP_k. We will not need it in what follows, but there does not seem to be an explicit graph-theoretic description of it published, and our work was greatly influenced by the paper [3], which is based on this construction.

First, here is a composition operation. We start with a graph A, and a stable subset S of A. Let $S = \{a_1, \ldots, a_s\}$ say, and for $1 \le i \le s$ let N_i be the set of neighbours of a_i in A.

Now take a graph consisting of $s+1$ isomorphic copies of $A \setminus S$, say $A_0, \ldots, A_s$, pairwise disjoint and with no edges between them. For $0 \le i, j \le s$, let the isomorphism from $A \setminus S$ to A_i map N_j to N_{ij}. Now add to this $3s^2$ new vertices, namely x_{ij}, y_{ij}, z_{ij} for all i, j with $1 \le i, j \le s$. Also add edges so that x_{ij}, y_{ij} are both adjacent to every vertex in $N_{0,i}$, and x_{ij}, z_{ij} are both adjacent to every vertex in N_{ij}, and $y_{ij}z_{ij}$ an edge, for $1 \le i, j \le s$. Let G be the resulting graph, and let T be the set

$$\{x_{ij}, y_{ij}\ : 1 \le i, j \le s\}.$$

We say that (G, T) is obtained by *composing* (A, S) *with itself.*

To define SP_k let SP_1 be the complete graph K_2, and let $T_1 \subseteq V(SP_1)$ with $|T_1| = 1$. Inductively let (SP_{k+1}, T_{k+1}) be obtained by composing (SP_k, T_k) with itself. It is easy to check that SP_k has no triangles, and for every colouring of SP_k with any number of colours, some vertex in T_k has neighbours of k different colours, and in particular $\chi(SP_k) \ge k+1$. Moreover, there are graphs H such that no subdivision of H appear as an induced subgraph of any SP_k, as discussed in the previous section. SP_k is the only construction known to the authors with this property. Indeed, the following very wild statement might be true as far as we know:

2.1 Conjecture: *For all $m, i, \nu \geq 0$ there exists n such that if G has $\omega(G) \leq \nu$ and $\chi(G) > n$, then either some (≥ 1)-subdivision of K_m appears in G as an induced subgraph, or SP_i appears in G as an induced subgraph.*

We have little faith in this conjecture; indeed we cannot prove it even for graphs G that are themselves induced subgraphs of some SP_k. We could make it more plausible by weakening it to: "For all $i, \nu \geq 0$ there exists n such that if G has $\omega(G) \leq \nu$ and $\chi(G) > n$, then some subdivision of SP_i appears in G as an induced subgraph", and indeed then we think it might well be true; but first we should disprove the stronger form.

3 Two routing lemmas

If X, Y are subsets of the vertex set of a graph G, we say

- X is *complete* to Y if $X \cap Y = \emptyset$ and every vertex in X is adjacent to every vertex in Y;
- X is *anticomplete* to Y if $X \cap Y = \emptyset$ and every vertex in X is nonadjacent to every vertex in Y; and
- X *covers* Y if $X \cap Y = \emptyset$ and every vertex in Y has a neighbour in X.

(If $X = \{v\}$ we say v is complete to Y instead of $\{v\}$, and so on.)

Throughout the paper, we will be applying various forms of Ramsey's theorem. Here is one that contains all that we need.

3.1 *For all integers $k, n, \alpha, \beta \geq 0$ there exists $R(k, n, \alpha, \beta) \geq n$ with the following property. Let A, B be disjoint sets, both of cardinality at least $R(k, n, \alpha, \beta)$. Let E be the set of all sets $X \subseteq A \cup B$ with $|X \cap A| = \alpha$ and $|X \cap B| = \beta$. If we partition E into k subsets, then there exist $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = n$ such that all the sets $X \in E$ with $X \subseteq A' \cup B'$ belong to the same subset.*

Before we begin the main proofs, we prove two lemmas which will be applied later. We are trying to prove that certain graphs G with bounded clique number contain a subdivision of some fixed graph H as an induced subgraph. This is true if G has an induced subgraph which is a proper subdivision of $K_{\mu,\mu}$ for appropriate μ; and so we might as well confine ourselves to graphs G that do not contain (as an induced subgraph) any proper subdivision of $K_{\mu,\mu}$, for some fixed μ. This is a little more than we actually need. For integers $\lambda \geq 2$ and $\mu, \nu \geq 0$, let us say that G is (λ, μ, ν)-*restricted* if $\omega(G) \leq \nu$, and no induced subgraph of G is a proper $(\leq \lambda)$-subdivision of $K_{\mu,\mu}$.

Let G, H be graphs. An *impression* of H in G is a map η with domain $V(H) \cup E(H)$, such that:

- $\eta(v) \in V(G)$ for each $v \in V(H)$;
- for all distinct $u, v \in V(H)$, $\eta(u) \neq \eta(v)$ and $\eta(u), \eta(v)$ are nonadjacent in G;
- for every edge $e = uv$ of H, $\eta(e)$ is a path of G with ends $\eta(u), \eta(v)$;
- if $e, f \in E(H)$ have no common end then $V(\eta(e))$ is anticomplete to $V(\eta(f))$.

The *order* of an impression η is the maximum length of the paths $\eta(e)$ $(e \in E(H))$.

Our first lemma is:

3.2 *For all $\lambda \geq 1$ and $\mu, \nu \geq 0$, there exists m such that if G is (λ, μ, ν)-restricted then there is no impression of $K_{m,m}$ in G of order at most $\lambda + 1$.*

Proof. We proceed by induction on λ. If $\lambda > 1$ choose m_4 such that the theorem is satisfied with λ replaced by $\lambda - 1$ and m by m_4, and if $\lambda = 1$ let $m_4 = 0$. Let

$$\begin{aligned} m_3 &= \max(m_4 + 1, \mu, \nu + 2) \\ m_2 &= R(3^{\lambda^2}, m_3, 2, 1) \\ m_1 &= R(3^{\lambda^2}, m_2, 1, 2) \\ m &= R(\lambda, m_1, 1, 1). \end{aligned}$$

We claim that m satisfies the theorem. For suppose that η is an impression of $K_{m,m}$ in G of order at most $\lambda + 1$.

(1) *$\{\eta(v) : v \in V(H)\}$ is a stable set of G, and if $e \in E(H)$ and $v \in V(H)$ is not incident with e, then $\eta(v)$ does not belong to $\eta(e)$, and has no neighbours in $V(\eta(e))$.*

The first is immediate from the definition of impression. For the second, if $e \in E(H)$ and $v \in V(H)$ not incident with e, then there is an edge f of H incident with v and with no common end with e, and since $V(\eta(e))$ is anticomplete to $V(\eta(f))$, it follows in particular that $\eta(v)$ does not belong to $\eta(e)$, and has no neighbours in $V(\eta(e))$. This proves (1).

Also we might as well assume that each path $\eta(e)$ is an induced path in G. Let (A, B) be a bipartition of $H = K_{m,m}$. There are only λ possibilities for the length of each path $\eta(e)$ ($e \in E(H)$); and so by 3.1, there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| = |B_1| = m_1$ such that the paths $\eta(ab)$ all have the same length, for all $a \in A_1$ and $b \in B_1$. Let this common length be ℓ; thus $2 \leq \ell \leq \lambda + 1$. Let us number the vertices of each path $\eta(ab)$ ($a \in A_1, b \in B_1$) as $p_{ab0}, p_{ab1}, \ldots, p_{ab\ell}$ in order, where $p_{ab0} = \eta(a)$ and $p_{ab\ell} = \eta(b)$.

Take an ordering of B, denoted by $<$. For each $a \in A$ and all $b, b' \in B$ with $b < b'$, let us say the *first pattern* of (a, b, b') is the set of all pairs (i, j) with $1 \leq i, j \leq \ell - 1$ such that $p_{abi} = p_{ab'j}$; and the *second pattern* of (a, b, b') is the set of all pairs (i, j) with $1 \leq i, j \leq \ell - 1$ such that $p_{abi}, p_{ab'j}$ are distinct and adjacent in G. There are only 3^{λ^2} possibilities for the first and second patterns; so by 3.1 there exist $A_2 \in A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2| = m_2$, such that all the triples (a, b, b') (for $a \in A_2$ and $b, b' \in B_2$ with $b < b'$) have the same first patterns and they all have the same second patterns. Let these patterns be Π_1, Π_2 say.

Similarly, by exchanging A, B, choosing an ordering $<$ of A_2 and repeating the argument, we deduce that there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ with $|A_3| = |B_3| = m_3$, and sets $\Pi_3, \Pi_4 \subseteq \{1, \ldots, \ell - 1\}^2$ such that for all $a, a' \in A_3$ with $a < a'$ and $b \in B_4$, $p_{abi} = p_{a'bj}$ if and only if $(i, j) \in \Pi_3$, and $p_{abi}, p_{a'bj}$ are different and adjacent if and only if $(i, j) \in \Pi_4$.

(2) $\Pi_1, \Pi_2 = \emptyset$.

For suppose that there exists $(i, j) \in \Pi_1 \cup \Pi_2$. By reversing the order on B if necessary, we may assume that $i \leq j$. Choose $b_0 \in B_3$, minimal under the ordering of B. For each $a \in A_3$ and

$b \in B_3 \setminus \{b_0\}$, let

$$Q(ab) = \{p_{abj}, p_{ab(j+1)}, , \ldots, p_{ab\ell}\}.$$

Since $(i,j) \in \Pi_1 \cup \Pi_2$, it follows that for each $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, there is a path P_{ab} of G with ends p_{ab_0i}, b and with vertex set a subset of $\{p_{ab_0i}\} \cup Q(ab)$. For each $b \in B_3 \setminus \{b_0\}$ let $\eta'(b) = \eta(b)$; for each $a \in A_3$, let $\eta'(a) = p_{ab_0i}$; and for every edge ab of $H = K_{m,m}$ with $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, let $\eta'(ab) = P_{ab}$. We claim that η' is an impression of K_{m_3+1,m_3} in G. To see this, note first that the vertices $\eta'(a)$ ($a \in A_3$) are all distinct; for choose $b \in B_3 \setminus \{b_0\}$, and let $a, a' \in A_3$ be distinct. Then p_{ab_0i} is equal or adjacent to p_{abj}, but $p_{a'b_0i}$ is different from and nonadjacent to p_{abj} since $V(\eta(a'b_0)), V(\eta(ab))$ are anticomplete, from the definition of an impression. Consequently p_{ab_0i} is different from $p_{a'b_0i}$. If $(i,i) \in \Pi_4$, then all the vertices p_{ab_0i} ($a \in A_3$) are pairwise adjacent, contradicting that $\omega(G) \le \nu$; so $(i,i) \notin \Pi_4$, and the vertices $\eta'(a)$ ($a \in A_3$) are pairwise nonadjacent. Also for each $a \in A_3$ and $b \in B_3 \setminus \{b_0\}$, $\eta'(a)$ is different from and nonadjacent to $\eta'(b)$ by (1). Thus the first three conditions for an impression are satisfied. For the final condition, we must check that if $a, a' \in A_3$ are distinct and $b, b' \in B_3 \setminus \{b_0\}$ are distinct, then $V(P_{ab})$ is anticomplete to $V(P_{a'b'})$. We recall that $V(P_{ab}) \subseteq \{p_{ab_0i}\} \cup Q(ab)$, where $Q(ab)$ is a subset of the vertex set of $\eta(ab)$, and $V(P_{a'b'}) \subseteq \{p_{a'b_0i}\} \cup Q(a'b')$. We have seen that $p_{ab_0i}, p_{a'b_0i}$ are distinct and nonadjacent, so, exchanging a, a' and b, b' if necessary, it suffices to show that $V(P_{ab})$ is anticomplete to $Q(a'b')$. But $V(P_{ab})$ is a subset of $V(\eta(ab_0)) \cup V(\eta(ab))$, and both the latter sets are anticomplete to $V(\eta(a'b')) \supseteq Q(a'b')$. This proves that η' is an impression as claimed.

Since $m_3 - 1 \ge m_4$, the inductive hypothesis on λ implies that the order of η' is at least $\lambda + 1$. But its order is at most $\ell - j + 1$ if $(i,j) \in \Pi_2$, and at most $\ell - j$ if $(i,j) \in \Pi_1$. Since $\ell \le \lambda + 1$ and $j \ge 1$, we deduce that $j = 1$, and $\ell = \lambda + 1$; and so $i = 1$, since $i \le j$, and $(1,1) \in \Pi_2$. Choose $a \in A_3$; then all the vertices p_{ab1} ($b \in B_3 \setminus \{b_0\}$) are distinct and pairwise adjacent, contradicting that $\omega(G) \le \nu$. This proves (2).

Similarly $\Pi_3, \Pi_4 = \emptyset$. But then G contains an ℓ-subdivision of K_{m_3,m_3}, contradicting that G is (λ, μ, ν)-restricted. This proves 3.2. ▮

The second lemma is:

3.3 *For all $\mu, \nu \ge 0$, there exists m with the following property. Let G be $(1, \mu, \nu)$-restricted, and let $X \subseteq V(G)$ with $|X| \ge m$. Then there exist distinct nonadjacent $x, x' \in X$ such that every vertex of G adjacent to both x, x' has at least one more neighbour in X.*

Proof. Choose m_4 such that 3.2 holds with m replaced by m_4. Let

$$\begin{aligned} m_3 &= \max(m_4, \nu + 1); \\ m_2 &= R(4, m_3, 2, 2); \\ m_1 &= 2m_2; \\ m &= R(2, m_1, 2, 0). \end{aligned}$$

We claim that m satisfies the theorem. For suppose that G, X are as in the theorem, and for all distinct nonadjacent $x, x' \in X$ there exists $w(x, x')$ adjacent to both x, x' and nonadjacent to all other vertices in X. Since $\omega(G) \le \nu < m_1$, there is a stable subset X_1 of X with $|X_1| = m_1$, by 3.1. It follows that all the vertices $w(x, x')$ ($x, x' \in M_1, x \neq x'$) are distinct from one another and distinct from the vertices in M_1. Choose two disjoint subsets A_2, B_2 of X_1, both of cardinality m_2. Take an ordering of A_2 and of B_2, both denoted by $<$. Let E be the set of all quadruples (a, a', b, b') such that $a, a' \in A$, $a < a'$, and $b, b' \in B$ and $b < b'$. For all $(a, a', b, b') \in E$, we say the *first pattern* of (a, a', b, b') is 1 or 0 depending whether $w(a, b), w(a', b')$ are adjacent or not; and the *second pattern* is 1 or 0 depending whether $w(a, b'), w(a', b)$ are adjacent or not. There are four possible choices of first and second pattern; so by 3.1 there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ with $|A_3| = |B_3| = m_3$, such that, if E_3 denotes the set of $(a, a', b, b') \in E$ with $a, a' \in A_3$ and $b, b' \in B_3$, then

- either $w(a, b), w(a', b')$ are adjacent for all $(a, a', b, b') \in E_3$, or $w(a, b), w(a', b')$ are nonadjacent for all $(a, a', b, b') \in E_3$; and
- either $w(a, b'), w(a', b)$ are adjacent for all $(a, a', b, b') \in E_3$, or $w(a, b'), w(a', b)$ are nonadjacent for all $(a, a', b, b') \in E_3$.

