A local strengthening of Reed's ω, Δ, and χ conjecture for quasi-line graphs

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Abstract

Reed's ω, Δ, χ conjecture proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$; it is known to hold for all claw-free graphs. In this paper we consider a local strengthening of this conjecture. We prove the local strengthening for line graphs, then note that previous results immediately tell us that the local strengthening holds for all quasi-line graphs. Our proofs lead to polytime algorithms for constructing colourings that achieve our bounds: $O(n^2)$ for line graphs and $O(n^3 m^2)$ for quasi-line graphs. For line graphs, this is faster than the best known algorithm for constructing a colouring that achieves the bound of Reed's original conjecture.

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1 Introduction

All graphs and multigraphs we consider in this paper are finite. Loops are permitted in multigraphs but not graphs. Given a graph G with maximum degree $\Delta(G)$ and clique number $\omega(G)$, the chromatic number $\chi(G)$ is trivially bounded above by $\Delta(G)+1$ and below by $\omega(G)$. Reed's ω, Δ, χ conjecture proposes, roughly speaking, that $\chi(G)$ falls in the lower half of this range:

Conjecture 1 (Reed)**.** *For any graph G,*

$$\chi(G) \leq \left\lceil \tfrac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

One of the first classes of graphs for which this conjecture was proven is the class of line graphs [6]. This result was then extended to quasi-line graphs [4, 5] and then claw-free graphs [4] by King and Reed (we will define these graph classes shortly). In his thesis, King proposed a local strengthening of Reed's conjecture. For a vertex v, let $\omega(v)$ denote the size of the largest clique containing v.

Conjecture 2 (King [4])**.** *For any graph G,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \tfrac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

Even for line graphs this would be tight, as evidenced by the strong product of C_5 and K_ℓ for any positive ℓ; this is the line graph of the multigraph constructed by replacing each edge of C_5 by ℓ parallel edges.

There are several pieces of evidence that lend credence to Conjecture 2. First is the fact that the fractional relaxation holds. This was noted by McDiarmid [7], and the full proof appears in [4] §2.2:

Theorem 3 (McDiarmid)**.** *For any graph G,*

$$\chi_f(G) \leq \max_{v \in V(G)} \left(\tfrac{1}{2}(d(v) + 1 + \omega(v)) \right) .$$

The second piece of evidence for Conjecture 2 is that the result holds for claw-free graphs with stability number at most three [4]. However, for the remaining classes of claw-free graphs, which are constructed as a generalization of line graphs [2], the conjecture has remained open. In this paper we prove that Conjecture 2 holds for line graphs. We then show that we can extend this result to quasi-line graphs in the same way that Conjecture 1 was extended from line graphs to quasi-line graphs in [5]. Our main result is:

Theorem 4. *For any quasi-line graph G,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \tfrac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

Given a multigraph G, the *line graph of* G, denoted $L(G)$, is the graph with vertex set $V(L(G)) = E(G)$ in which two vertices of $L(G)$ are adjacent precisely if their corresponding edges in H share an endpoint. We say that a graph G' is a *line graph* if for some multigraph G, $L(G)$ is isomorphic to G'. A graph G is *quasi-line* if every vertex v is *bisimplicial*, i.e. the neighbourhood of v induces the complement of a bipartite graph. A graph G is *claw-free* if it contains no induced $K_{1,3}$. Observe that every line graph is quasi-line and every quasi-line graph is claw-free.

2 Proving the local strengthening for line graphs

In order to prove Conjecture 2 for line graphs, we prove an equivalent statement in the setting of edge colourings of multigraphs. Given distinct adjacent vertices u and v in a multigraph G, we let $\mu_G(uv)$ denote the number of edges between u and v. We let $t_G(uv)$ denote the maximum, over all vertices $w \notin \{u, v\}$, of the number of edges with both endpoints in $\{u, v, w\}$. That is,

$$t_G(uv) := \max_{w \in N(u) \cap N(v)} \left(\mu_G(uv) + \mu_G(uw) + \mu_G(vw)\right).$$

We omit the subscripts when the multigraph in question is clear.

Observe that given an edge e in G with endpoints u and v, the degree of uv in $L(G)$ is $d(u) + d(v) - \mu(uv) - 1$. And since any clique in $L(G)$ containing e comes from the edges incident to u, the edges incident to v, or the edges in a triangle containing u and v, we can see that $\omega(v)$ in $L(G)$ is equal to $\max\{d(u), d(v), t(uv)\}$. Therefore we prove the following theorem, which, aside from the algorithmic claim, is equivalent to proving Conjecture 2 for line graphs:

Theorem 5. *Let G be a multigraph on m edges, and let*

$$\gamma_l'(G) := \max_{uv \in E(G)} \Big\lceil \max\Big\{ d(u) + \tfrac{1}{2}(d(v) - \mu(vu)),\ d(v) + \tfrac{1}{2}(d(u) - \mu(uv)), \tfrac{1}{2}(d(u) + d(v) - \mu_G(uv) + t(uv)) \Big\} \Big\rceil . \quad (1)$$

Then $\chi'(G) \leq \gamma_l'(G)$, and we can find a $\gamma_l'(G)$-edge-colouring of G in $O(m^2)$ time.

The most intuitive approach to achieving this bound on the chromatic index involves assuming that G is a minimum counterexample, then characterizing $\gamma_l'(G)$-edge-colourings of $G - e$ for an edge e. We want an algorithmic result, so we will have to be a bit more careful to ensure that we can modify partial $\gamma_l'(G)$-edge-colourings efficiently until we find one that we can extend to a complete $\gamma_l'(G)$-edge-colouring of G.

