Pure pairs. II. Excluding all subdivisions of a graph

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544

Alex Scott[2]
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[3] and Sophie Spirkl
Princeton University, Princeton, NJ 08544

October 28, 2017; revised February 3, 2020

[1]Supported by NSF grant DMS-1550991. This material is based upon work supported in part by the U. S. Army Research Laboratory and the U. S. Army Research Office under grant number W911NF-16-1-0404.
[2]Supported by a Leverhulme Trust Research Fellowship
[3]Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.

Abstract

We prove for every graph H there exists $\epsilon > 0$ such that, for every graph G with $|G| \geq 2$, if no induced subgraph of G is a subdivision of H, then either some vertex of G has at least $\epsilon|G|$ neighbours, or there are two disjoint sets $A, B \subseteq V(G)$ with $|A|, |B| \geq \epsilon|G|$ such that no edge joins A and B. It follows that for every graph H, there exists $c > 0$ such that for every graph G, if no induced subgraph of G or its complement is a subdivision of H, then G has a clique or stable set of cardinality at least $|G|^c$. This is related to the Erdős-Hajnal conjecture.

1 Introduction

For a graph G, we write $\omega(G), \alpha(G)$ for the cardinalities of the largest clique and largest stable set in G respectively. The number of vertices of G is denoted by $|G|$, and $\overline{G}$ denotes the complement graph of G. If $v \in V(G)$, $N(v)$ denotes the set of neighbours of v. Subsets A, B of $V(G)$ are *complete* if $A \cap B = \emptyset$ and every vertex of A is adjacent to every vertex of B, and *anticomplete* if $A \cap B = \emptyset$ and no vertex in A has a neighbour in B. A pair (A, B) of subsets of $V(G)$ is *pure* if A is either complete or anticomplete to B. For graphs G, H, we say G is *H-free* if no induced subgraph of G is isomorphic to H. (All graphs in this paper are finite and have no loops or parallel edges.) An *ideal* of graphs is a class of graphs closed under isomorphism and under taking induced subgraphs; and an ideal is *proper* if it is not the class of all graphs.

It is well-known from Ramsey theory [14] that every graph G contains a clique or stable set of size at least $\frac{1}{2}\log|G|$. On the other hand, there are graphs G with no clique or stable set of size more than $2\log|G|$ [11] (in fact, most graphs have this property). The celebrated Erdős-Hajnal conjecture asserts that H-free graphs have *much* larger cliques or stable sets. Let us say that an ideal $\mathcal{I}$ has the *Erdős-Hajnal property* if there is some $\epsilon > 0$ such that every graph $G \in \mathcal{I}$ has a clique or stable set of size at least $|G|^\epsilon$. The Erdős-Hajnal conjecture [12, 13] is the following:

1.1 Conjecture: *For every graph H, the ideal of H-free graphs has the Erdős-Hajnal property.*

A related but stronger property for an ideal is that every graph in the ideal contains a pure pair of linear-sized sets. More formally, let us say that an ideal $\mathcal{I}$ has the *strong Erdős-Hajnal property* if there is some $\epsilon > 0$ such that every graph $G \in \mathcal{I}$ with at least two vertices contains a pure pair of sets that both have size at least $\epsilon|G|$. It is easy to show that if an ideal has the strong Erdős-Hajnal property then it has the Erdős-Hajnal property (see [1, 16]; or section 3 below). But the reverse implication does not hold. In fact, if the ideal of H-free graphs has the strong Erdős-Hajnal property then both H and $\overline{H}$ are forests (to show that H must be a forest, suppose that H contains a cycle of length k, take a random graph $G \in \mathcal{G}(n,p)$, where p is chosen so that $np \to \infty$ and $(np)^k = o(n)$, and delete one vertex from each cycle of length k; for $\overline{H}$, take complements). Thus the ideal of H-free graphs does not have the strong Erdős-Hajnal property for any graph H with more than four vertices. In this paper, we are interested in which ideals do have the strong Erdős-Hajnal property.

An ideal is characterized by the minimal induced subgraphs that it does not contain. If an ideal is defined by a finite number of excluded induced subgraphs, then the random graph construction shows that one of them must be a forest and one of them must be the complement of a forest. An important result of this type is due to Bousquet, Lagoutte and Thomassé [3]. Improving on earlier work of Chudnovsky and Zwols [10] and Chudnovsky and Seymour [9], they showed that for every path P, the ideal of graphs with no induced P or $\overline{P}$ has the strong Erdős-Hajnal property. More recently, Liebenau, Pilipczuk, Seymour and Spirkl [18] (improving on an earlier result of Choromanski, Falik, Patel and Pilipczuk [4]) showed that if T is a subdivision of a caterpillar then the ideal of graphs with no induced T or $\overline{T}$ has the strong Erdős-Hajnal property. They further conjectured that the same statement holds for any forest T. This conjecture is proved by the current authors in [8], completing the classification for ideals defined by a finite number of excluded induced subgraphs.

What about ideals that are not defined by a finite number of excluded induced subgraphs? A breakthrough result in this direction was proved by Bonamy, Bousquet and Thomassé [2], who showed that for every k the ideal of graphs G such that neither G nor $\overline{G}$ contains an induced cycle of length at least k has the strong Erdős-Hajnal property. In other words, we exclude induced subdivisions of the cycle C_k from both G and $\overline{G}$. In this paper, we prove a very substantial extension of this result.