Suppose that $w(a, b), w(a', b')$ are adjacent for all $(a, a', b, b') \in E_3$. Choose

$$a_1 < a_2 < \cdots < a_{\nu+1} \in A_3$$
$$b_1 < b_2 < \cdots < b_{\nu+1} \in B_3$$

(this is possible since $m_3 \geq \nu+1$); then the vertices $w(a_1, b_1), w(a_2, b_2), \ldots, w(a_{\nu+1}, b_{\nu+1})$ are pairwise adjacent, contradicting that $\omega(G) \leq \nu$. So the nonadjacency alternative holds in the first bullet above, and similarly nonadjacency holds in the second bullet. Let (A', B') be a bipartition of K_{m_3,m_3}, and choose η mapping A' onto A and B' onto B; and for all $a' \in A'$ and $b' \in B'$, let $\eta(a'b')$ be the path of G with vertex set $\{a, w(a, b), b\}$ where $a = \eta(a')$ and $b = \eta(b')$. Then η is an impression of K_{m_3,m_3} in G, of order 2, and the result follows from 3.2. This proves 3.3. ∎

4 Multicovers

A *levelling* in a graph G is a sequence of pairwise disjoint subsets $(L_0, L_1, \ldots, L_k)$ of $V(G)$ such that

- $|L_0| = 1$;
- for $1 \leq i \leq k$, L_{i-1} covers L_i; and
- for $0 \leq i < j \leq k$, if $j > i + 1$ then L_i is anticomplete to L_j.

If $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ is a levelling, L_k is called the *base* of $\mathcal{L}$, and the vertex in L_0 is the *apex* of $\mathcal{L}$, and $L_0 \cup \cdots \cup L_k$ is the *union* of $\mathcal{L}$, denoted by $V(\mathcal{L})$. If $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ and $\mathcal{L}' = (L'_0, L'_1, \ldots, L'_k)$ are levellings, we say that $\mathcal{L}'$ is *contained in* $\mathcal{L}$ if $L'_i \subseteq L_i$ for $0 \leq i \leq k$.

Let $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ be a levelling in G with $k \geq 1$, and let $C \subseteq V(G) \setminus V(\mathcal{L})$. We say that $\mathcal{L}$ is a *k-cover for C* if L_k covers C, and $L_0, \ldots, L_{k-1}$ are anticomplete to C. Let $\mathcal{L} = (L_0, \ldots, L_k)$ be a k-cover of C, with apex x say. If $z \in C$, then z has a neighbour in L_k, and that vertex has a neighbour in L_{k-1}, and so on; and hence there is a path between z and x of length $k+1$, with exactly one vertex in each of $L_0, \ldots, L_k$. Moreover, this path is induced; we call such a path an *$\mathcal{L}$-radius* for z.

For $C \subseteq V(G)$, a *k-multicover for C* in G is a family $\mathcal{M} = (\mathcal{L}_i : i \in I)$, where I is a set of integers, such that

- for $1 \le i \le m$, $\mathcal{L}_i$ is a k-cover for C;
- for $1 \le i < j \le m$, $V(\mathcal{L}_i)$ is disjoint from $V(\mathcal{L}_j)$;
- for all $i, j \in I$ with $i < j$, every vertex in $V(\mathcal{L}_i)$ with a neighbour in $V(\mathcal{L}_j)$ belongs to the base of $\mathcal{L}_i$.

We denote the union of the sets $V(\mathcal{L}_i)$ $(i \in I)$ by $V(\mathcal{M})$. We call $|I|$ the *magnitude* of the multicover. Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ and $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be k-multicovers in G for C and for C', respectively, where $C' \subseteq C$. If $I' \subseteq I$, and $\mathcal{L}'_i$ is contained in $\mathcal{L}_i$ for each $i \in I'$, we say that $\mathcal{M}'$ is *contained in* $\mathcal{M}$.

Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a k-multicover for C in G. Let $z \in V(G) \setminus (V(\mathcal{M}) \cup C)$, and for each $i \in I$ let S_i be an induced path of G between z and the apex x_i say of $\mathcal{L}_i$, such that

- z has no neighbours in $V(\mathcal{M}) \cup C$;
- for each $i \in I$, $V(S_i) \cap (V(\mathcal{M}) \cup C) = \{x_i\}$; and
- for each $i \in I$, every vertex in $V(\mathcal{M}) \cup C$ with a neighbour in $V(S_i) \setminus \{x_i\}$ belongs to $V(\mathcal{L}_i)$.

(We do not require the paths S_i to be pairwise internally disjoint; they may intersect one another arbitrarily.) We say that the family $(S_i\ : i \in I)$ is a *tick* on $(\mathcal{M}, C)$, and z is its *head*, and its *order* is the maximum length of the paths S_i $(i \in I)$. We need to prove the following.

4.1 *For all $k \ge 2$ and $\mu, \nu, \tau, m', c' \ge 0$ there exist $m, c \ge 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^k(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M}$ be a k-multicover for C with magnitude m. Then there exist $C' \subseteq C$ with $\chi(C') \ge c'$, and a k-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with magnitude m', and a tick $(S_i\ : i \in I)$ on $(\mathcal{M}', C')$ of order at most $k + 4$, such that $V(S_i) \subseteq V(\mathcal{M})$ for each $i \in I$.*

The proof breaks into two cases, depending whether $k = 2$ or not. In this section we handle the easier case $k \ge 3$, and postpone $k = 2$ until the next section. When $k \ge 3$, a stronger statement holds, the following:

4.2 *For all $k \ge 3$ and $\tau, m, c' \ge 0$ there exists $c \ge 0$ with the following property. Let G be a graph such that $\chi^k(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a k-multicover for C, with $|I| = m$. Then there exist $C' \subseteq C$ with $\chi(C') \ge c'$, and a k-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with magnitude m, and a tick $(S_i\ : i \in I)$ on $(\mathcal{M}', C')$ with head $z \in C \setminus C'$, such that for each $i \in I$, S_i has length $k + 1$, and $V(S_i) \subseteq V(\mathcal{L}_i) \cup \{z\}$ (and so the paths S_i $(i \in I)$ are pairwise disjoint except for z).*

Proof. Let $c = c' + (mk + 1)\tau$, and let G, C and $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be as in the theorem. Let x_i be the apex of $\mathcal{L}_i$ for each $i \in I$, and let $X = \{x_i : i \in I\}$. For each $i \in I$, let C_i be the set of vertices in C with distance at most k from x_i in G. Then by hypothesis, $\chi(C_i) \le \tau$; let D be the set of vertices in C that do not belong to the union of the sets C_i $(i \in I)$. It follows that $\chi(D) > c - m\tau$. Since $c \ge m\tau$, there exists $z \in D$; choose some such z. For each $i \in I$ let S_i be some $\mathcal{L}_i$-radius for z.

(1) *For all distinct $i, i' \in I$, $x_{i'}$ has no neighbours in $V(S_i)$.*

Suppose that some $x_{i'}$ is adjacent to a vertex in S_i. Since S_i has length $k+1$, and the distance from $x_{i'}$ to z is at least $k+1$ (because $z \notin C_{i'}$), and X is stable, it follows that $x_{i'}$ is adjacent to the neighbour of x_i in S_i; but this contradicts that $\mathcal{M}$ is a mulitcover, since $k \geq 3$. This proves (1).

Let S be the union of the sets $V(S_i)$ $(i \in I)$. Thus $|S| = mk+1$. Let C' be the set of vertices in C with distance at least $k+1$ in G from every vertex in S. Since $X \subseteq S$ it follows that $C' \subseteq D$, and $z \in D \setminus C'$, and $\chi(C') > c - (mk+1)\tau = c'$. For each $j \in I$, let $\mathcal{L}_j = (L_{0j}, \dots, L_{kj})$ say, and for $0 \leq i \leq k$ let L'_{ij} be the set of vertices $v \in L_{ij}$ such that some $\mathcal{L}_j$-radius contains both v and a vertex in C'; and let $\mathcal{L}'_j = (L'_{0j}, \dots, L'_{kj})$. Then $\mathcal{L}'_j$ is a k-covering for C'; let $\mathcal{M}' = (\mathcal{L}'_j : j \in I)$, and then $\mathcal{M}'$ is a k-multicover for C' contained in $\mathcal{M}$. We claim that it satisfies the theorem. Certainly $z \in C \setminus C'$.

(2) *$V(S_i) \cap V(\mathcal{M}') = \{x_i\}$ for each $i \in I$.*

For suppose that $u \in V(S_j) \cap V(\mathcal{M}')$, and choose $j' \in I$ such that $u \in V(\mathcal{L}'_{j'})$. Since $V(S_j) \subseteq V(\mathcal{L}_j)$ and $V(\mathcal{L}'_{j'}) \subseteq V(\mathcal{L}_{j'})$, it follows that $V(\mathcal{L}_j)$ is not disjoint from $V(\mathcal{L}_{j'})$, and so $j' = j$. Since $u \in V(\mathcal{L}'_j)$, there exists i with $0 \leq i \leq k$ such that $u \in L'_{ij}$; and so the distance in G between u and some vertex in C' is at most $k+1-i$. But from the definition of C', since $u \in S$ it follows that this distance is at least $k+1$, and so $i = 0$, that is, $u = x_j$. This proves (2).

(3) *For each $j \in I$, if some $u \in V(S_j)$ is adjacent to some $v \in V(\mathcal{M}') \cup C'$ then $v \in V(\mathcal{L}'_j)$.*

Assume that $u \in V(S_j)$ and $v \in V(\mathcal{M}') \cup C'$ are adjacent. Since $u \in S$ and so has distance at least $k+1$ from every vertex in C', it follows that $v \notin C'$, and so $v \in V(\mathcal{L}'_{j'})$ for some $j' \in I'$. Choose i such that $v \in L'_{ij'}$; then the distance in G between v and some vertex in C' is at most $k+1-i$, and so the distance between u and some vertex in C' is at most $k+2-i$. Since this distance is at least $k+1$, it follows that $i \leq 1$, and so v is equal to or adjacent to $x_{j'}$, and in either case v does not belong to the base of $\mathcal{L}_{j'}$. If u belongs to the base of $\mathcal{L}_j$, then u is adjacent to z (because only one vertex in S_j belongs to the base of $\mathcal{L}_j$, namely the neighbour of z); and since $i \leq 1$, and therefore the distance between u and $x_{j'}$ in G is at most 2, it follows that the distance between z and $x_{j'}$ is at most 3, contrary to the definition of D (since $k \geq 3$). Thus u does not belong to the base of $\mathcal{L}_j$; and since $\mathcal{M}$ is a multicover, it follows that $j = j'$. This proves (3).

From (1), (2) and (3) it follows that $(S_i \ : i \in I)$ is a tick on $(\mathcal{M}', C')$. This proves 4.2. ∎

5 Extracting ticks from 2-multicovers

In this section we prove 4.1 when $k = 2$. We will need the following lemma, proved in [5]:

5.1 *Let $\mathcal{A}$ be a set of nonempty subsets of a finite set V, and let $k \geq 0$ be an integer. Then either:*

- *there exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$;*
- *there are k distinct members $A_1, \dots, A_k \in \mathcal{A}$, and for all i, j with $1 \leq i < j \leq k$ an element $v_{ij} \in V$, such that for all $h, i, j \in \{1, \dots, k\}$ with $i < j$, $v_{ij} \in A_h$ if and only if $h \in \{i, j\}$; or*

- *there exists $X \subseteq V$ with $|X| \le 11(k+4)^5$ such that $X \cap A \neq \emptyset$ for all $A \in \mathcal{A}$.*

The idea of using 5.1 in this context is due to Bousquet and Thomassé [1]. We use it to prove the following.

5.2 *For all $\mu, \nu \ge 0$, there exists $m \ge 0$ with the following property. Let G be $(1, \mu, \nu)$-restricted, and let $X \subseteq V(G)$, such that every two vertices in X have distance at most 2 in G. Then there exists $Y \subseteq V(G)$ with $|Y| \le m$ such that every vertex in $X \setminus Y$ has a neighbour in Y.*

Proof. Choose k such that 3.3 holds with m replaced by k, and let $m = 11(k+4)^5$. We claim that m satisfies the theorem; for let G, X be as in the theorem. For each $x \in X$, let $N[x]$ be the set of all vertices equal to or adjacent in G to x, and let $\mathcal{A}$ be the set $\{N[x] : x \in X\}$. By hypothesis, no two members of $\mathcal{A}$ are disjoint. Suppose that there are k distinct members $A_1, \dots, A_k \in \mathcal{A}$, and for all i, j with $1 \le i < j \le k$ a vertex $v_{ij} \in V(G)$, such that for all $h, i, j \in \{1, \dots, k\}$ with $i < j$, $v_{ij} \in A_h$ if and only if $h \in \{i, j\}$. For $1 \le i \le k$, let $A_i = N[x_i]$; then for all i, j with $1 \le i < j \le k$, either x_i, x_j are adjacent in G or there is a vertex adjacent to x_i and to x_j, and nonadjacent to all other vertices in $\{x_1, \dots, x_k\}$. But this is impossible from the choice of k.

From 5.1 we deduce that there exists $Y \subseteq V$ with $|Y| \le 11(k+4)^5 = m$ such that $Y \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. But then every vertex in X either belongs to Y or has a neighbour in Y. This proves 5.2. ∎

If $\mathcal{M} = (\mathcal{L}_i : i \in I)$ is a 2-multicover of C, and $i, j \in I$ are distinct, and $z \in C$, let P, Q be $\mathcal{L}_i$- and $\mathcal{L}_j$-radii for z respectively; then $P \cup Q$ is a path of G (not necessarily induced), and we call such a path an $(\mathcal{L}_i, \mathcal{L}_j)$-*diameter*. We need another lemma.

5.3 *For all $\mu, \nu, \tau, c' \ge 0$ and $m > 0$ there exist $c \ge 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^2(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a 2-multicover for C with $|I| = m$. Let x_i be the apex of $\mathcal{L}_i$ for $i \in I$. Let $k \in I$ be maximum. For each $g \in I \setminus \{k\}$, there exist*

- *a subset $I' \subseteq I \setminus \{k\}$ with $|I'| \ge m/2$ and with $\{i \in I : i \le g\} \subseteq I'$;*
- *a subset $C' \subseteq C$ with $\chi(C') > c'$;*
- *for each $i \in I'$, a 2-cover $\mathcal{L}'_i$ for C' contained in $\mathcal{L}_i$; and*
- *an $(\mathcal{L}_g, \mathcal{L}_k)$-diameter S, such that $V(S)$ is anticomplete to C', and $V(S)$ is anticomplete to $V(\mathcal{L}'_i)$ for each $i \in I' \setminus \{g\}$, and $V(S) \cap V(\mathcal{L}_g) = \{x_g\}$.*

Proof. Choose m_0 such that 5.2 holds with m replaced by m_0. Let $c = \max(m_0\tau, 12\tau + c'2^{m+1})$. We claim that c satisfies the theorem. For let $G, C, \mathcal{M} = (\mathcal{L}_i : i \in I), k, g$ be as in the theorem, where $\mathcal{L}_i = (\{x_i\}, A_i, B_i)$ for each $i \in I$, say. We may assume that every vertex in B_g has a neighbour in C, because any other vertex of B_g can be deleted. Suppose that $Y \subseteq V(G)$, and every vertex in $B_g \setminus Y$ has a neighbour in Y. Then every vertex in C has distance at most two from a vertex in Y, and so $\chi(C) \le |Y|\tau$; and since $\chi(C) > c$, it follows that $|Y| > c\tau^{-1} > m_0$. From 5.2, there exist $y_1, y_2 \in B_g$ with distance at least three in G. Choose $z_1, z_2 \in C$ adjacent to y_1, y_2 respectively. Let S_1 be an $(\mathcal{L}_g, \mathcal{L}_k)$-diameter containing y_1 and z_1, and choose S_2 for y_2, z_2 similarly. The union of S_1 and S_2 has at most 12 vertices, and so the set of vertices in C with distance at most two from a vertex in

$S_1 \cup S_2$ has chromatic number at most 12τ. Consequently there exists $C_1 \subseteq C$ with $\chi(C_1) > c - 12\tau$ such that every vertex in C_1 has distance at least three from every vertex in $S_1 \cup S_2$. For $1 \le i \le g$, let $\mathcal{L}'_i$ be the levelling $(\{x_i\}, A'_i, B'_i)$, where B'_i is the set of vertices in B_i with a neighbour in C_1, and A'_i is the set of vertices in A_i with a neighbour in B'_i. Then $V(S_1 \cup S_2) \cap V(\mathcal{L}'_g) = \{x_g\}$, because every vertex in C_1 has distance at least three from $S_1 \cup S_2$. Also $V(S_1 \cup S_2)$ is anticomplete to $V(\mathcal{L}'_i)$ if $i < g$, since every vertex in $V(\mathcal{L}_i)$ with a neighbour in $S_1 \cup S_2$ belongs to B_i (from the definition of a 2-multicover) and hence does not belong to B'_i (because vertices in B'_i have neighbours in C_1 and therefore have no neighbours in $S_1 \cup S_2$).

