LARGE RAINBOW MATCHINGS IN GENERAL GRAPHS

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1. Introduction

Let $\mathcal{C} = (C_1, \ldots, C_m)$ be a system of sets. The range of an injective partial choice function from $\mathcal{C}$ is called a *rainbow set*, and is also said to be *multicolored* by $\mathcal{C}$. If ϕ is such a partial choice function and $i \in dom(\phi)$ we say that C_i *colors* $\phi(i)$ in the rainbow set. If the elements of C_i are sets then a rainbow set is said to be a *(partial) rainbow matching* if its range is a matching, namely it consists of disjoint sets.

Definition 1.1. For integers n, m let $f(n, m)$ (respectively $g(n, m)$) be the minimal number k such that any family $\mathcal{M} = (M_1, \ldots, M_k)$ of matchings of size n in a bipartite (respectively, general) graph has a partial rainbow matching of size m.

A greedy choice shows that $g(n, 2n-1) = n$. In [1] it was conjectured that $f(n, n+1) = n$, namely every family of n matchings of size $n+1$ has a rainbow matching of size n. If true, this would yield by a simple argument that $f(n, n-1) \leq n$. The current best result in this direction is $f(n, \lceil \frac{3}{2}n \rceil) = n$.

A strange jump occurs here: while possibly $f(n, n+1) = n$, if we take matchings of size one less, namely n, we need to take $2n-1$ of them to obtain a rainbow matching of size n. Namely, $f(n, n) = 2n-1$.

Example 1.2. To show that $f(n, n) > 2n-2$ take $M_i, 1 \leq i \leq n-1$ to be all equal to one of the two perfect matchings in C_{2n} and $M_i,\ i \leq 2n-2$ to be all equal to the other perfect matching. Clearly, this system does not have a rainbow matching.

The fact that $2n-1$ matchings suffice was essentially proved by Drisko [6]:

Theorem 1.3. *Let A be an $m \times n$ matrix in which the entries of each row are all distinct. If $m \geq 2n-1$, then A has a transversal, namely a set of n distinct entries with no two in the same row or column.*

In [1] this was formulated in the rainbow matching setting, and given a short proof:

Theorem 1.4. *Any family $\mathcal{M} = (M_1, \ldots, M_{2n-1})$ of matchings of size n in a bipartite graph possesses a rainbow matching.*

In [3] it was shown that Example 1 is the only instance in which $2n-2$ matchings do not suffice. In [4] Theorem 1.4 was strengthened, using topological methods:

Theorem 1.5. *If $M_i,\ i = 1, 2n-1$ are matchings in a bipartite graphs satisfying $|M_i| = \min(i, n)$ for all $i \leq 2n-1$ then there exists a rainbow matching of size n.*

The conjecture we wish to study in this paper is due to Barát, Gyárfás and Sárközy:

Conjecture 1.6. [5] *For n even $g(n, n) = 2n$, and for n odd $g(n, n) = 2n-1$.*

Example 1.7. The following example shows that for n even $f(n, n) \geq 2n$, namely $2n-1$ matchings of size n in a graph do not necessarily have a rainbow matching of size n. Let $n = 2k$. Number the vertices of C_{2n} as $v_1, v_2, \ldots, v_{4n}$, and let K be the matching $\{v_1v_3, v_2v_4, v_5v_7, v_6v_8 \ldots, v_{4n-3}v_{4n-1}, v_{4n-2}v_{4k}\}$. Let $M_0 = K$, let $\mathcal{M}$ be the family consisting of K and of $n-1$ copies of each of the two matchings of size n in C_{2n}. Then

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$\mathcal{M}$ does not have a rainbow matching of size n. If there was, it would have to contain an edge from K, and without loss of generality this edge is v_1v_3. But then no edge can be chosen from any other matching in $\mathcal{M}$ that contains the vertex v_2.

Question: is this the only example? (As mentioned above, in the bipartite case Example 1 is the unique example showing sharpness of Drisko's theorem).

We shall prove:

Theorem 1.8. *$g(n,n) \leq 3n-2$ for all n.*

2. Preliminaries and notation

We shall use the following notation regarding paths. The first vertex on a path P is denoted by $in(P)$, and its last vertex by $ter(P)$. The edge set of P is denoted by $E(P)$, and its vertex set by $V(P)$. Given a family of paths $\mathcal{P}$, we write $E[\mathcal{P}] = \bigcup_{P\in\mathcal{P}} E(P)$. For a path P and a vertex v on it, we denote by Pv the part of P up to and including v, and by vP the part from v (including v) and on. If P, Q are paths such that $in(Q) = ter(P)$ we write PQ for the trail (namely not necessarily simple path) resulting from the concatenation of P and Q.