1.2 *For every graph H, the ideal of graphs G such that neither G nor $\overline{G}$ contains an induced subdivision of H has the strong Erdős-Hajnal property.*

If instead we take the ideal of all graphs G that do not contain an induced subdivision of H, then in general such ideals need not have the strong strong Erdős-Hajnal property. For instance, the ideal of all graphs that do not contain an induced subdivision of a cycle of length six does not have the strong Erdős-Hajnal property, because it includes the ideal of complements of all triangle-free graphs.

As an immediate corollary of 1.2 we obtain the following.

1.3 *For every graph H, there exists $c > 0$ such that for every graph G, one of the following holds:*

- *G or its complement contains an induced subdivision of H;*
- *G contains a clique or stable set of size at least $|G|^c$.*

We can say a little more than 1.2. If an ideal satisfies the strong Erdős-Hajnal property then we know that every graph in the ideal has a pair of large sets that are either complete or anticomplete, but we may not be able to choose which (for instance, consider the ideal consisting of all vertex-disjoint unions of cliques). However, a theorem of Rödl [20] (discussed in section 3) allows us to assume that our graph is either quite sparse or quite dense. We can then deduce 1.2 from the following significantly stronger "one-sided" result.

1.4 *For every graph H, there exists $c > 0$ such that every graph G with at least two vertices and at most $c|G|^2$ edges satisfies one of the following:*

- *G contains an induced subdivision of H;*
- *there are two anticomplete subsets of $V(G)$, both of size at least $c|G|$.*

In fact, we will prove an even more general result: we will show it is enough to consider induced subdivisions where the edges in a specified path are not subdivided; and we will prove a version of the result (stated as 2.5) that works when the graph is weighted. We introduce the necessary definitions and state results formally in the next section. We discuss Rödl's theorem and its application in section 3, and then give the proof of 2.5 over the next four sections. An important feature of the proof is to divide the problem into two cases: in one case, we may assume that all small balls have small mass; in the other, we may assume that a significant mass is always concentrated in a small ball. After some initial work, these cases are handled separately in sections 6 and 7. We conclude, in the final section, with some applications, and a discussion of the relationship between the Erdős-Hajnal conjecture and questions about χ-boundedness.

2 Statement of results

Every proper ideal is contained in the ideal of H-free graphs for some H. Thus 1.1 can be reformulated as:

2.1 Conjecture: *For every proper ideal $\mathcal{I}$, there exists $c > 0$ such that every graph $G \in \mathcal{I}$ satisfies $\omega(G)\alpha(G) \geq |G|^c$.*

For $\epsilon > 0$, let us say a graph G is *ϵ-coherent* if

- $|G| \geq 2$;
- $|N(v)| < \epsilon|G|$ for each $v \in V(G)$; and
- $\min(|A|, |B|) < \epsilon|G|$, for every two anticomplete sets $A, B \subseteq V(G)$.

As we explain in section 3, 2.1 is equivalent to the following:

2.2 Conjecture: *For every proper ideal $\mathcal{I}$ there exist $\epsilon > 0$ and $c > 0$ such that every ϵ-coherent graph $G \in \mathcal{I}$ satisfies $\omega(G)\alpha(G) \geq |G|^c$.*

Let us say an ideal is *incoherent* if for some $\epsilon > 0$, no member of $\mathcal{I}$ is ϵ-coherent; and *coherent* if there is no such ϵ.

Let H be a graph and let P be a subgraph of H. Let J be a graph obtained from H by subdividing at least once every edge of H not in $E(P)$, and not subdividing the edges in $E(P)$. We call such a graph J (and graphs isomorphic to it) a *P-filleting* of H. Our main result states:

2.3 *Let H be a graph and let P be a path of H. Then every coherent ideal contains a P-filleting of H.*

Here are some consequences of 2.3.

- By setting $H = K_t$ and $|P| = 1$, it follows that the ideal of graphs with no induced subgraph a subdivision of K_t is incoherent.
- Let P be a path of H, let $k \geq 1$ be an integer, and let H_k be obtained from H by subdividing every edge not in $E(P)$ exactly k times. Since every coherent ideal contains a P-filleting of H_k, it follows that every coherent ideal contains a P-filleting of H where every edge not in P is subdivided at least k times.
- If T is a tree such that some path P of T contains all vertices of degree at least three (such a tree is called a *caterpillar subdivision*), it follows that every coherent ideal contains T (this is the main theorem of [18]). To see this, observe that since T is a caterpillar subdivision, every P-filleting of T contains a copy of T as an induced subgraph.

It might be possible to strengthen 2.3, to the following:

2.4 Conjecture: *Let H be a graph and let P be a forest of H. Then every coherent ideal contains a P-filleting of H.*

If so, this would be best possible, in the sense that the same conclusion does not hold if P contains a cycle of H (because there are coherent ideals in which every graph has girth at least any fixed integer). It would imply the main theorem of [8] in the same way that 2.3 implies the main theorem of [18].

