Packing seagulls

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Abstract

A *seagull* in a graph is an induced three-vertex path. When does a graph G have k pairwise vertex-disjoint seagulls? This is NP-complete in general, but for graphs with no stable set of size three we give a complete solution. This case is of special interest because of a connection with Hadwiger's conjecture which was the motivation for this research; and we deduce a unification and strengthening of two theorems of Blasiak [2] concerned with Hadwiger's conjecture.

Our main result is that a graph G (different from the five-wheel) with no three-vertex stable set contains k disjoint seagulls if and only if

- $|V(G)| \geq 3k$
- G is k-connected,
- for every clique C of G, if D denotes the set of vertices in $V(G) \setminus C$ that have both a neighbour and a non-neighbour in C then $|D| + |V(G) \setminus C| \geq 2k$, and
- the complement graph of G has a matching with k edges.

We also address the analogous fractional and half-integral packing questions, and give a polynomial time algorithm to test whether there are k disjoint seagulls.

1 Hadwiger's conjecture and seagulls

Hadwiger's conjecture from 1943 asserts [5] that every graph with chromatic number t contains K_t as a minor. (All graphs in this paper are finite and have no loops or parallel edges.) One special case is when we restrict to graphs G with $\alpha(G) \leq 2$ (we denote by $\alpha(G)$ the cardinality of the largest stable set of vertices). Since the chromatic number for such graphs is at least half the number of vertices, Hadwiger's conjecture implies:

1.1 Conjecture. *Let G be a graph with $\alpha(G) \leq 2$, and let $t = \lceil |V(G)|/2 \rceil$; then G contains K_t as a minor.*

This conjecture remains open, and seems to be very challenging, and perhaps false. (Note that if the conjecture 1.1 is true for all graphs G with $\alpha(G) \leq 2$, then Hadwiger's conjecture is also true for these graphs. For a proof see [6].)

For graphs G with $\alpha(G) \leq 2$, we are looking for t pairwise disjoint connected subgraphs (where $t = \lceil |V(G)|/2 \rceil$) that pairwise touch (two disjoint subgraphs or subsets *touch* if there is an edge between them), to give a K_t minor. Perhaps even more is true; that there exist t such subgraphs, each with either one or two vertices. We call this the *matching version of 1.1.*

In an attempt to make some progress, Jonah Blasiak [2] added more hypotheses. He proved:

1.2 *Let G be a graph with $\alpha(G) \leq 2$ and with $|V(G)|$ even. If either*

- *$V(G)$ is the union of three cliques, or*
- *there is a list of k cliques such that every vertex belongs to strictly more than $k/3$ of them*

then G satisfies 1.1, and indeed satisfies the matching version of 1.1.

But what if $|V(G)|$ is odd (and G still satisfies the other hypotheses of 1.2, and in particular $|V(G)| = 2t - 1$)? In this case 1.1 implies that there is a K_t minor, but Blasiak's method only yields a K_{t-1} minor, and trying to gain the missing $+1$ has been the focus of a fair amount of effort. For the matching version of 1.1, gaining the missing $+1$ is still open; but in this paper we prove 1.1 itself under a common generalization of Blasiak's alternative extra hypotheses. We prove the following.

1.3 *Let G be a graph with $\alpha(G) \leq 2$, and let $t = \lceil |V(G)|/2 \rceil$. If some clique in G has cardinality at least $|V(G)|/3$, then G has a K_t minor.*

To prove this we use the next result, which has been proved independently by several authors (see [6]):

1.4 *Let G be a graph with $\alpha(G) \leq 2$, and let $t = \lceil |V(G)|/2 \rceil$. If G is not t-connected then G satisfies 1.1, and indeed satisfies the matching version of 1.1.*

A *seagull* in G is a subset $S \subseteq V(G)$ with $|S| = 3$ such that exactly one pair of vertices in S are nonadjacent in G, and a *singleton* in G is a subset of $V(G)$ with cardinality one. Then 1.3 follows immediately from 1.4 and the next result, proved in section 2.

1.5 *Let G be a graph with $\alpha(G) \leq 2$, let $t = \lceil |V(G)|/2 \rceil$, and let G be t-connected. If the largest clique Z in G satisfies $|V(G)|/3 \leq |Z| \leq t$, then there are $t - |Z|$ pairwise disjoint seagulls in $V(G) \setminus Z$, and consequently G has a K_t minor.*

This is a consequence of our main result, which we explain next. If $X, Y \subseteq V(G)$ are disjoint, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y, and X is *anticomplete* to Y if no vertex in X has a neighbour in Y. We say a vertex v is complete (or anticomplete) to a set Y if $\{v\}$ is, and v is *mixed* on Y if $v \in V(G) \setminus Y$ and v is neither complete nor anticomplete to Y. If C is a nonempty clique in G, let A, B, D be the sets of vertices in $V(G) \setminus C$ that are complete to C, anticomplete to C, and mixed on C respectively. Thus (A, B, C, D) is a partition of $V(G)$, called the *associated partition* of C. We define the *capacity* $\mathrm{cap}(C)$ of C to be $|D| + |A \cup B|/2$. A *five-wheel* is a six-vertex graph obtained from a cycle of length five by adding one new vertex adjacent to every vertex of the cycle. An *antimatching* in G is a matching in the complement graph $\overline{G}$ of G. Our main result is the following.

1.6 *Let G be a graph with $\alpha(G) \leq 2$, and let $k \geq 0$ be an integer, such that if $k = 2$ then G is not a five-wheel. Then G has k pairwise disjoint seagulls if and only if*

- $|V(G)| \geq 3k$
- *G is k-connected,*
- *every clique of G has capacity at least k, and*
- *G admits an antimatching of cardinality k.*

Proof of 1.5, assuming 1.6. Let $n = |V(G)|$ and $t = \lceil n/2 \rceil$; thus $|Z| \geq n/3$, and we may assume that $|Z| < t$, and so $|Z| < n/2$. Let $k = t - |Z|$, and $H = G \setminus Z$. We claim that there are k disjoint seagulls in H.

To see this, we verify the hypotheses of 1.6. Since G is t-connected, it follows that H is k-connected. Now

$$|V(H)| = n - |Z| = 2(n + 1 - 2t) + 3(|Z| - n/3) + 4(t - |Z|) - 2 \geq 4(t - |Z|) - 2 = 4k - 2$$

since $2t \leq n + 1$ and $|Z| \geq n/3$. Suppose that $|V(H)| < 3k$; then $4k - 2 < 3k$, and so $k = 1$, and since $|V(H)| \geq (n+1)/2$ it follows that $n \leq 3$ and $|Z| = 1$, contradicting that Z is the largest clique in G. Thus $|V(H)| \geq 3k$.

Suppose that C is a clique of H, with capacity (in H) less than k. Let its associated partition be (A, B, C, D) (so (A, B, C, D, Z) is a partition of $V(G)$). Thus $|D| + |A \cup B|/2 \leq k - 1/2$, and so $|A| + |B| + 2|D| \leq 2k - 1$. Consequently

$$2t - 1 + |D| \leq |V(G)| + |D| = |A| + |B| + |C| + 2|D| + |Z| \leq 2k - 1 + |C| + |Z| \leq 2k - 1 + 2|Z| = 2t - 1,$$

and so we have equality throughout; and in particular $D = \emptyset$ and $|C| = |Z|$. Thus C is a largest clique in G, and hence $A = \emptyset$ (since if $v \in A$ then $C \cup \{v\}$ is a clique); and since there are no edges between B and C, and H is k-connected, it follows that $B = \emptyset$. Thus $C \cup Z = V(G)$, which is impossible since $|C| \leq |Z| < n/2$. This proves that every clique of H has capacity at least k.

We claim that H has an antimatching of cardinality at least k. For suppose not, and let M be a maximal antimatching in H of cardinality at most $k - 1$. Hence there are at least $|V(H)| - 2(k-1)$ vertices in H not incident with any edge of M, and the maximality of M implies that these vertices are pairwise adjacent in H. Consequently H has a clique of cardinality at least $|V(H)| - 2(k-1)$, and the maximality of $|Z|$ implies that $|V(H)| - 2(k-1) \le |Z|$, and so $n \le 2|Z| + 2(k-1) = 2t - 2$, a contradiction. This proves that H has an antimatching of cardinality at least k.

Finally, suppose that $k = 2$ and H is a 5-wheel. Then $n = |Z| + 6$, and since $t - |Z| = k = 2$, it follows that $n = t + 4$. Since $2t - 1 \le n \le 2t$, we deduce that $2t - 1 \le t + 4 \le 2t$, that is, $4 \le t \le 5$. If $t = 4$ then $n = 8$ and $|Z| = 2$, contradicting that $|Z| \ge n/3$. Thus $t = 5$ and $n = 9$, and $|Z| = 3$. Let D be the induced cycle of length five in H. Since Z is a maximum clique of G, it follows that G has no K_4 subgraph; and so for each edge uv of D, at most one member of Z is adjacent to both u, v. Since D has only five edges, there exists $z \in Z$ such that $\{z, u, v\}$ is a clique for at most one edge uv of D. But then there are two nonadjacent vertices of D both nonadjacent to z, contradicting that $\alpha(G) \le 2$.

From 1.6, we deduce that there are k disjoint seagulls in H. These, together with the singletons of Z, form a K_t minor (note that they pairwise touch since every vertex has a neighbour in every seagull, because $\alpha(G) \le 2$). This proves 1.5. ∎

In the final section, we prove a result analogous to 1.6 for packing seagulls fractionally, and derive a polynomial time algorithm to test whether G has k disjoint seagulls (for graphs G with no three-vertex stable set). What about the problem of deciding whether a general graph has k disjoint seagulls? If G is the line graph of some graph H, and $|E(H)| = 3k$, then deciding whether G has k disjoint seagulls is the same as deciding whether the edges of H can be partitioned into paths of length three; and the latter was shown to be NP-complete by Dor and Tarsi [3]. Consequently the problem of deciding whether a general graph has k disjoint seagulls is also NP-complete, and therefore there is no analogue of 1.6 for general graphs unless NP = co-NP.

2 Finding disjoint seagulls

In this section we prove 1.6. The "only if" half is easy and we leave it to the reader. To prove the "if" half, suppose that it is false, and choose G, k that fail to satisfy 1.6, with $|V(G)| + k$ minimum. Throughout this section G, k are fixed with these properties, which we summarize for convenient reference:

2.1 *G is a graph with $\alpha(G) \le 2$, and $k \ge 0$ is an integer, such that*

1. *$|V(G)| \ge 3k$,*
2. *G is k-connected,*
3. *every clique of G has capacity at least k,*
4. *G admits an antimatching of cardinality k,*
5. *there do not exist k seagulls in G, pairwise disjoint, and*
6. *every pair G', k' with $|V(G')| + k' < |V(G)| + k$ satisfies 1.6.*

We prove a series of results about G, k that will eventually lead to a contradiction.

Let A, B be disjoint subsets of $V(H)$, where H is a graph. A *matching between A and B* is a set of edges of G, each with one end in A and the other in B, and pairwise with no common ends.

A *matching of A into B* is a matching between A and B of cardinality $|A|$; and if such a matching exists, we say that A can be *matched* into B. We use similar terminology for antimatchings. (An *antiedge* means an edge of $\overline{G}$.) We need the following lemma.

2.2 *Let H be a graph, and let $A, B \subseteq V(H)$ be disjoint. Let $p, q \geq 0$ be integers such that $p+q \leq |B|$ and $p, q \leq |A|$. Suppose that either $q = |A|$ or there is a matching between A and B of cardinality p, and either $p = |A|$ or there is an antimatching between A, B of cardinality q. Suppose also that for every nonempty subset $Y \subseteq B$, the number of vertices in A that are mixed on Y is at least $|Y|+p+q-|A|-|B|$. Then there exist disjoint $P, Q \subseteq B$, with $|P| = p$ and $|Q| = q$, such that there is a matching of P into A and an antimatching of Q into A.*

Proof. We claim that there is a matching between A and B of cardinality p. If $q < |A|$ then this is a hypothesis of the theorem, so we assume that $q = |A|$. To show the desired matching exists, it suffices by König's theorem to show that for every $X \subseteq A \cup B$, if $A \setminus X$ is anticomplete to $B \setminus X$ then $|X| \geq p$. Let $Y = B \setminus X$. If $Y = \emptyset$ then $|X| \geq |B| \geq p$ as required, so we may assume that $Y \neq \emptyset$. Every vertex in A mixed on Y belongs to $X \cap A$, and so by hypothesis $|X \cap A| \geq |Y|+p+q-|A|-|B|$, that is, $|X| \geq p$ as required. This proves our claim that there is a matching between A and B of cardinality p. Similarly there is an antimatching of cardinality q.

