LARGE RAINBOW MATCHINGS IN GENERAL GRAPHS

RON AHARONI, ELI BERGER, MARIA CHUDNOVSKY, DAVID HOWARD, AND PAUL SEYMOUR

1. Introduction

Let $C = (C_1, \ldots, C_m)$ be a system of sets. The range of an injective partial choice function from $C$ is called a rainbow set, and is also said to be multicolored by $C$. If $\phi$ is such a partial choice function and $i \in \text{dom}(\phi)$ we say that $C_i$ colors $\phi(i)$ in the rainbow set. If the elements of $C_i$ are sets then a rainbow set is said to be a (partial) rainbow matching if its range is a matching, namely it consists of disjoint sets.

Definition 1.1. For integers $n, m$ let $f(n, m)$ (respectively $g(n, m)$) be the minimal number $k$ such that any family $\mathcal{M} = (M_1, \ldots, M_k)$ of matchings of size $n$ in a bipartite (respectively, general) graph has a partial rainbow matching of size $m$.

A greedy choice shows that $g(n, 2n - 1) = n$. In [1] it was conjectured that $f(n, n + 1) = n$, namely every family of $n$ matchings of size $n + 1$ has a rainbow matching of size $n$. If true, this would yield by a simple argument that $f(n, n - 1) \leq n$. The current best result in this direction is $f(n, \lceil \frac{3}{2}n \rceil) = n$.

A strange jump occurs here: while possibly $f(n, n + 1) = n$, if we take matchings of size one less, namely $n$, we need to take $2n - 1$ of them to obtain a rainbow matching of size $n$. Namely, $f(n, n) = 2n - 1$.

Example 1.2. To show that $f(n, n) > 2n - 2$ take $M_i, 1 \leq i \leq n - 1$ to be all equal to one of the two perfect matchings in $C_{2n}$ and $M_i, i \leq 2n - 2$ to be all equal to the other perfect matching. Clearly, this system does not have a rainbow matching.

The fact that $2n - 1$ matchings suffice was essentially proved by Drisko [6]:

Theorem 1.3. Let $A$ be an $m \times n$ matrix in which the entries of each row are all distinct. If $m \geq 2n - 1$, then $A$ has a transversal, namely a set of $n$ distinct entries with no two in the same row or column.

In [1] this was formulated in the rainbow matching setting, and given a short proof:

Theorem 1.4. Any family $\mathcal{M} = (M_1, \ldots, M_{2n - 1})$ of matchings of size $n$ in a bipartite graph possesses a rainbow matching.

In [3] it was shown that Example 1 is the only instance in which $2n - 2$ matchings do not suffice. In [4] Theorem 1.4 was strengthened, using topological methods:

Theorem 1.5. If $M_i, i = 1, 2n - 1$ are matchings in a bipartite graphs satisfying $|M_i| = \min(i, n)$ for all $i \leq 2n - 1$ then there exists a rainbow matching of size $n$.

The conjecture we wish to study in this paper is due to Barát, Gyárfás and Sárközy:

Conjecture 1.6. [5] For $n$ even $g(n, n) = 2n$, and for $n$ odd $g(n, n) = 2n - 1$.

Example 1.7. The following example shows that for $n$ even $f(n, n) \geq 2n$, namely $2n - 1$ matchings of size $n$ in a graph do not necessarily have a rainbow matching of size $n$. Let $n = 2k$. Number the vertices of $C_{2n}$ as $v_1, v_2, \ldots, v_{4n}$, and let $K$ be the matching \{ $v_1v_3, v_2v_4, v_5v_7, v_6v_8, \ldots, v_{4n-3}v_{4n-1}$, $v_{4n-2}v_{4k}$ \}. Let $M_0 = K$, let $\mathcal{M}$ be the family consisting of $K$ and of $n - 1$ copies of each of the two matchings of size $n$ in $C_{2n}$. Then
Lemma 2.4. Let $\mathcal{M}$ does not have a rainbow matching of size $n$. If there was, it would have to contain an edge from $K$, and without loss of generality this edge is $v_1v_3$. But then no edge can be chosen from any other matching in $\mathcal{M}$ that contains the vertex $v_2$.