We begin by defining, for a vertex v, a *fan hinged at* v. Let e be an edge incident to v, and let $v_1, \ldots, v_\ell$ be a set of distinct neighbours of v with e between v and v_1. Let $c : E \setminus \{e\} \to \{1, \ldots, k\}$ be a proper edge colouring of $G \setminus \{e\}$ for some fixed k. Then $F = (e; c; v; v_1, \ldots, v_\ell)$ is a *fan* if for every j such that $2 \leq j \leq \ell$, there exists some i less than j such that some edge between v and v_j is assigned a colour that does not appear on any edge incident to v_i (i.e. a colour *missing* at v_i). We say that F is *hinged at* v. If there is no $u \notin \{v, v_1, \ldots, v_\ell\}$ such that $F' = (e; c; v; v_1, \ldots, v_\ell, u)$ is a fan, we say that F is a *maximal fan.* The *size* of a fan refers to the number of neighbours of the hinge vertex contained in the fan (in this case, ℓ). These fans generalize Vizing's fans, originally used in the proof of Vizing's theorem [12]. Given a partial k-edge-colouring of G and a vertex w, we say that a colour is *incident to* w if the colour appears on an edge incident to w. We use $\mathcal{C}(w)$ to denote the set of colours incident to w, and we use $\bar{\mathcal{C}}(w)$ to denote $[k] \setminus \mathcal{C}(w)$.

Fans allow us to modify partial k-edge-colourings of a graph (specifically those with exactly one uncoloured edge). We will show that if $k \geq \gamma_l'(G)$, then either every maximal fan has size 2 or we can easily find a k-edge-colouring of G. We first prove that we can construct a k-edge-colouring of G from a partial k-edge-colouring of $G - e$ whenever we have a fan for which certain sets are not disjoint.

Lemma 6. *For some edge e in a multigraph G and positive integer k, let c be a k-edge-colouring of $G - e$. If there is a fan $F = (e; c; v; v_1, \ldots, v_\ell)$ such that for some j, $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$, then we can find a k-edge-colouring of G in $O(k + m)$ time.*

Proof. Let j be the minimum index for which $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j)$ is nonempty. If $j = 1$ then the result is trivial, since we can extend c to a proper k-edge-colouring of G. Otherwise $j \geq 2$ and we can find j in $O(m)$ time. We define e_1 to be e. We then construct a function $f : \{2, \ldots, \ell\} \to \{1, \ldots \ell - 1\}$ such that for each i, (1) $f(i) < i$ and (2) there is an edge e_i between v and v_i such that $c(e_i)$ is missing at $v_{f(i)}$. We can find this function in $O(k + m)$ time by building a list of the earliest v_i at which each colour is missing, and computing f for increasing values of i starting at 2. While doing so we also find the set of edges $\{e_i\}_{i=2}^{\ell}$.

We construct a k-edge-colouring c_j of $G - e_j$ from c by shifting the colour $c(e_j)$ from e_j to $e_{f(j)}$, shifting the colour $c(e_{f(j)})$ from $e_{f(j)}$ to $e_{f(f(j))}$, and so on, until we shift a colour to e. We now have a k-edge-colouring c_j of $G - e_j$ such that some colour is missing at both v and v_j. We can therefore extend c_j to a proper k-edge-colouring of G in $O(k + m)$ time. □

Lemma 7. *For some edge e in a multigraph G and positive integer k, let c be a k-edge-colouring of $G - e$. If there is a fan $F = (e; c; v; v_1, \ldots, v_\ell)$ such that for some i and*

j satisfying $1 \leq i < j \leq \ell$, $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$, then we can find v_i and v_j in $O(k+m)$ time, and we can find a k-edge-colouring of G in $O(k+m)$ time.

Proof. We can easily find i and j in $O(k+m)$ time if they exist. Let α be a colour in $\bar{\mathcal{C}}(v)$ and let β be a colour in $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j)$. Note that by Lemma 6, we can assume $\alpha \in \mathcal{C}(v_i) \cap \mathcal{C}(v_j)$ and $\beta \in \mathcal{C}(v)$.

Let $G_{\alpha,\beta}$ be the subgraph of G containing those edges coloured α or β. Every component of $G_{\alpha,\beta}$ containing v, v_i, or v_j is a path on ≥ 2 vertices. Thus either v_i or v_j is in a component of $G_{\alpha,\beta}$ not containing v. Exchanging the colours α and β on this component leaves us with a k-edge-colouring of $G - e$ in which either $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_i) \neq \emptyset$ or $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$. This allows us to apply Lemma 6 to find a k-edge-colouring of G. We can easily do this work in $O(m)$ time. □

The previous two lemmas suggest that we can extend a colouring more easily when we have a large fan, so we now consider how we can extend a fan that is not maximal. Given a fan $F = (e; c; v; v_1, \ldots, v_\ell)$, we use $d(F)$ to denote $d(v) + \sum_{i=1}^{\ell} d(v_i)$.

Lemma 8. *For some edge e in a multigraph G and integer $k \geq \Delta(G)$, let c be a k-edge-colouring of $G - e$ and let F be a fan. Then we can extend F to a maximal fan $F' = (e; c; v; v_1, v_2, \ldots, v_\ell)$ in $O(k + d(F'))$ time.*

Proof. We proceed by setting $F' = F$ and extending F' until it is maximal. To this end we maintain two colour sets. The first, $\mathcal{C}$, consists of those colours appearing incident to v but not between v and another vertex of F'. The second, $\bar{\mathcal{C}}_{F'}$, consists of those colours that are in $\mathcal{C}$ and are missing at some fan vertex. Clearly F' is maximal if and only if $\bar{\mathcal{C}}_{F'} = \emptyset$. We can perform this initialization in $O(k + d(F))$ time by counting the number of times each colour in $\mathcal{C}$ appears incident to a vertex of the fan.