Let M_1 be the set of all subsets of B that can be matched into A, and let M_2 be the set of all subsets of B that can be antimatched into A. Thus M_1, M_2 are matroids (regarded as a set of independent subsets), of ranks at least p, q respectively. For $Y \subseteq B$, let $N_1(Y), N_2(Y)$ be the sets of all vertices in A with a neighbour in Y and with a nonneighbour in Y, respectively. Thus from König's theorem, for $i = 1, 2$ the rank function r_i of M_i is given by $r_i(X) = \min_{Y \subseteq X} |N_i(Y)|+|X \setminus Y|$.

We claim that there exist $X_1 \in M_1$ and $X_2 \in M_2$ with $|X_1 \cup X_2| \geq p+q$. To see this, it suffices from the matroid union theorem [4] to show that for all $X \subseteq B$, $r_1(X) + r_2(X) + |B \setminus X| \geq p+q$; that is, for all $X \subseteq B$ and all $Y_1, Y_2 \subseteq X$,

$$|N_1(Y_1)| + |X \setminus Y_1| + |N_2(Y_2)| + |X \setminus Y_2| + |B \setminus X| \geq p+q,$$

which can be rewritten as $|N_1(Y_1)|+|N_2(Y_2)|+|X|-|Y_1|-|Y_2| \geq p+q-|B|$. Let $Y = Y_1 \cap Y_2$; then the left side of the previous inequality is at least $|N_1(Y)| + |N_2(Y)| - |Y|$, since $|X| - |Y_1| - |Y_2| \geq -|Y|$. It therefore suffices to show that $|N_1(Y)| + |N_2(Y)| - |Y| \geq p+q-|B|$ for all $Y \subseteq B$. If $Y = \emptyset$ this is true, since $|B| \geq p+q$, and so we may assume that $Y \neq \emptyset$; and so $N_1(Y) + N_2(Y) = A$. Let $M(Y)$ be the set of vertices in A that are mixed on Y; then $|N_1(Y)| + |N_2(Y)| = |A| + |M(Y)|$. We must therefore show that $|A| + |M(Y)| - |Y| \geq p+q-|B|$ for all nonempty $Y \subseteq B$. But this is a hypothesis of the theorem. This proves our claim.

Let $X_1 \in M_1$ and $X_2 \in M_2$ with $|X_1 \cup X_2| \geq p+q$. Since the matroids have ranks at least p and at least q respectively, we may choose X_1, X_2 with $|X_1 \cup X_2| \geq p+q$, and $|X_1| \geq p$, and $|X_2| \geq q$. But now we can choose $P \subseteq X_1$ and $Q \subseteq X_2$ to satisfy the theorem. This proves 2.2. ∎

We also frequently need the following.

2.3 *Let H be a graph and let $n \geq 0$ be an integer, with $|V(H)| \geq 2n$. Then either H has a matching with cardinality n, or there exists $X \subseteq V(H)$ such that:*

- *$H \setminus X$ has exactly $|X| + |V(H)| - 2n + 2$ components*

- *for every component C of $H \setminus X$, and every vertex $v \in V(C)$, $C \setminus \{v\}$ has a perfect matching (and consequently $|V(C)|$ is odd, and if $|V(C)| > 1$ then C is not bipartite), and*
- $|X| \leq n - 1$.

Proof. Suppose that the matching does not exist. By the "Tutte-Berge formula" [1], there exists $X \subseteq V(H)$ such that $H \setminus X$ has more than $|X| + |V(H)| - 2n$ odd components, where an "odd" component means a component with an odd number of vertices. Choose such a set X maximal.

Since the number of odd components of $H \setminus X$ has the same parity as $|V(H)|+|X|$, it follows that $H \setminus X$ has at least $|X|+|V(H)|-2n+2$ odd components. If $H \setminus X$ has more than $|X|+|V(H)|-2n+2$ odd components, then since $|V(H)| \geq 2n$ it follows that $H \setminus X$ has at least one odd component C, and choosing $v \in V(C)$ and replacing X by $X \cup \{v\}$ contradicts the maximality of X. Consequently $H \setminus X$ has exactly $|X| + |V(H)| - 2n + 2$ odd components.

Let C be a component of $H \setminus X$. Let $v \in V(C)$, and suppose that $C \setminus \{v\}$ has no perfect matching. By Tutte's theorem, there exists $Y \subseteq V(C) \setminus \{v\}$ such that $C \setminus (Y \cup \{v\})$ has at least $|Y| + 1$ odd components; but then $H \setminus (X \cup Y \cup \{v\})$ has at least $(|X| + |V(H)| - 2n + 2) - 1 + (|Y| + 1)$ odd components, and so more than $|X \cup Y \cup \{v\}| + |V(H)| - 2n$, contrary to the maximality of X. This proves that $C \setminus \{v\}$ has a perfect matching. Consequently C is odd, and if $|V(C)| > 1$ then C is not bipartite.

Finally, since $H \setminus X$ has $|X| + |V(H)| - 2n + 2$ components, it follows that $H \setminus X$ has at least $|X| + |V(H)| - 2n + 2$ vertices, and so $|V(H)| - |X| \geq |X| + |V(H)| - 2n + 2$, that is, $|X| \leq n - 1$. This proves 2.3. ∎

2.4 *G is $(k + 1)$-connected.*

Proof. Suppose not; then since G is k-connected by 2.1.2, there is a partition (L, M, R) of $V(G)$ such that $L, R \neq \emptyset$, and L is anticomplete to R, and $|M| = k$. If $|L|, |R| \geq k$, then there are k vertex-disjoint paths between L and R, and each includes a seagull, contrary to 2.1.5. Thus we may assume that $|L| = k - x$ say, where $1 \leq x \leq k - 1$. Since $|V(G)| \geq 3k$ by 2.1.1, it follows that $|R| \geq k + x$. Since G is k-connected, there is a matching of cardinality k between M and R. Suppose that there is a nonempty subset $C \subseteq R$ such that fewer than $|C|+k+x-|M|-|R|$ vertices in M are mixed on C. Then C is a clique; let the associated partition be (A, B, C, D) say. Thus $R \setminus C \subseteq A$, and $L \subseteq B$, and $D \subseteq M$, and $|D| < |C| + k + x - |M| - |R|$. But C has capacity

$$|D|/2 + |V(G) \setminus C|/2 < (|C| + k + x - |M| - |R|)/2 + (|L| + |M| + |R| - |C|)/2 = k$$

since $|L| = k - x$, contrary to 2.1.2. Thus from 2.2, there exist disjoint $P, Q \subseteq R$ with $|P| = k$ and $|Q| = x$, such that P is matchable into M and Q is antimatchable into M. Let $M = \{m_1, \ldots, m_k\}$, and let $P = \{p_1, \ldots, p_k\}$ and $Q = \{q_1, \ldots, q_x\}$, where $p_i m_i$ is an edge for $1 \leq i \leq k$, and $q_i m_i$ is an antiedge for $1 \leq i \leq x$. Since G is k-connected, we may number L as $\{l_1, \ldots, l_{k-x}\}$ such that $l_i m_{x+i}$ is an edge for $1 \leq i \leq k - x$. But then

$$\{m_i, p_i, q_i\} \ (1 \leq i \leq x), \{l_i, m_{x+i}, p_{x+i}\} \ (1 \leq i \leq k - x)$$

are k disjoint seagulls, a contradiction. This proves 2.4. ∎

2.5 *Every clique of G has capacity at least $k+1/2$, and if $|V(G)| > 3k$ then every clique has capacity at least $k+1$.*

Proof. Suppose that C is a clique with $2\,\mathrm{cap}(C) \le |V(G)| - k$ and with $\mathrm{cap}(C) \le k + 1/2$, chosen with $\mathrm{cap}(C)$ minimum. Let (A, B, C, D) be the associated partition; thus $|D| + (|A| + |B|)/2 = \mathrm{cap}(C)$, and $|D| \le k$. Since $2|D| + |A| + |B| = 2\,\mathrm{cap}(C)$, and $|A| + |B| + |C| + |D| = |V(G)|$, it follows that $|C| = |D| + |V(G)| - 2\,\mathrm{cap}(C) \ge |D| + k$.

(1) *There are $|D|$ pairwise disjoint seagulls included in $C \cup D$, each with exactly one vertex in D.*

Suppose that there exists a nonempty subset $Y \subseteq C$ such that fewer than $|Y| + |D| - |C|$ vertices in D are mixed on Y. Since the only vertices mixed on Y belong to D, it follows that $2\,\mathrm{cap}(Y) < (|Y| + |D| - |C|) + (|V(G)| - |Y|)$. But $|C| = |D| + |V(G)| - 2\,\mathrm{cap}(C)$, and so $\mathrm{cap}(Y) < \mathrm{cap}(C)$, a contradiction. Thus there is no such Y. Moreover, since $|C| \ge |D| + k$ and $|D| \le k$, it follows that $|C| \ge 2|D|$. Hence the claim follows from 2.2, taking $p = q = |D|$. This proves (1).

Let H be the graph with vertex set $A \cup B$, in which distinct u, v are adjacent if either $u, v \in A$ and u, v are nonadjacent in G, or exactly one of u, v is in A and u, v are adjacent in G.

(2) *There is a matching in H of cardinality $k - |D|$.*

For suppose not. Since $|A| + |B| \ge 2(k - |D|)$, 2.3 implies that there exists $X \subseteq V(H)$ with $|X| \le k - |D| - 1$ such that $H \setminus X$ has $|X| + |V(H)| - 2(k - |D| - 1)$ odd components; and every component of $H \setminus X$ is odd, and every component of $H \setminus X$ with more than one vertex is not bipartite. Since $|X| \le k - |D| - 1$, it follows that $|D \cup X| < k$, and since G is k-connected, we deduce that $G \setminus (D \cup X)$ is connected.

Suppose that some component P of $H \setminus X$ contains a vertex in B. Since $\alpha(G) < 3$, no other component contains two vertices of A that are nonadjacent in G; and so every other component of $H \setminus X$ is bipartite, and therefore has only one vertex. Since $G \setminus (D \cup X)$ is connected, some vertex in A belongs to the union of all components of $H \setminus X$ that contain a vertex in B; and so P has more than one vertex. Consequently P is not bipartite, and so contains two vertices of A that are nonadjacent in G; and therefore no other component has a vertex in B. Thus $B \subseteq V(P) \cup X$. Since $H \setminus X$ has $|X| + |V(H)| - 2(k - |D| - 1)$ odd components, it follows that

$$|A \setminus (X \cup P)| \ge |X| + |V(H)| - 2(k - |D| - 1) - 1.$$

Let $C' = C \cup (A \setminus (X \cup P))$. Then

$$|C'| \ge |C| + |X| + |V(H)| - 2(k - |D| - 1) - 1 = |V(G)| + |X| - 2k + 1 + |D|$$

The only vertices mixed on C' belong to $X \cup D$, and so

$$2\,\mathrm{cap}(C') \le |X \cup D| + |V(G)| - |C'| \le |X \cup D| + |V(G)| - (|V(G)| + |X| - 2k + 1 + |D|) = 2k - 1,$$

a contradiction.

Thus no component of $H \setminus X$ meets B, and so $B \subseteq X$. Since G has an antimatching of cardinality k, there is an antimatching of cardinality at least $k-|D|-|X|$ in $G \setminus (D \cup X)$. Since C is complete to all other vertices of $G \setminus (D \cup X)$, it follows that there is an antimatching of cardinality $k-|D|-|X|$ in $G \setminus (C \cup D \cup X)$. Hence at least $2(k-|D|-|X|)$ vertices are incident with antiedges in this antimatching; but for every odd component of $H \setminus X$, at least one vertex is not incident with an antiedge of the antimatching. Consequently

$$|A \setminus X| \geq 2(k-|D|-|X|) + |X| + |V(H)| - 2(k-|D|-1),$$

that is, $|A \setminus X| \geq -|X| + |A| + |B| + 2$, which is impossible. This proves (2).