For every subset $X \subseteq V(G)$, let $\mu(X)$ be some real number, satisfying

- $\mu(\emptyset) = 0$ and $\mu(V(G)) = 1$, and $\mu(X) \leq \mu(Y)$ for all X, Y with $X \subseteq Y$; and
- $\mu(X \cup Y) \leq \mu(X) + \mu(Y)$ for all disjoint sets X, Y.

We call such a function μ a *mass* on G, and we call the pair (G, μ) a *massed graph*. For instance, we could take $\mu(X) = |X|/|G|$, or $\mu(X) = \chi(G[X])/\chi(G)$, where χ denotes chromatic number. (It is sometimes convenient to speak of the "mass" of a set X, meaning $\mu(X)$.) For $\epsilon > 0$ let us say a massed graph (G, μ) is *ϵ-coherent* if

- $\mu(\{v\}) < \epsilon$ for every vertex v;
- $\mu(N(v)) < \epsilon$ for every vertex v; and
- $\min(\mu(A), \mu(B)) < \epsilon$ for every two anticomplete sets of vertices A, B.

In [18] it was found that the main theorem of that paper could be extended to massed graphs: that for every caterpillar subdivision T, there exists $\epsilon > 0$ such that for every ϵ-coherent massed graph (G, μ), some induced subgraph of G is isomorphic to T. We will show that 2.3 admits the same extension to masses. We will prove:

2.5 *For every graph H and path P of H, there exists $\epsilon > 0$ such that for every ϵ-coherent massed graph (G, μ), some induced subgraph of G is a P-filleting of H.*

Proof of 2.3, assuming 2.5. Let H be a graph and let P be a path of H. Let $\mathcal{I}$ be a coherent ideal; we must show that $\mathcal{I}$ contains a P-filleting of H. Choose $\epsilon > 0$ satisfying 2.5. By reducing ϵ, we may assume that $\epsilon < 1/2$. Since $\mathcal{I}$ is coherent, some $G \in \mathcal{I}$ is ϵ-coherent. Define $\mu(X) = |X|/|G|$ for every $X \subseteq V(G)$. Then (G, μ) is a massed graph. We claim it is ϵ-coherent. To see this we must check the three conditions in the definition of ϵ-coherence for massed graphs. The second and third follow immediately from the second and third conditions in the definition of ϵ-coherence for graphs, but the first is not so clear. For the first we must show that $\mu(\{v\}) < \epsilon$ for every vertex v; that is, $\epsilon|G| > 1$. To see this, there are two cases. If all vertices in G are pairwise adjacent, then since $|N(v)| < \epsilon|G|$ for each $v \in V(G)$, it follows that $|G| - 1 < \epsilon|G|$, and so $(1 - \epsilon)|G| < 1$, which is impossible since $|G| \geq 2$ and $\epsilon \leq 1/2$. If some two vertices u, v of G are nonadjacent, then $\{u\}, \{v\}$ are anticomplete sets, and so $1 < \epsilon|G|$ since G is ϵ-coherent. This proves that (G, μ) is ϵ-coherent. Consequently, from 2.5, some induced subgraph of G is a P-filleting of H. This proves 2.3. ∎

3 Rödl's theorem

Before we go on, let us prove the equivalence of 2.1 and 2.2. This is routine, and no doubt well-known to those familiar with the field, but we give the proof anyway. Certainly 2.1 implies 2.2, and for the converse we use an invaluable tool due to Rödl [20], the following.

3.1 *For every graph H and all $\epsilon > 0$ there exists $\delta > 0$ such that for every H-free graph G, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, every vertex in X has degree less than $\epsilon|X|$.*

3.1 implies:

3.2 *If $\mathcal{I}$ is an ideal such that $\mathcal{I}$ and the ideal of complements of members of $\mathcal{I}$ are both incoherent, then there exists $c > 0$ such that for all $G \in \mathcal{I}$ with $|G| > 1$, there is a pure pair (A, B) in G with $|A|, |B| \geq c|G|$.*

Proof. Let $\mathcal{I}_2$ be the ideal of complements of members of $\mathcal{I}$, and choose $\epsilon > 0$ such that no member of $\mathcal{I} \cup \mathcal{I}_2$ is ϵ-coherent. Choose a graph H not in $\mathcal{I}$; and choose $\delta > 0$ to satisfy 3.1, with H, ϵ as given. Let $c = \delta\epsilon$. Now let $G \in \mathcal{I}$ with $|G| > 1$. We claim there is a pure pair (A, B) in G with $|A|, |B| \geq c|G|$. If $c|G| \leq 1$, then we may take A, B to be disjoint singleton sets, and this is possible since $|G| > 1$. Thus we may assume that $c|G| > 1$. Since G does not contain H, by the choice of δ there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, say G', every vertex in X has degree less than $\epsilon|X|$. Thus $|X| \geq \delta|G| \geq c|G| > 1$. Since $G' \in \mathcal{I} \cup \mathcal{I}_2$, it follows that G' is not ϵ-coherent; and so there exist $A, B \subseteq X$, anticomplete, with $|A|, |B| \geq \epsilon|X| \geq c|G|$, as claimed. This proves 3.2. ∎

Another consequence of 3.1 is:

3.3 *Let $\mathcal{I}$ be a proper ideal, let $\epsilon, c_0 > 0$, and suppose that for every $G \in \mathcal{I}$, if one of $G, \overline{G}$ is ϵ-coherent then $\omega(G)\alpha(G) \geq |G|^{c_0}$. Then $\mathcal{I}$ satisfies 2.1.*