Now suppose we have $F' = (e; c; v; v_1, v_2, \ldots, v_\ell)$, along with sets $\mathcal{C}$ and $\bar{\mathcal{C}}_{F'}$, which we may assume is not empty. Take an edge incident to v with a colour in $\bar{\mathcal{C}}_F$; call its other endpoint $v_{\ell+1}$. We now update $\mathcal{C}$ by removing all colours appearing between v and $v_{\ell+1}$. We update $\bar{\mathcal{C}}_{F'}$ by removing all colours appearing between v and $v_{\ell+1}$, and adding all colours in $\mathcal{C} \cap \bar{\mathcal{C}}(v_{\ell+1})$. Set $F' = (e; c; v; v_1, v_2, \ldots, v_{\ell+1})$. We can perform this update in $d(v_{\ell+1})$ time; the lemma follows. □

We can now prove that if $k \geq \gamma_l'(G)$ and we have a maximal fan of size 1 or at least 3, we can find a k-edge-colouring of G in $O(k+m)$ time.

Lemma 9. *For some edge e in a multigraph G and positive integer $k \geq \gamma_l'(G)$, let c be a k-edge-colouring of $G - e$ and let $F = (e; c; v; v_1)$ be a fan. If F is a maximal fan we can find a k-edge-colouring of G in $O(k+m)$ time.*

Proof. If $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$ is nonempty, then we can easily extend the colouring of $G - e$ to a k-edge-colouring of G. So assume $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$ is empty. Since $k \geq \gamma_l'(G) \geq 1$, $\bar{\mathcal{C}}(v_1)$ is nonempty. Therefore there is a colour in $\bar{\mathcal{C}}(v_1)$ appearing on an edge incident to v whose other endpoint, call it v_2, is not v_1. Thus $(e; c; v; v_1, v_2)$ is a fan, contradicting the maximality of F. □

Lemma 10. *For some edge e in a multigraph G and positive integer $k \geq \gamma_l'(G)$, let c be a k-edge-colouring of $G - e$ and let $F = (e; c; v; v_1, v_2, \ldots, v_\ell)$ be a maximal fan with $\ell \geq 3$. Then we can find a k-edge-colouring of G in $O(k + m)$ time.*

Proof. Let v_0 denote v for ease of notation. If the sets $\bar{\mathcal{C}}(v_0), \bar{\mathcal{C}}(v_1), \ldots, \bar{\mathcal{C}}(v_\ell)$ are not all pairwise disjoint, then using Lemma 6 or Lemma 7 we can find a k-edge-colouring of G in $O(m)$ time. We can easily determine whether or not these sets are pairwise disjoint in $O(k + m)$ time. Now assume they are all pairwise disjoint; we will exhibit a contradiction, which is enough to prove the lemma.

The number of missing colours at v_i, i.e. $|\bar{\mathcal{C}}(v_i)|$, is $k - d(v_i)$ if $2 \leq i \leq \ell$, and $k - d(v_i) + 1$ if $i \in \{0, 1\}$. Since F is maximal, any edge with one endpoint v_0 and the other endpoint outside $\{v_0, \ldots, v_\ell\}$ must have a colour not appearing in $\cup_{i=0}^{\ell} \bar{\mathcal{C}}(v_i)$. Therefore

$$\left(\sum_{i=0}^{\ell} k - d(v_i) \right) + 2 + \left(d(v_0) - \sum_{i=1}^{\ell} \mu(v_0 v_i) \right) \quad \leq \quad k. \tag{2}$$

Thus

$$\ell k + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \quad \leq \quad \sum_{i=1}^{\ell} d(v_i). \tag{3}$$

But since $k \geq \gamma_l'(G)$, (1) tells us that for all $i \in [\ell]$,

$$d(v_i) + \tfrac{1}{2}(d(v_0) - \mu(v_0 v_i)) \quad \leq \quad k \tag{4}$$

Thus substituting for k tells us

$$\sum_{i=1}^{\ell} \frac{d(v_0) + 2d(v_i) - \mu(v_0 v_i)}{2} + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \quad \leq \quad \sum_{i=1}^{\ell} d(v_i).$$

So

$$\begin{aligned} 2 + \tfrac{1}{2}\ell d(v_0) - \tfrac{3}{2} \sum_{i=1}^{\ell} \mu(v_0 v_i) \quad &\leq \quad 0 \\ 2 + \tfrac{1}{2}\ell d(v_0) \quad &\leq \quad \tfrac{3}{2} \sum_{i=1}^{\ell} \mu(v_0 v_i) \\ \tfrac{\ell}{2} d(v_0) \quad &< \quad \tfrac{3}{2} d(v_0). \end{aligned}$$

This is a contradiction, since $\ell \geq 3$. ☐

We are now ready to prove the main lemma of this section.

Lemma 11. *For some edge e_0 in a multigraph G and positive integer $k \geq \gamma_l'(G)$, let c_0 be a k-edge-colouring of $G-e$. Then we can find a k-edge-colouring of G in $O(k+m)$ time.*

As we will show, this lemma easily implies Theorem 5. We approach this lemma by constructing a sequence of overlapping fans of size two until we can apply a previous lemma. If we cannot do this, then our sequence results in a cycle in G and a set of partial k-edge-colourings of G with a very specific structure that leads us to a contradiction.

Proof. We postpone algorithmic considerations until the end of the proof.

Let v_0 and v_1 be the endpoints of e_0, and let $F_0 = (e_0; c_0; v_1; v_0, u_1, \ldots, u_\ell)$ be a maximal fan. If $|\{u_1, \ldots, u_\ell\}| \neq 1$ then we can apply Lemma 9 or Lemma 10. More generally, if at any time we find a fan of size three or more we can finish by applying Lemma 10. So assume $\{u_1, \ldots, u_\ell\}$ is a single vertex; call it v_2.

Let $\bar{\mathcal{C}}_0$ denote the set of colours missing at v_0 in the partial colouring c_0, and take some colour $\alpha_0 \in \bar{\mathcal{C}}_0$. Note that if α_0 does not appear on an edge between v_1 and v_2 then we can find a fan $(e_0; c_0; v_1; v_0, v_2, u)$ of size 3 and apply Lemma 10 to complete the colouring. So we can assume that α_0 does appear on an edge between v_1 and v_2.