Now we shall choose one of S_1, S_2 to satisfy the other requirements of the theorem. For each $j \in I \setminus \{k\}$ with $j > g$ and each $v \in C_1$, let P_{jv} be an $\mathcal{L}_j$-radius for v. For $x \in C_1$ for the moment. Now P_{jv} has length three; let its vertices be x_j-a_{jv}-b_{jv}-v in order. Since $v \in C_1$ and therefore has distance three from every vertex in $S_1 \cup S_2$, it follows that b_{jv} has no neighbour in $S_1 \cup S_2$; but a_{jv} might have neighbours in $S_1 \cup S_2$. From the definition of a multicover, every neighbour of a_{jv} in $S_1 \cup S_2$ is one of y_1, y_2; and since y_1, y_2 have distance at least three in G, a_{jv} is not adjacent to them both. Consequently $V(P_{jv})$ is anticomplete to at least one of S_1, S_2. Choose $I_v \subseteq I \setminus \{k\}$ including $\{i \in I : i \le g\}$, with $|I_v| \ge m/2$, such that for one of S_1, S_2 (say S_v), each of the paths P_{jv} ($j \in I_v, j > g$) is anticomplete to S_v. There are only 2^{m+1} possibilities for the pair (S_v, I_v); and so there exists $C' \subseteq C_1$ with $\chi(C') \ge \chi(C_1)2^{-m-1} > c'$, and one of S_1, S_2, say S, and a set I', such that $S_v = S$ and $I_v = I'$ for all $v \in C'$. For each $j \in I \setminus \{k\}$ with $j > g$, let $\mathcal{L}'_j$ be the levelling $(\{x_j\}, A'_j, B'_j)$, where $A'_j = \{a_{jv} : v \in C'\}$ and $B'_j = \{b_{jv} : v \in C'\}$. Then the theorem is satisfied. This proves 5.3. ∎

We deduce:

5.4 *For all $\mu, \nu, \tau, c' \ge 0$, and $t > 0$, and $m \ge t2^t$, there exist $c \ge 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^2(G) \le \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a 2-multicover for C with $|I| = m$. Let $k \in I$ be maximum. Then there exist*

- *a subset $I' \subseteq I \setminus \{k\}$ with $|I'| \ge m2^{-t}$; $I' = \{i_1, \dots, i_n\}$ say, where $i_1 < i_2 < \dots < i_n$;*
- *a subset $C' \subseteq C$ with $\chi(C') > c'$;*
- *for each $i \in I'$, a 2-cover $\mathcal{L}'_i$ for C', contained in $\mathcal{L}_i$;*
- *for each $i \in \{i_1, \dots, i_t\}$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C', and $V(S_i)$ is anticomplete to $V(\mathcal{L}'_j)$ for all $j \in I' \setminus \{i\}$, and $V(S_i) \cap V(\mathcal{L}'_i) = \{x_i\}$.*

Proof. We assume first that $t = 1$. Choose c such that 5.3 is satisfied. Choose $g \in I$, minimum; then the result follows from 5.3. Thus the result holds if $t = 1$.

We fix μ, ν, τ, m, and proceed by induction on t (assuming $m \ge t2^t$). Thus we assume that $t > 1$ and the result holds with t replaced by $t - 1$. Choose c'' such that 5.3 is satisfied with c replaced by c'' (and the given value of m). Let c have the value that satisfies the theorem with t, c' replaced by $t - 1, c''$; we claim that c satisfies the theorem.

For let G, C and $\mathcal{M} = (\mathcal{L}_i : i \in I), k$ be as in the theorem, where $|I| = m \ge t2^t$. From the inductive hypothesis, there exist

- a subset $I'' \subseteq I \setminus \{k\}$ with $|I''| \ge m2^{1-t}$; $I'' = \{i_1, \dots, i_n\}$ say, where $i_1 < i_2 < \dots < i_n$;

- a subset $C'' \subseteq C$ with $\chi(C'') > c''$;
- for each $i \in I''$, a 2-cover $\mathcal{L}''_i$ for C'', contained in $\mathcal{L}_i$;
- for each $i \in \{i_1, \dots, i_{t-1}\}$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C'', and $V(S_i)$ is anticomplete to $V(\mathcal{L}''_j)$ for all $j \in I'' \setminus \{i\}$, and $V(S_i) \cap V(\mathcal{L}''_i) = \{x_i\}$.

Let $\mathcal{L}''_k = \mathcal{L}_k$. Thus $\mathcal{M}'' = (\mathcal{L}''_i : i \in I'' \cup \{k\})$ is a 2-multicover of C'', contained in $\mathcal{M}$. Also $n \geq 2t$, since $n \geq m2^{1-t}$ and $m \geq t2^t$. From 5.3 applied to $\mathcal{M}''$ taking $g = i_t$, we deduce that there exist

- a subset $I' \subseteq I''$ with $|I'| \geq (|I''| + 1)/2 \geq m2^{-t}$ and with $\{i_1, \dots, i_t\} \subseteq I'$;
- a subset $C' \subseteq C''$ with $\chi(C') > c'$;
- for each $i \in I'$, a 2-cover $\mathcal{L}'_i$ for C' contained in $\mathcal{L}''_i$;
- an $(\mathcal{L}''_{i_t}, \mathcal{L}''_k)$-diameter S_{i_t} (which is therefore also an $(\mathcal{L}_{i_t}, \mathcal{L}_k)$-diameter), such that $V(S_{i_t})$ is anticomplete to C', and $V(S_{i_t})$ is anticomplete to $V(\mathcal{L}'_i)$ for all $i \in I' \setminus \{i_t\}$, and $V(S_{i_t}) \cap V(\mathcal{L}'_{i_t}) = \{x_{i_t}\}$.

But then I', C', $\mathcal{L}'_i$ $(i \in I')$, and the paths S_i $(i \in \{i_1, \dots, i_t\})$ satisfy the theorem. This proves 5.4. ▮

Now we prove the main result of this section, the case of 4.1 for 2-multicovers:

5.5 *For all $\mu, \nu, \tau, c' \geq 0$ and $m' > 0$ there exist $m, c \geq 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph such that $\chi^2(G) \leq \tau$. Let $C \subseteq V(G)$ with $\chi(C) > c$, and let $\mathcal{M}$ be a 2-multicover for C, with magnitude m. Then there exist $C' \subseteq C$ with $\chi(C') \geq c'$, and a 2-multicover $\mathcal{M}'$ for C' contained in $\mathcal{M}$ with magnitude m', and a tick $(S_i : i \in I)$ on $(\mathcal{M}', C')$ of order at most 6, such that $V(S_i) \subseteq V(\mathcal{M})$ for each $i \in I$.*

Proof. Let $m = m'2^{m'}$ and let c satisfy 5.4 with this choice of m. We claim that m, c satisfy the theorem. For let G, C and $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be as in the theorem. For each $i \in I$, let $\mathcal{L}_i = (\{x_i\}, A_i, B_i)$ say.

Let $k \in I$ be maximum. We may assume that $|I| = m'2^{m'}$. By 5.4 applied to $\mathcal{M}$, there exist

- a subset $I' \subseteq I \setminus \{k\}$ with $|I'| = t = |I|2^{-t}$;
- a subset $C' \subseteq C''$ with $\chi(C') > c'$;
- for each $i \in I'$, a 2-cover $\mathcal{L}'_i$ for C', contained in $\mathcal{L}_i$;
- for each $i \in I'$, an $(\mathcal{L}_i, \mathcal{L}_k)$-diameter S_i, such that $V(S_i)$ is anticomplete to C', and $V(S_i) \cap V(\mathcal{L}'_i) = \{x_i\}$, and $V(S_i)$ is anticomplete to $V(\mathcal{L}'_j)$ for all $j \in I' \setminus \{i\}$.

Let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$. Then $\mathcal{M}'$ is a 2-multicover of C', and $(S_i : i \in I')$ is a tick on $(\mathcal{M}', C')$ of order at most six, with head x_k. This proves 5.5. ▮

This therefore also completes the proof of 4.1. Let us apply it before we go on. By starting with a k-multicover $\mathcal{M} = (\mathcal{L}_i : i \in I)$ with $|I|$ large enough, for a set C with chromatic number large enough, and applying 4.1 repeatedly, we obtain a sequence of subsets of I, each a subset of its predecessor, and a sequence of multicovers, each contained in its predecessor, and a sequence of ticks all on the last multicover of the sequence $\mathcal{M}'$ say. The ticks are vertex-disjoint except for their vertices in $V(\mathcal{M}')$. There may be edges between them, but if say $(S_i : i \in I)$ and $(T_i : i \in I)$ are two of these ticks, and some vertex in S_i is adjacent to some vertex in T_j, then $i = j$. Consequently we have obtained an impression of $K_{n,n}$ of order at most $k+4$, with n large, which is impossible if G is $(k+3, \mu, \nu)$-restricted. We deduce:

5.6 *For all $k \geq 2$ and $\mu, \nu, \tau \geq 0$ there exist $m, c \geq 0$ with the following property. Let G be a $(k+3, \mu, \nu)$-restricted graph such that $\chi^k(G) \leq \tau$. If $C \subseteq V(G)$ with $\chi(C) > c$, then there is no k-multicover of C in G with magnitude m.*

In [4] we proved an analogue of 5.6 for $k = 1$, but it only applies to "independent" 1-multicovers. Let us say a 1-multicover $\mathcal{M} = (\mathcal{L}_i : i \in I)$ is *independent* if for all $i, j \in I$ with $i < j$, the apex of $\mathcal{L}_j$ has no neighbour in the base of $\mathcal{L}_i$. (Thus, any edge between $V(\mathcal{L}_i)$ and $V(\mathcal{L}_j)$ is between the two bases.) A warning: in [4] we used the term "multicover" to mean what in this paper is called an independent 1-multicover. The result of [4] that we need is the following.

5.7 *For all $\mu, \nu, \tau \geq 0$ there exist $m, c \geq 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph, such that $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \nu$. If $C \subseteq V(G)$ with $\chi(C) > c$, then there is no independent 1-multicover of C in G with magnitude m.*

6 Reducing control

In this section we prove a result of great importance (for us), the following.

6.1 *Let $\mu, \nu \geq 0$ and $\rho \geq 2$. Every ρ-controlled class of $(\rho+2, \mu, \nu)$-restricted graphs is 2-controlled.*

Proof. The result is trivial for $\rho = 2$, and we proceed by induction on ρ. Let $\rho \geq 3$, and let $\mathcal{C}$ be a ρ-controlled class of $(\rho+2, \mu, \nu)$-restricted graphs. Let ϕ be nondecreasing such that every graph in $\mathcal{C}$ is (ρ, ϕ)-controlled. Let $\mathcal{C}^+$ be the class of all induced subgraphs of graphs in $\mathcal{C}$. The graphs in $\mathcal{C}^+$ are also (ρ, ϕ)-controlled and $(\rho+2, \mu, \nu)$-restricted.

Let $\tau \geq 0$, and let $\mathcal{D}$ be the set of all graphs $H \in \mathcal{C}^+$ with $\chi^{\rho-1}(H) \leq \tau$. Let m, c satisfy 5.6, setting $k = \rho - 1$. Then from 5.6 we have:

(1) *If $G \in \mathcal{D}$, and $C \subseteq V(H)$ with $\chi(C) > c$, then there is no $(\rho-1)$-multicover of C in H with magnitude m.*

Define $c_0 = c$, and inductively $c_t = \phi(c_{t-1} + \tau)$ for $t \geq 1$. We claim:

(2) *For $0 \leq t \leq m$, if $H \in \mathcal{D}$, and $C \subseteq V(H)$ with $\chi(C) > c_t$, then there is no $(\rho-1)$-multicover of C in H with magnitude $m - t$.*

For this is true if $t = 0$, by (1). Let $t \geq 1$ with $t \leq m$, and suppose that the claim holds for

$t-1$. Let $H \in \mathcal{D}$, and $C \subseteq V(H)$ with $\chi(C) > c_t$, and suppose that $\mathcal{M}$ is a $(\rho-1)$-multicover of C in H with magnitude $m-t$. Let $\mathcal{M} = (\mathcal{L}_i : i \in \{1, \dots, m-t\})$ say. Let $J = H[C]$. Since H is (ρ, ϕ)-controlled, $\chi(J) \le \phi(\chi^\rho(J))$, and therefore

$$\phi(c_{t-1} + \tau) = c_t < \phi(\chi^\rho(J)).$$

Consequently $c_{t-1} + \tau < \chi^\rho(J)$ since ϕ is nondecreasing; that is, $c_{t-1} + \tau < \chi(N_J^\rho[v])$ for some vertex v of J. Since $\chi(N_J^{\rho-1}[v]) \le \tau$, it follows that $c_{t-1} < \chi(N_J^\rho(v))$. For $0 \le i \le \rho$, let L_i be the set of vertices in $V(J)$ with distance i from v in J. In particular, $N_J^\rho(v) = L_\rho$, and so $\chi(L_\rho) > c_{t-1}$. Now $(L_0, \dots, L_\rho)$ is a levelling in J and hence in H; let $\mathcal{L}_{m-t+1} = (L_0, \dots, L_{\rho-1})$. Then $\mathcal{L}_{m-t+1}$ is a $(\rho-1)$-cover of L_ρ, and so $(\mathcal{L}_i : i \in \{1, \dots, m-t+1\})$ is a $(\rho-1)$-multicover of L_ρ in H with magnitude $m-t+1$. Since $\chi(L_\rho) > c_{t-1}$, this contradicts the inductive hypothesis. Consequently there is no such $\mathcal{M}$. This completes the inductive proof of (2).

(3) *If $H \in \mathcal{D}$, then $\chi(H) \le c_m$.*

This follows from (2) by setting $t = m$.

(4) *For all $\tau \ge 0$, there exists $\phi'(\tau)$ such that $\chi(H) \le \phi'(\chi^{\rho-1}(H))$ for all $H \in \mathcal{C}^+$.*

At the start of the proof we made an arbitrary choice of τ, and all the subsequent variables in (2) and (3) (such as $\mathcal{D}, m$ and the sequence $c_0, c_1, \dots$) depend on τ. In particular, c_m is a function of τ, say $\phi'(\tau)$. If $H \in \mathcal{C}^+$, then setting $\tau = \chi^{\rho-1}(H)$ in (3) implies that $\chi(H) \le \phi'(\chi^{\rho-1}(H))$. This proves (4).

We may assume that ϕ' is nondecreasing. Then (4) asserts that all graphs in $\mathcal{C}$ are $(\rho-1, \phi')$-controlled, and so $\mathcal{C}$ is $(\rho-1)$-controlled, and hence 2-controlled, from the inductive hypothesis. This proves 6.1. ∎

Let us deduce 1.7, and before that, here is a useful lemma.