Let F be a matching in a graph, and let K be a set of edges disjoint from F. A path P is said to be *$K - F$-alternating* if every odd-numbered edge of P belongs to K and every even-numbered edge belongs to F. If there is no restriction on the odd edges of P then we just say that it is F-alternating. If both $in(P)$ and $ter(P)$ do not belong to $\bigcup F$ then P is said to be *augmenting*. The origin of the name is that in such a case $E(P) \triangle F$ is a matching larger than F. The converse is also well known to be true:

Lemma 2.1. *If F, G are matchings and $|G| > |F|$ then $E(F) \cup E(G)$ contains an F-alternating augmenting path.*

Proof. Viewed as a multigraph, the connected components of $E(F) \cup E(G)$ are cycles (possibly digons) and paths that alternate between G and F edges. Since $|G| > |F|$ one of these paths contains more edges from G than from F, and is thus F-augmenting. □

Definition 2.2. Let F be a matching, let K be a set of edges disjoint from F, and let a be any vertex. A vertex $v \in \bigcup M$ is said to be *oddly K-reachable* (resp. *evenly K-reachable*) from a if there exists an odd (respectively even) $K-F$-alternating path starting with an edge $ab \in K$ and ending at v. Note that being an odd alternating path means ending with an edge from K, and being an even alternating path means ending with an edge of F. Let $OR(a, K, F)$ be the set of vertices oddly reachable from a, $ER(a, K, F)$ the set of vertices evenly reachable from a, and let $DR(a, K, F) = OR(a, K, F) \cap ER(a, K, F)$. We say that v is *oddly K-reachable* (respectively *evenly K-reachable*) if it is oddly (respectively evenly) reachable from some vertex not belonging to $\bigcup F$.

Note that there exists a $K - F$ augmenting alternating path if and only if $OR(K, F) \not\subseteq \bigcup F$.

Definition 2.3. A graph G is called *hypomatchable* if $G - v$ has a perfect matching for every $v \in V(G)$.

Lemma 2.4. *Let F be a matching in a graph G, let $K = E \setminus F$, and suppose that $V(G) \setminus \bigcup F$ consists of a single vertex a. Then a vertex x belongs to $ER(a, K, F)$ if and only if $G - x$ has a perfect matching.*

Proof. Suppose that there exists a matching M of $G-x$. Then the $F-M$-alternating path starting at x with an edge of F must terminate at a with an edge of M, wich proves that $x \in OR(a, K, F)$. If $x \in OR(a, K, F)$ then taking L to be the odd $a - x$ F-alternating path reaching x and letting $M = F \triangle L$ yields a perfect matching of $G - x$.

□

Note that $x \in OR(a, K, F)$ if and only if $F(x) \in ER(a, K, F)$. Hence the lemma implies:

Corollary 2.5. *Let F be a matching in a graph G, let $K = E(G) \setminus F$, and let a be the single vertex in $V(G) \setminus \bigcup F$. Then G is hypomatchable if and only if $V(G) = DR(a, K, F)$.*

3. Snick-Berry switches

Let G be a graph, let F be a matching in it, and write $K = E \setminus F$. The pair (G, F) is called a *snick-berry tree* if it can be obtained from a rooted tree T with root r as follows. Subdivide every edge $e = st$ of T, where s is the vertex nearer to r, by a vertex $m(e)$. Replace each vertex s of the original tree by a hypomatchable graph $H(s)$, such that $F \restriction H(s)$ matches all vertices apart from a single vertex $r(s)$. For every descendant t of s connect some vertex $v \in H(s)$ different from $r(s)$ to $m(st)$ by an edge of K, and connect $m(st)$ to $r(t)$ by an edge of F. The sets $V_t = V(H(t))$ are called *islands*. We say that T *guides* the snick-berry tree.

A pair (G, F) of a graph G and a matching F in it is called a *snick-berry switch*, or SBS for short, if each of its connected components is of one of two types: a snick-berry tree, or a component on which F induces a perfect matching.

Theorem 3.1. *Let $G = (V, E)$ be a graph, let F be a matching in G, and let $K = E \setminus F$. Suppose that:*

(1) F is a matching of maximal size in G, and
(2) For every $L \subsetneqq K$ we have $OR(L, F) \subsetneqq OR(K, F)$.

