The Roots of the Independence Polynomial of a Clawfree Graph

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Abstract

The *independence polynomial* of a graph G is the polynomial $\Sigma_A\ x^{|A|}$, summed over all independent subsets $A \subseteq V(G)$. We prove that if G is clawfree, then all the roots of its independence polynomial are real. This extends a theorem of Heilmann and Lieb [12], answering a question posed by Hamidoune [11] and Stanley [15].

1 Introduction

All graphs in this paper are finite and simple. An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence polynomial* of a graph G is the polynomial

$$I(G,x) = \Sigma_A \, x^{|A|},$$

where the sum is over all independent subsets $A \subseteq V(G)$. (See for instance [2, 3, 4, 5, 9, 10, 13] for work on these polynomials.)

Given a graph H, its *line graph* $L(H)$ is the graph whose vertex set is the set of edges of H, and two vertices are adjacent if they share an end in H. In [12] Heilmann and Lieb proved that if G is a line graph, then all the roots of $I(G,x)$ are real. This property does not hold for all graphs, since the independence polynomial of a claw is

$$1 + 4x + 3x^2 + x^3$$

and not all its roots are real (a *claw* is the graph with vertex set $\{v_1, v_2, v_3, v_4\}$ and three edges v_1v_2, v_1v_3, v_1v_4.)

A graph G is said to be *clawfree* if no induced subgraph of it is a claw. The main result of this paper is the following, answering a question of Hamidoune [11] that was later posed as a conjecture by Stanley [15].

1.1 *If G is clawfree then all roots of $I(G,x)$ are real.*

Since all line graphs are clawfree, this extends the result of [12].

2 Proof of the main result

Let $f_1(x), \dots, f_k(x)$ be polynomials in one variable with real coefficients. We say they are *compatible* if for all $c_1, \dots, c_k \geq 0$, all the roots of the polynomial $\Sigma_{i=1}^k c_i f_i(x)$ are real.

For $v \in V(G)$ we denote by $N[v]$ the union of $\{v\}$ and the set of neighbours of v in G, and for an induced subgraph H of G let $N_H[v] = N[v] \cap V(H)$. For $A \subseteq V(G)$ we denote by $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$. Our proof of 1.1 is essentially by induction on the number of vertices, making use of the following fundamental identity, the proof of which is clear:

2.1 *If v is a vertex of a graph G, then $I(G,x) = I(G \setminus \{v\}, x) + xI(G \setminus N[v], x)$.*

When we apply this, there is a problem; while we would know inductively that all roots of $G \setminus \{v\}$ and $G \setminus N[v]$ are real, how can we deduce that all roots of G are real? It is not true that the sum of two polynomials with all roots real again has all roots real. The trick is to prove a stronger statement inductively, that certain pairs of polynomials are compatible. Thus we would know inductively that all pairs of the appropriate polynomials in the smaller graph are compatible; but it seems that we have not gained anything, because when we try to apply 2.1, we find that we need to know that all *quadruples* of these polynomials are compatible. But there is a general lemma that if all pairs of certain polynomials are compatible then so are all larger subsets of them, as we explain next.

If $f_1(x), \dots, f_k(x)$ are as before, we say they are *pairwise compatible* if for all $i, j \in \{1, \dots, k\}$, the polynomials $f_i(x)$ and $f_j(x)$ are compatible. In this section we assume the truth of the following lemma, which we prove in section 3. (This lemma is new as far as we know, but is closely related to a result of Bartlett, Hollot and Lin [1].)

2.2 *Let $f_1(x), \dots, f_k(x)$ be pairwise compatible polynomials with positive leading coefficients. Then $f_1(x), \dots, f_k(x)$ are compatible.*

We start with some lemmas. A *clique* in G is a set of vertices all pairwise adjacent. The following is a useful extension of 2.1 (again the proof is clear).