Let e_1 denote the edge between v_1 and v_2 given colour α_0 in c_0. We construct a new colouring c_1 of $G - e_1$ from c_0 by uncolouring e_1 and assigning e_0 colour α_0. Let $\bar{\mathcal{C}}_1$ denote the set of colours missing at v_1 in the colouring c_1. Now let $F_1 = (e_1; c_1; v_2; v_1, v_3)$ be a maximal fan. As with F_0, we can assume that F_1 exists and is indeed maximal. The vertex v_3 may or may not be the same as v_0.

Let $\alpha_1 \in \bar{\mathcal{C}}_1$ be a colour in $\bar{\mathcal{C}}_1$. Just as α_0 appears between v_1 and v_2 in c_0, we can see that α_1 appears between v_2 and v_3. Now let e_2 be the edge between v_2 and v_3 having colour α_1 in c_1. We construct a colouring c_2 of $G - e_2$ from c_1 by uncolouring e_2 and assigning e_1 colour α_1.

We continue to construct a sequence of fans $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$ for $i = 0, 1, 2, \ldots$ in this way, maintaining the property that $\alpha_{i+2} = \alpha_i$. This is possible because when we construct c_{i+1} from c_i, we make α_i available at v_{i+2}, so the set $\bar{\mathcal{C}}_{i+2}$ (the set of colours missing at v_{i+2} in the colouring c_{i+2}) always contains α_i. We continue constructing our sequence of fans until we reach some j for which $v_j \in \{v_i\}_{i=0}^{j-1}$, which will inevitably happen if we never find a fan of size 3 or greater. We claim that $v_j = v_0$ and j is odd. To see this, consider the original edge-colouring of $G - e_0$ and note that for $1 \leq i \leq j-1$, α_0 appears on an edge between v_i and v_{i+1} precisely if i is odd, and α_1 appears on an edge between v_i and v_{i+1} precisely if i is even. Thus since the edges of colour α_0 form a matching, and so do the edges of colour α_1, we indeed have $v_j = v_0$

and j odd. Furthermore $F_0 = F_j$. Let C denote the cycle $v_0, v_1, \dots, v_{j-1}$. In each colouring, α_0 and α_1 both appear $(j-1)/2$ times on C, in a near-perfect matching. Let H be the sub-multigraph of G consisting of those edges between v_i and v_{i+1} for $0 \le j \le j-1$ (with indices modulo j). Let A be the set of colours missing on at least one vertex of C, and let H_A be the sub-multigraph of H consisting of e_0 and those edges receiving a colour in A in c_0 (and therefore in any c_i).

Suppose $j = 3$. If some colour is missing on two vertices of C in c_0, c_1, or c_2, we can easily find a k-edge-colouring of G since any two vertices of C are the endpoints of e_0, e_1, or e_2. We know that every colour in $\bar{\mathcal{C}}_0$ appears between v_1 and v_2, and every colour in $\bar{\mathcal{C}}_1$ appears between v_2 and $v_3 = v_0$. Therefore $|E(H_A)| = |A| + 1$. Our construction tells us that every colour in $\bar{\mathcal{C}}_0$ appears between v_1 and v_2, and every colour in $\bar{\mathcal{C}}_1$ appears between v_2 and $v_3 = v_0$. Therefore

$$\begin{aligned}
2\gamma_l'(G) &\ge d_G(v_0) + d_G(v_1) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&= d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&\ge d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_{H_A}(v_0v_1) - \mu_{H_A}(v_0v_1) \\
&\ge 2|E(H_A)| + 2(k - |A|) \\
&> 2|A| + 2(k - |A|) = 2k
\end{aligned}$$

This is a contradiction since $k \ge \gamma_l'(G)$. We can therefore assume that $j \ge 5$.

Let β be a colour in $A \setminus \{\alpha_0, \alpha_1\}$. If β is missing at two consecutive vertices v_i and v_{i+1} then we can easily extend c_i to a k-edge-colouring of G. Bearing in mind that each F_i is a maximal fan, we claim that if β is not missing at two consecutive vertices then either we can easily k-edge-colour G, or the number of edges coloured β in H_A is at least twice the number of vertices at which β is missing in any c_i.

To prove this claim, first assume without loss of generality that $\beta \in \bar{\mathcal{C}}_0$. Since β is not missing at v_1, β appears on an edge between v_1 and v_2 for the same reason that α_0 does. Likewise, since β is not missing at v_{j-1}, β appears on an edge between v_{j-1} and v_{j-2}. Finally, suppose β appears between v_1 and v_2, and is missing at v_3 in c_0. Then let e_β be the edge between v_1 and v_2 with colour β in c_0. We construct a colouring c_0' from c_0 by giving e_2 colour β and giving e_β colour α_1 (i.e. we swap the colours of e_β and e_2). Thus c_0' is a k-edge-colouring of $G - e_0$ in which β is missing at both v_0 and v_1. We can therefore extend $G - e_0$ to a k-edge-colouring of G. Thus if β is missing at v_3 or v_{j-3} we can easily k-edge-colour G. We therefore have at least two edges of H_A coloured β for every vertex of C at which β is missing, and we do not double-count edges. This proves the claim, and the analogous claim for any colour in A also holds.

Now we have

$$\sum_{i=0}^{j-1} \mu_{H_A}(v_iv_{i+1}) = |E(H_A)| \ > \ 2\sum_{i=0}^{j-1} (k - d_G(v_i))\,. \tag{5}$$

Therefore taking indices modulo j, we have

$$\sum_{i=0}^{j-1} \left(d_G(v_i) + \tfrac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2})\right) \ > \ jk. \tag{6}$$

Therefore there exists some index i for which

$$d_G(v_i) + \tfrac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2}) \quad > \quad k. \tag{7}$$

Therefore

$$k \geq d_G(v_i) + \tfrac{1}{2}\mu_G(v_{i+1}v_{i+2}) \quad > \quad k. \tag{8}$$

This is a contradiction, so we can indeed find a k-edge-colouring of G. It remains to prove that we can do so in $O(k+m)$ time.