Then the pair (G, F) is an SBS.

Proof. It suffices to show that if G satisfies the conditions of the theorem and is connected, then it is a snitch-berry tree. If G consists of a single edge belonging to F then the lemma is true, with the tree being empty. So, we may assume that this is not the case.

We construct the tree T guiding the snitch-berry tree inductively, by adding at the i-th stage an island V_{t_i} with a hypomatchable graph $H(t_i)$. We call the tree obtained after adding the i-th island T_i. The inductive assumption will be that for each island V_t in T_i we have:

(a) $V_t = DR(r(t), K, F)$.
(b) $r(t) \in OR(a, K, F) \setminus ER(a, K, F)$.

If $OR(K, F) = \emptyset$ then by condition (2) $K = \emptyset$, meaning that F is a perfect matching in G (actually, with the assumption of connectedness, a single edge), and the theorem is true. So, we may assume that $OR(K, F) \neq \emptyset$. This means that there exists a vertex $a \notin \bigcup F$. Define t_1 as r, the root of T, and let T_1 be the tree consisting of the single vertex t_1. Let $r(t_1) = a$, and let $V_{t_1} = DR(a, K, F)$.

Suppose that T_i has been defined. If $\bigcup\{V_t \mid t \in V(T_i)\} = V$ then we halt the construction and let $T = T_i$. Otherwise, choose an edge xy where $x \in V_s,\ s \in V(T_i)$ and $y \notin \bigcup\{V_t \mid t \in V(T_i)\}$. By its choice, $y \in OR(r(s), K, F)$ and since by the induction hypothesis $V_s = DR(r(s), K, F)$ and $r(s) \in OR(a, K, F)$, we have $y \in OR(a, K, F)$. By the inductive assumption $y \notin DR(r(s), K, F)$, meaning that $y \notin ER(a, K, F)$, proving (b) for T_{i+1}.

By condition (1), $y \in \bigcup F$. Let z be the vertex connected by F to y, obtain T_{i+1} by adding a descendant t_{i+1} of s to T_i, and let $V_{t_{i+1}} = DR(z, K, F)$. Let $z = r(t_{i+1})$. By (2) above, there is no other edge, except for xy, that connects y with any V_t, $t \in V(T_i)$. Also, there is no edge connecting V_r to $V_{t_{i+1}}$, since such an edge would generate a $K - F$ alternating path showing that $y \in DR(a, K, F)$, implying that $y \in V_r$, contrary to the choice of y.

By the construction, for every $t \in V(T)$ and every vertex $v \in V_t$ there exists an even $K - F$ alternating path $EP(v)$ from a to v going only through islands V_s, for s belonging to the path in T from r to t, and the bridges between them. Also, for every $v \in V_t$ there exists an even $K - F$-alternating path $EQ(v)$ from $r(t)$ to v.

Finally, we have to show that if an edge $uv \in E(G)$ satisfies $u \in V_s$ and $v \notin V_s$ then either

(1) $v = m(st)$ for a direct descendant t of s, or
(2) $u = r(s)$ and $v = m(ps)$ for the father p of s in the tree T.

Suppose, to the contrary, that there exists an edge uv contradicting this assertion. There are two cases to consider:

- $v \in V_t$ for some $t \in V(T)$. Let p be the father of t. Then $EP(u)$ concatenated with $\overleftarrow{EQ(v)}$ shows that $m(pt) \in ER(a, K, F)$, contrary to (b) above.
- $v = m(pq)$ where p is the father of the vertex q of T. This cannot happen, because the deletion of uv does not reduce $OR(a, K, F)$, contrary to assumption (2) in the theorem.

□

Remark 3.2. From the proof it follows that if there exists an edge joining a vertex in V_s and V_t where s is not a descendant of t then $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$.