2.3 *Let K be a clique of a graph G. Then*

$$I(G, x) = I(G \setminus K, x) + \Sigma_{k \in K} x I(G \setminus N[k], x).$$

A clique K is *simplicial* if for every $k \in K$ the set $N[k] \setminus K$ is a clique.

2.4 *Let G be a clawfree graph and let K be a simplicial clique in G. Then $N[k] \setminus K$ is a simplicial clique in $G \setminus K$ for all $k \in K$.*

Proof. Since K is a simplicial clique, $N[k] \setminus K$ is a clique for all $k \in K$. Suppose for some $k \in K$ and some $n \in N[k] \setminus K$, the set $N[n] \setminus (K \cup N[k])$ is not a clique. Let x, y be two non-adjacent vertices in $N[n] \setminus (K \cup N[k])$. Then $\{n, k, x, y\}$ induces a claw in G, a contradiction. This proves 2.4. ∎

Let us say a graph G is *real-rooted* if for every induced subgraph H of G, all roots of $I(H, x)$ are real. We wish to prove that every clawfree graph is real-rooted, and we need some lemmas about real-rooted graphs.

2.5 *Let G be a real-rooted clawfree graph. Then:*

1. *for every two simplicial cliques K, L in G, the polynomials $I(G \setminus K, x)$ and $I(G \setminus L, x)$ are compatible,*
2. *for every simplicial clique K, the polynomials $I(G, x)$ and $xI(G \setminus K, x)$ are compatible.*

Proof. We prove both statements simultaneously by induction on $|V(G)|$. First we prove 2.5.1. Since G is real-rooted, we may assume that $K \cup L \neq \emptyset$. Write $H = G \setminus (K \cup L)$. By 2.3 applied to $G \setminus L$ and $K \setminus L$, we have

$$I(G \setminus L, x) = I(H, x) + \Sigma_{v \in K \setminus L} x I(H \setminus N_H[v], x)$$

and similarly

$$I(G \setminus K, x) = I(H, x) + \Sigma_{v \in L \setminus K} x I(H \setminus N_H[v], x).$$

By 2.2, in order to prove that $I(G \setminus K, x)$ and $I(G \setminus L, x)$ are compatible, it is enough to show that the polynomials

$$I(H, x),\ xI(H \setminus N_H[v], x),\ xI(H \setminus N_H[u], x)$$

are pairwise compatible for all $u, v \in K \cup L$, since they all have positive leading coefficients. Since $|V(H)| < |V(G)|$ and by 2.4, it follows inductively from 2.5.2 that $I(H, x)$ is compatible with $xI(H \setminus N_H[v], x)$ for $v \in K \cup L$; and from 2.5.1 that $xI(H \setminus N_H[u], x)$ and $xI(H \setminus N_H[v], x)$ are compatible for all $u, v \in K \cup L$. This proves 2.5.1.

To prove 2.5.2, let K be a simplicial clique in G. Since G is real-rooted, we may assume that $K \neq \emptyset$. By 2.3,

$$I(G, x) = I(G \setminus K, x) + x\Sigma_{k\in K} I(G \setminus N[k], x).$$

By 2.2, to prove that $I(G, x)$ and $xI(G \setminus K, x)$ are compatible, it is enough to prove that the polynomials

$$xI(G \setminus K, x),\ I(G \setminus K, x),\ xI(G \setminus N[k], x),\ xI(G \setminus N[k'], x)$$

are pairwise compatible for all $k, k' \in K$, since they all have positive leading coefficients.

Since G is real-rooted, it follows that the roots of $I(G\setminus K, x)$ are real, so $xI(G\setminus K, x)$ and $I(G\setminus K, x)$ are compatible. By 2.4 and 2.5.2 (applied inductively to $G \setminus K$), $I(G \setminus K, x)$ and $xI(G \setminus N[k], x)$ are compatible for all $k \in K$, and by 2.5.1 with $L = \emptyset$, $xI(G\setminus K, x)$ and $xI(G\setminus N[k], x)$ are compatible for all $k \in K$. Inductively by 2.5.1, $xI(G \setminus N[k], x)$ and $xI(G \setminus N[k'], x)$ are compatible for all $k, k' \in K$. This proves that

$$xI(G \setminus K, x),\ I(G \setminus K, x),\ xI(G \setminus N[k], x),\ xI(G \setminus N[k'], x)$$

are pairwise compatible for all $k, k' \in K$, and so completes the proof of 2.5.2 and therefore of 2.5. ∎

We use 2.5 to prove the following.