Let $X_1, \dots, X_{k-|D|}$ be the vertex sets of the edges of the matching in H given by (2). Let $S_1, \dots, S_{|D|}$ be the $|D|$ seagulls in (1). Since $|C| \geq |D| + k = 2|D| + (k-|D|)$, there are at least $k-|D|$ vertices in C that do not belong to these seagulls, say $c_1, \dots, c_{k-|D|}$. But then

$$S_1, \dots, S_{|D|}, X_i \cup \{c_i\} \ (1 \leq i \leq k-|D|)$$

are k disjoint seagulls, a contradiction. This proves 2.5. ∎

2.6 *G admits an antimatching of cardinality $k+1$.*

Proof. Suppose not; then by 2.3, there exists $X \subseteq V(G)$ with $|X| \leq k$ such that $\overline{G} \setminus X$ has m odd components, say $C_1, \dots, C_m$, where $m = |V(G)| + |X| - 2k$, and for $1 \leq i \leq m$, $|V(C_i)|$ is odd, and either $|V(C_i)| = 1$ or C_i is not bipartite, and for every vertex v of C_i, $C_i \setminus \{v\}$ has a perfect matching. Let $X = \{x_1, \dots, x_t\}$.

(1) *There exist distinct $i_1, \dots, i_t, j_1, \dots, j_t \in \{1, \dots, m\}$ such that for $1 \leq h \leq t$, x_h has a neighbour in C_{i_h} and has a nonneighbour in C_{j_h}.*

Let M_1 be the set of all subsets $Y \subseteq \{1, \dots, t\}$ such that there is an injective function $f : Y \to X$ satisfying that $f(y)$ has a neighbour in C_y for each $y \in Y$; and let M_2 be the set of all $Y \subseteq \{1, \dots, t\}$ such that there is an injective function $f : Y \to X$ satisfying that $f(y)$ has a nonneighbour in C_y for each $y \in Y$. We need to show that there exist a member of M_1 and a member of M_2 with union of cardinality $2t$. For $Y \subseteq \{1, \dots, m\}$, let $C(Y)$ denote $\bigcup_{y \in Y} V(C_y)$, and let $N_1(Y), N_2(Y)$ denote respectively the set of all vertices in X with a neighbour in $C(Y)$, and the set of all vertices in X with a nonneighbour in $C(Y)$. As in the proof of 2.2, it suffices to show that for all $Y \subseteq \{1, \dots, m\}$,

$$|N_1(Y)| + |N_2(Y)| - |Y| \geq 2t - m.$$

If $Y = \emptyset$, the claim holds since $m \geq 2t$ (because $t \leq k$ and $|V(G)| \geq 3k$). Thus we may assume that $Y \neq \emptyset$, and so $|N_1(Y)| + |N_2(Y)| = |X| + |M(Y)|$, where $M(Y)$ denotes the set of all vertices in X with a neighbour and a nonneighbour in $C(Y)$. Hence we must show that $|Y| - |M(Y)| \leq m - t = |V(G)| - 2k$ for all nonempty subsets $Y \subseteq \{1, \dots, m\}$.

Suppose first that $C(Y)$ is not a clique of G. Then no vertex of X is anticomplete to $C(Y)$, and so all the components C_y $(y \in Y)$ are also components of $\overline{G} \setminus M(Y)$. Thus $\overline{G} \setminus M(Y)$ has at least

$|Y|$ odd components; and since $\overline{G}$ has a matching of cardinality k, it follows that $\overline{G} \setminus M(Y)$ has at most $|M(Y)|+|V(G)|-2k$ odd components. Consequently $|Y| \leq |M(Y)|+|V(G)|-2k$ as required.

Thus we may assume that $C(Y)$ is a clique of G. All vertices of G mixed on Y belong to $M(Y)$, and so $2\,\mathrm{cap}(C(Y)) \leq |M(Y)|+|V(G)|-|C(Y)|$. Since $2\,\mathrm{cap}(C(Y)) \geq 2k$, it follows that $2k \leq |M(Y)|+|V(G)|-|C(Y)|$. But since $C(Y)$ is a clique of G, it follows that each C_y ($y \in Y$) is also a clique of G, and hence has only one vertex; and so $|C(Y)| = |Y|$. Thus we have shown that $2k \leq |M(Y)|+|V(G)|-|Y|$, as required. This proves (1).

(2) *For each component C of $\overline{G} \setminus X$, if $|V(C)| > 1$ and C is not a cycle of length five, then there exists $c \in V(C)$ and a perfect matching M of $C \setminus \{c\}$ such that c is nonadjacent (in C) to both ends of some edge of M.*

For since $|V(C)| > 1$, there exist $u, v \in V(C)$, adjacent in C. From the choice of X, both $C \setminus \{u\}$ and $C \setminus \{v\}$ have perfect matchings; and by taking the union of two such perfect matchings, we deduce that there is an odd cycle D of C such that $C \setminus V(D)$ has a perfect matching. By choosing D minimal it follows that D is a hole of C. Let M be a perfect matching of $C \setminus V(D)$. Since $\alpha(G) < 3$, D has length at least five. Suppose it has length at least seven, and let d_1-d_2-$\cdots$-d_5 be a path of D. Then the edge d_4d_5 is contained in a perfect matching of $C \setminus \{d_1\}$, and d_1 is nonadjacent in C to both d_4, d_5, so the claim holds. Hence we may assume that D has length five. Suppose that $M \neq \emptyset$, and let pq be an edge of M. For each vertex $d \in V(D)$, there is a perfect matching of $C \setminus \{d\}$ containing pq, and so we may assume that d is adjacent to one of p, q; and so one of p, q is adjacent to at least three of the five vertices in D, which is impossible since C has no triangles. Thus $M = \emptyset$, and so C is a cycle of length five, contrary to the hypothesis. This proves (2).

If $X \neq \emptyset$ and $i \in \{1, \ldots, m\}$, we may choose $i_1, \ldots, i_t, j_1, \ldots, j_t$ as in (1) with

$$i \in \{i_1, \ldots, i_t, j_1, \ldots, j_t\}.$$

(To see this, assume that $i \neq i_1, \ldots, i_t, j_1, \ldots, j_t$. If x_1 has a neighbour in C_i we may replace i_1 by i, and otherwise we may replace j_1 by i.) In particular, if $X \neq \emptyset$ and $t < k$, then since $\overline{G} \setminus X$ has $|V(G)| - t$ vertices and only $|V(G)| + t - 2k$ components, at least one of the components has more than one vertex, and so we may assume that $i_h = h$ and $j_h = t + h$ for $1 \leq h \leq t$ and one of $C_1, \ldots, C_{2t}$ has more than one vertex. From (2), for $1 \leq i \leq m$ we may choose $c_i \in V(C_i)$ and a perfect matching of $C_i \setminus \{c_i\}$, such that

- for $1 \leq i \leq t$, c_i, x_i are adjacent
- for $t+1 \leq i \leq 2t$, c_i, x_{i-t} are nonadjacent
- for $i > 2t$, if C_i has more than one vertex and is not a cycle of length five then c_i is nonadjacent (in C_i) to both ends of some edge of M_i
- if $0 < t < k$, at least one of $C_1, \ldots, C_{2t}$ has more than one vertex.

Now $M_1 \cup \cdots \cup M_m$ is a matching of $\overline{G}$ and hence an antimatching of G; and its cardinality is $(|V(G)| - m - t)/2 = k - t$, since it covers all vertices except one of each C_i. If there is an injective map $f : M \to \{c_{2t+1}, \ldots, c_m\}$ such that $f(e)$ is adjacent to both ends of the antiedge e for each

$e \in M$, then the union of $\{f(e)\}$ with the set of ends of e is a seagull, for each $e \in M$, and these $k-t$ seagulls, together with the t seagulls $\{x_i, c_i, c_{t+i}\}$ $(1 \leq i \leq t)$ are k disjoint seagulls, a contradiction. Thus by Hall's theorem, there is a nonempty subset $M' \subseteq M$ such that $|N| \leq |M'| - 1$, where N is the set of all c_i with $2t < i \leq m$ such that c_i is adjacent to both ends of some member of M'. Let I be the set of $i \in \{1, \ldots, m\}$ such that some edge of C_i belongs to M'. Let $i \in I$, and let $e \in M' \cap E(C_i)$. Since C_i is a component of $\overline{G} \setminus X$, it follows that every vertex in $V(G) \setminus (X \cup V(C_i))$ is complete to $V(C_i)$ and in particular, each c_j with $j \neq i$ is adjacent to both ends of e; and so $\{j : 2t+1 \leq j \leq m, j \neq i\} \subseteq N$. We deduce that $|N| \geq m - 2t - 1$, and so $|M'| \geq m - 2t$. But $|M'| \leq |M| = k - t \leq m - 2t$, and so we have equality throughout, and in particular $M' = M$, and $|V(G)| = 3k$, and $i > 2t$, and $N = \{j : 2t+1 \leq j \leq m, j \neq i\}$. Since this holds for all $i \in I$, we deduce that $I \cap \{1, \ldots, 2t\} = \emptyset$, and $I = \{i\}$; let $i = m$ say. Consequently every edge of M belongs to C_m, and so $C_1, \ldots, C_{m-1}$ each have only one vertex. From the last bulletted statement above, it follows that $t = 0$ or $t = k$. If $t = k$ then $M = \emptyset$, which is impossible; so $t = 0$, and therefore $X = \emptyset$ and $|M| = k$.

Suppose that C_m is not a five-cycle. Then by the third bulletted statement above, there is an antiedge $p_k q_k$ of M such that $\{p_k, q_k, c_k\}$ is a seagull. Let the other antiedges of M be $p_i q_i$ for $i = 1, \ldots, k-1$. Then $\{p_i, q_i, c_i\}$ $(1 \leq i \leq k)$ are k disjoint seagulls, a contradiction.

This proves that C_m is a five-cycle, and so $|M| = 2$, and therefore $k = 2$, and $|V(G)| = 6$, and $m = 2$; but then G is a five-wheel, a contradiction. This proves 2.6. ∎

2.7 *G has exactly $3k$ vertices.*

Proof. Suppose that $|V(G)| > 3k$, and let $v \in V(G)$. Let $G' = G \setminus \{v\}$. By 2.4, G' is k-connected, and by 2.6, G' has an antimatching of cardinality k. By 2.5, since $|V(G)| > 3k$, it follows that every clique in G has capacity at least $k+1$, and therefore every clique in G' has capacity at least k. But since G does not have k disjoint seagulls, the same holds for G', and therefore from 2.1.6 it follows that $k = 2$ and G' is a five-wheel. Since this holds for every vertex v of G, it follows that G has at least two vertices of degree at least five (for otherwise we could delete a vertex leaving no vertex of degree five), and so (by deleting some third vertex) it follows that for some choice of v, $G \setminus \{v\}$ has at least two vertices of degree at least four, and therefore is not a five-wheel, a contradiction. This proves 2.7. ∎

2.8 *Every clique has capacity at least $k+1$.*

Proof. Suppose that C is a clique with capacity less than $k+1$, and therefore with capacity $k + 1/2$, by 2.5. Let (A, B, C, D) be the associated partition; thus $|A| + |B| + 2|D| = 2k+1$, that is, $|C| = |D| + k - 1$, since $|V(G)| = 3k$ by 2.7.

(1) $|D| \leq k - 1$.

For suppose not; then $|D| = k$ and $|A| + |B| = 1$. If $A = \emptyset$ and $|B| = 1$ then G is not $(k+1)$-connected, contrary to 2.4. If $B = \emptyset$ and $|A| = 1$ then $C \cup A$ is a clique with capacity at most $|D| = k$, contrary to 2.5. This proves (1).