Given the colouring c_i, we can construct the fan $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$ and determine whether or not it is maximal in $O(k + d(F_i))$ time. If it is not maximal, we can complete the k-edge-colouring of G in $O(m)$ time; this will happen at most once throughout the entire process. Therefore we will either complete the colouring or construct our cycle of fans $F_0, \ldots, F_{j-1}$ in $O(\sum_{i=0}^{j-1}(k + d(F_i)))$ time. This is not the desired bound, so suppose there is an index i for which $k > d(F_i)$. In this case we certainly have two intersecting sets of available colours in F_i, so we can apply Lemma 6 or 7 when we arrive at F_i, and find the k-edge-colouring of G in $O(k+m)$ time. If no such i exists, then $jk = O(\sum_{i=0}^{j-1}(d(F_i))) = O(m)$, and we indeed complete the construction of all fans in $O(k+m)$ time.

Since each F_i is a maximal fan, in c_0 there must be some colour $\beta \notin \{\alpha_0, \alpha_1\}$ missing at two consecutive vertices v_i and v_{i+1}, otherwise we reach a contradiction. We can find this β and i by going around the cycle of fans and comparing $\bar{\mathcal{C}}_i$ and $\bar{\mathcal{C}}_{i+1}$, and since this is trivial if $|\bar{\mathcal{C}}_i| + |\bar{\mathcal{C}}_{i+1}| > d(v_i) + d(v_{i+1})$ we can find β and i in $O(k+m)$ time, after which it is easy to construct the k-edge-colouring of G from c_i. □

We now complete the proof of Theorem 5.

Proof of Theorem 5. Order the edges of G $e_1, \ldots, e_m$ arbitrarily and let $k = \gamma_l'(G)$, which we can easily compute in $O(nm)$ time. For $i = 0, \ldots, m$, let G_i denote the subgraph of G on edges $\{e_j \mid j \leq i\}$. Since G_0 is empty it is vacuously k-edge-coloured. Given a k-edge-colouring of G_i, we can find a k-edge-colouring of G_{i+1} in $O(k+m)$ time by applying Lemma 11. Since $k = \gamma_l'(G) = O(m)$, the theorem follows. □

This gives us the following result for line graphs, since for any multigraph G we have $|V(L(G))| = |E(G)|$:

Theorem 12. *Given a line graph G on n vertices, we can find a proper colouring of G using $\gamma_l(G)$ colours in $O(n^2)$ time.*

This is faster than the algorithm of King, Reed, and Vetta [6] for $\gamma(G)$-colouring line graphs, which is given an improved complexity bound of $O(n^{5/2})$ in [4], §4.2.3.

3 Extending the result to quasi-line graphs

We now leave the setting of edge colourings of multigraphs and consider vertex colourings of simple graphs. As mentioned in the introduction, we can extend Conjecture 2 from line graphs to quasi-line graphs using the same approach that King and Reed used to extend Conjecture 1 from line graphs to quasi-line graphs in [5]. We do not require the full power of Chudnovsky and Seymour's structure theorem for quasi-line graphs [1]. Instead, we use a simpler decomposition theorem from [2].

3.1 The structure of quasi-line graphs

We wish to describe the structure of quasi-line graphs. If a quasi-line graph does not contain a certain type of homogeneous pair of cliques, then it is either a circular interval graph or built as a generalization of a line graph – where in a line graph we would replace each edge with a vertex, we now replace each edge with a linear interval graph. We now describe this structure more formally.

3.1.1 Linear and circular interval graphs

A *linear interval graph* is a graph $G = (V, E)$ with a *linear interval representation*, which is a point on the real line for each vertex and a set of intervals, such that vertices u and v are adjacent in G precisely if there is an interval containing both corresponding points on the real line. If X and Y are specified cliques in G consisting of the $|X|$ leftmost and $|Y|$ rightmost vertices (with respect to the real line) of G respectively, we say that X and Y are *end-cliques* of G. These cliques may be empty.

Accordingly, a *circular interval graph* is a graph with a *circular interval representation*, i.e. $|V|$ points on the unit circle and a set of intervals (arcs) on the unit circle such that two vertices of G are adjacent precisely if some arc contains both corresponding points. Circular interval graphs are the first of two fundamental types of quasi-line graph. Deng, Hell, and Huang proved that we can identify and find a representation of a circular or linear interval graph in $O(m)$ time [3].

3.1.2 Compositions of linear interval strips

We now describe the second fundamental type of quasi-line graph.

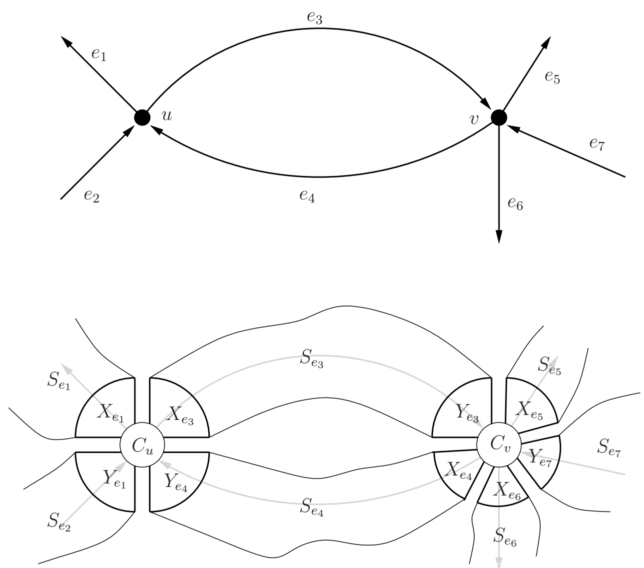

Figure 1: We compose a set of strips $\{(S_e, X_e, Y_e) \mid e \in E(H)\}$ by joining them together on their end-cliques. A hub clique C_u will arise for each vertex $u \in V(H)$.