2.6 *Let G be a clawfree graph, and let $v \in V(G)$ such that $G\setminus v$ is real-rooted. Then the polynomials $I(G \setminus v, x)$ and $xI(G \setminus N[v], x)$ are compatible.*

Proof. The proof is by induction on $|V(G)|$. If v has no neighbours, then by 2.1 $I(G \setminus N[v], x) = I(G\setminus v, x)$, and since $G\setminus v$ is real-rooted, it follows that $I(G\setminus v, x)$ and $xI(G\setminus N[v], x)$ are compatible. We may therefore assume that there is a vertex u adjacent to v. Let $H = G \setminus (N[u] \cap N[v])$.

(1) *$N_H[u]$ and $N_H[v]$ are both simplicial cliques in H.*

For $N_H[u]$ is a clique, since $\{u, v, x, y\}$ does not induce a claw for $x, y \in N_H[u]$. Suppose that for some $w \in N_H[u]$ there exist two non-adjacent vertices $s, t \in N_H[w] \setminus N_H[u]$. Then $\{w, u, s, t\}$ induces a claw in G, a contradiction. So $N_H[u]$, and similarly $N_H[v]$, is a simplicial clique in H. This proves (1). By 2.1 applied to $G \setminus \{v\}$,

$$I(G \setminus \{v\}, x) = I(G \setminus \{u, v\}, x) + xI(G \setminus N[u], x),$$

so to complete the proof of the theorem, by 2.2 it is enough to show that the polynomials

$$xI(G \setminus N[v], x),\ I(G \setminus \{u, v\}, x),\ xI(G \setminus N[u], x)$$

are pairwise compatible. But the pairs $xI(G \setminus N[v], x)$, $I(G \setminus \{u, v\}, x)$ and $xI(G \setminus N[u], x)$, $I(G \setminus \{u, v\}, x)$ are compatible by the inductive hypothesis applied, respectively, to the graphs $G \setminus \{u\}$ and $G \setminus \{v\}$; and $xI(G \setminus N[u], x)$, $xI(G \setminus N[v], x)$ are compatible by (1) and the first assertion of 2.5, since $I(G \setminus N[u], x) = I(H \setminus N_H[u], x)$ and $I(G \setminus N[v], x) = I(H \setminus N_H[v], x)$. This completes the proof of 2.6. ∎

Proof of 1.1
We proceed by induction on $|V(G)|$. We may assume that there exists $v \in V(G)$; and from the inductive hypothesis, $G \setminus \{v\}$ is real-rooted. By 2.6, $I(G \setminus \{v\}, x), xI(G \setminus N[v], x)$ are compatible; and since $I(G, x) = I(G \setminus \{v\}, x) + xI(G \setminus N[v], x)$ by 2.1, it follows that all roots of $I(G, x)$ are real. This proves 1.1. ∎

3 Proof of the lemma

It remains, therefore, to prove 2.2, and that is the goal of this section. We start by proving some general facts about compatible pairs of polynomials.

3.1 *If the polynomials f and g are compatible then so are their derivatives.*

Proof. We note that between every two real roots of a polynomial h there is a root of its derivative h' (counting according to multiplicity), and so if all roots of h are real then the same holds for h'. Thus the fact that f and g are compatible implies that their derivatives are compatible. This proves 3.1. ∎

If f is a polynomial, we denote its degree by $\deg(f)$.