Proof. Let $\mathcal{I}_2$ be the ideal of complements of members of $\mathcal{I}$. Then every ϵ-coherent graph $G \in \mathcal{I} \cup \mathcal{I}_2$ satisfies $\omega(G)\alpha(G) \geq |G|^{c_0}$. Choose $H \notin \mathcal{I}$; choose δ such that 3.1 holds; and choose c with $0 < c \leq c_0/2$ such that $\delta^{2c} \geq 1/2$ and $(\epsilon\delta)^c \geq 1/2$. We prove by induction on $|G|$ that for every graph $G \in \mathcal{I} \cup \mathcal{I}_2$, we have $\omega(G)\alpha(G) \geq |G|^c$. If $|G| \leq 1$ the claim is trivial, and if $2 \leq |G| \leq \delta^{-2}$ then the claim holds, since

$$\omega(G)\alpha(G) \geq 2 \geq \delta^{-2c} \geq |G|^c.$$

Thus we may assume that $|G| > \delta^{-2}$. By 3.1, since one of $G, \overline{G}$ is H-free, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, every vertex in X has degree at most $\epsilon|X|$. By replacing G by $\overline{G}$ if necessary, we may assume that every vertex in X has degree at most $\epsilon|X|$ in $G[X]$.

Now $(\delta|G|)^{c_0} \geq (\delta|G|)^{2c} \geq |G|^c$, since $c_0 \geq 2c$ and $|G| > \delta^{-2}$. Thus if $G[X]$ is ϵ-coherent, then

$$\omega(G[X])\alpha(G[X]) \geq |X|^{c_0} \geq (\delta|G|)^{c_0} \geq |G|^c$$

as required. If $G[X]$ is not ϵ-coherent, there exist two anticomplete subsets A, B of X such that $|A|, |B| \geq \epsilon|X|$. By the inductive hypothesis, $\omega(G[A])\alpha(G[A]) \geq |A|^c$, and the same for B, and since $\alpha(G) \geq \alpha(G[A]) + \alpha(G[B])$ and $\omega(G) \geq \omega(G[A]), \omega(G[B])$, it follows that

$$\omega(G)\alpha(G) \geq \omega(G[A])\alpha(G[A]) + \omega(G[B])\alpha(G[B]) \geq |A|^c + |B|^c \geq 2(\epsilon|X|)^c \geq 2(\epsilon\delta|G|)^c \geq |G|^c.$$

This proves 3.3. ∎

Let us show that if 2.2 holds (for all proper ideals), then so does 2.1. Let $\mathcal{I}$ be a proper ideal, and let $\mathcal{I}_2$ be the ideal of complements of members of $\mathcal{I}$. By applying 2.2 to $\mathcal{I}$ and to $\mathcal{I}_2$, there exist ϵ, c_0 such that every ϵ-coherent graph $G \in \mathcal{I}$ satisfies $\omega(G)\alpha(G) \geq |G|^{c_0}$, and the same for $\mathcal{I}_2$. But then the result follows from 3.3.

Another useful consequence of 3.3 is the following:

3.4 *If $\mathcal{I}$ is an ideal such that $\mathcal{I}$ and the ideal of complements of members of $\mathcal{I}$ are both incoherent, then $\mathcal{I}$ satisfies 2.1.*

Proof. Let $\mathcal{I}_2$ be the ideal of complements of members of $\mathcal{I}$, and choose $\epsilon > 0$ such that no member of $\mathcal{I} \cup \mathcal{I}_2$ is ϵ-coherent. Then the result follows from 3.3. ∎

Next let us deduce the claims of section 1. It is possible to prove 1.4 ⇒ 1.2 ⇒ 1.3, as we said in the introduction, but it is more convenient to derive them all directly from results proved in this section.

Proof of 1.2, assuming 2.3. Let H be a graph, and let $\mathcal{I}$ be the ideal of all graphs G such that neither G nor $\overline{G}$ contains an induced subdivision of H. Thus $\mathcal{I} = \mathcal{I}_2$, where $\mathcal{I}_2$ is the ideal of complements of members of $\mathcal{I}$. By 2.3, $\mathcal{I}$ is incoherent. By 3.2, there exists $c > 0$ such that for all $G \in \mathcal{I}$ with $|G| > 1$, there is a pure pair (A, B) in G with $|A|, |B| \geq c|G|$. Hence $\mathcal{I}$ has the strong Erdős-Hajnal property. This proves 1.2. ∎

Proof of 1.3, assuming 2.3. Let H be a graph, and let $\mathcal{I}$ be the ideal of all graphs G such that neither G nor $\overline{G}$ contains an induced subdivision of H. As before, $\mathcal{I} = \mathcal{I}_2$, where $\mathcal{I}_2$ is the ideal of complements of members of $\mathcal{I}$, and $\mathcal{I}$ is incoherent. By 3.4, $\mathcal{I}$ satisfies 2.1. ∎