A *linear interval strip* (S, X, Y) is a linear interval graph S with specified end-cliques X and Y. We compose a set of strips as follows. We begin with an underlying directed multigraph H, possibly with loops, and for every every edge e of H we take a linear interval strip (S_e, X_e, Y_e). For $v \in V(H)$ we define the *hub clique* C_v as

$$C_v = \left(\bigcup\{X_e \mid e \text{ is an edge out of } v\}\right) \cup \left(\bigcup\{Y_e \mid e \text{ is an edge into } v\}\right).$$

We construct G from the disjoint union of $\{S_e \mid e \in E(H)\}$ by making each C_v a clique; G is then a *composition of linear interval strips* (see Figure 1). Let G_h denote the subgraph of G induced on the union of all hub cliques. That is,

$$G_h = G[\cup_{v \in V(H)} C_v] = G[\cup_{e \in E(H)} (X_e \cup Y_e)].$$

Compositions of linear interval strips generalize line graphs: note that if each S_e satisfies $|S_e| = |X_e| = |Y_e| = 1$ then $G = G_h = L(H)$.

3.1.3 Homogeneous pairs of cliques

A pair of disjoint nonempty cliques (A, B) in a graph is a *homogeneous pair of cliques* if $|A| + |B| \geq 3$, every vertex outside $A \cup B$ is adjacent to either all or none of A, and every vertex outside $A \cup B$ is adjacent to either all or none of B. Furthermore (A, B) is *nonlinear* if G contains an induced C_4 in $A \cup B$ (this condition is equivalent to insisting that the subgraph of G induced by $A \cup B$ is a linear interval graph).

3.1.4 The structure theorem

Chudnovsky and Seymour's structure theorem for quasi-line graphs [2] tells us that any quasi-line graph not containing a clique cutset is made from the building blocks we just described.

Theorem 13. *Any quasi-line graph containing no clique cutset and no nonlinear homogeneous pair of cliques is either a circular interval graph or a composition of linear interval strips.*

To prove Theorem 4, we first explain how to deal with circular interval graphs and nonlinear homogeneous pairs of cliques, then move on to considering how to decompose a composition of linear interval strips.

3.2 Circular interval graphs

We can easily prove Conjecture 2 for circular interval graphs by combining previously known results. Niessen and Kind proved that every circular interval graph G satisfies $\chi(G) = \lceil \chi_f(G) \rceil$ [8], so Theorem 3 immediately implies that Conjecture 2 holds for circular interval graphs. Furthermore Shih and Hsu [10] proved that we can optimally colour circular interval graphs in $O(n^{3/2})$ time, which gives us the following result:

Lemma 14. *Given a circular interval graph G on n vertices, we can $\gamma_l(G)$-colour G in $O(n^{3/2})$ time.*

3.3 Nonlinear homogeneous pairs of cliques

There are many lemmas of varying generality that tell us we can easily deal with nonlinear homogeneous pairs of cliques; we use the version used by King and Reed [5] in their proof of Conjecture 1 for quasi-line graphs:

Lemma 15. *Let G be a quasi-line graph on n vertices containing a nonlinear homogeneous pair of cliques (A, B). In $O(n^{5/2})$ time we can find a proper subgraph G' of G such that G' is quasi-line, $\chi(G') = \chi(G)$, and given a k-colouring of G' we can find a k-colouring of G in $O(n^{5/2})$ time.*

It follows immediately that no minimum counterexample to Theorem 4 contains a nonlinear homogeneous pair of cliques.

3.4 Decomposing: Clique cutsets

Decomposing graphs on clique cutsets for the purpose of finding vertex colourings is straightforward and well understood.

For any monotone bound on the chromatic number for a hereditary class of graphs, no minimum counterexample can contain a clique cutset, since we can simply "paste together" two partial colourings on a clique cutset. Tarjan [11] gave an $O(nm)$-time algorithm for constructing a clique cutset decomposition tree of any graph, and noted that given k-colourings of the leaves of this decomposition tree, we can construct a k-colouring of the original graph in $O(n^2)$ time. Therefore if we can $\gamma_l(G)$-colour any quasi-line graph containing no clique cutset in $O(f(n,m))$ time for some function f, we can $\gamma_l(G)$-colour any quasi-line graph in $O(f(n,m)+nm)$ time.

If the multigraph H contains a loop or a vertex of degree 1, then as long as G is not a clique, it will contain a clique cutset.

3.5 Decomposing: Canonical interval 2-joins

A canonical interval 2-join is a composition by which a linear interval graph is attached to another graph. Canonical interval 2-joins arise from compositions of strips, and can be viewed as a local decomposition rather than one that requires knowledge of a graph's global structure as a composition of strips.

Given four cliques X_1, Y_1, X_2, and Y_2, we say that $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ is an *interval 2-join* if it satisfies the following:

- $V(G)$ can be partitioned into nonempty V_1 and V_2 with $X_1 \cup Y_1 \subseteq V_1$ and $X_2 \cup Y_2 \subseteq V_2$ such that for $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge precisely if $\{v_1, v_2\}$ is in $X_1 \cup X_2$ or $Y_1 \cup Y_2$.
- $G|V_2$ is a linear interval graph with end-cliques X_2 and Y_2.

If we also have X_2 and Y_2 disjoint, then we say $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ is a *canonical interval 2-join*. The following decomposition theorem is a straightforward consequence of the structure theorem for quasi-line graphs:

Theorem 16. *Let G be a quasi-line graph containing no nonlinear homogeneous pair of cliques. Then one of the following holds.*

- *G is a line graph*
- *G is a circular interval graph*
- *G contains a clique cutset*
- *G admits a canonical interval 2-join.*

Therefore to prove Theorem 4 it only remains to prove that a minimum counterexample cannot contain a canonical interval 2-join. Before doing so we must give some notation and definitions.