3.2 *If f, g are compatible polynomials with positive leading coefficients then $|\deg(f) - \deg(g)| \leq 1$.*

Proof. The proof is by induction on $\max\{\deg(f), \deg(g)\}$. Assume first that one of f, g is a constant function, say $f(x) = c$. Since the leading coefficient of f is positive, it follows that $c > 0$. Then adding a large enough multiple of f to g produces a polynomial with at most one real root (since the leading coefficient of g is positive). Since the pair f, g is compatible it follows that $\deg(g) \leq 1$ and the result holds.

So we may assume that both f and g have degree at least 1. By 3.1 the derivatives f' and g' of f and g respectively are compatible. Both f' and g' have positive leading coefficients and it follows inductively that

$$|\deg(f) - \deg(g)| = |\deg(f') - \deg(g')| \leq 1.$$

This proves 3.2. ∎

When $f(x)$ is a polynomial, we denote by $n_f(x)$ the number of real roots of $f(x)$ that lie in the interval $[x, \infty)$ (counted with their multiplicities). We say that f and g *agree* at $a \in \Re$ if $f(a)$ and $g(a)$ are either both positive or both negative (and both non-zero).

3.3 *If f and g are compatible polynomials that agree at a and b for some $a < b \in \Re$, then*

$$n_f(b) - n_f(a) = n_g(b) - n_g(a).$$

Proof. For $0 \leq t \leq 1$ let $p_t(x) = tf(x) + (1-t)g(x)$. Since f and g agree at a and b, for all t the polynomial $p_t(x)$ has no roots on the boundary of the interval $[a, b]$. For $0 \leq t \leq 1$ the roots of p_t move continuously with t in the complex plane, and therefore in the real line, since all the roots of p_t are real because f and g are compatible. In particular, the number of roots of $p_t(x)$ in the open interval (a, b) is independent of t. But the polynomials $p_0(x)$ and $p_1(t)$ have $n_g(b) - n_g(a)$ and $n_f(b) - n_f(a)$ roots respectively in this interval. This proves 3.3. ∎

3.4 *Let f, g be compatible polynomials with positive leading coefficients. Then $|n_f(x) - n_g(x)| \le 1$ for all $x \in \Re$.*

Proof. The proof is by induction on $\max\{\deg(f), \deg(g)\}$. Since the common roots of f and g contribute the same amount to n_f and n_g we may assume that f, g have no common roots by factoring out the greatest common divisor (the compatibility property is preserved). Suppose that $n_f(x_0) - n_g(x_0) \ge 2$ for some $x_0 \in \Re$. We may assume that x_0 is a root of f, and indeed x_0 is the largest root such that $n_f(x_0) - n_g(x_0) \ge 2$. Since x_0 is a root of f, it is not a root of g.

First we claim that $n_f(x_0) - n_g(x_0) = 2$. Assume for a contradiction that $n_f(x_0) - n_g(x_0) \ge 3$. Since between every two real roots of a polynomial there is a real root of its derivative, it follows that $n_{f'}(x_0) = n_f(x_0) - 1$ and $n_{g'}(x_0) \le n_g(x_0)$. Thus

$$n_{f'}(x_0) - n_{g'}(x_0) \ge n_f(x_0) - n_g(x_0) - 1 \ge 2,$$

contrary to the inductive hypothesis. This proves that $n_f(x_0) - n_g(x_0) = 2$.

Choose y_1 strictly larger than all roots of f and all roots of g. Since f and g both have positive leading coefficients, it follows that f and g agree at y_1. For the same reason and since $n_f(x_0) - n_g(x_0)$ is even, we can choose $y_2 < x_0$ such that f and g agree at y_2 and have no roots in the interval $[y_2, x_0)$. But now

$$n_f(y_2) - n_f(y_1) \ne n_g(y_2) - n_g(y_1),$$

contrary to 3.3. This proves 3.4. ∎

For two monotone non-increasing sequences $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ of real numbers, we say that the first *interleaves* the second if $n \le m \le n+1$ and $(a_1, b_1, a_2, b_2, \ldots)$ is another monotone non-increasing sequence. (This does not imply that the second sequence interleaves the first.)