6.2 *Let $\rho \ge 2$, and let $\mathcal{C}$ be a class of graphs, such that for all $\nu \ge 0$, the class $\mathcal{C}_\nu$ of graphs $G \in \mathcal{C}$ with $\omega(G) \le \nu$ is ρ-controlled. Then $\mathcal{C}$ is ρ-controlled.*

Proof. For each $\nu \ge 0$, let ϕ_ν be a function such that each graph G in $\mathcal{C}_\nu$ is (ρ, ϕ_ν)-controlled. For $c \ge 0$, let $\psi(c) = \max_{\nu \le c} \phi_\nu(c)$. We claim that $\mathcal{C}$ is (ρ, ψ)-controlled. For let $G \in \mathcal{C}$, and let H be an induced subgraph of G such that $\chi(H) > \psi(c)$, for some c. Let $\nu = \omega(G)$. If $\nu > c$, then choose a clique X of H with $|X| > c$, and choose $v \in X$; then X belongs to $N_H^\rho[v]$, and so $\chi^\rho(H) \ge |X| > c$ as required. Thus we may assume that $\nu \le c$, and so $\chi(H) > \phi_\nu(c)$. Since G is (ρ, ϕ_ν)-controlled, it follows that $\chi^\rho(H) > c$ as required. This proves 6.2. ∎

Now we prove 1.7, which we restate.

6.3 *Let $\mu \ge 0$ and $\rho \ge 2$, and let $\mathcal{C}$ be a ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \dots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is 2-controlled.*

Proof. Let $\mathcal{D}$ be the class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph. Let $\nu \geq 0$, and let $\mathcal{D}_\nu$ be the class of all graphs $G \in \mathcal{D}$ with $\omega(G) \leq \nu$. Every graph in $\mathcal{D}_\nu$ is therefore $(\rho+2, \mu, \nu)$-restricted, and so $\mathcal{D}_\nu$ is 2-controlled by 6.1. From 6.2 it follows that $\mathcal{D}$ is 2-controlled. This proves 6.3. ∎

7 Clique control

Now we come to the second part of the paper, in which we handle 2-controlled graphs. We will follow the approach taken in [4]; and in particular, it will be helpful to introduce a refinement of control, called "clique-control". If X is a clique with $|X| = \xi$ we call X a *ξ-clique.* We denote by $N^1_G(X)$ the set of all vertices in $V(G) \setminus X$ that are complete to X; and by $N^2_G(X)$ the set of all vertices in $V(G) \setminus X$ with a neighbour in $N^1(X)$ and with no neighbour in X. When $X = \{v\}$ we write $N^i_G(v)$ for $N^i_G(X)$ $(i = 1, 2)$. We are assuming that in every induced subgraph H of large χ, there is a vertex v such that $N^2_H(v)$ also has large χ; and perhaps the same is true for cliques larger than singletons. For instance, it may or may not be true that in every induced subgraph H of large χ, there is a 2-clique X such that $N^2_H(X)$ also has large χ. If this is false, we can find induced subgraphs H of arbitrarily large χ such that $N^2_H(X)$ has bounded χ for all 2-cliques X, and we focus on these subgraphs. If it is true, then we ask the same question for triples, and so on; we must soon hit a clique-size for which the answer is "false", because none of our graphs have a clique larger than ν. Let us say this more precisely.

If $\mathcal{C}$ is a class of graphs, we denote by $\mathcal{C}^+$ the class of all induced subgraphs of members of $\mathcal{C}$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, and let $\xi \geq 1$ be an integer. We say a graph G is *(ξ, ϕ)-clique-controlled* if for every induced subgraph H of G and every integer $n \geq 0$, if $\chi(H) > \phi(n)$ then there is a ξ-clique X of H such that $\chi(N^2(X)) > n$. Roughly, this means that in every induced subgraph H of large chromatic number, there is a ξ-clique X with $N^2_H(X)$ of large chromatic number. We say a class of graphs $\mathcal{C}$ is *ξ-clique-controlled* if there is a nondecreasing function ϕ such that every graph in $\mathcal{C}$ is (ξ, ϕ)-clique-controlled. Since we are now concerned with a 2-controlled class of graphs, it is 1-clique-controlled (with some care for the difference between $N^1(v)$ and $N^1[v]$); and since there is an upper bound on the clique number of all these graphs, there is certainly a largest ξ such that the class is ξ-clique-controlled, and the next result exploits this. (Note that this works in the opposite direction from control; there we chose ρ *minimum* such that the class was ρ-controlled.)

7.1 *Let $\nu \geq 1$ and $\tau_1 \geq 0$, and let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is 2-controlled;*
- *$\omega(G) \leq \nu$ for each $G \in \mathcal{C}$;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$; and*
- *there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number.*

Then there exist ξ with $1 \leq \xi \leq \nu$ and $\tau_2 \geq 0$ with the following properties:

- *$\mathcal{C}$ is ξ-clique-controlled; and*

- *for all $c \geq 0$ there is a graph $H \in \mathcal{C}^+$ with $\chi(H) > c$, such that $\chi(N^2_H(X)) \leq \tau_2$ for every $(\xi+1)$-clique X of H.*

Proof. Suppose that $\mathcal{C}$ is ν-clique-controlled, and choose a function ϕ such that every graph in $\mathcal{C}$ is (ν, ϕ)-clique-controlled. Let $c = \phi(0)$; then by hypothesis, there exists $G \in \mathcal{C}$ with $\chi(G) > c$. From the definition of (ν, ϕ)-clique-controlled, there is a ν-clique X in G with $\chi(N^2(X)) > 0$, which is impossible since $N^1(X) = \emptyset$ (because $\omega(G) \leq \nu$).

This proves that $\mathcal{C}$ is not ν-clique-controlled. We claim that $\mathcal{C}$ is 1-clique-controlled. For choose ϕ such that every graph in $\mathcal{C}$ is $(2, \phi)$-controlled; and let $\phi'(c) = \phi(c + \tau + 1)$ for each $c \geq 0$. We claim that every $G \in \mathcal{C}$ is $(1, \phi)$-clique-controlled. For let $c \geq 0$, and let H be an induced subgraph of $G \in \mathcal{C}$, with $\chi(H) > \phi'(c)$. Then $\chi(H) > \phi(c + \tau + 1)$, and since G is $(2, \phi)$-controlled, it follows that $\chi^2(H) > c + \tau_1 + 1$. Hence there is a vertex v of H such that $\chi(N^2_H[v]) > c + \tau_1 + 1$. Now $\chi(N^1_H[v]) \leq \tau_1 + 1$, since the subgraph of H induced on $N^1_H(v)$ has clique number at most $\nu - 1$. Consequently $\chi(N^2_H(v)) > c$. This proves that $\mathcal{C}$ is 1-clique-controlled.

Choose ξ maximum such that $\mathcal{C}$ is ξ-clique-controlled; then $1 \leq \xi < \nu$. Suppose that for all $\kappa \geq 0$, there exists m_κ such that for every $G \in \mathcal{C}$ and every induced subgraph H of G with $\chi(H) > m_\kappa$, there is a $(\xi+1)$-clique X of H with $\chi(N^2_H(X)) > \kappa$. Then G is $(\xi+1, \phi')$-clique-controlled, where we define $\phi'(\kappa) = m_\kappa$ for each $\kappa \geq 0$ (having arranged that $m_0 \leq m_1 \leq \ldots$). Consequently $\mathcal{C}$ is $(\xi+1)$-clique-controlled, a contradiction.

Thus there exists $\kappa \geq 0$ such that for all c, there are graphs $H \in \mathcal{C}^+$ such that $\chi(H) > c$ and $\chi(N^2_H(X)) \leq \kappa$ for every $(\xi+1)$-clique X of H. Let $\tau_2 = \kappa$. This proves 7.1. ▮

We already mentioned independent 1-multicovers earlier; now we need a generalization, using ξ-cliques instead of singletons. Let G be a graph, and $X, N, W, C \subseteq V(G)$, such that

- X, N, C are pairwise disjoint, and $X, N, C \subseteq W$
- X is a ξ-clique;
- X is complete to N;
- X is anticomplete to C; and
- N covers C.

We say that the triple $\mathcal{L} = (X, N, W)$ is a *ξ-clique-cover* of C. We write $X(\mathcal{L}) = X$, $N(\mathcal{L}) = N$, and $W(\mathcal{L}) = W$. (The purpose of the sets W is not yet evident; they will become important when we talk about skew pairs in the next section, but in this section they are just carried along.)

A *ξ-clique-multicover* of C is a family $(\mathcal{L}_i : i \in I)$ of ξ-clique-covers of C, where I is a set of integers, such that for all $i, j \in I$ with $i < j$:

- $W(\mathcal{L}_j) \subseteq W(\mathcal{L}_i)$; and
- X_i is anticomplete to W_j.

Its *magnitude* is $|I|$, and its *chromatic number* is $\chi(C)$.

For $i, j \in I$ with $i < j$, we say that the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is *independent (with respect to C)* if there exists $x_j \in X(\mathcal{L}_j)$ such that no vertex in $N(\mathcal{L}_i)$ with a neighbour in C is adjacent to x_j. A ξ-clique-multicover $\mathcal{M} = (\mathcal{L}_i : i \in I)$ of C is *independent* if all its pairs $(\mathcal{L}_i, \mathcal{L}_j)$ (where $j > i$) are independent.

For brevity, let us say a graph G is (ξ, ζ, c)*-free* if for each $C \subseteq V(G)$ with $\chi(C) > c$, there is no independent ξ-multicover of C with magnitude ζ.

We proved in [4], where these objects (with the sets W removed) were called "ξ-cables of type 1 and length $|I|$" (the proof is immediate from 5.7 via Ramsey's theorem), that

7.2 *For all $\tau_1, \mu, \nu \geq 0$ and $\xi \geq 1$, there exist $m \geq 1$ and $c \geq 0$ with the following property. Let G be a $(1, \mu, \nu)$-restricted graph, such that $\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \nu$. Then G is (ξ, m, c)-free.*

Let ϕ be nondecreasing, and let $\xi, \zeta \geq 0$. We say that G is (ξ, ζ, ϕ)*-multiclique-controlled* if for every induced subgraph H of G and all $c \geq 0$, if $\chi(H) > \phi(c)$ then H is not (ξ, ζ, c)-free. We say a class of graphs is (ξ, ζ)*-clique-controlled* if there is a function ϕ such that all graphs in the class are (ξ, ζ, ϕ)-multiclique-controlled.

7.3 *For all $\tau_1, \mu, \nu \geq 0$ and $\xi \geq 1$, there exists $\zeta_0 \geq 1$ with the following property. Let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is (ξ, ζ_0)-multiclique-controlled;*
- *every graph in $\mathcal{C}$ is $(1, \mu, \nu)$-restricted; and*
- *$\chi(H) \leq \tau_1$ for all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof. Let m, c' satisfy 7.2 (with c replaced by c'), and let $\zeta_0 = m$; and suppose that $\mathcal{C}$ is a class of graphs that is (ξ, ζ_0)-controlled, and all graphs $G \in \mathcal{C}$ are $(1, \mu, \nu)$-restricted, and $\chi(H) \leq \tau_1$ for all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$. Choose a function ϕ such that all graphs in $\mathcal{C}$ are (ξ, ζ_0, ϕ)-multiclique-controlled. We claim that $c = \phi(c')$ satisfies the theorem. If there exists $G \in \mathcal{C}$ with $\chi(G) > \phi(c')$, then from the definition of "(ξ, ζ_0, ϕ)-multiclique-controlled", G is not (ξ, ζ_0, c')-free, contrary to 7.2. Consequently every graph in $\mathcal{C}$ has chromatic number at most $\phi(c') = c$. This proves 7.3. ∎

Thus, from 7.3, for all $\tau_1, \mu, \nu \geq 0$ and $\xi \geq 1$, there exists ζ_0 such that for every ξ-clique-controlled class $\mathcal{C}$ of $(1, \mu, \nu)$-restricted graphs such that $\chi(H) \leq \tau_1$ for every all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$, if there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number, then there is a maximum ζ such that $\mathcal{C}$ is (ξ, ζ)-multiclique-controlled, and $\zeta < \zeta_0$. That motivates the following.

7.4 *For all $\xi, \zeta \geq 1$, let $\mathcal{C}$ be a class of graphs that is (ξ, ζ)-multiclique-controlled and not $(\xi, \zeta + 1)$-multiclique-controlled. Then there exists τ_3 such that for all c, some graph in $\mathcal{C}^+$ has chromatic number more than c, and is $(\xi, \zeta + 1, \tau_3)$-free.*

Proof. Choose ϕ such that every graph in $\mathcal{C}$ is (ξ, ζ, ϕ)-multiclique-controlled. If for all $\sigma \geq 0$ there exists m_σ such that no $H \in \mathcal{C}^+$ with $\chi(H) > m_\sigma$ is $(\xi, \zeta + 1, \sigma)$-free, then, defining $\phi'(\sigma) = m_\sigma$ (and having arranged that $m_0 \leq m_1 \leq m_2 \leq \ldots$), it follows that every graph in $\mathcal{C}$ is $(\xi, \zeta + 1, \phi')$-multiclique-controlled, and hence $\mathcal{C}$ is $(\xi, \zeta+1)$-multiclique-controlled, a contradiction. Consequently, for some σ there is no such m_σ; that is, there exists τ_3 as in the theorem. This proves 7.4. ∎

In our search for the graphs in our class that contain trees of chandeliers, we will focus on the induced subgraphs mentioned in 7.4. We will show the following, in later sections. (A "tree of lamps" is defined later, and is closely related to a tree of chandeliers).

7.5 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2, \tau_3 \geq 0$. Let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled;*
- *$\chi(N_G^2(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi + 1)$-clique X in G;*
- *every member of $\mathcal{C}$ is $(\xi, \zeta + 1, \tau_3)$-free; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c.

Before we begin the proof of 7.5, let us assume its truth and unravel the various inductions implicit in 7.4, 7.3 and 7.1.

7.6 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2 \geq 0$, and let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi + 1)$-clique X in G;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then $\mathcal{C}$ is $(\xi, \zeta + 1)$-multiclique-controlled.

Proof (assuming 7.5): Suppose that $\mathcal{C}$ is not $(\xi, \zeta + 1)$-multiclique-controlled, and let τ_3 be as in 7.4. Let $\mathcal{D}$ be the class of all $(\xi, \zeta+1, \tau_3)$-free graphs in $\mathcal{C}^+$. By 7.4 applied to $\mathcal{C}$, there are graphs in $\mathcal{D}$ with arbitrarily large chromatic number. But by 7.5 applied to $\mathcal{D}$, there exists c such that every graph in $\mathcal{D}$ has chromatic number at most c, a contradiction. Thus $\mathcal{C}$ is $(\xi, \zeta + 1)$-multiclique-controlled. This proves 7.6. ∎

7.7 *Let $\tau_1, \tau_2, \mu, \nu \geq 0$ and $\xi \geq 1$, and let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is ξ-clique-controlled;*
- *all graphs in $\mathcal{C}$ are $(1, \mu, \nu)$-restricted;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi + 1)$-clique X in G; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 7.5): Let ζ_0, c be as in 7.3. Now $\mathcal{C}$ is $(\xi, 1)$-multiclique-controlled, and so for all ζ with $1 \leq \zeta < \zeta_0$, it follows from 7.6 that $\mathcal{C}$ is $(\xi, \zeta + 1)$-multiclique-controlled, and hence (ξ, ζ_0)-multiclique-controlled. By 7.3, all graphs in $\mathcal{C}$ have chromatic number at most c. This proves 7.7. ∎

7.8 *Let $\tau_1, \mu, \nu \geq 0$, and let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is 2-controlled;*
- *all graphs in $\mathcal{C}$ are $(1, \mu, \nu)$-restricted;*
- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \nu$; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 7.5): Suppose that there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number, and let ξ, τ_2 be as in 7.1. Let $\mathcal{D}$ be the class of all graphs $H \in \mathcal{C}^+$ such that $\chi(N_H^2(X)) \leq \tau_2$ for every $(\xi + 1)$-clique X of H. Then from 7.1, $\mathcal{D}$ is ξ-clique-controlled, and for all $c \geq 0$ there is a graph $H \in \mathcal{D}$ with $\chi(H) > c$, contrary to 7.7 applied to $\mathcal{D}$. This proves 7.8. ∎

Finally, we deduce:

7.9 *Let $\mu, \nu \geq 0$, and let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\mathcal{C}$ is 2-controlled;*
- *all graphs in $\mathcal{C}$ are $(1, \mu, \nu)$-restricted; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c.