4. Multicolored alternating paths and proof of Theorem 1.8

Theorem 4.1. *Let F be a matching, let K be a set of edges disjoint from F such that there is no $K - F$ augmenting F-alternating path. If A is an augmenting F-alternating path then there exists an edge $e \in E(A) \setminus F$ such that $OR(K \cup \{e\}, F) \supsetneqq OR(K, F)$.*

Proof. Let G be the graph on V whose edge set is $K \cup F$. By the assumption that there is no $K - F$ augmenting alternating path, F is a maximal matching in G. Clearly, if the theorem is true when K is replaced by a subset L with $OR(L, F) = OR(K, F)$ then it is true also for K. Thus we may assume that condition (2) in Theorem 3.1 holds. By this theorem it follows that the pair (G, F) is an SBS. Since we may clearly assume that G is connected, it is in fact a snick-berry tree, guided by some tree T. Suppose that there exits an edge $e = uv$ of A between two distinct islands V_s and V_t $(s, t \in V(T))$. One of s, t, say s, is not a descendant of the other. By Remark 3.2 it follows that $r(t) \in OR(K \cup \{e\}, F) \setminus OR(K, F)$, which validates the theorem.

Thus we may assume that there is no such edge. Let V_q be the last island visited by A. Since A terminates at a non $\bigcup F$ vertex, it must leave V_q at some point, and by the above the edge of A leaving V_q must be of the form $xm(uv)$ for some vertex $x \in V_q$ and an edge uv of T. Then its next edge must be $m(uv)v$, reaching the island V_p where $v = r(p)$, contradicting the assumption that V_q is the last island visited by A. □

Given a family $\mathcal{P}$ of F-alternating paths, an F-alternating path P is said to be $\mathcal{P}$*-multicolored* if $E(P) \setminus F$ is a partial rainbow set of the family $E(Q),\ Q \in \mathcal{P}$.

Corollary 4.2. *If $|\mathcal{P}| > 2|F|$ then there exists an augmenting $\mathcal{P}$-multicolored F-alternating path.*

Proof. By Theorem 4.1 we can construct sets of edges K_i, where $K_0 = \emptyset$ and $K_i = K_{i-1} \cup \{e_i\}$, $e_i \in E(P_i)$, and $OR(K_{i+1}, F) \supsetneqq OR(K_i, F)$. Since there are only $2|F|$ vertices in $\bigcup F$, at some point $OR(K_i, F)$ will contain a vertex not in $\bigcup F$, meaning that there exists an augmenting $K_i - F$-alternating path P, which by the inductive construction of the sets K_i is $\mathcal{P}$-multicolored. □

Finally, we derive Theorem 1.8 from Corollary 4.2. We have to show that given $3n-2$ matchings $M_i,\ i \leq 3n-2$ there exists a partial rainbow matching of size n. Let F be a rainbow matching of maximal size, and let $|F| = k$. We wish to show that $k = n$. Suppose to the contrary that $k < n$. Then there are at least $2k-1$ matchings M_i not represented in F. Each of these generates an augmenting F-alternating path P_i, and by the corollary, there is an augmenting multicolored F-alternating path P using edges from the paths P_i. Then $F \triangle E(P)$ is a partial rainbow matching of size $k+1$, contradicting the maximality property of k.

Remark 4.3. In [3] it was shown that in the bipartite case Corollary 4.2 only demands $|\mathcal{P}| > |F|$. In the case of general graphs Corollary 4.2 is sharp - $2|F| - 1$ matchings do not suffice. The example is essentially the same as Example 1.7. Let F be a matching $\{u_iv_i \mid i \leq k-1\} \cup \{xy\}$, let $P_1, \ldots, P_k$ all be the same matching $\{xu_1\} \cup \{v_{k-1}y\} \cup \{v_iu_{i+1} \mid i \leq k-2\}$ and let $P_{k+1}, \ldots, P_{2k}$ all be equal to the matching $\{xv_1\} \cup \{u_{k-1}y\} \cup \{u_iv_{i+1} \mid i \leq k-2\}$. Example 1.7 consists of the matchings M_i, together with the matching F.

5. A conjectured scrambled version

Should the sets M_i in Drisko's theorem be matchings? What happens when we take $2n-1$ matchings of size n each, and scramble them, so as to obtain another system of sets of edges, each of size n? We conjecture that there still must exist a rainbow matching of size n. By König's edge coloring theorem this is equivalent to the following:

Conjecture 5.1. *Any system $E_1, \ldots, E_{2n-1}$ of sets of edges in a bipartite graph, each of size n and satisfying $\Delta(\bigcup E_i) \leq 2n-1$, has a rainbow matching of size n.*

In [2] a weaker version was proved, using topological methods:

Theorem 5.2. *Let $d \geq n^2$ and let $E_1, \ldots, E_d$ be sets of edges of size n in a bipartite graph, each of size n, and assume that $\Delta(\bigcup E_i) \leq d$. Then the sets have a rainbow matching of size n.*

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