We actually need to bound a refinement of $\gamma_l(G)$. Given a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ in G with an appropriate partitioning V_1 and V_2, let G_1 denote $G|V_1$, let G_2 denote $G|V_2$ and let H_2 denote $G|(V_2 \cup X_1 \cup Y_1)$. For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in H_2 containing v and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^j(H_2)$ as $\max_{v \in H_2} \lceil d_G(v) + 1 + \omega'(v) \rceil$ (here the superscript j denotes *join*). Observe that $\gamma_l^j(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G|(X_2 \cup Y_2))$.

The following lemma is due to King and Reed and first appeared in [4]; we include the proof for the sake of completeness.

Lemma 17. *Let G be a graph on n vertices and suppose G admits a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$. Then given a proper l-colouring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l-colouring of G in $O(nm)$ time.*

Since $\gamma_l^j(H_2) \leq \gamma_l(G)$, this lemma implies that no minimum counterexample to Theorem 4 contains a canonical interval 2-join.

It is easy to see that a minimum counterexample cannot contain a simplicial vertex (i.e. a vertex whose neighbourhood is a clique). Therefore in a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ in a minimum counterexample, all four cliques X_1, Y_1, X_2, and Y_2 must be nonempty.

Proof. We proceed by induction on l, observing that the case $l = 1$ is trivial. We begin by modifying the colouring so that the number k of colours used in both X_1 and Y_1 in the l-colouring of G_1 is maximal. That is, if a vertex $v \in X_1$ gets a colour that is not seen in Y_1, then every colour appearing in Y_1 appears in $N(v)$. This can be done in $O(n^2)$ time. If l exceeds $\gamma_l^j(H_2)$ we can just remove a colour class in G_1 and apply induction on what remains. Thus we can assume that $l = \gamma_l^j(H_2)$ and so if we apply induction we must remove a stable set whose removal lowers both l and $\gamma_l^j(H_2)$.

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the colouring of G_1 to an l-colouring of G in one step. In other cases we remove a colour class in G_1 together with vertices in G_2 such that everything we remove is a stable set, and when we remove it we reduce $\gamma_l^j(v)$ for every $v \in H_2$; after doing this we apply induction on l. Notice that if $X_1 \cap Y_1 \neq \emptyset$ and there are edges between X_2 and Y_2 we may have a large clique in H_2 which contains some but not all of X_1 and some but not all of Y_1; this is not necessarily obvious but we deal with it in every applicable case.

Case 1. $Y_1 \subseteq X_1$.

H_2 is a circular interval graph and X_1 is a clique cutset. We can $\gamma_l(H_2)$-colour H_2 in $O(n^{3/2})$ time using Lemma 14. By permuting the colour classes we can ensure that this colouring agrees with the colouring of G_1. In this case $\gamma_l(H_2) \leq \gamma_l^j(H_2) \leq l$ so we are done. By symmetry, this covers the case in which $X_1 \subseteq Y_1$.

Case 2. $k = 0$ and $l > |X_1| + |Y_1|$.

Here X_1 and Y_1 are disjoint. Take a stable set S greedily from left to right in G_2. By this we mean that we start with $S = \{v_1\}$, the leftmost vertex of X_2, and we move along the vertices of G_2 in linear order, adding a vertex to S whenever doing so will leave S a stable set. So S hits X_2. If it hits Y_2, remove S along with a colour class in G_1 not intersecting $X_1 \cup Y_1$; these vertices together make a stable set. If $v \in G_2$ it is easy to see that $\gamma_l^j(v)$ will drop: every remaining vertex in G_2 either loses two neighbours or is in Y_2, in which case S intersects every maximal clique containing v. If $v \in X_1 \cup Y_1$ then since X_1 and Y_1 are disjoint, $\omega'(v)$ is either $|X_1| + |X_2|$ or $|Y_1| + |Y_2|$; in either case $\omega'(v)$, and therefore $\gamma_l^j(v)$, drops when S and the colour class are removed. Therefore $\gamma_l^j(H_2)$ drops, and we can proceed by induction.

If S does not hit Y_2 we remove S along with a colour class from G_1 that hits Y_1 (and therefore not X_1). Since $S \cap Y_2 = \emptyset$ the vertices together make a stable set. Using the same argument as before we can see that removing these vertices drops both l and $\gamma_l^j(H_2)$, so we can proceed by induction.

Case 3. $k = 0$ and $l = |X_1| + |Y_1|$.

Again, X_1 and Y_1 are disjoint. By maximality of k, every vertex in $X_1 \cup Y_1$ has at least $l - 1$ neighbours in G_1. Since $l = |X_1| + |Y_1|$ we know that $\omega'(X_1) \leq |X_1| + |Y_1| - |X_2|$ and $\omega'(Y_1) \leq |X_1| + |Y_1| - |Y_2|$. Thus $|Y_1| \geq 2|X_2|$ and similarly $|X_1| \geq 2|Y_2|$. Assume without loss of generality that $|Y_2| \leq |X_2|$.

We first attempt to l-colour $H_2 - Y_1$, which we denote by H_3, such that every colour in Y_2 appears in X_1 – this is clearly sufficient to prove the lemma since we can permute the colour classes and paste this colouring onto the colouring of G_1 to get a proper l-colouring of G. If $\omega(H_3) \leq l - |Y_2|$ then this is easy: we can $\omega(H_3)$-colour the vertices of H_3, then use $|Y_2|$ new colours to recolour Y_2 and $|Y_2|$ vertices of X_1. This is possible since Y_2 and X_1 have no edges between them.

Define b as $l - \omega(H_3)$; we can assume that $b < |Y_2|$. We want an $\omega(H_3)$-colouring of H_3 such that at most b colours appear in Y_2 but not X_1. There is some clique $C = \{v_i, \ldots, v_{i+\omega(H_3)-1}\}$ in H_3; this clique does not intersect X_1 because $|X_1 \cup X_2| \leq l - \frac{1}{2}|Y_1| \leq l - |Y_2| < l - b$. Denote by v_j the leftmost neighbour of v_i. Since $\gamma_l^j(v_i) \leq l$, it is clear that v_i has at most $2b$ neighbours outside C, and since $b < |Y_2| \leq \frac{1}{2}|X_1|$ we can be assured that $v_i \notin X_2$. Since $\omega(H_3) > |Y_2|$, $v_i \notin Y_2$.