Proof of 1.4, assuming 2.3. Let H be a graph, and let $\mathcal{I}$ be the ideal of all graphs that contain no induced subdivision of H. Let P be a one-vertex path of H. By 2.3, every coherent ideal contains a P-filleting of H, and so $\mathcal{I}$ is incoherent. Choose $\epsilon > 0$ such that no member of $\mathcal{I}$ is ϵ-coherent, and let $c = \epsilon/9$. We claim that c satisfies 1.4. Let G be a graph with $|G| > 1$ and $|E(G)| \leq c|G|^2$ that contains no induced subdivision of H. Let Y be the set of vertices of G with degree at least $\epsilon|G|/2$. Then $|Y|\epsilon|G|/2 \leq 2|E(G)| \leq 2c|G|^2$, and so $|Y| < |G|/2$. Let $X = V(G) \setminus Y$; so $|X| > |G|/2$, and so $|X| \geq 2$ since $|G| \geq 2$. Every vertex of $G[X]$ has degree in $G[X]$ less than $\epsilon|G|/2 \leq \epsilon|X|$. Since $G[X] \in \mathcal{I}$, $G[X]$ is not ϵ-coherent, and so there exist anticomplete subsets A, B of X with $|A|, |B| \geq \epsilon|X| \geq c|G|$, as required. This proves 1.4. ∎

4 Some preliminaries

In order to prove 2.5, we might as well assume that P is a Hamilton path of H. To see this, let P have vertices $v_1, \ldots, v_k$ in order and let the remaining vertices of H be $v_{k+1}, \ldots, v_n$. Add new vertices $u_{k+1}, \ldots, u_n$ to H, where each u_i is adjacent to v_{i-1} and v_i; let the new graph be H' and let P' be the path with vertices

$$v_1, \ldots, v_k, u_{k+1}, v_{k+1}, u_{k+2}, \ldots, u_n, v_n.$$

Then P' is a Hamilton path of H', and if the theorem holds for (H', P') then it holds for (H, P). We will therefore assume that P is a Hamilton path of H. In order to find a P-filleting of H as an induced subgraph of G, we need to find an induced path Q say of G, with the same number of vertices as P, such that certain pairs of vertices of Q are joined by induced paths in G, pairwise disjoint and disjoint from Q (except for their ends), such that their union with Q is induced in G. (In particular, there must be no edges of G between their interiors.)

A *pairing* Π in a graph G is a set of pairwise disjoint subsets of $V(G)$, each of cardinality one or two; and let $V(\Pi)$ be the union of the members of Π. If $X \subseteq V(G)$, a *pairing of* X means a pairing Π with $V(\Pi) = X$. A pairing Π of X is *feasible* in G if for each $e \in \Pi$ with $|e| = 2$ there is an induced path P_e of G joining the two members of e, and for each $e \in \Pi$ with $|e| = 1$, P_e is the one-vertex path with vertex set e, such that for all distinct $e, f \in \Pi$, the sets $V(P_e), V(P_f)$ are anticomplete.

An induced subgraph of G isomorphic to T is called a *copy* of T in G. Let us say a *caterpillar* is a tree in which some path contains all vertices with degree more than one. Its *leaves* are its vertices of degree one. A *leaf-pairing* of T means a pairing of the set of leaves of T. A *caterpillar in* G means an induced subgraph of G that is a caterpillar. Let T be a caterpillar in G, and let Π be a leaf-pairing of T. Let X be the set of all vertices in $V(G) \setminus V(T)$ with no neighbours in $V(T) \setminus V(\Pi)$. The pairing Π is *feasible* in G *relative to* T if Π is feasible in $G[X \cup V(\Pi)]$. Thus, another way to pose the problem of 2.5 is to say that we are given a caterpillar T with a leaf-pairing, and we are searching for a copy T' of T in G such that the corresponding leaf-pairing of T' is feasible in G relative to T'.

Let T be a caterpillar in G. We say T is *versatile* in G if *every* leaf-pairing of T is feasible in G relative to T. In order to prove 2.5 it therefore suffices to prove the following strengthening.

4.1 *For every caterpillar T, there exists $\epsilon > 0$ such that for every ϵ-coherent massed graph (G, μ), there is a versatile copy of T in G.*

If u is a vertex of a graph H, we denote by $N^r(u)$ the set of all vertices v of G such that the distance between u, v is exactly r, and $N^r[u]$ the set of v such that this distance is at most r. For $r \geq 1$ an integer, and $\epsilon > 0$, let us say a massed graph (G, μ) is (ϵ, r)-*coherent* if

- $\mu(N^r[v]) < \epsilon$ for every vertex v; and
- $\min(\mu(A), \mu(B)) < \epsilon$ for every two anticomplete sets of vertices A, B.

(Thus, $(\epsilon, 1)$-coherent $\Rightarrow$ ϵ-coherent $\Rightarrow$ $(2\epsilon, 1)$-coherent.)

The proof of 4.1 breaks into two parts. Given a caterpillar T, we will first prove the statement of 4.1 for massed graphs (G, μ) that are (ϵ, r)-coherent (for some appropriate value of r depending on T but not on G); and then we will use this to prove 4.1 in general. The first part is more difficult, and carried out in section 6.

5 Finding a caterpillar

We need a result which is a modification of the main theorem (2.6) of [18]. The main idea of its proof is exactly that of [18], but we need several minor changes, and it seemed best to prove the whole thing again. First we need a lemma (also proved in [18]). We say $X \subseteq V(G)$ is *connected* if $G[X]$ is connected.