Proof (assuming 7.5): We proceed by induction on ν. We may assume that $\nu \geq 1$ and the result holds for $\nu - 1$. Let $\mathcal{D}$ be the class of all $H \in \mathcal{C}^+$ with $\omega(H) < \nu$. Thus all graphs in $\mathcal{D}$ are $(1, \mu, \nu - 1)$-restricted, and so by the inductive hypothesis, there exists τ_1 such that all graphs in $\mathcal{D}$ have chromatic number at most τ_1. By 7.8, there exists c such that all graphs in $\mathcal{C}$ have chromatic number at most c. This proves 7.9. ∎

We see that 1.5 is an immediate consequence of 7.9. Let us prove 1.6, which we restate:

7.10 *For all $\rho \geq 2$, every forest of chandeliers is pervasive in every ρ-controlled class.*

Proof (assuming 7.5): Let $\mathcal{C}$ be a ρ-controlled class, let T be a forest of chandeliers, and let $\nu, \ell \geq 0$. We must show that there exists c such that for every graph $G \in \mathcal{C}$ with $\omega(G) \leq \nu$ and $\chi(G) > c$, there is an induced subgraph of G isomorphic to an $(\geq \ell)$-subdivision of T. Let T_1 be an ℓ-subdivision of T; then T_1 is also a forest of chandeliers. Choose a tree of lamps T' such that some subdivision of T_1 is an induced subgraph of T' (that this is always possible is discussed after the definition of "tree of lamps", later), and choose $\mu \geq 0$ such that some subdivision of T_1 is an induced subgraph of $K^1_{\mu,\mu}$ (and hence each of $K^2_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ contains some $(\geq \ell)$-subdivision of T as an induced subgraph). Let $\mathcal{C}$ be a ρ-controlled class, and let $\mathcal{D}$ be the class of graphs in $\mathcal{C}$ with clique number at most ν such that no induced subgraph is an $(\geq \ell)$-subdivision of T. It follows that every graph in $\mathcal{D}$ is $(\rho + 2, \mu, \nu)$-restricted, and hence $\mathcal{D}$ is 2-controlled by 6.1. By 7.9 applied to $\mathcal{D}$ and T', the members of $\mathcal{D}$ have bounded chromatic number. This proves 7.10. ∎

8 Skew pairs

If $v \in V(G)$ and $Z, W \subseteq V(G)$, and $\beta \geq 0$ and $\xi > 0$, we say that v is *(β,ξ)-earthed via (Z,W)* if there is a ξ-clique $X \subseteq V(G)$ with $v \in X$, such that $\chi(M) > \beta$, where M is the set of all vertices in $W \setminus X$ that are anticomplete to X and have a neighbour in Z that is complete to X.

Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G. The purpose of the sets $W(\mathcal{L}_i)$ is to enable the following definition. For $i, j \in I$ with $i < j$, and $\beta \geq 0$, let Z be the set of vertices in $N(\mathcal{L}_i)$ that are anticomplete to $C \cup \bigcup_{k \in I, k>j} W(\mathcal{L}_k)$; we say that the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is *β-skew (with respect to $\mathcal{M}, C$)* if

- every vertex in $N(\mathcal{L}_i) \setminus Z$ is complete to $X(\mathcal{L}_j)$; and
- every vertex in $N(\mathcal{L}_j)$ is (β,ξ)-earthed via $(Z, W(\mathcal{L}_j))$.

(Note that whether a pair $(\mathcal{L}_i, \mathcal{L}_j)$ is independent depends on C but not on the other members of $\mathcal{M}$; but whether the pair is β-skew depends both on C and on the other members of $\mathcal{M}$.) We say that $\mathcal{M}$ is *β-skewed with respect to C* if $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C$ for all $i, j \in I$ with $i < j$. We stress that, with $\mathcal{M}, C$ as before, if $i, j \in I$ with $i < j$, then the pair $(\mathcal{L}_i, \mathcal{L}_j)$ is itself a ξ-clique-multicover of C, and might be β-skewed with respect to C (as a ξ-clique-multicover of C) without being β-skew with respect to $\mathcal{M}, C$; so for a pair in a larger multicover, the statements that it is β-skew and β-skewed have different meanings.

Let (X, N, W) be a ξ-cover of C, and let $N' \subseteq N$. If every vertex in $N \setminus N'$ has a neighbour in C, we say that (X, N', W) is a *C-residue* of (X, N, W). Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C, and let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be a ξ-clique-multicover of C'. We say that $\mathcal{M}'$ is an *$(\mathcal{M}, C)$-residue covering C';* if

- $I' \subseteq I$;
- $C' \subseteq C$; and
- $\mathcal{L}'_i$ is a C-residue of $\mathcal{L}_i$ for each $i \in I'$.

If $I' = I$, it is said to be *spanning.* We need:

8.1 *Let $\mathcal{M} = (\mathcal{L}_i : i \in I)$ be a ξ-clique-multicover of C in G, and let $\mathcal{M}' = (\mathcal{L}'_i : i \in I')$ be an $(\mathcal{M}, C)$-residue covering $C' \subseteq C$. For all $i, j \in I'$ with $i < j$, if $(\mathcal{L}_i, \mathcal{L}_j)$ is independent with respect to C then $(\mathcal{L}'_i, \mathcal{L}'_j)$ is independent with respect to C'; and if $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C$ then $(\mathcal{L}'_i, \mathcal{L}'_j)$ is β-skew with respect to $\mathcal{M}', C'$.*

Proof. Let $i, j \in I'$ with $i < j$, such that $(\mathcal{L}_i, \mathcal{L}_j)$ is independent with respect to C. Consequently there exists $x_j \in X(\mathcal{L}_j)$ such that no vertex in $N(\mathcal{L}_i)$ has a neighbour in C and is adjacent to x_j; and so no vertex in $N(\mathcal{L}'_i)$ has a neighbour in C' and is adjacent to x_j. Thus $(\mathcal{L}'_i, \mathcal{L}'_j)$ is independent with respect to C'.

Let Z be the set of vertices in $N(\mathcal{L}_i)$ that are anticomplete to $C \cup \bigcup_{k \in I, k>j} W(\mathcal{L}_k)$, and let Z' be the set of vertices in $N(\mathcal{L}'_i)$ that are anticomplete to $C' \cup \bigcup_{k \in I', k>j} W(\mathcal{L}'_k)$. Then $Z \subseteq Z'$ (note that $Z \subseteq N(\mathcal{L}'_i)$ since $\mathcal{L}'_i$ is a C-residue of $\mathcal{L}_i$, and no vertex in Z has a neighbour in C).

Now assume that $(\mathcal{L}_i, \mathcal{L}_j)$ is β-skew with respect to $\mathcal{M}, C$. Thus $N(\mathcal{L}_i) \setminus Z$ is complete to $X(\mathcal{L}_j)$, and so every vertex in $N(\mathcal{L}'_i) \setminus Z'$ is complete to $X(\mathcal{L}'_j)$. Also, every vertex in $N(\mathcal{L}_j)$ is (β,ξ)-earthed via $(Z, W(\mathcal{L}_j))$, and so every vertex in $N(\mathcal{L}'_j)$ is (β,ξ)-earthed via $(Z', W(\mathcal{L}'_j))$. Thus $(\mathcal{L}'_i, \mathcal{L}'_j)$ is β-skew with respect to $\mathcal{M}', C'$. This proves 8.1. ∎

Note also that if $C'' \subseteq C' \subseteq C$ and $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are ξ-clique-multicovers of C, C', C'' respectively, and $\mathcal{M}'$ is an $(\mathcal{M}, C)$-residue, and $\mathcal{M}''$ is an $(\mathcal{M}', C')$-residue, then $\mathcal{M}''$ is an $(\mathcal{M}, C)$-residue (we leave the proof to the reader). We call this *transitivity of residues.* A set $V \subseteq V(G)$ is *β-homogeneous via* (Z, W) if either every vertex in V is (β, ξ)-earthed via (Z, W), or none are.

If $\mathcal{L} = (\mathcal{L}_i : i \in I)$ is a ξ-clique-multicover of C in G, a pair $(\mathcal{L}_i, \mathcal{L}_j)$ is *β-tidy with respect to $\mathcal{M}, C$* if it is either independent with respect to C or β-skew with respect to $\mathcal{M}, C$. If every pair in $\mathcal{M}$ is β-tidy, we say that $\mathcal{M}$ is *β-tidied (with respect to C).* Again, for a pair in a larger ξ-clique-multicover, the statements that it is β-tidy and β-tidied have different meanings. Our next goal is to get rid of the untidy pairs. We begin with ξ-clique-multicovers of magnitude two.

8.2 *Let $\xi > 0$ and $\tau_1, \tau_2, \beta \geq 0$; and let G be such that $\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$, and $\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G. Define $\gamma = \beta + \tau_2 + \xi\tau_1 + 1$. Let $\mathcal{L} = (\mathcal{L}_a, \mathcal{L}_b)$ be a ξ-clique-multicover of C in G, where $a < b$, $\mathcal{L}_i = (X_i, N_i, W_i)$ for $i = a, b$, and $\chi(C) > (\xi+1)\gamma$. Suppose that $X_b \cup N_b$ is γ-homogeneous via (N_a, W_b). Then there exist $C' \subseteq C$ with $\chi(C') \geq \chi(C)/(\xi+1)$, and a C-residue $\mathcal{L}'_a$ of $\mathcal{L}_a$ covering C', such that $(\mathcal{L}'_a, \mathcal{L}_b)$ is β-tidied with respect to C'.*

Proof. For each $x \in X_b$, let Y_x be the set of vertices in N_a that are adjacent to x and have a neighbour in C, and let C_x be the set of vertices in C with a neighbour in $N_a \setminus Y_x$. Suppose that there exists $x \in X_b$ with $\chi(C_x) \geq \chi(C)/(\xi+1)$. Let $\mathcal{L}'_a = (X_a, N_a \setminus Y_x, W_a)$; then $(\mathcal{L}'_a, \mathcal{L}_b)$ is an $(\mathcal{M}, C)$-residue covering C_x, and is independent with respect to C_x, and therefore $(\mathcal{L}'_a, \mathcal{L}_b)$ is β-tidied with respect to C_x, and the theorem is satisfied.

Thus we may assume that $\chi(C_x) < \chi(C)/(\xi+1)$ for each $x \in X$. Let C' be the set of all vertices in C that are not in any of the sets C_x $(x \in X_b)$. It follows that $\chi(C') \geq \chi(C) - \chi(C)\xi/(\xi+1) = \chi(C)/(\xi+1)$. Let U be the set of vertices in N_a that are complete to X_b. Thus every vertex in C' has no neighbour in any of the sets Y_x $(x \in X_b)$, and therefore all its neighbours in N_a belong to U.

We claim that $\mathcal{M}$ itself, with C', satisfy the theorem in this case. To show this we must show that $(\mathcal{L}_a, \mathcal{L}_b)$ is β-skew with respect to $\mathcal{M}, C'$. Since $\chi(C') \geq \chi(C)/(\xi+1) > \gamma$, and every vertex in C' has a neighbour in U, and this neighbour is therefore complete to the ξ-clique X_b, it follows that every vertex in X_b is γ-earthed via (N_a, C'), and therefore via (N_a, W_b). Since $X_b \cup N_b$ is γ-homogeneous via (N_a, W_b), it follows that every vertex in N_b is also γ-earthed via (N_a, W_b).

Let $v \in N_b$. From the definition of "β-earthed", there is a ξ-clique $X \subseteq V(G)$ with $v \in X$, such that $\chi(M) > \gamma$, where M is the set of all vertices in $W_b \setminus X$ that are anticomplete to X and have a neighbour in N_a that is complete to X. Let $M' \subseteq M$ be the set of vertices in $M \setminus X_b$ with a neighbour in U that is complete to X, and with no neighbour in X_b; then since v is complete to X_b, and so by hypothesis $\chi(N^2(X_b \cup \{v\})) \leq \tau_2$, it follows that $\chi(M') \leq \tau_2$. (Note that $M' \subseteq N^2(X_b \cup \{v\})$). Also the set M'' of vertices in M that either belong to X_b or have a neighbour in X_b has chromatic number at most $\xi\tau_1 + 1$. Hence

$$\chi(M \setminus (M' \cup M'')) > \gamma - \tau_2 - \xi\tau_1 - 1 = \beta,$$

and consequently v is β-earthed via $(N_a \setminus U, W_b)$. Since no vertex in $N_a \setminus U$ has a neighbour in C', it follows that $(\mathcal{L}_a, \mathcal{L}_b)$ is β-skew with respect to $\mathcal{M}, C'$. This proves 8.2. ∎

From 8.2 we deduce the following (we remark that it was in order to prove this result and its consequence 8.5 that we introduced the concept of being (ξ, ζ)-multiclique-controlled).

8.3 *Let $\xi > 0$ and $\tau_1, \tau_2, \tau_3, \beta \ge 0$, and ϕ a nondecreasing function. For all $c' \ge 0$ there exists $c \ge 0$ with the following property. Let G be such that*

- *$\chi(H) \le \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \le \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}_1 = (X_1, N_1, W_1)$ be a ξ-clique-cover of C in G, where $\chi(C) > c$. Then there exist $X_2, N_2, W_2 \subseteq C$ and $C' \subseteq W_2$ such that $\mathcal{L}_2 = (X_2, N_2, W_2)$ is a ξ-clique-cover of C', and $\chi(C') > c'$, and there is a C-residue $\mathcal{L}_1'$ of $\mathcal{L}_1$ such that $(\mathcal{L}_1', \mathcal{L}_2)$ is β-skewed with respect to C'.