We now colour H_3 from left to right, modulo $\omega(H_3)$. If at most b colours appear in Y_2 but not X_1 then we are done, otherwise we will "roll back" the colouring, starting at v_i. That is, for every $p \geq i$, we modify the colouring of H_3 by giving v_p the colour after the one that it currently has, modulo $\omega(H_3)$. Since v_i has at most $2b$ neighbours behind it, we can roll back the colouring at least $\omega(H_3) - 2b - 1$ times for a total of $\omega(H_3) - 2b$ proper colourings of H_3.

Since $v_i \notin Y_2$ the colours on Y_2 will appear in order modulo $\omega(H_3)$. Thus there are $\omega(H_3)$ possible sets of colours appearing on Y_2, and in $2b + 1$ of them there are at most b colours appearing in Y_2 but not X_1. It follows that as we roll back the colouring of H_3 we will find an acceptable colouring.

Henceforth we will assume that $|X_1| \geq |Y_1|$.

Case 4. $0 < k < |X_1|$.

Take a stable set S in $G_2 - X_2$ greedily from left to right. If S hits Y_2, we remove S from G, along with a colour class from G_1 intersecting X_1 but not Y_1. Otherwise, we remove S along with a colour class from G_1 intersecting both X_1 and Y_1. In either case it is a simple matter to confirm that $\gamma_l^j(v)$ drops for every $v \in H_2$ as we did in Case 2. We proceed by induction.

Case 5. $k = |Y_1| = |X_1| = 1$.

In this case $|X_1| = k = 1$. If G_2 is not connected then X_1 and Y_1 are both clique cutsets and we can proceed as in Case 1. If G_2 is connected and contains an l-clique, then there is some $v \in V_2$ of degree at least l in the l-clique. Thus $\gamma_l^j(H_2) > l$, contradicting our assumption that $l \geq \gamma_l^j(H_2)$. So $\omega(G_2) < l$. We can $\omega(G_2)$-colour G_2 in linear time using only colours not appearing in $X_1 \cup Y_1$, thus extending the l-colouring of G_1 to a proper l-colouring of G.

Case 6. $k = |Y_1| = |X_1| > 1$.

Suppose that k is not minimal. That is, suppose there is a vertex $v \in X_1 \cup Y_1$ whose closed neighbourhood does not contain all l colours in the colouring of G_1. Then we can change the colour of v and apply Case 4. So assume k is minimal.

Therefore every vertex in X_1 has degree at least $l + |X_2| - 1$. Since $X_1 \cup X_2$ is a clique, $\gamma_l^j(H_2) \geq l \geq \frac{1}{2}(l + |X_2| + |X_1| + |X_2|)$, so $2|X_2| \leq l - k$. Similarly, $2|Y_2| \leq l - k$, so $|X_2| + |Y_2| \leq l - k$. Since there are $l - k$ colours not appearing in $X_1 \cup Y_1$, we can $\omega(G_2)$-colour G_2, then permute the colour classes so that no colour appears in both $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Thus we can extend the l-colouring of G_1 to an l-colouring of G.

These cases cover every possibility, so we need only prove that the colouring can be found in $O(nm)$ time. If k has been maximized and we apply induction, k will stay maximized: every vertex in $X_1 \cup Y_1$ will have every remaining colour in its closed neighbourhood except possibly if we recolour a vertex in Case 6. In this case the overlap in what remains is $k-1$, which is the most possible since we remove a vertex from X_1 or Y_1, each of which has size k. Hence we only need to maximize k once. We can determine which case applies in $O(m)$ time, and it is not hard to confirm that whenever we extend the colouring in one step our work can be done in $O(nm)$ time. When we apply induction, i.e. in Cases 2, 4, and possibly 6, all our work can be done in $O(m)$ time. Since $l < n$ it follows that the entire l-colouring can be completed in $O(nm)$ time. □

3.6 Putting the pieces together

We can now prove an algorithmic version of Theorem 4.

Theorem 18. *Let G be a quasi-line graph on n vertices and m edges. Then we can find a proper colouring of G using $\gamma(G)$ colours in $O(n^3m^2)$ time.*

Proof. We proceed by induction on n. As already explained, we need only consider graphs containing no clique cutsets since $n^3m^2 \geq nm$. We begin by applying Lemma 15 at most m times in order to find a quasi-line subgraph G' of G such that $\chi(G) = \chi(G')$, and given a k-colouring of G', we can find a k-colouring of G in $O(n^2m^2)$ time. We must now colour G'.

If G' is a circular interval graph we can determine this and $\gamma_l(G)$-colour it in $O(n^{3/2})$ time. If G' is a line graph we can determine this in $O(m)$ time using an algorithm of Roussopoulos [9], then $\gamma_l(G)$-colour it in $O(n^2)$ time. Otherwise, G' must admit a canonical interval 2-join. In this case Lemma 6.18 in [4], due to King and Reed, tells us that we can find such a decomposition in $O(n^2m)$ time.

This canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ leaves us to colour the induced subgraph G_1 of G', which has at most $n-1$ vertices and is quasi-line. Given a $\gamma_l(G)$-colouring of G_1 we can $\gamma_l(G)$-colour G' in $O(nm)$ time, then reconstruct the $\gamma_l(G)$-colouring of G in $O(n^2m^2)$ time. The induction step takes $O(n^2m^2)$ time and reduces the number of vertices, so the total running time of the algorithm is $O(n^3m^2)$. □

Remark: With some care and using more sophisticated results on decomposing quasi-line graphs, we believe it should be possible to reduce the running time of the entire algorithm to $O(m^2)$.

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