Question: is this the only example? (As mentioned above, in the bipartite case Example 1 is the unique example showing sharpness of Drisko's theorem).

We shall prove:

**Theorem 1.8.** $g(n, n) \leq 3n - 2$ for all $n$.

2. Preliminaries and notation

We shall use the following notation regarding paths. The first vertex on a path $P$ is denoted by $in(P)$, and its last vertex by $ter(P)$. The edge set of $P$ is denoted by $E(P)$, and its vertex set by $V(P)$. Given a family of paths $\mathcal{P}$, we write $E[\mathcal{P}] = \bigcup_{P \in \mathcal{P}} E(P)$. For a path $P$ and a vertex $v$ on it, we denote by $Pv$ the part of $P$ up to and including $v$, and by $vP$ the part from $v$ (including $v$) and on. If $P, Q$ are paths such that $in(Q) = ter(P)$ we write $PQ$ for the trail (namely not necessarily simple path) resulting from the concatenation of $P$ and $Q$.

Let $F$ be a matching in a graph, and let $K$ be a set of edges disjoint from $F$. A path $P$ is said to be $K - F$-alternating if every odd-numbered edge of $P$ belongs to $K$ and every even-numbered edge belongs to $F$. If there is no restriction on the odd edges of $P$ then we just say that it is $F$-alternating. If both $in(P)$ and $ter(P)$ do not belong to $\bigcup F$ then $P$ is said to be augmenting. The origin of the name is that in such a case $E(P) \triangle F$ is a matching larger than $F$. The converse is also well known to be true:

**Lemma 2.1.** If $F, G$ are matchings and $|G| > |F|$ then $E(F) \cup E(G)$ contains an $F$-alternating augmenting path.

**Proof.** Viewed as a multigraph, the connected components of $E(F) \cup E(G)$ are cycles (possibly digons) and paths that alternate between $G$ and $F$ edges. Since $|G| > |F|$ one of these paths contains more edges from $G$ than from $F$, and is thus $F$-augmenting.

**Definition 2.2.** Let $F$ be a matching, let $K$ be a set of edges disjoint from $F$, and let $a$ be any vertex. A vertex $v \in \bigcup M$ is said to be oddly $K$-reachable (resp. evenly $K$-reachable) from $a$ if there exists an odd (respectively even) $K - F$-alternating path starting with an edge $ab \in K$ and ending at $v$. Note that being an odd alternating path means ending with an edge from $K$, and being an even alternating path means ending with an edge of $F$. Let $OR(a, K, F)$ be the set of vertices oddly reachable from $a$, $ER(a, K, F)$ the set of vertices evenly reachable from $a$, and let $DR(a, K, F) = OR(a, K, F) \cap ER(a, K, F)$. We say that $v$ is oddly $K$-reachable (respectively evenly $K$-reachable) if it is oddly (respectively evenly) reachable from some vertex not belonging to $\bigcup F$.

Note that there exists a $K - F$ augmenting alternating path if and only if $OR(K, F) \not\subseteq \bigcup F$.

**Definition 2.3.** A graph $G$ is called hypomatchable if $G - v$ has a perfect matching for every $v \in V(G)$.

**Lemma 2.4.** Let $F$ be a matching in a graph $G$, let $K = E \setminus F$, and suppose that $V(G) \setminus \bigcup F$ consists of a single vertex $a$. Then a vertex $x$ belongs to $ER(a, K, F)$ if and only if $G - x$ has a perfect matching.

**Proof.** Suppose that there exists a matching $M$ of $G - x$. Then the $F - M$-alternating path starting at $x$ with an edge of $F$ must terminate at $a$ with an edge of $M$, which proves that $x \in OR(a, K, F)$. If $x \in OR(a, K, F)$ then taking $L$ to be the odd $a - x F$-alternating path reaching $x$ and letting $M = F \triangle L$ yields a perfect matching of $G - x$.