(2) *There are $|D|$ pairwise disjoint seagulls included in $C \cup D$, each with exactly one vertex in D.*

Suppose that there exists a nonempty subset $Y \subseteq C$ such that fewer than $|Y| + |D| - |C|$ vertices in D are mixed on Y. Since the only vertices mixed on Y belong to D, it follows that

$$2\,\mathrm{cap}(Y) < (|Y| + |D| - |C|) + (|V(G)| - |Y|) = |V(G)| - |C| + |D| = 2\,\mathrm{cap}(C) = 2k + 1,$$

contrary to 2.5. Thus there is no such Y. Moreover, since $|C| = |D| + k - 1$ and $|D| \leq k - 1$, it follows that $|C| \geq 2|D|$. Hence the claim follows from 2.2, taking $p = q = |D|$. This proves (2).

Let H be the graph with vertex set $A \cup B$ in which distinct vertices u, v are adjacent if either $u, v \in A$ and u, v are nonadjacent in G, or exactly one of u, v is in A and u, v are adjacent in G.

(3) *There is a matching in H of cardinality $k - |D|$.*

For suppose not. Since $|A| + |B| \geq 2(k - |D|)$, 2.3 implies that there exists $X \subseteq V(H)$ with $|X| \leq k - |D| - 1$ such that $H \setminus X$ has $|X| + |V(H)| - 2(k - |D| - 1)$ odd components, and every component of $H \setminus X$ is odd, and every component of $H \setminus X$ with more than one vertex is not bipartite. Since $|X| \leq k - |D| - 1$, it follows that $|D \cup X| < k$, and since G is k-connected, we deduce that $G \setminus (D \cup X)$ is connected.

Suppose that some component P of $H \setminus X$ contains a vertex in B. Since $\alpha(G) < 3$, no other component contains two vertices of A that are nonadjacent in G; and so every other component of $H \setminus X$ is bipartite, and therefore has only one vertex. Since $G \setminus (D \cup X)$ is connected, some vertex in A belongs to the union of all components of $H \setminus X$ that contain a vertex in B; and so P has more than one vertex. Consequently P is not bipartite, and so contains two vertices of A that are nonadjacent in G; and therefore no other component has a vertex in B. Thus $B \subseteq V(P) \cup X$. Let $C' = C \cup (A \setminus (X \cup P))$. Thus

$$|C'| \geq |C| + |X| + |V(H)| - 2(k - |D| - 1) - 1 = |V(G)| + |X| - 2k + 1 + |D|$$

since $H \setminus X$ has at least $|X| + |V(H)| - 2(k - |D| - 1)$ odd components. The only vertices mixed on C' belong to $X \cup D$, and so

$$2\,\mathrm{cap}(C') \leq |X \cup D| + |V(G)| - |C'| \leq |X \cup D| + |V(G)| - (|V(G)| + |X| - 2k + G1 + |D|) = 2k - 1,$$

a contradiction.

Thus no component of $H \setminus X$ meets B, and so $B \subseteq X$. Since G has an antimatching of cardinality k, there is an antimatching of cardinality at least $k - |D| - |X|$ in $G \setminus (D \cup X)$. Since C is complete to all other vertices of $G \setminus (D \cup X)$, it follows that there is an antimatching of cardinality $k - |D| - |X|$ in $G \setminus (C \cup D \cup X)$. Hence at least $2(k - |D| - |X|)$ vertices are incident with antiedges in this antimatching; but for every odd component of $H \setminus X$, at least one vertex is not incident with an antiedge of the antimatching. Consequently

$$|A \setminus X| \geq 2(k - |D| - |X|) + |X| + |V(H)| - 2(k - |D| - 1),$$

that is, $|A \setminus X| \geq -|X| + |A| + |B| + 2$, which is impossible. This proves (3).

If M is a matching as in (3), then $2|M| = 2(k-|D|) = |V(H)|-1$, and so there is a unique vertex w of H not incident in H with any edge of M. We call w the *free vertex of M*. Let $c_1, \dots, c_{k-|D|-1}$ be the vertices of C in none of $S_1, \dots, S_{|D|}$.

(4) $D = \emptyset$.

For suppose not. Let the $|D|$ seagulls of (2) be $S_1, \dots, S_{|D|}$ say, and let $S_1 = \{d_1, u, v\}$, where $d_1 \in D$ and $u, v \in C$ and u is adjacent to d_1. Choose M as in (3), with free vertex w say. Let $X_1, \dots, X_{k-|D|}$ be the sets of ends of the edges of M. If w is mixed on $\{d_1, u\}$ then $\{w, d_1, u\}$ is a seagull, and

$$\{w, d_1, u\}, S_2, \dots, S_{|D|}, X_i \cup \{c_i\}\ (1 \le i \le k-|D|-1), X_{k-|D|} \cup \{v\}$$

are k disjoint seagulls, a contradiction; so w is not mixed on $\{d_1, u\}$. Since $w \notin D$, w is also not mixed on $\{u, v\}$; and so w is not mixed on $\{d_1, v\}$. Since $\alpha(G) < 3$ and d_1, v are nonadjacent, we deduce that w is adjacent to both d_1, v; but then

$$\{w, d_1, v\}, S_2, \dots, S_{|D|}, X_i \cup \{c_i\}\ (1 \le i \le k-|D|-1), X_{k-|D|} \cup \{u\}$$

are k disjoint seagulls, again a contradiction. This proves (4).

From (4) we deduce that $C = \{c_1, \dots, c_{k-1}\}$.

(5) *For every choice of the matching M as in (3), if w is the free vertex of M and X is the set of ends of some edge of M then $X \cup \{w\}$ is not a seagull of G.*

For let the edges of M have sets of ends $X_1, \dots, X_k$ say, and suppose that $X_k \cup \{w\}$ is a seagull of G. Then

$$X_i \cup \{c_i\}\ (1 \le i \le k-1), X_k \cup \{w\}$$

are k disjoint seagulls, a contradiction. This proves (5).

(6) *For every k-edge matching of H, its free vertex is not in B.*

For let $B = \{b_0, b_1, \dots, b_s\}$, and suppose that for some k-edge matching M of H, its free vertex is b_0. There are exactly s edges of M with an end in B, say a_ib_i $(1 \le i \le s)$. Let $t = k - s$, and let the remaining edges of M be p_jq_j $(1 \le j \le t)$. Thus p_j, q_j are nonadjacent in G for $1 \le j \le t$, and

$$A = \{a_1, \dots, a_s\} \cup \{p_1, \dots, p_t\} \cup \{q_1, \dots, q_t\}.$$

Let N be the set of all vertices in A that are complete to B in G. For $1 \le i \le s$, since b_0 is adjacent to b_i, (5) implies that b_0 is adjacent to a_i. For $1 \le i \le s$, by applying (5) to the matching M' of H with free vertex b_i obtained from M by replacing a_ib_i by a_ib_0, we deduce that b_i is complete to $\{a_1, \dots, a_s\}$. Consequently $a_1, \dots, a_s \in N$. By (5), for $1 \le j \le t$ b_0 is nonadjacent to one of p_j, q_j, and similarly (replacing M by the matching M' above) it follows that no vertex in B is adjacent in G to both p_j, q_j. In particular, at least one of p_j, q_j does not belong to N.

Suppose that $p_1, q_1 \notin N$. Let B_0, B_1 be the sets of vertices in B adjacent to p_1 and adjacent to q_1 respectively. Since no vertex in B is adjacent to both p_1, q_1, it follows that $B_0 \cap B_1 = \emptyset$; since $\alpha(G) \leq 2$ it follows that $B_0 \cup B_1 = B$; and since $p_1, q_1 \notin N$ it follows that $B_0, B_1 \neq \emptyset$.

We claim that p_1, q_1 are both complete to $\{a_1, \dots, a_s\}$. For suppose that q_1 is nonadjacent to a_1 say. Since $\{a_1, \dots, a_s\}$ is complete to B, there is symmetry between the members of B, and so we may assume that $b_0 \in B_0$ and $b_1 \in B_1$. But then

$$(M \setminus \{a_1b_1, p_1q_1\}) \cup \{q_1a_1, p_1b_0\}$$

is a k-edge matching of H with free vertex b_1; and yet $\{b_1, p_1, b_0\}$ is a seagull contrary to (5). This proves that p_1, q_1 are both complete to $\{a_1, \dots, a_s\}$.

Suppose that $|B_0| \geq 2$, and let $b_0, b_1 \in B_0$ say. Then

$$(M \setminus \{p_1q_1\}) \cup \{p_1b_0\}$$

is a k-edge matching of H with free vertex q_1, and yet $\{q_1, a_1, b_1\}$ is a seagull, contrary to (5). So $|B_0| = |B_1| = 1$, and so $s = 1$. Let $B_0 = \{b_0\}$ and $B_1 = \{b_1\}$ say.

Suppose that $k \geq 3$, and hence $t \geq 2$. Suppose first $p_2 \in N$. Since no vertex in B is adjacent to both p_2, q_2 it follows that q_2 is nonadjacent to both b_0, b_1, and therefore adjacent to both p_1, q_1, since $\alpha(G) < 3$. But then

$$(M \setminus \{p_2q_2\}) \cup \{p_2b_0\}$$

is a k-edge matching of H with free vertex q_2, and yet $\{q_2, p_1, q_1\}$ is a seagull contrary to (5). This proves that $p_2 \notin N$, and similarly $p_j, q_j \notin N$ for $2 \leq j \leq t$, and so $N = \{a_1, \dots, a_s\}$. We may assume that p_t is adjacent to b_0 and q_t to b_1. But then

$$\{p_tb_0, q_tb_1\} \cup \{p_jq_j \ : 1 \leq j \leq t-1\}$$

is a k-edge matching of H with free vertex a_1, and yet $\{a_1, p_1, q_1\}$ is a seagull, contrary to (5). Thus $k = 2$; but then G is a five-wheel, a contradiction.

This proves that at least one of $p_1, q_1 \in N$; so we may assume that $p_j \in N$ and $q_j \notin N$ for $1 \leq j \leq t$. Let $P = \{p_1, \dots, p_t\}$ and $Q = \{q_1, \dots, q_t\}$. Since no vertex in B is adjacent to both p_j, q_j, it follows that Q is anticomplete to B, and so Q is a clique. Now

$$(M \setminus \{p_1q_1\}) \cup \{p_1b_0\}$$

is a k-edge matching of H with free vertex q_1, and so $\{q_1, a_i, b_i\}$ is not a seagull for $1 \leq i \leq s$, and $\{q_1, p_j, q_j\}$ is not a seagull for $2 \leq j \leq t$. Consequently q_1 is anticomplete to N, and therefore Q is anticomplete to $N \cup B$.

But then no vertex of G is mixed on N, and yet $|N| = k$ since $N = \{a_1, \dots, a_s, p_1, \dots, p_t\}$ and $s + t = k$; and so $\text{cap}(N) = k$ since $|V(G)| = 3k$, contrary to 2.5. This proves (6).

(7) *For every k-edge matching of H, its free vertex is not in A.*

For suppose that M is a k-edge matching of H with free vertex in A. Let the edges of H with one end in B be a_ib_i $(1 \leq i \leq s)$, and let those with both ends in A be p_jq_j $(1\ e_j \leq t)$, where $s + t = k$ and $B = \{b_1, \dots, b_s\}$. Let the free vertex be a_0 where

$$A = \{a_0, a_1, \dots, a_s, p_1, \dots, p_t, q_1, \dots, q_t\}.$$

If a_0, a_1 are nonadjacent, then $(M \setminus \{a_1b_1\}) \cup \{a_0a_1\}$ is a k-edge matching of H with free vertex $b_1 \in B$, contrary to (6). So a_0 is complete to $a_1, \dots, a_s$, and therefore to B by (5). For $1 \leq i \leq s$, by replacing a_ib_i by a_0b_i we deduce that a_i is complete to $\{a_0, a_1, \dots, a_s\} \setminus \{a_i\}$ and to B, and so $\{a_0, a_1, \dots, a_s\} \cup B$ is a clique.