If f is a polynomial of degree d with all roots real, let $r_1 \ge \ldots \ge r_d$ be the roots of f. We call the sequence $(r_1, \ldots, r_d)$ the *root sequence* of f. Let $f_1(x), \ldots, f_k(x)$ be polynomials with positive leading coefficients and all roots real. A *common interleaver* for $f_1(x), \ldots, f_k(x)$ is a sequence that interleaves the root sequence of each f_i.

We observe the following (the proof is easy and we leave it to the reader).

3.5 *Let $f(x), g(x)$ be polynomials with all roots real. They have a common interleaver if and only if $|n_f(x) - n_g(x)| \le 1$ for all $x \in \Re$.*

We now complete the proof of 2.2, by proving it in the strengthened form below. If $k = 2$ and f_1, f_2 have the same degree, this was essentially proved by Dedieu [6, 7] (thanks to Alan Sokal for bringing this reference to our attention).

3.6 *Let $f_1(x), \ldots, f_k(x)$ be polynomials with positive leading coefficients and all roots real. The following four statements are equivalent:*

1. *$f_1, \ldots, f_k$ are pairwise compatible,*
2. *for all s, t such that $1 \le s < t \le k$, the polynomials f_s, f_t have a common interleaver,*
3. *$f_1, \ldots, f_k$ have a common interleaver,*
4. *$f_1, \ldots, f_k$ are compatible.*

Proof. For $1 \le i \le k$ let $d_i = \deg(f_i)$ and let $d = \max_{1\le i\le k} d_i$. Let $(r^i_1, \dots, r^i_{d_i})$ be the root sequence of f_i. If $d_i \ge 1$, let $I^i_1, \dots, I^i_{d_i+1}$ be the intervals of $\Re$ defined as follows: $I^i_1 = [r^i_1, \infty), I^i_{d_i+1} = (-\infty, r^i_{d_i}]$ and $I^i_j = [r^i_j, r^i_{j-1}]$ for $2 \le j \le d_i$. If $d_i = 0$ we set $I^i_1 = \Re$. We note that a sequence $(p_1, \dots, p_m)$ interleaves the root sequence of f_i if and only if $d_i - 1 \le m \le d_i$ and $p_j \in I^i_j$ for $i \le j \le m$.

3.6.1 implies 3.6.2.

For let $1 \le s < t \le k$ and let $d^* = min\{d_s, d_t\}$. By 3.2 $d^* \ge \max\{d_s, d_t\} - 1$. It is enough to prove that for $1 \le j \le d^* + 1$ the intersection $I^s_j \cap I^t_j$ is non-empty. Suppose not, and let j be minimum such that $I^s_j \cap I^t_j = \emptyset$. Since the leading coefficients of both f_s and f_t are positive, $j \ge 2$. From the symmetry we may assume that $r^s_{j-1} \le r^t_{j-1}$, and so r^t_j exists and $r^s_{j-1} < r^t_j$. But then $n_{f_t}(r^t_j) = j$ and $n_{f_s}(r^t_j) \le j - 2$, contrary to 3.4. This proves that 3.6.1 implies 3.6.2.

3.6.2 implies 3.6.3.

For it follows from 3.6.2 that for all $s, t \in \{1, \dots, k\}$ and all $j \in \{1, \dots, d\}$, the intersection $I^s_j \cap I^t_j$ is non-empty. From the Helly property of linear intervals we deduce that $\bigcap_{i=1}^k I^i_j \ne \emptyset$ for all $j \in \{1, \dots, d\}$. For $1 \le j \le d$ choose $p_j \in \bigcap_{i=1}^k I^i_j$; then $(p_1, \dots, p_d)$ is a common interleaver for $f_1, \dots, f_k$. This proves that 3.6.2 implies 3.6.3.