Note that $x \in OR(a, K, F)$ if and only if $F(x) \in ER(a, K, F)$. Hence the lemma implies:

**Corollary 2.5.** Let $F$ be a matching in a graph $G$, let $K = E(G) \setminus F$, and let $a$ be the single vertex in $V(G) \setminus \bigcup F$. Then $G$ is hypomatchable if and only if $V(G) = DR(a, K, F)$. 
3. **Snick-Berry switches**

Let \( G \) be a graph, let \( F \) be a matching in it, and write \( K = E \setminus F \). The pair \((G, F)\) is called a **snick-berry tree** if it can be obtained from a rooted tree \( T \) with root \( r \) as follows. Subdivide every edge \( e = st \) of \( T \), where \( s \) is the vertex nearer to \( r \), by a vertex \( m(e) \). Replace each vertex \( s \) of the original tree by a hypomatchable graph \( H(s) \), such that \( F \cap H(s) \) matches all vertices apart from a single vertex \( r(s) \). For every descendant \( t \) of \( s \) connect some vertex \( v \in H(s) \) different from \( r(s) \) to \( m(st) \) by an edge of \( K \), and connect \( m(st) \) to \( r(t) \) by an edge of \( F \). The sets \( V_t = V(H(t)) \) are called **islands**. We say that \( T \) **guides** the snick-berry tree.

A pair \((G, F)\) of a graph \( G \) and a matching \( F \) in it is called a **snick-berry switch**, or SBS for short, if each of its connected components is of one of two types: a snick-berry tree, or a component on which \( F \) induces a perfect matching.

**Theorem 3.1.** Let \( G = (V, E) \) be a graph, let \( F \) be a matching in \( G \), and let \( K = E \setminus F \). Suppose that:

1. \( F \) is a matching of maximal size in \( G \), and
2. For every \( L \subseteq K \) we have \( \text{OR}(L, F) \subseteq \text{OR}(K, F) \).

Then the pair \((G, F)\) is an SBS.

**Proof.** It suffices to show that if \( G \) satisfies the conditions of the theorem and is connected, then it is a snick-berry tree. If \( G \) consists of a single edge belonging to \( F \) then the lemma is true, with the tree being empty. So, we may assume that this is not the case.

We construct the tree \( T \) guiding the snick-berry tree inductively, by adding at the \( i \)-th stage an island \( V_t \) with a hypomatchable graph \( H(t) \). We call the tree obtained after adding the \( i \)-th island \( T_i \). The inductive assumption will be that for each island \( V_t \) in \( T_i \) we have:

(a) \( V_t = \text{DR}(r(t), K, F) \).
(b) \( r(t) \in \text{OR}(a, K, F) \setminus \text{ER}(a, K, F) \).

If \( \text{OR}(K, F) = \emptyset \) then by condition (2) \( K = \emptyset \), meaning that \( F \) is a perfect matching in \( G \) (actually, with the assumption of connectedness, a single edge), and the theorem is true. So, we may assume that \( \text{OR}(K, F) \neq \emptyset \). This means that there exists a vertex \( a \notin \bigcup F \). Define \( t_1 \) as \( r \), the root of \( T \), and let \( T_1 \) be the tree consisting of the single vertex \( t_1 \). Let \( r(t_1) = a \), and let \( V_t = \text{DR}(a, K, F) \).

Suppose that \( T_i \) has been defined. If \( \bigcup \{V_t \mid t \in V(T_i)\} = V \) then we halt the construction and let \( T = T_i \). Otherwise, choose an edge \( xy \) where \( x \in V_s \), \( s \in V(T_i) \) and \( y \notin \bigcup \{V_t \mid t \in V(T_i)\} \). By its choice, \( y \in \text{OR}(r(s), K, F) \) and since by the induction hypothesis \( V_s = \text{DR}(r(s), K, F) \) and \( r(s) \in \text{OR}(a, K, F) \), we have \( y \in \text{OR}(a, K, F) \). By the inductive assumption \( y \notin \text{DR}(r(s), K, F) \), meaning that \( y \notin \text{ER}(a, K, F) \), proving (b) for \( T_{i+1} \).