Since a_0 is adjacent to at least one of p_j, q_j (since $\alpha(G) < 3$) and not to both (by (5)), we may assume that a_0, p_j are adjacent and a_0, q_j are nonadjacent for $1 \leq j \leq t$. Let $P = \{p_1, \dots, p_t\}$ and $Q = \{q_1, \dots, q_t\}$. For $1 \leq j \leq t$, by the argument above applied to the matching $(M \setminus \{p_jq_j\}) \cup \{a_0q_j\}$ we deduce that p_j is complete to $\{a_0, a_1, \dots, a_s\} \cup B$, and so P is complete to $\{a_0, a_1, \dots, a_s\} \cup B$. By (5) applied to $(M \setminus \{a_1b_1\}) \cup \{a_0b_1\}$ it follows that q_1 is nonadjacent to a_1, and similarly Q is anticomplete to $\{a_0, \dots, a_s\}$.

Suppose that p_1, q_2 are adjacent. By (5) applied to $(M \setminus \{p_1q_1\}) \cup \{a_0q_1\}$, it follows that $\{p_1, p_2, q_2\}$ is not a seagull, and so p_1, p_2 are nonadjacent. If $s > 0$ then

$$(M \setminus \{p_1q_1, p_2q_2, a_1b_1\}) \cup \{a_0q_1, a_1q_2, p_1p_2\}$$

is a k-edge matching of H with free vertex $b_1 \in B$, contrary to (6). Thus $B = \emptyset$; but then G has no antimatching of cardinality $k + 1$ contrary to 2.6. This proves that p_1, q_2 are nonadjacent and similarly P is anticomplete to Q.

Hence $P \cup \{a_0, \dots, a_s\}$ is a clique. It has cardinality $k + 1$, and yet no vertex is mixed on it, and so has capacity less than k, a contradiction. This proves (7).

From (3), (6) and (7) we have a contradiction. This proves 2.8. ∎

2.9 $k \geq 3$.

Proof. Certainly $k > 1$; suppose that $k = 2$, and so $|V(G)| = 6$. Since $\alpha(G) < 3$ and $|V(G)| = 6$, there are three pairwise adjacent vertices say b_1, b_2, b_3. Since there is an antimatching of cardinality three by 2.6, we may assume that $V(G) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$, where a_i, b_i are nonadjacent for $i = 1, 2, 3$. Each of a_1, a_2, a_3 has a neighbour in $\{b_1, b_2, b_3\}$ since G is 3-connected by 2.4; so from the symmetry we may assume that a_1b_2 and a_2b_3 are edges. Since $\{a_1, b_1, b_2\}$ is a seagull, it follows that $\{a_2, a_3, b_3\}$ is not a seagull, since there do not exist two disjoint seagulls; and so a_2, a_3 are nonadjacent. Since G is three-connected, we deduce that a_2 is adjacent to a_1, b_1 and a_3 is adjacent to a_1, b_1, b_2. But then $\{a_1, a_2, b_1\}, \{a_3, b_2, b_3\}$ are two disjoint seagulls, a contradiction. This proves 2.9. ∎

2.10 *If H is a graph with $\alpha(H) < 3$, and S is a seagull of H such that $H \setminus S$ is a five-wheel, then H has three disjoint seagulls.*

Proof. Let $S = \{p, q, r\}$ where q is adjacent to p, r, and let $H \setminus S$ be a five-wheel, with vertex set $\{c_1, \dots, c_5, w\}$, where w is adjacent to $c_1, \dots, c_5$ and c_1-c_2-$\cdots$-c_5-c_1 are the vertices of a cycle in order. Suppose for a contradiction that H does not have three disjoint seagulls.

Suppose that $c_1, \dots, c_5$ are not mixed on S. Since c_i is adjacent to one of p, r (because $\alpha(H) < 3$) it follows that $\{c_1, \dots, c_5\}$ is complete to S. But then $\{c_1, c_3, w\}, \{c_2, c_4, q\}, \{c_5, p, r\}$ are three disjoint seagulls, a contradiction.

Thus we may assume that some c_i is mixed on S; and hence some c_i is mixed on $\{p,q\}$ or mixed on $\{q,r\}$, and from the symmetry we may assume that c_1 is mixed on $\{p,q\}$. Thus $\{c_1,p,q\}$ is a seagull. Now r is adjacent to one of c_2,c_5, since $\alpha(H)<3$, and from the symmetry we may assume that r,c_2 are adjacent. Since $\{c_1,p,q\},\{c_3,c_4,c_5\}$ are seagulls, it follows that $\{r,c_2,w\}$ is not a seagull, and so r,w are adjacent. Because of the seagulls $\{c_1,p,q\},\{c_2,c_3,c_4\}$ it follows similarly that r,c_5 are adjacent; and the seagulls $\{c_1,p,q\},\{c_3,w,c_5\}$ imply that r,c_4 are nonadjacent, and similarly r,c_3 are nonadjacent.

We have shown then that if c_i is mixed on $\{p,q\}$ then r,c_i have the same neighbour set in $\{c_1,\ldots,c_5,w\}\setminus\{c_i\}$; and in particular i is unique. Thus we may assume that $c_2,\ldots,c_5$ are not mixed on $\{p,q\}$. Since c_3,c_4 are nonadjacent to r and therefore adjacent to p (because $\alpha(H)<3$), it follows that c_3,c_4 are adjacent to q. Hence c_3,c_4 are both mixed on $\{q,r\}$, contrary to what we already showed (with p,r exchanged). This proves 2.10. ∎

Let us say a vertex of G is *big* if its degree is at least $2k-1$.

2.11 *Every two big vertices of G are adjacent.*

Proof. Suppose that $u,v\in V(G)$ are nonadjacent big vertices.

(1) *There are $k-1$ seagulls in $G\setminus\{u,v\}$, pairwise disjoint.*

To show this, it suffices by 2.1.6 to check that $G\setminus\{u,v\}$ is $(k-1)$-connected and has an anti-matching of cardinality $k-1$, and every clique in $G\setminus\{u,v\}$ has capacity at least $k-1$ in $G\setminus\{u,v\}$, and if $k=3$ then $G\setminus\{u,v\}$ is not a five-wheel. We check these in turn. First, by 2.4, G is $(k+1)$-connected and so $G\setminus\{u,v\}$ is $(k-1)$-connected. Second, by 2.6 G has an antimatching of cardinality $k+1$, and therefore $G\setminus\{u,v\}$ has an antimatching of cardinality $k-1$. Third, let C be a clique of $G\setminus\{u,v\}$. In G, C has capacity at least $k+1$, by 2.8, and so in $G\setminus\{u,v\}$, C has capacity at least $k-1$. Finally, if $k=3$ then $|V(G)|=9$, and so $G\setminus\{u,v\}$ has seven vertices and is therefore not a five-wheel. From 2.1.6 this proves (1).

Let $S_1,\ldots,S_{k-1}$ be the $k-1$ seagulls of (1). For $1\le i\le k-1$, we say that S_i is *tame* if for all $x\in\{u,v\}$, if x is complete to only one of $S_1,\ldots,S_{k-1}$ then x is not complete to S_i.

(2) *It is possible to choose $S_1,\ldots,S_{k-1}$ such that one of them is tame.*

Let I be the set of $i\in\{1,\ldots,k-1\}$ such that u is complete to S_i, and define J similarly with respect to v. If $k\ge 4$ then we may choose $i\in\{1,\ldots,k-1\}$ such that if $|I|=1$ then $i\notin I$, and if $|J|=1$ then $i\notin J$, and then S_i is tame. Thus we may assume that $k=3$, and $I=\{1\}$, $J=\{2\}$ say. Let w be the unique vertex of G not in $S_1\cup S_2$ and different from u,v. Since G does not have three disjoint seagulls, it follows that $\{u,v,w\}$ is not a seagull, so w is nonadjacent to at least one of u,v; and since $\alpha(G)<3$, w is adjacent to one of u,v. Thus we may assume that w is adjacent to v and not to u. Now there is a seagull $S_1'\subseteq S_1\cup\{w\}$ containing w. Thus S_1',S_2 are disjoint; u has a nonneighbour in S_1'; and v is complete to S_2, and so again the claim holds. This proves (2).

Choose $S_1,\ldots,S_{k-1}$ such that one of them is tame. Choose one of these seagulls, tame, such that in addition one of u,v has two nonneighbours in it if possible. Let this seagull be S_1 say. Since

G does not have k disjoint seagulls, it follows that $G \setminus S_1$ does not have $k-1$ disjoint seagulls. We use 2.1.6 to obtain a contradiction, as follows.

(3) *$G \setminus S_1$ is $(k-1)$-connected and has an antimatching of cardinality $k-1$.*

For suppose that $G \setminus S_1$ is not $(k-1)$-connected. Thus there is a partition (P, Q, R) of $V(G)$ such that $P, Q \neq \emptyset$, $|R| \leq k+1$, $S_1 \subseteq R$, and P is anticomplete to Q. Every seagull of G contains a vertex of R, and in particular $S_2, \ldots, S_{k-1}$ each contain a vertex of R, and S_1 contains three; and so $S_2, \ldots, S_{k-1}$ each have exactly one vertex in R, and $|R| = k+1$, and $R \subseteq S_1 \cup \cdots \cup S_{k-1}$, and therefore $u, v \notin R$. Since P, Q are cliques and u, v are nonadjacent, we may assume that $u \in P$ and $v \in Q$. Since $|R| = k+1$, 2.7 implies that $|P| + |Q| < 2k$, and so we may assume that $|P| < k$. Since u has degree at least $2k-1$, and $|P| + |R| \leq (k-1) + (k+1)$, it follows that $|P| = k-1$ and u is complete to R. Since $|V(G)| = 3k$, we deduce that $|Q| = k$. Now v has degree at least $2k-1$, and $|Q \cup R| = 2k+1$, and so v has only one nonneighbour in S_1. Since S_1 is tame, and u is complete to S_1, it follows that u is complete to one of $S_2, \ldots, S_{k-1}$, say S_2; and so $S_2 \subseteq P \cup R$. Since S_2 contains only one vertex of R, it follows that v has two nonneighbours in S_2. But S_2 is tame, since u is complete to S_1 and v is not complete to S_2; and yet neither of u, v has two nonneighbours in S_1, contrary to our choice of S_1. This proves that $G \setminus S_1$ is $(k-1)$-connected. Each of $S_2, \ldots, S_{k-1}$ includes a nonadjacent pair of vertices, and these pairs together with the pair uv form an antimatching of cardinality $k-1$ in $G \setminus S_1$. This proves (3).

(4) *Every clique of $G \setminus S_1$ has capacity at least $k-1$ in $G \setminus S_1$.*

For let C be a clique of $G \setminus S_1$, and let (A, B, C, D) be the associated partition of $V(G \setminus S_1)$. Suppose that C has capacity at most $k - 3/2$ in $G \setminus S_1$; thus, $2|D| + |A \cup B| \leq 2k-3$. Let w be the vertex of G not in $S_1, \ldots, S_{k-1}$ different from u, v, and let $T = \{u, v, w\}$. Let S be the union of $S_2, \ldots, S_{k-1}$. Since $S \cup T = V(G \setminus S_1)$, it follows that

$$2|D \cap S| + |(A \cup B) \cap S| + 2|D \cap T| + |(A \cup B) \cap T| \leq 2k-3.$$

Now each of $S_2, \ldots, S_{k-1}$ contains either a member of $D \setminus S_1$, or two members of $(A \cup B) \setminus S_1$; and so $2|D \cap S| + |(A \cup B) \cap S| \geq 2k-4$. Consequently $2|D \cap T| + |(A \cup B) \cap T| \leq 1$. Hence $D \cap T = \emptyset$. Moreover, since u, v are nonadjacent, they do not both belong to C, so we may assume that $u \in A \cup B$; and so $(A \cup B) \cap T = \{u\}$. Hence $v, w \in C$. Since u is not mixed on C and u, v are nonadjacent, we deduce that u is anticomplete to C and so $u \in B$. Every neighbour of u in G therefore belongs to $A \cup B \cup D \cup S_1$, and since $2|D| + |A \cup B| \leq 2k-3$ and u has degree at least $2k-1$, it follows that we have equality throughout; and in particular, $D = \emptyset$ and u is complete to $A \cup (B \setminus \{u\}) \cup S_1$, and each of $S_2, \ldots, S_{k-1}$ contains exactly two members of $A \cup B$. It follows that each of $S_2, \ldots, S_{k-1}$ contains a member of C, and so is not complete to u, contradicting that S_1 is tame. This proves (4).