5.1 *Let (G, μ) be an ϵ-coherent massed graph, and let $Y \subseteq V(G)$ with $\mu(Y) \geq 3\epsilon$. Then there is a connected subset $X \subseteq Y$ with $\mu(X) > \mu(Y) - \epsilon$.*

Proof. Let the vertex sets of the components of $G[Y]$ be $X_1, \ldots, X_k$ say. Choose $i \geq 1$ minimal such that $\mu(X_1 \cup \cdots \cup X_i) \geq \epsilon$. Since the sets $X_1 \cup \cdots \cup X_i$ and $X_{i+1} \cup \cdots \cup X_n$ are anticomplete, it follows that $\mu(X_{i+1} \cup \cdots \cup X_n) < \epsilon$; and from the minimality of i, $\mu(X_1 \cup \cdots \cup X_{i-1}) < \epsilon$. But

$$\mu(X_1 \cup \cdots \cup X_{i-1}) + \mu(X_i) + \mu(X_{i+1} \cup \cdots \cup X_n) \geq \mu(Y) \geq 3\epsilon,$$

and so $\mu(X_i) \geq \epsilon$. Since the sets X_i and $Y \setminus X_i$ are anticomplete, it follows that $\mu(Y \setminus X_i) < \epsilon$, and so $\mu(X_i) > \mu(Y) - \epsilon$. This proves 5.1. ∎

A *rooted caterpillar* is a tree T with a distinguished vertex h, called its *head*, such that some path of T with one end h contains all the vertices with degree more than one. A rooted caterpillar T with more than one vertex has a unique *predecessor* T' (up to isomorphism), defined as follows. Let h be the head of T.

- If h is adjacent to some leaf u of T, let T' be the rooted caterpillar obtained from T by deleting u, with the same head h.
- If h has no neighbours that are leaves, then h is a leaf; let its neighbour be u, and let T' be the rooted caterpillar obtained by deleting v, with head u.

Thus, every rooted caterpillar can be grown in canonical one-vertex steps from a one-vertex rooted caterpillar. If T is a rooted caterpillar with n vertices, say, let $T_1, \ldots, T_n$ be the rooted caterpillars such that $T_n = T$, and $|T_1| = 1$, and T_{i-1} is the predecessor of T_i for $2 \leq i \leq n$. We call $T_1, \ldots, T_n$ the *ancestors* of T.

Let $\mathcal{Y}$ be a set of pairwise disjoint subsets of $V(G)$, where G is a graph. Let N be a graph, and for each $v \in V(N)$ let $X_v \subseteq V(G)$. We say that the family X_v $(v \in V(N))$ is *$\mathcal{Y}$-spread* if for each $v \in V(N)$ there exists $Y_v \in \mathcal{Y}$ such that the sets Y_v $(v \in V(N))$ are all different, and $X_v \subseteq Y_v$ for each $v \in V(N)$.

If $A, B \subseteq V(G)$ are disjoint, we say A *covers* B if every vertex in B has a neighbour in A. Let (G, μ) be a massed graph. We say $X \subseteq V(G)$ is *δ-dominant* if $\mu(X \cup \bigcup_{x \in X} N(x)) \geq \delta$.

The distance in G between u, v is called the *G-distance between* u, v. For $X \subseteq V(G)$ and $v \in X$, let us say v is an *r-centre* of X if every vertex in X has $G[X]$-distance at most r from v (and consequently X is connected). Let us say a massed graph (G, μ) is *(δ, r)-focussed* if for every $Z \subseteq V(G)$ with $\mu(Z) \geq \delta$, there is a vertex $v \in Z$ with $\mu(N^r_{G[Z]}[v]) \geq \mu(Z)/2$.

Let N be the union of one or more rooted caterpillars with pairwise anticomplete vertex sets. (Thus each component of N has a head.) Let H be the set of heads of the components of N. Let (G, μ) be a massed graph. A *δ-realization* of N in G is an assignment of a subset $X_v \subseteq V(G)$ to each vertex $v \in V(N)$, satisfying the following conditions:

- the sets X_v $(v \in V(N))$ are pairwise disjoint;
- for every edge uv of N, if v lies on the path of N between u and the head of the component of N containing u, then X_u covers X_v;

- for all distinct $u, v \in V(N)$, if u, v are nonadjacent in N and not both in H then X_u, X_v are anticomplete; and
- for each $v \in H$, $\mu(X_v) \geq \delta$, and for each $v \in V(N) \setminus H$, X_v is connected and δ-dominant.

5.2 *Let T be a rooted caterpillar, let $\delta, \epsilon > 0$, let (G, μ) be an ϵ-coherent massed graph, and let $\mathcal{Y}$ be a set of disjoint subsets of $V(G)$ such that $|\mathcal{Y}| = 2^{|T|}$ and $\mu(Y) \geq 2^{2^{|T|}}(\delta + \epsilon)$ for each $Y \in \mathcal{Y}$. Then there is a $\mathcal{Y}$-spread δ-realization of T in G.*

If in addition $r \geq 0$ is an integer, $\epsilon \leq \delta/2$, and (G, μ) is (δ, r)-focussed, then there is a $\mathcal{Y}$-spread δ-realization $(X_v : v \in V(T))$ of T in G such that X_v has an r-centre for each $v \in V(T)$ except the head.