By condition (1), \( y \in \bigcup F \). Let \( z \) be the vertex connected by \( F \) to \( y \), obtain \( T_{i+1} \) by adding a descendant \( t_{i+1} \) of \( s \) to \( T_i \), and let \( V_{t_{i+1}} = \text{DR}(z, K, F) \). Let \( z = r(t_{i+1}) \). By (2) above, there is no other edge, except for \( xy \), that connects \( y \) with any \( V_t \), \( t \in V(T_i) \). Also, there is no edge connecting \( V_r \) to \( V_{t_{i+1}} \), since such an edge would generate a \( K - F \) alternating path showing that \( y \in \text{DR}(a, K, F) \), implying that \( y \in V_r \), contrary to the choice of \( y \).

By the construction, for every \( t \in V(T) \) and every vertex \( v \in V_t \) there exists an even \( K - F \) alternating path \( \text{EP}(v) \) from \( a \) to \( v \) going only through islands \( V_s \), for \( s \) belonging to the path in \( T \) from \( r \) to \( t \), and the bridges between them. Also, for every \( v \in V_t \) there exists an even \( K - F \)-alternating path \( \text{EQ}(v) \) from \( r(t) \) to \( v \).

Finally, we have to show that if an edge \( uv \in E(G) \) satisfies \( u \in V_s \) and \( v \notin V_s \) then either

1. \( v = m(st) \) for a direct descendant \( t \) of \( s \), or
2. \( u = r(s) \) and \( v = m(ps) \) for the father \( p \) of \( s \) in the tree \( T \).

Suppose, to the contrary, that there exists an edge \( uv \) contradicting this assertion. There are two cases to consider:
• $v \in V_i$ for some $t \in V(T)$. Let $p$ be the father of $t$. Then $EP(u)$ concatenated with $\overline{EQ(v)}$ shows that $m(pt) \in ER(a, K, F)$, contrary to (b) above.

• $v = m(pq)$ where $p$ is the father of the vertex $q$ of $T$. This cannot happen, because the deletion of $uv$ does not reduce $OR(a, K, F)$, contrary to assumption (2) in the theorem.

\[ \square \]

Remark 3.2. From the proof it follows that if there exists an edge joining a vertex in $V_s$ and $V_t$ where $s$ is not a descendant of $t$ then $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$.

4. Multicolored alternating paths and proof of Theorem 1.8

Theorem 4.1. Let $F$ be a matching, let $K$ be a set of edges disjoint from $F$ such that there is no $K - F$ augmenting $F$-alternating path. If $A$ is an augmenting $F$-alternating path then there exists an edge $e \in E(A) \setminus F$ such that $OR(K \cup \{e\}, F) \supseteq OR(K, F)$.

Proof. Let $G$ be the graph on $V$ whose edge set is $K \cup F$. By the assumption that there is no $K - F$ augmenting alternating path, $F$ is a maximal matching in $G$. Clearly, if the theorem is true when $K$ is replaced by a subset $L$ with $OR(L, F) = OR(K, F)$ then it is true also for $K$. Thus we may assume that condition (2) in Theorem 3.1 holds. By this theorem it follows that the pair $(G, F)$ is an SBS. Since we may clearly assume that $G$ is connected, it is in fact a snick-berry tree, guided by some tree $T$. Suppose that there exists an edge $e = uv$ of $A$ between two distinct islands $V_s$ and $V_t$ ($s, t \in V(T)$). One of $s, t$, say $s$, is not a descendant of the other. By Remark 3.2 it follows that $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$, which validates the theorem.

Thus we may assume that there is no such edge. Let $V_q$ be the last island visited by $A$. Since $A$ terminates at a non $\bigcup F$ vertex, it must leave $V_q$ at some point, and by the above the edge of $A$ leaving $V_q$ must be of the form $xm(uv)$ for some vertex $x \in V_q$ and an edge $uv$ of $T$. Then its next edge must be $m(uv)v$, reaching the island $V_p$ where $v = r(p)$, contradicting the assumption that $V_q$ is the last island visited by $A$. \[ \square \]

Given a family $\mathcal{P}$ of $F$-alternating paths, an $F$-alternating path $P$ is said to be $\mathcal{P}$-multicolored if $E(P) \setminus F$ is a partial rainbow set of the family $E(Q), Q \in \mathcal{P}$.