Now from 2.10 $G \setminus S_1$ is not a five-wheel, so by 2.1.6 $G \setminus S_1$ has $k-1$ disjoint seagulls, and therefore G has k disjoint seagulls, a contradiction. This proves 2.11. ∎

If $\{p, q, r\}$ is a seagull, with q adjacent to p, r, we call p, r the *wings* of the seagull and q its *body*.

2.12 *Let S be a seagull in G, such that either it contains a big vertex, or there is no big vertex in $V(G)$. Then $G \setminus S$ has an antimatching of cardinality at least $k-1$.*

Proof. Suppose not; then since $|V(G\setminus S)| = 3k-3 \geq 2(k-1)$, 2.3 applied in $\overline{G}\setminus S$ implies that there exists $X \subseteq V(G)$ with $S \subseteq X$, such that $\overline{G}\setminus X$ has $(|X|-3)+(|V(G)|-3)-2(k-2) = |X|+k-2$ components, and for each component C of $\overline{G}\setminus X$ and each $v \in V(C)$, $C\setminus\{v\}$ has a perfect matching.

We claim that every vertex in $V(G)\setminus X$ is big. For let the components of $\overline{G}\setminus X$ be $C_1,\ldots,C_{|X|+k-2}$, let $1 \leq i \leq |X|+k-2$, with $|V(C_i)| = 2m+1$ say, and let $v \in C_i$. Let $Y = V(G)\setminus(X \cup V(C_i))$. Since $\overline{G}\setminus X$ has $|X|+k-2$ components, it follows that $|Y| \geq |X|+k-3$; but also $|Y| = |V(G)|-|X|-(2m+1) = 3k-|X|-2m-1$, and so, summing, we deduce that $2|Y| \geq (|X|+k-3)+(3k-|X|-2m-1) = 4k-2m-4$, and so $|Y| \geq 2k-m-2$. Since $C\setminus\{v\}$ has a perfect matching (which is an antimatching of G) and $\alpha(G) < 3$, it follows that v is adjacent in G to at least m vertices in C_i. Also, v is complete to Y; and v has a neighbour in $S \subseteq X$, since $\alpha(G) < 3$. Hence the degree of v is at least $m+|Y|+1 \geq m+(2k-m-2)+1 = 2k-1$. This proves our claim that that every vertex in $V(G)\setminus X$ is big.

By 2.11, all vertices in $V(G)\setminus X$ are pairwise adjacent, and so each C_i has only one vertex. Consequently $|X|+k-2 = |V(G)\setminus X|$, and so $|X| = k+1$ and $|V(G)\setminus X| = 2k-1$. By 2.8, $\mathrm{cap}(V(G)\setminus X) \geq k+1$, and therefore every vertex in X is mixed on $V(G)\setminus X$; and so no vertex in X is big, by 2.11. In particular, no vertex of S is big, and so there is no big vertex by hypothesis, and hence $X = V(G)$, which is impossible. This proves 2.12. ∎

A *cutset* in G is a subset $X \subseteq V(G)$ such that $G\setminus X$ is disconnected (and consequently has two components, both complete). We need the following lemma.

2.13 *Let M be a cutset of G, with $|M| = k+1$. Let A, B be the two components of $G\setminus M$. Then for every subset $P \subseteq M$ with $|P| \leq |B|$, there is a matching of P into B. Moreover, if $b \in B$ has a neighbour in P, then there is a matching with cardinality $|P|$ between P and a subset of B containing b.*

Proof. Suppose not; then by König's theorem, there is a subset $X \subseteq P \cup B$ with $|X| < |P|$ such that $P\setminus X$ is anticomplete to $B\setminus X$. But then $B\setminus X \neq \emptyset$, since $|X| < |P| \leq |B|$, and so $X \cup (M\setminus P)$ is a cutset. We deduce that $|X \cup (M\setminus P)| \geq k+1$ by 2.4, and so $|X \cup (M\setminus P)| \geq |M|$, that is, $|M\setminus P|+|X| \geq |M\setminus P|+|P|$, a contradiction. This proves the first assertion of 2.13, and the second follows easily. ∎

2.14 *Let C be a clique in G with $\mathrm{cap}(C) = k+1$, with associated partition (A,B,C,D), and suppose that $D \neq \emptyset$, and if some vertex of $V(G)$ is big, then there is a big vertex in D. Then there do not exist partitions (A_1,A_2) of A and (B_1,B_2) of B such that A_1 is anticomplete to B_2, and A_2 is anticomplete to B_1, and $B_1, B_2 \neq \emptyset$, and $|A_1|-|B_1| = |A_2|-|B_2|$.*

Proof. For suppose that such partitions exist. Thus $B_1 \cup B_2 = B$ is a clique, and is complete to D; A_1 is a clique, since it is anticomplete to $B_2 \neq \emptyset$; and similarly A_2 is a clique.

Let $|D| = d$. Since $\mathrm{cap}(C) = k+1$ and $|A_1|+|B_2| = |A_2|+|B_1|$, it follows that $|A_1|+|B_2| = k+1-d$, and

$$|C| = 3k-(|A|+|B|+|D|) = 3k-2(k+1-d)-d = d+k-2.$$

(1) $|A_1| = |B_1|$ *and* $|A_2| = |B_2|$.

For let $|A_1| - |B_1| = x$ say; then $|A_2| - |B_2| = x$, and $|A| - |B| = 2x$. Since $G \setminus (D \cup A)$ is disconnected (because $B \neq \emptyset$) we deduce that $|D| + |A| \geq k+1$ from 2.4, and so $|D| + |A_1| + |A_2| \geq |D| + |A_2| + |B_1|$, that is, $x \geq 0$. Since $|A_1| + |B_2| = |A_2| + |B_1| = k+1-d$, it follows that $|A| + |B| = 2(k+1-d)$. Consequently $(|B| + 2x) + |B| = 2(k+1-d)$, that is, $|B| = k+1-d-x$, and $|A| = k+1-d+x$. By hypothesis, there exists $v \in D$ such that if any vertex is big then v is big. Since $v \in D$, it has a nonneighour $u \in C$, and consequently u is not big, by 2.11. But u has $|C| + |A| - 1$ neighbours in $C \cup A$, and

$$|C| + |A| - 1 = (d+k-2) + (k+1-d+x) - 1 = 2k-2+x,$$

and so $x = 0$. This proves (1).

(2) *For each* $v \in D$, *there are* $d-1$ *disjoint seagulls each with one vertex in* $D \setminus \{v\}$ *and two in* C.

For if not, by 2.2 there is a subset $Y \subseteq C$ such that at most $|Y| + |D \setminus \{v\}| - |C| - 1 = |Y| - k$ vertices in $D \setminus \{v\}$ are mixed on Y, and hence only $|Y| - k + 1$ vertices in $V(G) \setminus Y$ are mixed on Y. But then $\text{cap}(Y) \leq (3k - |Y|)/2 + (|Y| - k + 1)/2 = k + 1/2$, contrary to 2.8. This proves (2).

(3) *For* $i = 1, 2$ *there is a matching in* G *of* B_i *into* A_i.

For let $i = 1$ say. Since $D \cup A_1 \cup B_2$ is a cutset in G of cardinality $k+1$, the claim follows from 2.13 and (1).

(4) A_1, A_2 *are complete to* D.

For suppose that $a_1 \in A_1$ is nonadjacent to $v_1 \in D$. By (3) there is a matching of B into A, say $X_0, \ldots, X_{k-d}$, where $v_1 \in X_0$. Let $S_1 = X_0 \cup \{v_1\}$; thus S_1 is a seagull. Let $D = \{v_1, \ldots, v_d\}$. Let $S_2, \ldots, S_d$ be seagulls as in (2) with $v_i \in S_i$ $(2 \leq i \leq d)$. Since $|C| = d+k-2$, there are $k-d$ vertices in C not in $S_2 \cup \cdots \cup S_d$, say $c_1, \ldots, c_{k-d}$. But then

$$S_1, \ldots, S_d, X_i \cup \{c_i\} \ (1 \leq i \leq k-d)$$

are k disjoint seagulls, a contradiction. This proves (4).

(5) *Let* H *be a graph with six vertices* $\{h_0, \ldots, h_5\}$, *in which* h_i *is adjacent to* h_{i+1} *for* $1 \leq i \leq 4$, *and* h_5 *is adjacent to* h_1, *and* h_0 *is adjacent to* h_2, h_3, h_4, h_5, *and the pairs* h_2h_5, h_1h_0 *may be edges, but all other pairs are nonadjacent. Then either* $V(H)$ *can be partitioned into two seagulls or* H *is a five-wheel.*

For if h_0, h_1 are nonadjacent then $\{h_0, h_1, h_2\}, \{h_3, h_4, h_5\}$ are two disjoint seagulls, so we may assume that h_0, h_1 are adjacent. If h_2, h_5 are adjacent then $\{h_0, h_1, h_3\}, \{h_2, h_4, h_5\}$ are two disjoint seagulls, and if h_2, h_5 are nonadjacent then H is a five-wheel. This proves (5).

Let $h_0 \in D$, and let $S_1, \ldots, S_{d-1}$ be seagulls as in (2), not containing h_0. Let the vertices of C not in $S_1, \ldots, S_{d-1}$ be $c_1, \ldots, c_{k-d}$. Let $X_1, \ldots, X_{k+1-d}$ be a matching between B and A, where $X_{k-d} \subseteq A_1 \cup B_1$ and $X_{k+1-d} \subseteq A_2 \cup B_2$. Let W be the union of the $k-2$ seagulls

$$S_1, \ldots, S_{d-1}, X_i \cup \{c_i\} \; (1 \leq i \leq k-d-1),$$

and let S be any one of these $k-2$ seagulls (since $k \geq 3$, this is possible). Then

$$V(G) \setminus W = \{c_{k-d}, h_0\} \cup X_{k-d} \cup X_{k+1-d},$$

and the subgraph H induced on these six vertices satisfies the hypotheses of (5). Since H does not have two disjoint seagulls (since G does not have k disjoint seagulls), (5) implies that H is a five-wheel. By 2.10, $V(H) \cup S$ can be partitioned into three seagulls, and so $V(G)$ has k disjoint seagulls, a contradiction. This proves 2.14. ∎

If S is a seagull, a seagull S' is *close* to S if $|S \cap S'| = 2$. A seagull S is a *king seagull* if it satisfies:

- either some vertex of S is big, or no vertex is big, and
- either two vertices of S are big, or no seagull close to S contains two big vertices.

2.15 *For every king seagull S, $G \setminus S$ is not $(k-1)$-connected.*

Proof. Let S be a king seagull, and suppose that $G \setminus S$ is $(k-1)$-connected. Let $G' = G \setminus S$. By 2.12 it follows that G' has an antimatching of cardinality at least $k-1$. By 2.10 G' is not a five-wheel. Since G' does not have $k-1$ disjoint seagulls, 2.1.6 implies that there is a clique C of G' with capacity in G' at most $k - 3/2$. Let (A, B, C, D) be the associated partition in G; thus $S \subseteq A \cup B \cup D$, and

$$|D \setminus S| + |(A \cup B) \setminus S|/2 \leq k - 3/2.$$

Consequently $\operatorname{cap}_G(C) \leq k + 3/2$; and if $S \not\subseteq D$, then $\operatorname{cap}_G(C) \leq k+1$, with equality only if $|S \cap D| = 2$. On the other hand, by 2.8, $\operatorname{cap}_G(C) \geq k+1$; so either $S \subseteq D$ and $\operatorname{cap}_G(C) \leq k+3/2$, or $|S \cap D| = 2$ and $\operatorname{cap}_G(C) = k+1$. In either case it follows that $|D| + (|A|+|B|)/2 \leq k+3/2-\delta/2-\epsilon/2$, that is,

$$|C| + (|A| + |B|)/2 \geq 2k - 3/2 + \delta/2 + \epsilon/2,$$

where $\delta = 1$ if the body of S is not in D, and 0 otherwise, and ϵ is the number of wings of S that are not in D.