Proof. Let $\gamma = \beta + \tau_2 + \xi\tau_1 + 1$; and let $c = 2\phi((\xi+1)^\zeta \max(c', \tau_3))$. We claim that c satisfies the theorem. For let G, C and $\mathcal{L}_1 = (X_1, N_1)$ be as in the theorem, with $\chi(C) > c$. Let D_1 be the set of vertices in C that are γ-earthed via (N_1, C), and $D_2 = C \setminus D_1$. Since $\chi(C) > 2\phi(\max(c', \tau_3)(\xi+1)^\zeta)$, one of D_1, D_2, say D, has chromatic number larger than $\phi(\max(c', \tau_3)(\xi+1)^\zeta)$. Since G is (ξ, ζ, ϕ)-multiclique-controlled, it follows that there is an independent ξ-clique-cover $(\mathcal{L}_i : 2 \le i \le \zeta+1)$ of some $C_1 \subseteq D$, with $\chi(C_1) > (\xi+1)^\zeta \max(c', \tau_3)$ and $W(\mathcal{L}_i) \subseteq D$ for $2 \le i \le \zeta+1$. Now D is γ-homogeneous via (N_1, C). Let $2 \le i \le \zeta+1$, and and let $\mathcal{L}_i' = (X(\mathcal{L}_i), N(\mathcal{L}_i), C_1)$. Then $X(\mathcal{L}_i') \cup N(\mathcal{L}_i')$ is γ-homogeneous via (N_1, C). By ζ applications of 8.2, to the pairs $(\mathcal{L}_1, \mathcal{L}_i')$ for $i = 2, \ldots, \zeta+1$ in turn, and successive subsets of C_1, we deduce that for $i = 2, \ldots, \zeta+1$ there exists $C_i \subseteq C_{i-1}$ with $\chi(C_i) > \chi(C_i)/(\xi+1)$, and a C_{i-1}-residue $\mathcal{L}_{1,i}$ of $\mathcal{L}_1$, such that the pairs $(\mathcal{L}_{1,i}, \mathcal{L}_i')$ are each β-tidied with respect to C_i. In particular, this is true when $i = \zeta+1$; let $C' = C_{\zeta+1}$ and $\mathcal{L}_1' = \mathcal{L}_{1,\zeta+1}$. Thus $\chi(C') > \max(c', \tau_3)$, and $\mathcal{L}_1'$ is a C-residue of $\mathcal{L}_1$. Suppose that each of the pairs $(\mathcal{L}_1', \mathcal{L}_i')$ is independent with respect to C', for $i = 2, \ldots, \zeta+1$; then also the pairs $(\mathcal{L}_1', \mathcal{L}_i)$ are independent with respect to C for $i = 2, \ldots, \zeta+1$, and so $(\mathcal{L}_1', \mathcal{L}_2, \ldots, \mathcal{L}_{\zeta+1})$ is an independent ξ-clique-multicover of C', which is impossible since $\chi(C') > \tau_3$. Thus there exists $i \in \{2, \ldots, \zeta+1\}$ such that $(\mathcal{L}_1', \mathcal{L}_i')$ is β-skewed with respect to C'. This proves 8.3. ∎

This implies:

8.4 *Let $\xi, t > 0$ and $\tau_1, \tau_2, \tau_3, \beta \ge 0$, and ϕ a nondecreasing function. For all $c' \ge 0$ there exists $c \ge 0$ with the following property. Let G be such that*

- *$\chi(H) \le \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \le \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}_1$ be a ξ-clique-cover of C in G, where $\chi(C) > c$. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a C-residue $\mathcal{L}_1'$ of $\mathcal{L}_1$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_t$ of C', such that

- *$\mathcal{M} = (\mathcal{L}_1', \mathcal{L}_2, \ldots, \mathcal{L}_t)$ is a ξ-clique-multicovering of C';*

- *$\mathcal{M}$ is β-tidied with respect to C'; and*
- *for $2 \le i \le t$, the pair $(\mathcal{L}'_1, \mathcal{L}_i)$ is β-skew with respect to $\mathcal{M}, C'$.*

Proof. The result is true when $t = 1$, taking $c' = c$; so we assume that $t > 1$ and the result holds for $t - 1$. Define $\gamma = \beta + \tau_2 + \xi\tau_1 + 1$. Choose n_2 such that setting $c = n_2$ satisfies 8.3 when c' is replaced by $(\xi + 1)^{t-2} \max(\gamma, c')$. Choose a value of c that satisfies the result with t, c' replaced by $t - 1, 2^{t-2}n_2$ respectively. We claim that c satisfies the theorem. For let G, C and $\mathcal{L}_1 = (X_1, N_1, W_1)$ be as in the theorem, with $\chi(C) > c$. From the choice of c, there exist $D_1 \subseteq C$ with $\chi(D_1) > 2^{t-2}n_2$, and a C-residue $\mathcal{L}'_1$ of $\mathcal{L}_1$ covering D_1, and ξ-clique-covers $\mathcal{L}_2, \dots, \mathcal{L}_{t-1}$ of D_1, such that

- $\mathcal{M}_1 = (\mathcal{L}'_1, \mathcal{L}_2, \dots, \mathcal{L}_{t-1})$ is a ξ-clique-multicovering of D_1;
- $\mathcal{M}_1$ is β-tidied with respect to D_1; and
- for $2 \le i \le t - 1$, the pair $(\mathcal{L}'_1, \mathcal{L}_i)$ is β-skew with respect to $\mathcal{M}_1, D_1$.

For $2 \le i \le t - 1$, let $\mathcal{L}_i = (X_i, N_i, W_i)$. For each such i, there is a partition of D_1 into two parts, both β-homogeneous via (N_i, D_1); and so there is a partition of D_1 into 2^{t-2} parts, each β-homogeneous via each of $(N_2 i, D_1), \dots, (N_{t-1}, D_1)$. Let D_2 be one of the parts, chosen such that $\chi(D_2) \ge 2^{2-t}\chi(D_1) > n_2$. Now $\mathcal{L}_1$ is a ξ-clique-cover of D_2, so by 8.3 and the choice of n_2, there is a D_2-residue $\mathcal{L}''_1$ of $\mathcal{L}'_1$, and $D_3 \subseteq D_2$ with $\chi(D_3) > (\xi + 1)^{t-2} \max(\gamma, c')$, and a ξ-clique-cover $\mathcal{L}_t = (X_t, N_t, W_t)$ of D_3, such that $W_t \subseteq D_2$, and the pair $(\mathcal{L}''_1, \mathcal{L}_t)$ is β-skewed with respect to D_3. Thus

$$\mathcal{M}_3 = (\mathcal{L}''_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_{t-1}, \mathcal{L}_t)$$

is a ξ-clique-multicover of D_3. Moreover, $\mathcal{L}''_1$ is a C-residue of $\mathcal{L}_1$, by the transitivity of residues.

(1) *Every pair of $\mathcal{M}_3$ is β-tidy with respect to $\mathcal{M}_3, D_3$ except possibly the pairs $(\mathcal{L}_i, \mathcal{L}_t)$ where $2 \le i \le t - 1$.*

To see this, there are three kinds of pairs to consider:

- The pair $(\mathcal{L}''_1, \mathcal{L}_i)$ where $2 \le i \le t - 1$: this pair is β-tidy with respect to $\mathcal{M}_1, D_1$, and since $W(\mathcal{L}_t) \subseteq D_1$, it is also β-tidy with respect to $\mathcal{M}_3, D_3$.
- The pair $(\mathcal{L}''_1, \mathcal{L}_t)$: this is chosen to be β-skewed with respect to D_3, and therefore β-tidy with respect to $\mathcal{M}_3, D_3$.
- The pair $(\mathcal{L}_i, \mathcal{L}_j)$ where $2 \le i < j \le t - 1$: again, this is β-tidy with respect to $\mathcal{M}_1, D_1$, and hence also with respect to $\mathcal{M}_3, D_3$ since $W(\mathcal{L}_t) \subseteq D_1$.

This proves (1).

Let $C_1 = D_3$. By $t - 2$ applications of 8.2, applied to the pairs $(\mathcal{L}_i, \mathcal{L}_t)$ and C_{i-1} for $2 \le i \le t - 1$ in turn, we deduce that for $2 \le i \le t - 1$ there exist $C_i \subseteq C_{i-1}$ with $\chi(C_i) \ge \chi(C_{i-1})/(\xi + 1)$, and a C_{i-1}-residue $\mathcal{L}'_i$ of $\mathcal{L}_i$ covering C_i, such that $(\mathcal{L}'_i, \mathcal{L}_t)$ is β-tidied with respect to C_i. It follows from 8.1 and (1) that

$$\mathcal{M} = (\mathcal{L}''_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_{t-1}, \mathcal{L}_t)$$

(setting $C' = C_{t-1}$) satisfies the theorem. This proves 8.4. ∎

By choosing t large enough in 8.4, and applying Ramsey's theorem to the sequence $(\mathcal{L}_2, \ldots, \mathcal{L}_t)$, we deduce since G is $(\xi, \zeta+1, \tau_3)$-free that the same result as 8.4 is true with "β-tidied" replaced by "β-skewed". This result is important enough that it deserves to be said explicitly:

8.5 *Let $\xi, t > 0$ and $\tau_1, \tau_2, \tau_3, \beta \geq 0$, and ϕ a nondecreasing function. For all $c' \geq 0$ there exists $c \geq 0$ with the following property. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}_1 = (X_1, N_1)$ be a ξ-clique-cover of C, where $\chi(C) > c$. Then there exist $C' \subseteq C$ with $\chi(C') > c'$, and a C-residue $\mathcal{L}'_1$ of $\mathcal{L}_1$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_t$ of C', such that

- *$\mathcal{M} = (\mathcal{L}'_1, \mathcal{L}_2, \ldots, \mathcal{L}_t)$ is a ξ-clique-multicovering of C';*
- *$\mathcal{M}$ is β-skewed with respect to C'.*

9 Finding a tree of lamps

Now we come to reap the benefit of all the complications of 8.5; we show that any graph satisfying the conditions of 8.5 contains any given tree of lamps as an induced subgraph, if the number t and the chromatic number are large enough.

Here at last is a definition of a tree of lamps. Start with a tree T, and select a vertex of T called the *root*; and take a map w from $V(T)$ into the set of positive integers, such that

- every vertex v different from the root has a unique neighbour u with $w(u) > w(v)$ (and consequently the w-value of the root is strictly larger than all the other values)
- there is a vertex v with $w(v) = 1$ (necessarily, either v is the root and $|V(T)| = 1$, or v is a leaf of T);
- for all vertices u, v with $u \neq v$, if $w(u) = w(v)$ then $w(u) = 1$.

We call such a function w a *height function* for T. Let $w(V(T))$ denote the set $\{w(v) : v \in V(T)\}$.

Now choose a set J of integers, each at least 1 and at most the w-value of the root, with $J \cap w(V(T)) = \{1\}$. For each $j \in J$, take a new vertex x_j; and make x_j adjacent to v for every edge uv of T such that $w(v) \leq j$ and $w(u) > j$. (If $|V(T)| = 1$, make x_1 adjacent to the root.) A graph constructed this way is called a *lamp*, and x_1 is its *plug*. Thus every chandelier is a lamp, but many lamps are not chandeliers.

Analogously to trees of chandeliers, we can make trees of lamps, by taking a new lamp, and attaching trees of lamps already constructed to this new lamp by their plugs. However, we are not permitted to attach anything to neighbours of the plug of the new lamp. Let us say this more precisely. A *spotlight* is a one-vertex graph, with plug its vertex. No tree of lamps has negative height; and the spotlight is the only tree of lamps of height zero. Inductively for $r > 0$, having defined trees of lamps of height $\leq r-1$ and their plugs, we proceed as follows. Let L be a lamp with plug ℓ. For each $v \in V(L)$, let Q_v be a tree of lamps of height at most $r-1$, such that all the graphs L and Q_v ($v \in V(L)$) are pairwise anticomplete, and such that if v is equal to or adjacent to ℓ, then Q_v is a spotlight. Now identify v with the plug of Q_v, for each $v \in V(L)$. (More precisely, add new edges joining v to every neighbour of the plug of Q_v, and then delete the plug of Q_v, for each $v \in V$.) Let the result be Q. Any such graph Q, with plug ℓ, is said to be a tree of lamps of height $\leq r$ (and so is the spotlight). We say G is a *forest of lamps* if every component is a tree of lamps.

We used earlier the fact that for every tree of chandeliers H, there is a tree of lamps J such that some subdivision of H is an induced subgraph of J. We leave it to the reader to verify this. We will show the following.

9.1 *Let $\xi, \zeta > 0$ and $\tau_1, \tau_2, \tau_3 \geq 0$, and ϕ a nondecreasing function. Let Q be a tree of lamps. Then there exists $c \geq 0$ with the following property. Let G be such that*

- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\chi(N^2(X)) \leq \tau_2$ for every $(\xi+1)$-clique X in G;*
- *G is (ξ, ζ, ϕ)-multiclique-controlled; and*
- *G is $(\xi, \zeta+1, \tau_3)$-free.*

Let $\mathcal{L}_1 = (X_1, N_1)$ be a ξ-clique-cover of C, where $\chi(C) > c$, and let $a_1 \in X_1$. Then there is an isomorphism from Q to an induced subgraph of G, mapping the plug of Q to a_1.

Proof. We proceed by induction on $|V(Q)|$. Certainly it is true if $|V(Q)| = 1$, so we assume that $|V(Q)| > 1$ and the result holds for all smaller trees of lamps. Since, up to isomorphism, there are only finitely many smaller trees of lamps, we can choose $c_0 \geq 0$ such that the theorem is true with c replaced by c_0 for every tree of lamps with at most $|V(Q)|-1$ vertices. Let $\beta = c_0 + |V(Q)|(\tau_2 + (\xi+1)\tau_1)$.

There is a lamp L with plug ℓ say, and trees of lamps Q_v ($v \in V(L)$) such that Q is obtained from L and the graphs Q_v ($v \in V(L)$) as in the definition above.

There is a tree T, a height function w, a set J of integers, and vertices x_j ($j \in J$) in L, as in the definition of a lamp. Choose w such that $w(v)$ is congruent to 1 modulo 3 for all v, and every member of J is also congruent to 1 modulo 3. Let q_0 be the root of T, and let $t = w(q_0)$. Choose c such that 8.5 holds with $c' = 0$. We claim that c satisfies the theorem.

Let $G, \mathcal{L}_1 = (X_1, N_1)$ and C be as in the theorem. By 8.5, there exist $C' \subseteq C$ with $\chi(C') > c'$, and a C-residue $\mathcal{L}'_1$ of $\mathcal{L}_1$ covering C', and ξ-clique-covers $\mathcal{L}_2, \ldots, \mathcal{L}_t$ of C', such that

- $\mathcal{M} = (\mathcal{L}'_1, \mathcal{L}_2, \ldots, \mathcal{L}_t)$ is a ξ-clique-multicovering of C'; and
- $\mathcal{M}$ is β-skewed with respect to C'.

For $1 \leq i \leq t$, let $\mathcal{L}_i = (X_i, N_i, W_i)$; and for $1 \leq i < j \leq t$, let $Z_{i,j}$ be the set of vertices in N_i that have a neighbour in W_j and are anticomplete to $C' \cup W_{j+1} \cup \cdots \cup W_t$. Thus the sets $Z_{i,i+1}, \ldots, Z_{i,t}$ are pairwise disjoint subsets of N_i, and no vertex of any of them has a neighbour in C'; let $Y_i = N_i \setminus (Z_{i,i+1} \cup \cdots \cup Z_{i,t})$. From the definition of "skew", it follows that

(1) *$Z_{i,k}$ is complete to X_j for $1 \le i < j < k \le t$, and Y_i is complete to X_j for $1 \le i < j \le t$.*

Also, every vertex in N_j is (β, ξ)-earthed via $(Z_{i,j}, W_j)$.

(2) *Let $1 \le i < j \le t$, and let $v \in X_h \cup N_h$ for some $h \in \{1, \dots, t\}$. Let M be the set of vertices in W_j that are equal or adjacent to v, or have a neighbour in $Z_{i,j}$ adjacent to v. Then $\chi(M) \le \tau_2 + (\xi+1)\tau_1$, unless either*

- *$h < i$ and $v \in Z_{h,i}$, or*
- *$h = i$ and $v \in X_i$, or*
- *$i < h < j$ and $v \in X_h$, or*
- *$h = j$.*

There are several cases, depending on the value of h relative to i, j. We may assume that v has a neighbour in $Z_{i,j}$, and so $h \le j$; and we may therefore assume that $h < j$.

Suppose first that some ξ-clique X is complete to both $\{v\}$ and $Z_{i,j}$. Since $N^2(X \cup \{v\}) \le \tau_2$ (because $X \cup \{v\}$ is a $(\xi + 1)$-clique), and X is complete to $Z_{i,j}$, it follows that the set of vertices in M that are adjacent to a neighbour of v in $Z_{i,j}$ and anticomplete to $X \cup \{v\}$ has chromatic number at most τ_2. But the chromatic number of the set of vertices in M that belong to or have a neighbour in $X \cup \{v\}$ is at most $(\xi + 1)\tau_1$; and so $\chi(M) \le \tau_2 + (\xi + 1)\tau_1$, and the result holds. Thus we may assume that there is no such X. In particular, v is not complete to X_i. Also, since we may assume that $v \notin X_i$, it follows that $h \ne i$; and so either $i < h < j$, or $h < i$.

If $i < h < j$, we may assume that $v \notin X_h$, and so $v \in N_h$; but then the ξ-clique X_h is complete to $\{v\}$ and to $Z_{i,j}$ by (1), a contradiction.