Proof. There are two cases: in one (let us call this the "focussed case") we have the additional hypotheses that $r \geq 0$ is an integer, $\epsilon \leq \delta/2$, and (G, μ) is (δ, r)-focussed; and in the other case (the "unfocussed case") we do not assume this. Let $p = 2^{|T|}$ and for $0 \leq i \leq p$ let $m_i = 2^i(\delta + \epsilon) - \epsilon$. Thus $m_0 = \delta$, and $m_{i+1} = 2m_i + \epsilon$ for $0 \leq i < p$.

If N is a disjoint union of rooted caterpillars, each isomorphic to an ancestor of T, we call N a *nursery*, and we define $\phi(N) = \sum_C 2^{|C|}$, where the sum is taken over all components C of N. Let N_p be the nursery with p components, each an isolated vertex. Thus $\phi(N_p) = 2p$, and since $2^p(\delta + \epsilon) \geq m_p$, the members of $\mathcal{Y}$ form a $\mathcal{Y}$-spread m_p-realization of N_p in G. Choose $k \leq p$ minimum such that there is a nursery N_k with k components and with $\phi(N_k) \geq 2p$, and there is a $\mathcal{Y}$-spread m_k-realization of N_k in G. Since $\phi(N_k) \geq 2p$ and each component of N_k is an ancestor of T, it follows that N_k has at least two components, and so $k \geq 2$. Suppose (for a contradiction) that each component of N_k is isomorphic to an ancestor of T different from T.

Let the components of N_k be $H_1, \ldots, H_k$, where $|H_1| \leq \cdots \leq |H_k|$, and for $1 \leq i \leq k$ let h_i be the head of H_i. Now for $1 \leq i \leq k$ there is an ancestor S_i of T such that H_i is the predecessor of S_i, since H_i is not isomorphic to T. We recall that S_i is obtained from H_i by adding a new leaf adjacent to h_i, and either keeping the same head, or making the new vertex the new head. Let I be the set of all $i \in \{1, \ldots, k\}$ such that H_i, S_i have different heads. If $I \neq \emptyset$, choose $i \in I$, maximum, and otherwise let $i = 1$.

Let $(X_v : v \in V(N_k))$ be a $\mathcal{Y}$-spread m_k-realization of N_k in G. We will choose $Z \subseteq X_{h_i}$ with $\mu(Z) \geq \epsilon$, and an ordering $\{z_1, \ldots, z_n\}$ of the elements of Z, but we treat the foccussed and unfocussed cases differently. Suppose first we are in the unfocussed case. Since $k \geq 2$ and hence $m_k \geq 3\epsilon$, 5.1 implies that there exists $Z \subseteq X_{h_i}$ with $\mu(Z) > \mu(X_{h_i}) - \epsilon \geq m_k - \epsilon$ such that Z is connected. Number the vertices of Z as $z_1, \ldots, z_n$ say, such that $\{z_1, \ldots, z_q\}$ is connected for $1 \leq q \leq n$.

In the focussed case, since (G, μ) is (δ, r)-focussed and $\mu(X_{h_i}) \geq m_k \geq \delta$, there exists $Z \subseteq X_{h_i}$ with $\mu(Z) > \mu(X_{h_i})/2$ and with an r-centre. Thus $\mu(Z) \geq \epsilon$ since $\mu(Z) \geq \mu(X_{h_i})/2 \geq m_k/2 \geq \delta/2 \geq \epsilon$, by hypothesis. Let z_1 be an r-centre of Z, and choose the ordering $z_1, \ldots, z_n$ of the vertices of Z in increasing order of $G[Z]$-distance from z_1.

Since $k \geq 2$, there exists $j \neq i$ with $1 \leq j \leq k$; and since $\mu(Z) \geq \epsilon$, the set of vertices in X_{h_j} with a neighbour in Z has mass more than $\mu(X_{h_j}) - \epsilon \geq m_{k-1}$. Consequently we may choose q with $0 \leq q \leq n$, minimum such that for some $j \in \{1, \ldots, k\} \setminus \{i\}$, the set of vertices in X_{h_j} with a neighbour in $\{z_1, \ldots, z_q\}$ has mass at least m_{k-1}. In particular, $\{z_1, \ldots, z_q\}$ is m_{k-1}-dominant, and $q \geq 1$.

- If $j < i$, it follows that $i \in I$. Let N_{k-1} be the graph obtained from N_k by adding the edge h_ih_j, and deleting all vertices in $V(H_j) \setminus \{h_j\}$. Let H'_i be the component of N_{k-1} that contains the edge h_ih_j, and let us assign its head to be h_j. Consequently H'_i is isomorphic to S_i, and so N_{k-1} is a nursery with $k-1$ components. Moreover, $|H_i| \geq |H_j|$ (because $i > j$), and so $\phi(N_{k-1}) \geq \phi(N_k)$.
- If $j > i$, it follows that $j \notin I$. Let N_{k-1} be the graph obtained from N_k by adding the edge h_ih_j, and deleting all vertices in $V(H_i) \setminus \{h_i\}$. Let H'_j be the component of N_{k-1} that contains the edge h_ih_j, and let us assign its head to be h_j. Thus H'_j is isomorphic to S_j, and again N_{k-1} is a nursery with $k-1$ components and $\phi(N_{k-1}) \geq \phi(N_k)$.