3.6.3 implies 3.6.4.

The proof is by induction on d. We may assume that no $x_0 \in \Re$ is a root of all $f_1, \dots, f_k$, for otherwise the theorem follows from the inductive hypothesis applied to the family $f_i/(x - x_0)$ $(1 \le i \le k)$, which still has a common interleaver by 3.5. Let $c_1, \dots, c_k$ be non-negative real numbers, and let $f = \Sigma^k_{i=1} c_i f_i$. We need to prove that all the roots of f are real. We may assume that $c_1, \dots, c_k$ are all positive.

Since $f_1, \dots, f_k$ have a common interleaver, it follows that $d - 1 \le d_i \le d$ for all $1 \le i \le k$, and so we may assume that there is a common interleaver with d terms, say $(p_1, \dots, p_d)$.

Let $1 \le i \le k$. Since the leading coefficient of f_i is positive and $(p_1, \dots, p_d)$ interleaves the root sequence of f_i, it follows that for all $j \in \{1, \dots, d\}$, $f_i(p_j) \ge 0$ if j is odd and $f_i(p_j) \le 0$ if j is even. Since no p_j is a common root of $f_1, \dots, f_k$, it follows that for $1 \le j \le d$ we have $f(p_j) > 0$ if j is odd and $f(p_j) < 0$ if j is even.

So for $1 \le j < d$ there exists r_j with $p_{j+1} < r_j < p_j$ such that $f(r_j) = 0$. Since f is a real polynomial of degree d (and therefore has an even number of non-real roots), it follows that all the roots of f are real. This proves that 3.6.3 implies 3.6.4.

Since 3.6.4 clearly implies 3.6.1, this completes the proof of 3.6. ∎

The equivalence of 3.6.1 and 3.6.4 is closely related to a result of Bartlett, Hollot and Lin [1]. They prove the same assertion (indeed, they weaken statement 3.6.1, just requiring the compatibility of all pairs of $f_1, \dots, f_k$ that are adjacent in the 1-skeleton of the boundary of the convex hull of $f_1, \dots, f_k$); but they require that $f_1, \dots, f_k$ all have the same degree.

If we remove the hypothesis that the leading coefficients of $f_1, \dots, f_k$ are all positive, 2.2 is no longer true. For example, the polynomials $x^2 - 2x, x^2 + 2x, 1 - x^2$ are pairwise compatible, but their sum does not have all roots real.

It is therefore natural to ask for an analogue of 3.6 without the assumption that the leading coefficients of the polynomials are all positive. We have already seen that statements 3.6.1 and 3.6.4 are not equivalent, but it is possible that statements 3.6.3 and 3.6.4 are equivalent under an appropriate modification of the definition of a common interleaver. We have not been able to decide this.

What if we ask for *all* linear combinations of the polynomials to have all roots real, instead of just all non-negative linear combinations? Let us say that a set $f_1, \ldots, f_k$ of polynomials is *strongly compatible* if for all $c_1, \ldots, c_k \in \Re$, all the roots of the polynomial $\Sigma_{i=1}^{k} c_i f_i(x)$ are real. When $k = 2$, there is an analogue of 3.6; a theorem of Obreschkoff [14] when there are no common roots, and of Dedieu [6] in general, asserts that f_1, f_2 are strongly compatible if and only if their degrees differ by at most one, and the root sequence of one of them interleaves that of the other. But there is no similar analogue for general k, because it is easy to see (thanks to David Moulton for pointing this out to us) that three polynomials are never strongly compatible unless they are linearly dependent. (Proof sketch: if two nonzero polynomials are strongly compatible, then their degrees differ by at most one; given three linearly independent polynomials f_1, f_2, f_3, where $\deg(f_1) \geq \deg(f_2), \deg(f_3)$, there is a nonzero linear combination of them of degree at most $\deg(f_1) - 2$; so this linear combination is not strongly compatible with f_1.)

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