Thus $h < i$. Then since v has a neighbour in $Z_{i,j}$, it follows that $v \in N_h$. Since v is not complete to X_i, (1) implies that $v \notin Y_h$; and therefore $v \in Z_{h,k}$ for some k. By (1), $k \le i$. Since v has a neighbour in N_i, it follows that $k = i$, as required. This proves (2).

Now we begin to construct the isomorphism η from Q to an induced subgraph of G. We recall that q_0 is the root of T; choose some vertex in N_t, and call it $\eta(q_0)$. At a general stage of the process, we will have defined $\eta(p)$ only for the vertices p in a subset $\mathrm{dom}(\eta)$ of $V(Q)$. We will ensure that η is injective, and for all $u, v \in \mathrm{dom}(\eta)$, u, v are adjacent in Q if and only if $\eta(u), \eta(v)$ are adjacent in G.

First we extend $\mathrm{dom}(\eta)$ to equal $V(T)$, in such a way that $\eta(p) \in N_{w(p)}$ for each $p \in V(T)$, by repeating the following process.

- Choose an integer n maximum such that $w(v) = n$ for some $v \in V(T) \notin \mathrm{dom}(\eta)$. (When this is not possible stop.)
- Let u be the neighbour of v in $\mathrm{dom}(\eta)$ (necessarily unique).
- Choose a vertex $y \in Z_{w(v),w(u)}$ adjacent to $\eta(u)$ and nonadjacent to all the vertices $\eta(p)(p \in \mathrm{dom}(\eta) \setminus \{u\})$. To see that this is possible, let $p \in \mathrm{dom}(\eta) \setminus \{u\}$. Since $w(p) > n$, and $\eta(p)$ belongs to $N_{w(p)}$, and therefore $w(p) \ne w(u)$, it follows from (2) that the set of vertices in $W_{w(u)}$

that have a neighbour in $Z_{w(v),w(u)}$ adjacent to $\eta(p)$ has chromatic number at most $\tau_2+(\xi+1)\tau_1$. Consequently the set of vertices in $W_{w(u)}$ that have a neighbour in $Z_{w(v),w(u)}$ with a neighbour in $\{\eta(p) : p \in \mathrm{dom}(\eta) \setminus \{u\}\}$ has chromatic number at most $|V(Q)|(\tau_2 + (\xi + 1)\tau_1)$. Since $\eta(u)$ is (β, ξ)-earthed via $(Z_{w(v),w(u)}, W_{w(u)})$, and $\beta > |V(Q)|(\tau_2 + (\xi + 1)\tau_1)$, there is at least one vertex $x \in W_{w(u)}$ that has a neighbour $y \in Z_{w(v),w(u)}$ adjacent to $w(u)$, and has no neighbour in $Z_{w(v),w(u)}$ that is adjacent to any of $\eta(p)(p \in \mathrm{dom}(\eta) \setminus \{u\})$. In particular, y is nonadjacent to all of $\eta(p)(p \in \mathrm{dom}(\eta) \setminus \{u\})$. This shows the existence of the vertex y as claimed.

- Define $\eta(v) = y$, and add v to $\mathrm{dom}(\eta)$.

Next we add all the vertices $x_j (j \in J)$ to $\mathrm{dom}(\eta)$, defining $\eta(x_j)$ to be some vertex in X_j for each $j \in J$, and in particular choosing $\eta(x_1) = a_1$. By (1), η still defines an isomorphism from $\mathrm{dom}(\eta)$ into $V(G)$.

Now we turn to adding the "pendant" trees of lamps $Q_v (v \in V(L))$. We proceed as follows.

- For $n = t, t-3, t-6, \ldots, 1$ in turn, do the following.

- If $n = 1$, then since all the Q_v are spotlights when $w(v) = 1$, the process stops. So we assume that $n \geq 2$.

- If $n \notin J$ and there is no $p \in V(T)$ with $w(p) = n$, continue to the next value of n.

- Suppose that there exists $u \in V(T)$ with $w(u) = n$. Then u is unique (since $n \geq 1$), and $n \notin J$. Now $\eta(u)$ is (β, ξ)-earthed via $(Z_{n-1,n}, W_n)$. For each $p \in \mathrm{dom}(\eta)$, either $p \in V(T)$, or $p = x_j$ for some $j \in J$, or p is a vertex of some Q_v where $w(v) > n$. In each case, let $f(p)$ be the chromatic number of the set of vertices in W_n that are either equal or adjacent to $\eta(p)$, or are adjacent to a neighbour of $\eta(p)$ in $Z_{n-1,n}$. We need to bound $f(p)$ in each case. In the first case, when $p \in V(T)$, it follows that $w(p)$ is congruent to 1 modulo 3, and so $w(p) \neq n-1$; and also $\mathrm{dom}(\eta) \cap Z_{h,n-1} = \emptyset$ for $h < n-1$, since $n-1$ is divisible by 3. So by (2), $f(p) \leq \tau_2 + (\xi+1)\tau_1$. In the second case, when $p = x_j$ for some $j \in J$, it follows that j is 1 modulo 3, and so $j \neq n-1$ and again (2) implies that $f(p) \leq \tau_2 + (\xi+1)\tau_1$. In the third case, when p is a vertex of some Q_v where $w(v) > n$, it follows that $\eta(p) \in W_{w(v)-1}$, and therefore is anticomplete to $Z_{n-1,n}$; and so $f(p) \leq \tau_1$. In total then, the set of vertices in W_n that either belong to or have a neighbour in $\{\eta(p) : p \in \mathrm{dom}(\eta)\}$, or have a neighbour in $Z_{n-1,n}$ which has a neighbour in this set, has chromatic number at most $|V(Q)|(\tau_2+(\xi+1)\tau_1) = \beta - c_0$. Consequently there exist $Z \subseteq Z_{n-1,n}$ and $W \subseteq W_n$, such that $Z \cup W$ is anticomplete to $\{\eta(p) : p \in \mathrm{dom}(\eta)\}$, and such that $\eta(u)$ is (c_0, ξ)-earthed via (Z, W). From the inductive hypothesis, there is an automorphism from Q_u to an induced subgraph of $G[Z \cup W \cup \{\eta(u)\}]$, mapping the plug of Q_u to $\eta(u)$. This provides the desired extension of η and $\mathrm{dom}(\eta)$ to include $V(Q_u)$. Then go to the next value of n.

- Finally, suppose that $n \in J$. Then $n < t$, and so there are vertices in N_{n+1}. Now every vertex in N_{n+1} is (β, ξ)-earthed via $(Z_{n-1,n+1}, W_{n+1})$, and since $\eta(x_n)$ is anticomplete to W_{n+1} and complete to $Z_{n-1,n+1}$ by (1), it follows that $\eta(x_n)$ is also (β, ξ)-earthed via $(Z_{n-1,n+1}, W_{n+1})$. But now we proceed as in the previous case, noting that $\mathrm{dom}(\eta) \cap N_{n+1} = \emptyset$ since $n+1$ is 2 modulo 3.

This completes the construction of the isomorphism, and so completes the proof of 9.1. ∎

We deduce 7.5, which we restate:

9.2 *Let $\xi, \zeta \geq 1$, and $\tau_1, \tau_2, \tau_3 \geq 0$. Let T be a tree of lamps. Let $\mathcal{C}$ be a class of graphs such that*

- *$\chi(H) \leq \tau_1$ for every $H \in \mathcal{C}^+$ with $\omega(H) < \omega(G)$;*
- *$\mathcal{C}$ is (ξ, ζ)-multiclique-controlled;*
- *$\chi(N^2_G(X)) \leq \tau_2$ for every $G \in \mathcal{C}$ and every $(\xi+1)$-clique X in G;*
- *every member of $\mathcal{C}$ is $(\xi, \zeta+1, \tau_3)$-free; and*
- *no graph in $\mathcal{C}$ contains T as an induced subgraph.*

Then there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c.

Proof. Choose ϕ such that every graph in $\mathcal{C}$ is (ξ, ζ, ϕ)-multiclique-controlled. Choose c' such that 9.1 is satisfied with c, Q replaced by c', T, and let $c = \phi(c')$. We claim that c satisfies the theorem. For let G be as in the theorem, and suppose that $\chi(G) > c$. Since $\chi(G) > \phi(c')$, there is a ξ-clique X_1 of G with $\chi(N^2(X_1)) > c'$. By 9.1, G contains T as an induced subgraph, a contradiction. This proves that $\chi(G) \leq c$, and so proves 7.5. ∎

10 Out of control

What can we say about the pervasive graphs for the class of all graphs? Any such graph must be a forest of chandeliers, and perhaps the converse is true, but we have no proof. We do have the following.

10.1 *Let H be a forest of chandeliers. Suppose that for all $\ell \geq 1$ and $\nu, \tau_1 \geq 0$ there exists $\rho \geq 2$, such that for all $\tau_2 \geq 0$, every graph G satisfying*

- *$\omega(G) \leq \nu$;*
- *$\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \nu$; and*
- *$\chi^\rho(G) \leq \tau_2$*

with sufficiently large chromatic number has an induced subgraph isomorphic to an $(\geq \ell)$-subdivision of H. Then H is pervasive in the class of all graphs.

Proof. For $\ell, \nu \geq 0$, let $\mathcal{C}_{\ell,\nu}$ be the class of all graphs G with $\omega(G) \leq \nu$, such that no induced subgraph of G is isomorphic to an $(\geq \ell)$-subdivision of H. To show that H is pervasive in the class of all graphs, we must show that for all $\ell, \nu \geq 0$, there is an upper bound on the chromatic numbers of the graphs in $\mathcal{C}_{\ell,\nu}$. For fixed ℓ we prove this by induction on ν. Thus we may assume that there exists τ_1 such that $\chi(H) \leq \tau_1$ for every graph in $\mathcal{C}_{\ell,\nu-1}$. Consequently all graphs in $\mathcal{C}_{\ell,\nu}$ satisfy the first two bullets in the theorem. Choose ρ as in the theorem.

For each $\tau_2 \geq 0$, from the hypothesis there is an upper bound $\phi(\tau_2)$ on the chromatic number of the graphs in $\mathcal{C}_{\ell,\nu}$ with $\chi^\rho(G) \leq \tau_2$; and we can choose ϕ nondecreasing. Consequently $\mathcal{C}_{\ell,\nu}$ is ρ-controlled. By 1.6, H is pervasive in $\mathcal{C}_{\ell,\nu}$. Since no graph in $\mathcal{C}_{\ell,\nu}$ contains an induced subgraph which is an $(\geq \ell)$-subdivision of H, it follows that there is an upper bound on the chromatic number of the graphs in $\mathcal{C}_{\ell,\nu}$. This completes the induction, and hence proves 10.1. ∎

Thus, if H is a forest of chandeliers for which we can show that the hypothesis of 10.1 holds, then H is pervasive. We have shown this for a few small graphs, such as the graph obtained from a triangle by adding two leaves to each of its three vertices (which eluded us for a long time), but it is not worth listing here all the little cases we can do. For the moment, let us just do an easy one (which nevertheless considerably extends 1.2), a restatement of 1.11:

10.2 *For all $n \geq 0$, the complete bipartite graph $K_{2,n}$ is pervasive in the class of all graphs.*

Proof. We may assume that $n \geq 2$. Let $\ell \geq 1$ and $\nu, \tau_1 \geq 0$, and let $\mathcal{C}$ be the class of graphs G such that

- no induced subgraph of G is an (ℓ)-subdivision of $K_{2,n}$;
- $\omega(G) \leq \nu$;
- $\chi(H) \leq \tau_1$ for every induced subgraph H of G with $\omega(H) < \nu$.

By 10.1, it suffices to prove that for some $\rho \geq 2$ and all $\tau_2 \geq 0$, every graph in $\mathcal{C}$ with $\chi^\rho(G) \leq \tau_2$ has bounded chromatic number.

Let $\rho = \ell + 2$; we will show that this satisfies the theorem. For let $\tau_2 \geq 0$, and let $G \in \mathcal{C}$ satisfy $\chi^\rho(G) \leq \tau_2$. Let $t = R(9, \max(\nu+1, n+1), 20)$; we will show that $\chi(G) \leq 2^t \tau_2$. For suppose not.

Choose a component C_1 of G with maximum chromatic number, and choose $z_1 \in V(C)$. Then there is a number k_1 such that the set X_1 of vertices in C_1 with distance exactly k_1 from z_1 has chromatic number at least $\chi(G)/2$. Let C_2 be a component of $G[X_1]$ with maximum chromatic number, and choose $z_2 \in C_2$; then similarly, there is a number k_2 such that the set of vertices in C_2 with distance (in $G[C_2]$) exactly k_2 from z_2 has chromatic number at least $\chi(X_1)/2 \geq \chi(G)/4$. By repeating this t times, we obtain:

- t numbers $k_1, \dots, k_t \geq 0$;
- $t+1$ connected subsets $C_1, \dots, C_t, C_{t+1}$ of $V(G)$, with $C_j \subseteq C_i$ for $1 \leq i < j \leq t$, such that $\chi(C_{i+1}) \geq \chi(C_i)/2$ for $1 \leq i \leq t$;
- t vertices $z_1, \dots, z_t$, where $z_i \in C_i$, such that every vertex in C_{i+1} has distance (in $G[C_i]$) exactly k_i from z_i.

In particular, $\chi(C_{t+1}) \geq \chi(G)2^{-t} > \tau_2$. For each i, since $\chi(C_{i+1}) > \tau_2$, it follows that $k_i > \rho$, and in particular, $k_i > 2$. For $1 \leq i < j \leq t$, let A_i be the set of vertices in C_i with distance (in $G[C_i]$) at most $k_i - 2$ from z_i, and let L_i be the set of vertices in C_i with distance $k_i - 1$ in $G[C_i]$ from z_i. Thus $z_i \in A_i$. Since for $1 \leq i < j \leq t$, every vertex in C_j has distance (in $G[C_i]$) exactly k_i from z_i, it follows that:

(1) *For $1 \leq i < j \leq t$, no vertex in A_i has a neighbour in $A_j \cup L_j \cup C_j$.*

(2) *For $1 \leq i \leq t$, $G[A_i]$ is connected, and every vertex in C_{i+1} has a neighbour in L_i, and every vertex in L_i has a neighbour in A_i.*

Choose $u \in C_{t+1}$. Since $\chi(C_{t+1}) > \tau_2$, there exists $v \in C_{t+1}$ such that the distance between u, v in G is at least $\rho + 1 = \ell + 3$. For $1 \leq i \leq t$, choose $u_i, v_i \in L_i$ adjacent to u, v respectively, and choose an induced path P_i with ends u_i, v_i and all its interior vertices in A_i. This is possible by (2).

We see that the paths $P_1, \dots, P_t$ are pairwise vertex-disjoint, and for each i the path u-u_i-P_i-v_i-v is induced. Also, the latter path has length at least $\rho + 1$, since the distance between u, v is at least $\rho + 1$, and so P_i has length at least $\rho - 1$.

For $1 \leq i < j \leq t$, there may be edges between $V(P_i)$ and $V(P_j)$; but by (1), no vertex of P_i has a neighbour in $V(P_j)$ except possibly u_i, v_i.

For $1 \leq i < j \leq t$, let the *type* of (i, j) be the pair (x, y), where

- $x = 2$ if u_i is adjacent to a vertex in the interior of P_j;
- $x = 1$ if u_j is the only neighbour of u_i in $V(P_j)$;
- $x = 0$ if u_i has no neighbours in $V(P_j)$.

and y is defined similarly using v_i, v_j. (Note that u_i, v_j are nonadjacent since the distance between u, v is at least four, so x is well-defined.)