Corollary 4.2. If $|\mathcal{P}| > 2|F|$ then there exists an augmenting $\mathcal{P}$-multicolored $F$-alternating path.

Proof. By Theorem 4.1 we can construct sets of edges $K_i$, where $K_0 = \emptyset$ and $K_i = K_{i-1} \cup \{e_i\}, e_i \in E(P_i)$, and $OR(K_{i+1}, F) \supseteq OR(K_i, F)$. Since there are only $2|F|$ vertices in $\bigcup F$, at some point $OR(K_i, F)$ will contain a vertex not in $\bigcup F$, meaning that there exists an augmenting $K_i - F$-alternating path $P_i$ by the inductive construction of the sets $K_i$ is $\mathcal{P}$-multicolored. \[ \square \]

Finally, we derive Theorem 1.8 from Corollary 4.2. We have to show that given $3n - 2$ matchings $M_i$, $i \leq 3n - 2$ there exists a partial rainbow matching of size $n$. Let $F$ be a rainbow matching of maximal size, and let $|F| = k$. We wish to show that $k = n$. Suppose to the contrary that $k < n$. Then there are at least $2k - 1$ matchings $M_i$ not represented in $F$. Each of these generates an augmenting $F$-alternating path $P_i$, and by the corollary, there is an augmenting multicolored $F$-alternating path $P$ using edges from the paths $P_i$. Then $F \triangle E(P)$ is a partial rainbow matching of size $k + 1$, contradicting the maximality property of $k$.

Remark 4.3. In [3] it was shown that in the bipartite case Corollary 4.2 only demands $|\mathcal{P}| > |F|$. In the case of general graphs Corollary 4.2 is sharp - $|F| - 1$ matchings do not suffice. The example is essentially the same as Example 1.7. Let $F$ be a matching $\{u_iv_i | i \leq k - 1\} \cup \{xy\}$, let $P_1, \ldots, P_k$ all be the same matching $\{xu_1\} \cup \{v_{k-1}y\} \cup \{v_iu_{i+1} | i \leq k - 2\}$ and let $P_{k+1}, \ldots, P_{2k}$ all be equal to the matching $\{xv_1\} \cup \{u_{k-1}y\} \cup \{u_iv_{i+1} | i \leq k - 2\}$. Example 1.7 consists of the matchings $M_i$, together with the matching $F$. 

5. A conjectured scrambled version

Should the sets $M_i$ in Drisko’s theorem be matchings? What happens when we take $2n - 1$ matchings of size $n$ each, and scramble them, so as to obtain another system of sets of edges, each of size $n$? We conjecture that there still must exist a rainbow matching of size $n$. By König’s edge coloring theorem this is equivalent to the following:

**Conjecture 5.1.** Any system $E_1, \ldots, E_{2n-1}$ of sets of edges in a bipartite graph, each of size $n$ and satisfying $\Delta(\bigcup E_i) \leq 2n - 1$, has a rainbow matching of size $n$.

In [2] a weaker version was proved, using topological methods:

**Theorem 5.2.** Let $d \geq n^2$ and let $E_1, \ldots, E_d$ be sets of edges of size $n$ in a bipartite graph, each of size $n$, and assume that $\Delta(\bigcup E_i) \leq d$. Then the sets have a rainbow matching of size $n$.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, TECHNION**

*E-mail address*, Ron Aharoni: raharoni@gmail.com

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA**

*E-mail address*, Eli Berger: eberger@haifa

**DEPARTMENT OF IEOR, COLUMBIA**

*E-mail address*, Maria Chudnovsky: mchudnov@columbia.edu

**DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY**

*E-mail address*, David Howard: dmhoward@colgate.edu

**DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY**

*E-mail address*, Paul Seymour: pds@math.princeton.edu