Suppose that every vertex in C is big. Then at least one vertex of S is big, and since this vertex is complete to C by 2.11, it belongs to A. Consequently the other two vertices of S belong to D; and there exist $u \in A \cap S$ and $v \in D \cap S$, adjacent. Now v has a nonneighbour w in C, and so $\{u, v, w\}$ is a seagull, close to S, and containing two big vertices. We deduce that two vertices of S are big, since it is a king seagull; but one of them is in D and therefore not complete to C, contrary to 2.11.

Thus there exists $c \in C$ that is not big. Since c has $|C| + |A| - 1$ neighbours in $C \cup A$, and has at least one neighbour in D if both wings of S are in D, it follows that the degree of c is at least $|C| + |A| - \epsilon$. Consequently $|C| + |A| - \epsilon \leq 2k - 2$ since c is not big. We deduce that

$$2k - 2 + \epsilon \geq |C| + |A| = (|A| - |B|)/2 + |C| + (|A| + |B|)/2 \geq (|A| - |B|)/2 + 2k - 3/2 + \delta/2 + \epsilon/2,$$

that is, $|B| - |A| \geq \delta + 1 - \epsilon$.

Now any vertex of $S \cap B$ is the body of S, since the other two vertices of S are in D and B is complete to D (because every vertex in D has a nonneighbour in C), and so $\delta \geq |S \cap B|$. Consequently every wing of S not in D belongs to A, and so $|A| \geq \epsilon$. It follows that $|B| \geq \delta + 1 > |S \cap B|$, and so some vertex of B is not in S. Hence $(D \cup A) \setminus S$ is a cutset of $G \setminus S$, and since the latter is $(k-1)$-connected, we deduce that $|D| + |A| - |S \cap (D \cup A)| \geq k - 1$, and so $|D| + |A| \geq k + 2 - |S \cap B| \geq k + 2 - \delta$. Since $|B| - |A| \geq \delta + 1 - \epsilon$, it follows that

$$k + 3/2 - \delta/2 - \epsilon/2 \geq |D| + (|A| + |B|)/2 = |D| + |A| + (|B| - |A|)/2 \geq (k + 2 - \delta) + (\delta + 1 - \epsilon)/2,$$

a contradiction. This proves 2.15. ∎

Suppose that $M \subseteq V(G)$ with $|M| = k + 1$ such that $G \setminus M$ is disconnected. Then $G \setminus M$ has exactly two components L, R say, both cliques (since $\alpha(G) < 3$). Moreover, $|L| + |R| = 2k - 1$ since $|V(G)| = 3k$, and so exactly one of $|L|, |R| < k$. If $|L| < k$ we call the triple (L, M, R) a *$(k+1)$-cut*, and we call L and R the *left* and *right sides* of the cut. A $(k+1)$-cut (L, M, R) is *central* if either there is no big vertex, or some vertex in M is big.

2.16 *Let (L, M, R) be a central $(k+1)$-cut. Then there is a big vertex in $V(G)$, but none in R.*

Proof. For suppose not. Then there is a central $(k+1)$-cut such that either there is no big vertex in $V(G)$ or there is a big vertex in the right side of this cut; choose such a central $(k+1)$-cut (L, M, R) with left side minimal. Choose $u \in R$ and $v \in M$, adjacent, as follows. If there is a big vertex in $V(G)$, then by hypothesis there is a big vertex (say u) in R, and from the definition of "central", there is a big vertex (say v) in M, and u, v are adjacent by 2.11. If there is no big vertex in $V(G)$, choose $v \in M$ arbitrarily; and since G is $(k+1)$-connected, v has a neighbour $u \in R$.

Let $|L| = k - x$, so $|R| = k + x - 1$. Since G is $(k+1)$-connected by 2.4, it follows that v has a neighbour $w \in L$, and so $\{u, v, w\}$ is a king seagull S say. By 2.15, $G \setminus S$ is not $(k-1)$-connected, and so there is a $(k+1)$-cut (L', M', R') with $S \subseteq M'$. Let A_{ij} $(1 \leq i, j \leq 3)$ be the partition of $V(G)$ into nine subsets where

$$\begin{aligned} A_{11} \cup A_{12} \cup A_{13} &= L' \\ A_{21} \cup A_{22} \cup A_{23} &= M' \\ A_{31} \cup A_{32} \cup A_{33} &= R' \\ A_{11} \cup A_{21} \cup A_{31} &= L \\ A_{12} \cup A_{22} \cup A_{32} &= M \\ A_{13} \cup A_{23} \cup A_{33} &= R. \end{aligned}$$

Thus $u \in A_{23}$, $v \in A_{22}$, and $w \in A_{21}$. For $1 \leq i, j \leq 3$ let $a_{ij} = |A_{ij}|$.

(1) $A_{33} \neq \emptyset$.

For suppose that $A_{33} = \emptyset$. Since $v, w \in M'$ and $|M'| = k + 1$, it follows that $a_{23} \leq k - 1$, and since $|R| \geq k$ we deduce that $R \not\subseteq M'$, and so $A_{13} \neq \emptyset$. Consequently $A_{11} = \emptyset$, since any vertex in A_{11} is both complete to A_{13} (since L' is a clique) and anticomplete to A_{13} (since M is a cutset). Since

G is $(k+1)$-connected, it follows that w has a neighbour in L', and so $A_{12} \neq \emptyset$. But $|M| = k+1$, and since $A_{12}, A_{22} \neq \emptyset$ we deduce that $a_{32} \leq k-1$. Since $|R'| \geq k$ it follows that $A_{31} \neq \emptyset$. Thus both $A_{12} \cup A_{22} \cup A_{23}$ and $A_{21} \cup A_{22} \cup A_{32}$ are cutsets, and so both have cardinality at least $k+1$; and yet the sum of their cardinalities is

$$a_{12} + a_{22} + a_{23} + a_{21} + a_{22} + a_{32} = |M| + |M'| = 2(k+1)$$

and so we have equality throughout. We deduce that $(A_{31}, A_{21} \cup A_{22} \cup A_{32}, R \cup L')$ is a central $(k+1)$-cut, contrary to the minimality of L. This proves (1).

Since $L \neq \emptyset$ it follows that R is a clique, and since L' is anticomplete to R' it follows that $A_{13} = \emptyset$, and similarly $A_{31} = \emptyset$. Suppose that $A_{11} \neq \emptyset$. Then both $A_{32} \cup A_{22} \cup A_{23}$ and $A_{12} \cup A_{22} \cup A_{21}$ are cutsets, and the sum of their cardinalities equals $|M| + |M'| = 2(k+1)$, and so $(A_{11}, A_{12} \cup A_{22} \cup A_{21}, R \cup R')$ is a central $(k+1)$-cut, contrary to the minimality of L. Thus $A_{11} = \emptyset$. Since $L' \neq \emptyset$ it follows that $A_{12} \neq \emptyset$.

Now A_{33} is a clique; let (A, B, A_{33}, D) be the associated partition. Since R is a clique it follows that $A_{23} \subseteq A$, and similarly $A_{32} \subseteq A$; and since L is anticomplete to R, $A_{21} \subseteq B$, and similarly $A_{12} \subseteq B$. Thus $D \subseteq A_{22}$, and so

$$2\,\mathrm{cap}(A_{33}) \leq a_{12} + a_{21} + a_{32} + a_{23} + 2a_{22} = |M| + |M'| = 2k+2,$$

and since $\mathrm{cap}(A_{33}) \geq k+1$ by 2.8, it follows that $A_{22} = D$. But then A_{33} violates 2.14, a contradiction. This proves 2.16. ∎

Finally we can prove our main result.

Proof of 1.6. Every vertex is in a seagull, since G is connected and not complete; and consequently there is a king seagull S_0. From 2.15, $G \setminus S_0$ is not $(k-1)$-connected, and so there is a central $(k+1)$-cut (L, M, R) with $S_0 \subseteq M$. Let $|L| = k - x$, so $|R| = k + x - 1$.

(1) *Every vertex in R has at least $x+1$ nonneighbours in M.*

For let $v \in R$. Thus v has $k + x - 2$ neighbours in R (because R is a clique), and since v is not big by 2.16, it follows that v has at most $2k - 2 - (k + x - 2) = k - x$ neighbours in M. Since $|M| = k+1$, the claim follows. This proves (1).

(2) *For every $m_0 \in M$, and every matching of $M \setminus \{m_0\}$ into R, if ab belongs to the matching then m_0 is adjacent to both or to neither of a, b.*

For let $M = \{m_0, \ldots, m_k\}$, and let $R = \{r_1, \ldots, r_{k+x-1}\}$, where m_i, r_i are adjacent for $1 \leq i \leq k$. For $1 \leq i \leq x-1$, r_{k+i} has at least $x+1$ nonneighbours in M by (1), and so we may choose an antimatching of cardinality $x-1$ between $\{r_{k+1}, \ldots, r_{k+x-1}\}$ and $\{m_1, \ldots, m_k\}$. Let $s = k - x + 1$. Thus we may assume that r_{k+i} is nonadjacent to m_{s+i} for $1 \leq i \leq x-1$. Let $S_i = \{m_{s+i}, r_{s+i}, r_{k+i}\}$, for $1 \leq i \leq x-1$. Thus $S_1, \ldots, S_{x-1}$ are disjoint seagulls, and $m_0, \ldots, m_s, r_1, \ldots, r_s$ are the vertices of $M \cup R$ that are not in the union of these seagulls. Suppose that m_0 is adjacent to exactly one of m_j, r_j, where $1 \leq j \leq k$. There are two cases, depending whether $j \leq s$ or $j > s$. Suppose first that

$j \leq s$, say $j = 1$. Since G is $(k+1)$-connected and $|L| = s-1$, there is a matching of cardinality $|L|$ between $\{m_2, \ldots, m_s\}$ and L, say $\{X_2, \ldots, X_s\}$ where $m_i \in X_i$ for $2 \leq i \leq s$. Then

$$\{m_0, m_1, r_1\}, X_i \cup \{r_i\}\ (2 \leq i \leq s), S_1, \ldots, S_{x-1}$$

are k disjoint seagulls, a contradiction. Thus $j > s$ and so $x \geq 2$; let $j = s+1$ say. By (1), r_{k+1} is nonadjacent to one of $m_1, \ldots, m_s$, say m_s. By 2.13 there is a matching $\{X_1, \ldots, X_{s-1}\}$ between L and $\{m_1, \ldots, m_{s-1}\}$ with $m_i \in X_i$ for $1 \leq i \leq s-1$. Then

$$\{m_0, m_{s+1}, r_{s+1}\}, \{m_s, r_s, r_{k+1}\}, X_i \cup \{r_i\}\ (1 \leq i \leq s-1), S_2, \ldots, S_{x-1}$$

are k disjoint seagulls, a contradiction. This proves (2).

(3) *Let $m_0, m_1, m_2, m_3 \in M$ be distinct. If m_0m_1, m_1m_2, m_2m_3 are edges, then either m_0 is adjacent to m_2, or m_1 is adjacent to m_3.*

For suppose not. Let $M = \{m_0, \ldots, m_k\}$. By 2.13 there is a matching of $M \setminus \{m_0\}$ into R; let $R = \{r_1, \ldots, r_{k+x-1}\}$, where m_i, r_i are adjacent for $1 \leq i \leq k$. By (2) m_0 is adjacent to r_1, and nonadjacent to r_2, and m_0 is adjacent to r_3 if and only if it is adjacent to m_3. Now

$$\{m_0r_1\} \cup \{m_ir_i\ : 2 \leq i \leq k\}$$

is also a matching, and so m_1 is adjacent to r_2 and not to r_3. Similarly,

$$\{m_0r_1, m_1r_2\} \cup \{m_ir_i\ : 3 \leq i \leq k\}$$

is a matching, and so m_2 is adjacent to r_3 and not to r_1. Also,

$$\{m_0r_1, m_1r_2, m_2r_3\} \cup \{m_ir_i\ : 4 \leq i \leq k\}$$

is a matching, and so m_3 is nonadjacent to r_2. Since $\{m_0, m_3, r_2\}$ is not a stable set, we deduce that m_0, m_3 are adjacent, and so m_0, r_3 are adjacent. But then

$$\{m_0r_3, m_1r_1, m_2r_2\} \cup \{m_ir_i\ : 4 \leq i \leq k\}$$

is a matching, and yet m_3 is adjacent to m_2 and not to r_2, contrary to (2). This proves (3).