There are only nine possible types, and since $t = R(9, \max(n + 1, \nu + 1), 2, 0)$, there exists $I \subseteq \{1, \dots, t\}$ with $|I| = \max(n + 1, \nu + 1)$ such that all the pairs (i, j) with $i, j \in I$ and $i < j$ have the same type (x, y) say. It $x = 1$ then all the vertices $u_i (i \in I)$ are pairwise adjacent, which is impossible since $|I| > \nu$. So $x \in \{0, 2\}$ and similarly $y \in \{0, 2\}$. If $(x, y) = (0, 0)$, then there are no edges joining any two of the paths $P_i (i \in I)$, and together with u, v they form an $(\geq \ell)$-subdivision of $K_{2,|I|}$, which is impossible. So we may assume that $x = 2$, exchanging x, y if necessary. Choose $i_0 \in I$ minimum. Thus u_{i_0} has a neighbour in the interior of P_i for each $i \in I \setminus \{i_0\}$. If $y = 0$, for each $i \in I \setminus \{i_0\}$ let Q_i be an induced path between u_{i_0} and v with interior in $V(P_i)$; these paths $Q_i (i \in I \setminus \{i_0\})$ each have length at least $\rho \geq \ell + 2$, and there are no edges between them, so they give an $(\geq \ell)$-subdivision of $K_{2,|I|-1}$, which is impossible. So $y = 2$. For each $i \in I \setminus \{i_0\}$ let Q_i be an induced path between u_{i_0} and v_{i_0} with interior in $V(P_i)$; these paths $Q_i (i \in I \setminus \{i_0\})$ each have length at least $\rho - 1 \geq \ell + 1$, and there are no edges between them, so they give an $(\geq \ell)$-subdivision of $K_{2,|I|-1}$, which is impossible.

Thus $\chi(G) \leq 2^t \tau_2$, as claimed. This proves 1.11. ∎

11 String graphs

A *curve* means a subset of the plane which is homeomorphic to the interval $[0, 1]$. Given a finite set C of curves in the plane, its *intersection graph* is the graph with vertex set C in which distinct $S, T \in C$ are adjacent if $S \cap T \neq \emptyset$; and the intersection graphs of sets of curves are called *string graphs*. Every string graph can be realized by a set of piecewise linear curves, and in this paper, a *string* means a piecewise linear curve. In this section we prove that the class of string graphs is 3-controlled, and consequently the theorems of this paper can be applied to the class. The proof that they are 3-controlled is a modification and simplification of an argument of McGuinness [9], who showed that a similar statement holds for a triangle-free subclass of string graphs satisfying another condition that we omit.

Let $(v_1, \dots, v_n)$ be a sequence of distinct vertices of a graph G. We say that $(v_1, \dots, v_n)$ has the *cross property* if for all h, i, j, k with $1 \le h < i < j < k \le n$, if P, Q are paths of G between v_h, v_j and between v_i, v_k respectively, then $V(P)$ is not anticomplete to $V(Q)$. We need the following.

11.1 *Let Δ be a closed disc in the plane, and let C be a finite set of strings all within Δ. Let C_1 be the set of members of C with nonempty intersection with the boundary of Δ. Then C_1 can be ordered as $\{v_1, \dots, v_n\}$ such that $(v_1, \dots, v_n)$ has the cross property in the string graph of C.*

Proof. Let G be the string graph of C. Choose a point $d \in bd(\Delta)$ such that every member of C_1 contains a point of $bd(\Delta) \setminus \{d\}$, and for each $x \in C_1$ choose a point $f(x) \in x \cap (bd(\Delta) \setminus \{d\})$. Number C_1 such that the points $f(x)$ $(x \in C_1)$ are in clockwise order, starting from d and breaking ties arbitrarily. Let the numbering of C_1 be $\{v_1, \dots, v_n\}$. If $1 \le h < i < j < k \le n$, and P is a path of G between v_h and v_j, then the union of the strings in $V(P)$ is an arcwise connected subset of Δ, containing $f(v_h)$ and $f(v_j)$; and therefore includes a string s with ends $f(v_h)$ and $f(v_j)$ (not necessarily in C) with $s \subseteq \Delta$. Similarly if Q is between v_i, v_k, there is a string t between $f(v_i)$ and $f(v_k)$. The strings s, t intersect, and so one of the strings in $V(P)$ has nonempty intersection with one of the strings in $V(Q)$. This proves 11.1. ∎

A *homomorphism* from a graph H to a graph G is a map $\eta : V(H) \to V(G)$, such that for all adjacent $u, v \in V(H)$, $\eta(u), \eta(v)$ are distinct and adjacent in G.

11.2 *Let G be a string graph. Then there is a graph H and $V = \{v_1, \dots, v_n\} \subseteq V(H)$, such that*

- *$(v_1, \dots, v_n)$ has the cross property in H;*
- *every vertex in $V(H) \setminus V$ has a neighbour in V;*
- *there is a homomorphism from H to G; and*
- *$\chi(H \setminus V) \ge \chi(G)/2$.*

Proof. We may assume that $\chi(G) \ge 3$ for otherwise the result is trivial. Choose a component D of G with maximum chromatic number, and let $z \in D$. For $i \ge 0$ let L_i be the set of vertices of D with distance i from z. Choose k such that $\chi(L_k) \ge \chi(G)/2$. Thus $k \ne 0$, and if $k = 1$ then let H be the subgraph induced on $L_0 \cup L_1$, and let $n = 1$ and $v_1 = z$, and the theorem holds. So we may assume that $k \ge 2$. Let D' be a component of $G[L_k]$ with maximum chromatic number. The union of the set of strings in D' is a closed arcwise connected subset of the plane, say S_1; and also the union of the strings in $L_0 \cup \dots \cup L_{k-2}$ is nonnull, closed and arcwise connected, say S_2; and $S_1 \cap S_2 = \emptyset$. Consequently there is a closed disc Δ in the plane disjoint from S_2 and with S_1 in its interior. Moreover, we can choose Δ such that for each string in L_{k-1}, its intersection with Δ is the disjoint union of a finite set of strings. Let V be the set of all strings s such that s is a component of the intersection with Δ of a string in L_{k-1}, and let H be the intersection graph of the set of strings $V \cup L_k$. For each $s \in V$, we claim that $s \cap bd(\Delta) \ne \emptyset$. For there exists $t \in L_{k-1}$ such that s is a component of $t \cap \Delta$; then since t is adjacent in G to a vertex in S_2, and consequently $t \cap S_2 \ne \emptyset$, it follows that every component of $t \cap \Delta$ has nonempty intersection with $bd(\Delta)$, and in particular, $s \cap bd(\Delta) \ne \emptyset$ as claimed. The map $\eta : V(H) \to V(G)$ mapping each string in $V(H)$ to the string in $V(G)$ of which it is a component, is a homomorphism. Moreover, let $r \in V(H) \setminus V = L_k$; we claim that r is adjacent in H to a vertex in V. For let $t \in L_{k-1}$ be adjacent to r in G; then $r \cap t \neq \emptyset$, and since $r \subseteq S_1$, it follows that $r \cap s \neq \emptyset$ for some $s \in V$. Consequently r is adjacent in H to a vertex in V. The result follows from 11.1. This proves 11.2. ▮

Finally we need:

11.3 *Let H be a graph, let $V \subseteq V(H)$, and let $V = \{v_1, \ldots, v_n\}$ where $(v_1, \ldots, v_n)$ has the cross property in H. Assume also that every vertex in $V(H) \setminus V$ has a neighbour in V. Then*

$$\chi^3(H) \geq \chi(H \setminus V)/20.$$

Proof. Let $\kappa = \chi^3(H)$, and suppose that $\chi(H \setminus V) > 20\kappa$. We may assume that H is connected (by choosing a component of H with maximum chromatic number, and working inside that). For each $i \geq 0$, let L_i be the set of vertices of H with distance exactly i from v_1. Choose k such that $\chi(L_k \setminus V) \geq \chi(H \setminus V)/2$. Thus $\chi(L_k \setminus V) > 10\kappa$. Since every vertex in L_k has a neighbour in V, there are disjoint subsets $X_1, \ldots, X_n$ of $L_k \setminus V$ with union $L_k \setminus V$, such that every vertex in X_i is adjacent to v_i for $1 \leq i \leq n$. Consequently $\chi(X_i) \leq \kappa$ for $1 \leq i \leq n$.

(1) *There exist a, b, c, d with $1 \leq a < b < c < d \leq n$, such that there is a path of length three between v_a, v_d, and both its internal vertices belong to $L_k \setminus V$, and the subgraph of H induced on $\bigcup_{b \leq i \leq c} X_i$ has chromatic number more than 4κ.*

For $0 \leq h \leq j \leq n$, let $Y(h,j) = \bigcup_{h<i\leq j} X_i$. Let $i_0 = 0$. Inductively, having defined i_{j-1}, choose i_j with $i_{j-1} \leq i_j \leq n$ minimal such that $\chi(Y(i_{j-1}, i_j)) > 4\kappa$, if such a choice is possible; and otherwise let $i_j = n$ and stop. Let this process stop with $j = t$ and $i_t = n$ say. For $1 \leq j < t$, the minimality of i_j implies that $\chi(Y(i_{j-1}, i_j)) \leq 5\kappa$, since $\chi(X_{i_j}) \leq \kappa$. Also $\chi(Y(i_{t-1}, i_t)) \leq 4\kappa$ since the sequence stopped. Since each of $Y(i_0, i_1), Y(i_1, i_2), \ldots, Y(i_{t-1}, i_t)$ has chromatic number at most 5κ, and $\chi(L_k \setminus V) > 10\kappa$, there exist h, k with $1 \leq h \leq k \leq t$ and $h + 2 \leq k$ such that there is an edge between Y_{i_{h-1}, i_h} and Y_{i_{k-1}, i_k}. Choose j with $h < j < k$; then, taking $b = i_{j-1} + 1$ and $c = i_j$, and choosing $a \leq i_{j-1}$ and $d > i_j$ such that there is an edge between X_a and X_d, this proves (1).

Choose a, b, c, d as in (1), and let Q be a path between v_a, v_d of length three.

(2) *For each $v \in \bigcup_{b \leq i \leq c} X_i$, there is a vertex q of Q such that the distance between v, q is at most three.*

Since $v \in L_k$, there is a path P between v_1, v of length k. Let its vertices be p_0-p_1-$\cdots$-p_k in order, where $p_0 = v_1$ and $p_k = v$. Choose e with $b \leq e \leq c$ such that v is adjacent to v_e. Then there is a path of H between v_e, v_1 with interior included in $V(P)$. By the cross property, there is a vertex $q \in V(Q)$ that either belongs to $V(P) \cup \{v_e\}$ or has a neighbour in $V(P) \cup \{v_e\}$. Now since the interior vertices of Q belong to L_k, it follows that for $0 \leq i \leq k-3$, $p_i \notin V(Q)$ and has no neighbour in $V(Q)$. So q equals or is adjacent to one of $p_{k-2}, p_{k-1}, p_k = v, v_e$. In each case the distance between v, q is at most three. This proves (2).

Since the subgraph of H induced on $\bigcup_{b \leq i \leq c} X_i$ has chromatic number more than 4κ, (2) implies that for one of the four vertices of Q, say q, $\chi(N^3[q]) > \kappa$, a contradiction. Thus $\chi(H \setminus V) \leq 20\kappa$. This proves 11.3. ▮

From 11.2 and 11.3, we deduce:

11.4 *For every string graph G, $\chi(G) \le 40\chi^3(G)$.*

Proof. Let G be a string graph, and choose H and V as in 11.2. Thus $\chi(H \setminus V) \ge \chi(G)/2$. By 11.3, $\chi^3(H) \ge \chi(H \setminus V)/20$, and so $\chi^3(H) \ge \chi(G)/40$. But $\chi^3(G) \ge \chi^3(H)$ since there is a homomorphism from H to G. This proves 11.4. ∎

In particular, the class of string graphs is 3-controlled. Since no string graph has an induced subgraph which is a proper subdivision of $K_{3,3}$, 6.1 implies 1.8, which we restate:

11.5 *The class of string graphs is 2-controlled.*

Consequently the theorems of this paper apply to string graphs, and in particular, 7.9 implies 1.9, which we restate:

11.6 *Let $\nu \ge 0$, and let H be a tree of lamps. Then there exists c such that every string graph with clique number at most ν and chromatic number greater than c contains H as an induced subgraph.*

12 Linearity

In this paper we proved many theorems of the form "For all integers $c' \ge 0$ there exists $c \ge 0$ with the following property...", and the reader may have noticed that in each case, we were able to give an explicit formula for c in terms of c' (and other fixed parameters), and the dependence of c on c' is linear. While it seemed not worth the trouble to mention this linearity at each step, it also seems a pity just to ignore it, so let us see what adjustments we need to retain it. First, let us say a class of graphs $\mathcal{C}$ is *linearly ρ-controlled* if there is a linear nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is (ρ, ϕ)-controlled. Then we check that all the claims in this paper about ρ-controlled classes are also true for linearly ρ-controlled classes. For instance, 1.7 becomes

12.1 *Let $\mu \ge 0$ and $\rho \ge 2$, and let $\mathcal{C}$ be a linearly ρ-controlled class of graphs. The class of all graphs in $\mathcal{C}$ that do not contain any of $K^1_{\mu,\mu}, \ldots, K^{\rho+2}_{\mu,\mu}$ as an induced subgraph is linearly 2-controlled.*

If we wished, we could make the analogous modifications for clique-control and multiclique-control, and then all the results of the paper would have linear analogues. Note that some of these linear analogues are not strengthenings of the original, because for instance, 12.1 needs the stronger hypothesis that $\mathcal{C}$ is linearly ρ-controlled.

Conveniently, 11.4 implies that the class of string graphs is indeed linearly 3-controlled, and so by 12.1, it is also linearly 2-controlled. This answers a question of Bartosz Walczak (private communication).

13 Acknowledgement

We would like to thank Sean McGuinness for his advice on string graphs.

References

[1] Nicolas Bousquet and Stéphan Thomassé, "Scott's induced subdivision conjecture for maximal triangle-free graphs", *Combinatorics, Probability and Computing* 21 (2012), 512–514.

[2] James P. Burling, "On coloring problems of families of prototypes", PhD thesis, University of Colorado, Boulder, 1965.

[3] Jérémie Chalopin, Louis Esperet, Zhentao Li and Patrice Ossona de Mendez, "Restricted frame graphs and a conjecture of Scott", http://arxiv.org/abs/1406.0338 (manuscript June 2014).

[4] Maria Chudnovsky, Alex Scott and Paul Seymour, "Induced subgraphs of graphs with large chromatic number. III. Long holes", *Combinatorica*, to appear, arXiv:1506.02232.

[5] Guo-Li Ding, Paul Seymour and Peter Winkler, "Bounding the vertex cover number of a hypergraph", *Combinatorica* 14 (1994), 1–12.

[6] A. Gyárfás, "On Ramsey covering-numbers", *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), 10.

[7] A. Gyárfás, "Problems from the world surrounding perfect graphs", *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* 19 (1987), 413–441.

[8] Benjamin Lévêque, Frédéric Maffray and Nicolas Trotignon, "On graphs with no induced subdivision of K4", *J. Combinatorial Theory, Ser. B*, 102 (2012), 924–947.

[9] Sean McGuinness, "Colouring arcwise connected sets in the plane II", *Graphs and Combinatorics*, 17 (2001), 135–148.

[10] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T. Trotter and Bartosz Walczak, "Triangle-free intersection graphs of line segments with large chromatic number", *J. Combinatorial Theory, Ser. B*, 105 (2014), 6–10.

[11] A.D. Scott, "Induced trees in graphs of large chromatic number", *J. Graph Theory* 24 (1997), 297–311.

[12] Alex Scott and Paul Seymour, "Induced subgraphs of graphs with large chromatic number. I. Odd holes", *J. Combinatorial Theory, Ser. B*, to appear, arXiv:1410.4118.

[13] D.P. Sumner, "Subtrees of a graph and chromatic number", in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.