(4) *Let $m_0, m_1, m_2, m_3 \in M$ be distinct. If m_0, m_3 are nonadjacent, then some other pair of m_0, m_1, m_2, m_3 are nonadjacent.*

For suppose that the other five pairs are all edges. Let $R = \{r_1, \ldots, r_{k+x-1}\}$, where m_i, r_i are adjacent for $1 \leq i \leq k$ (this is possible by 2.13). By (2) m_0 is adjacent to r_1, r_2, and nonadjacent to r_3. From the matching

$$\{m_0r_1\} \cup \{m_ir_i\ : 2 \leq i \leq k\}$$

it follows that m_1 is adjacent to r_2, r_3, and similarly m_2 is adjacent to r_1, r_3. From the matching

$$\{m_0r_1, m_1r_2, m_2r_3\} \cup \{m_ir_i\ : 4 \leq i \leq k\}$$

it follows that m_3 is adjacent to r_2 and not to r_1. But m_0 is adjacent to m_2 and not to r_3, and so the matching

$$\{m_1r_1, m_2r_3, m_3r_2\} \cup \{m_ir_i\ : 4 \leq i \leq k\}$$

violates (2). This proves (4).

Let H be the subgraph of G induced on M.

(5) *There exists $m_0 \in M$ such that $H \setminus \{m_0\}$ is disconnected and m_0 is complete to $M \setminus \{m_0\}$.*

We recall that $S_0 \subseteq M$, and so H is connected, and not complete. Hence there exists $X \subseteq V(H)$ such that $H \setminus X$ is disconnected. Choose X minimal, and let A, B be the two components of $H \setminus X$. They are both cliques since $\alpha(H) < 3$. Since H is connected it follows that $X \neq \emptyset$. For each $v \in X$, x has a neighbour in A and a neighbour in B; and if v is not complete to A then there is a seagull with one wing v and the other two vertices in A, and this together with a neighbour of v in B violates (3). Thus X is complete to both A, B. If $|X| > 1$ then (3) or (4) is violated (depending whether two vertices in X are nonadjacent or adjacent); so $|X| = 1$ and the claim holds. This proves (5).

Let m_0 be as in (5). We claim that m_0 is complete to R; for suppose that $v \in R$ is nonadjacent to m_0. Since $\alpha(G) < 3$ and M is not a clique, it follows that v has a neighbour in M, and hence in $M \setminus \{m_0\}$; and so by 2.13 there is a matching of cardinality k between $M \setminus \{m_0\}$ and some subset of R containing v, contrary to (2). Thus m_0 is complete to R.

Let M_1, M_2 be the vertex sets of the two components of $H \setminus \{m_0\}$. For $i = 1, 2$, let R_i be the set of vertices in R that are complete to M_i, and let N_i be the set of vertices in R with a neighbour in M_i. Thus $R_i \subseteq N_i$, and since $\alpha(G) < 3$, it follows that $R_1 \cup R_2 = R$. Suppose that $N_1 \cap N_2 \neq \emptyset$. Hence by 2.13 there is a matching of cardinality k between $M \setminus \{m_0\}$ and a subset of R that contains a vertex in $N_1 \cap N_2$.

Let $M = \{m_0, m_1, \ldots, m_k\}$, and let $R = \{r_1, \ldots, r_{k+x-1}\}$ where m_i, r_i are adjacent for $1 \leq i \leq k$, and say $r_1 \in N_1 \cap N_2$; let r_1 be adjacent to m_2 say, where $m_1 \in M_1$ and $m_2 \in M_2$. Thus m_2 is adjacent to r_1 and not to m_1, contrary to (2) applied to

$$\{m_0r_2, m_1r_1\} \cup \{m_ir_i\ : 3 \leq i \leq k\}.$$

This proves that $N_1 \cap N_2 = \emptyset$; and so $R_i = N_i$ for $i = 1, 2$.

By (1), $|M_1| \geq x+1$. Now $|M| = k+1$, and so $|M_2| = k - |M_1| \leq k-x-1$. Since $M_2 \cup \{m_0\} \cup R_1$ is a cutset and G is $(k+1)$-connected, it follows that $(k-x-1)+1+|R_1| \geq k+1$, that is, $|R_1| \geq x+1$. Similarly $|M_2|, |R_2| \geq x+1$. Since every vertex in $M_1 \cup M_2$ has a nonneighbour in R, it follows that $M_1 \cup M_2$ is complete to L.

Let us say a seagull is *migratory* if it contains m_0 and a vertex of L. Suppose that there is a migratory king seagull. By 2.15 there is a cutset X in $G \setminus \{m_0\}$ of cardinality k that contains a vertex in L. Now $G \setminus \{m_0\}$ is k-connected, and so X is a minimal cutset of $G \setminus \{m_0\}$; and it follows that X is the union of some of the sets L, M_1, R_1, M_2, R_2. In particular $L \subseteq X$, and yet $|M_1|, |M_2|, |R_1|, |R_2| \geq x+1$, a contradiction. It follows that there is no migratory king seagull.

Now m_0 is big, since it is complete to $R \cup M_1 \cup M_2$ and the latter has cardinality $k+x-1+k \geq 2k$. We deduce that no vertex in L is big (for any big vertex in L would be adjacent to m_0 and therefore would be contained in a migratory king seagull, since m_0 has a neighbour in R). By 2.16, no vertex in R is big. If m_0 is the only big vertex, then every migratory seagull is a king seagull, and there is a migratory seagull since m_0 has a neighbour in L and one in R, a contradiction. Thus we may assume that there is a big vertex $m_1 \in M_1$. By 2.11 there are no big vertices in M_2, and so every big vertex belongs to $M_1 \cup \{m_0\}$. If m_0 has a nonneighbour $v \in L$, then $\{m_0, m_1, v\}$ is a migratory king seagull, a contradiction. Thus m_0 is complete to L. Let $u \in L$ and $v \in R_1$; then $\{u, v, m_0\}$ is a migratory king seagull since no close seagull contains two big vertices, a contradiction. This proves 1.6. ∎

3 Fractional packing

So far we have been concerned with whether G contains k disjoint seagulls. By a *fractional packing* (of seagulls) we mean an assignment of a real number $q(S) \geq 0$ to each seagull S of G, such that for every vertex v, the sum of $q(S)$ for all seagulls S containing v is at most one. The *value* of a fractional packing q is the sum of $q(S)$ over all seagulls S. (Thus there is a fractional packing q of value k such that $q(S) \in \{0, 1\}$ for every seagull S if and only if G has k disjoint seagulls.) In this section we find an analogue of 1.6 for fractional packing.

A *half-integral packing* means a fractional packing q such that $q(S) \in \{0, 1/2, 1\}$ for every seagull S. We prove the following:

3.1 *Let G be a graph with $\alpha(G) < 3$, and let $k \geq 0$ be a real number. The following are equivalent:*

- *There is a fractional packing of value at least k*
- *There is a half-integral packing of value at least k*
- *$|V(G)| \geq 3k$, and G has connectivity at least k, and for every clique C, $\operatorname{cap}(C) \geq k$.*

Proof. We show first that the first statement implies the third. Thus, let q be a fractional packing of value at least k. Then

$$3k \leq 3\sum_S q(S) = \sum_S \sum_{v \in S} q(S) = \sum_{v \in V(G)} \sum_{S \ni v} q(S) \leq \sum_{v \in V(G)} 1 = |V(G)|,$$

(where the summations subscripted by S are over all seagulls S), and so $|V(G)| \geq 3k$. If X is a cutset of G, then every seagull contains a vertex of X, and so

$$k \leq \sum_S q(S) \leq \sum_S q(S)|S \cap X| = \sum_{v \in X} \sum_{S \ni v} q(S) \leq \sum_{v \in X} 1 = |X|,$$

and consequently the connectivity of G is at least k. If C is a clique, let (A, B, C, D) be the associated partition; then every seagull either contains a vertex in D or two in $A \cup B$, and so

$$k \leq \sum_S q(S) \leq \sum_S (|S \cap D| + |S \cap (A \cup B)|/2) q(S) = \sum_{v \in D} \sum_{S \ni v} q(S) + \sum_{v \in A \cup B} \sum_{S \ni v} q(S)/2$$

$$\leq \sum_{v \in D} 1 + \sum_{v \in A \cup B} 1/2 = |D| + |A \cup B|/2 = \operatorname{cap}(C).$$

Consequently $\operatorname{cap}(C) \geq k$. Thus the first statement implies the third.

Clearly the second implies the first, so it remains to show that the third implies the second. Let G be as in the third statement. Let G' be the graph obtained from G by replacing every vertex v by two adjacent vertices a_v, b_v, and for every edge uv of G making $\{a_u, b_u\}$ complete to $\{a_v, b_v\}$. Thus $|V(G')| \geq 6k$, and it is easy to see that the connectivity of G' is twice that of G, and so at least $2k$. Moreover, for every clique C' of G', let C be the clique of G where $v \in C$ if and only if $C' \cap \{a_v, b_v\} \neq \emptyset$. Let D be the set of all vertices in $V(G) \setminus C$ that are mixed on C, and let D' be the set of all vertices in $V(G') \setminus C'$ that are mixed on C'. For every vertex $u \in D$, both a_u, b_u belong to D', and so $|D'| \geq 2|D|$. Since $|C'| \leq 2|C|$, it follows that

$$|V(G') \setminus C'| + |D'| \geq 2(|V(G) \setminus C| + |D|),$$

and so the capacity of C' in G' is at least twice that of C in G, and so at least $2k$.

We claim that G' has an antimatching of cardinality at least $2k$. For let H be obtained from G' by making a_u, a_v adjacent and making b_u, b_v adjacent, for all distinct $u, v \in V(G)$. Thus $\overline{H}$ is bipartite. If $\overline{H}$ has a matching of cardinality at least $2k$ then our claim holds, so by König's theorem we may assume that there is a set $X \subseteq V(H)$ with $|X| < 2k$ such that $H \setminus X$ is complete. Let A be the set of $v \in V(G)$ such that $a_v \in X$, and let B be the set of $v \in V(G)$ such that $b_v \in X$. Thus $|A| + |B| < 2k$; and for all $u \in V(G) \setminus A$ and $v \in V(G) \setminus B$, u is adjacent to v in G. Let $C = V(G) \setminus (A \cup B)$; then $(A \setminus B) \cup (B \setminus A)$ is complete to C, and so $\operatorname{cap}(C) \leq (|A| + |B|)/2 < k$, a contradiction. This proves that G' has an antimatching of cardinality at least $2k$. Moreover, G' is not a five-wheel, since no two vertices in a five-wheel have the same neighbour sets. From 1.6 it follows that there are at least $2k$ seagulls of G', pairwise disjoint. For every seagull S' of G', there are three vertices $v \in V(G)$ such that $S' \cap \{a_v, b_v\} \neq \emptyset$, and these three vertices form a seagull in G. Consequently there is a list of at least $2k$ seagulls in G (possibly with repetition) such that every vertex is in at most two of them; and hence the second statement holds. This proves 3.1. ∎

What about the algorithmic question – can we decide in polynomial time whether there are k disjoint seagulls in G (still assuming $\alpha(G) < 3$)? 1.6 gives us four conditions that would decide this question if we could check whether the conditions hold; and three of them are easy to check. However, we do not see how to check in polynomial time whether $\operatorname{cap}(C) \geq k$ for each clique C (although it might well be polynomial-time checkable in our view). Nevertheless, we can proceed indirectly, as follows. Using linear programming methods (the ellipsoid method), we can check in polynomial time whether there is a fractional packing of value k, since this is a linear programming problem of size polynomial in $|V(G|$. Consequently, we can check in polynomial time whether the third statement of 3.1 holds, by 3.1. This comprises three of the four conditions of 1.6 that we need to verify, including the difficult one; so we can indeed check the hypotheses of 1.6 in polynomial time, and thereby decide whether there are k disjoint seagulls in polynomial time. One would expect a direct, combinatorial algorithm, rather than the use of the ellipsoid method, but this is not yet discovered.

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