For each $v \in V(N_{k-1})$ define X'_v as follows:

- if $v \neq \{h_1, \dots, h_k\}$ let $X'_v = X_v$;
- let $X'_{h_i} = \{z_1, \dots, z_q\}$;
- let X'_{h_j} be the set of vertices in X_{h_j} with a neighbour in $\{z_1, \dots, z_q\}$;
- for $1 \leq \ell \leq k$ with $\ell \neq i, j$, let X'_{h_ℓ} be the set of vertices in X_{h_ℓ} with no neighbour in $\{z_1, \dots, z_q\}$.

We see that X'_{h_i} covers X'_{h_j}, and has no edges to X'_{h_ℓ} for $1 \leq \ell \leq k$ with $\ell \neq i, j$; and X'_{h_i} is connected and m_{k-1}-dominant; and in the focussed case, X'_{h_i} has an r-centre. Moreover, $\mu(X'_{h_j}) \geq m_{k-1}$. Let $1 \leq \ell \leq k$ with $\ell \neq i, j$; then, since $q \geq 1$ and from the choice of q, the mass of the set of vertices in X_{h_ℓ} with a neighbour in $\{z_1, \dots, z_{q-1}\}$ is less than m_{k-1}, and hence $\mu(X_{h_\ell} \setminus X'_{h_\ell}) < m_{k-1} + \epsilon$. Since $\mu(X_{h_\ell}) \geq m_k$ and $m_k = 2m_{k-1} + \epsilon$, it follows that $\mu(X'_{h_\ell}) \geq m_{k-1}$. Thus $(X'_v : v \in V(N_{k-1}))$ is a $\mathcal{Y}$-spread m_{k-1}-realization of N_{k-1} in G, contrary to the minimality of k since $\phi(N_{k-1}) \geq \phi(N_k) \geq 2p$.

Consequently some component of N_k is isomorphic to T; but then the theorem holds (since $m_k \geq \delta$). This proves 5.2. ∎

6 If no small ball has large mass

In this section we prove 4.1 assuming that no ball with bounded radius has large mass. We need first:

6.1 *Let $k \geq 0$ be an integer, and let $\epsilon, \kappa > 0$ such that $\kappa + 4\epsilon \leq 1$ and $(k-1)\kappa \leq 1$. Let (G, μ) be an ϵ-coherent massed graph. Then there are $2k+1$ subsets $A_1, \dots, A_k, B_1, \dots, B_k, C$ of $V(G)$, pairwise disjoint, with the following properties:*

- *for $1 \leq i \leq k$, A_i is connected and covers B_i;*
- *for $1 \leq i \leq k$, A_i, C are anticomplete;*
- *for all distinct $i, j \in \{1, \dots, k\}$, A_i is anticomplete to $A_j \cup B_j$;*
- *$\mu(C) \geq 1 - 3k\epsilon$; and*
- *for $1 \leq i \leq k$, the set of vertices in C covered by B_i has mass at least $\kappa - 3k\epsilon$.*

Proof. We proceed by induction on k; the result is trivial for $k = 0$, taking $C = V(G)$, so we assume $k \geq 1$. Consequently we may assume that there are $2k-1$ subsets $A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1}, C'$ of $V(G)$, pairwise disjoint, with the following properties:

- for $1 \leq i \leq k-1$, A_i is connected and covers B_i;
- for $1 \leq i \leq k-1$, A_i, C' are anticomplete;
- for all distinct $i, j \in \{1, \ldots, k-1\}$, A_i is anticomplete to $A_j \cup B_j$;
- $\mu(C') \geq 1 - 3(k-1)\epsilon$; and
- for $1 \leq i \leq k-1$, the set of vertices in C' covered by B_i has mass at least $\kappa - 3(k-1)\epsilon$.

Choose these subsets such that, in addition, $|B_1| + \cdots + |B_{k-1}|$ is minimum. For $1 \leq i \leq k-1$, let C_i be the set of vertices in C' covered by B_i. Thus $\mu(C_i) \geq \kappa - 3(k-1)\epsilon$, and from the minimality of $|B_1| + \cdots + |B_{k-1}|$, it follows that $\mu(C_i) \leq \kappa - (3k-4)\epsilon$. Let $D = C' \setminus (C_1 \cup \cdots \cup C_{k-1})$. Thus

$$\mu(D) \geq 1 - 3(k-1)\epsilon - (k-1)(\kappa - (3k-4)\epsilon) = (1 - (k-1)\kappa) + (k-1)(3k-7)\epsilon.$$

We claim that $\mu(D) \geq 3\epsilon$. If $k = 1$, the above implies that $\mu(D) = 1$, and if $k = 2$, the above implies that $\mu(D) \geq 1 - \kappa - \epsilon$; and so in either case $\mu(D) \geq 3\epsilon$, since $\kappa + 4\epsilon \leq 1$. If $k \geq 3$, then $(k-1)(3k-7) \geq 3$ (indeed, ≥ 4), and so the same displayed inequality implies that $\mu(D) \geq 3\epsilon$ since $1 - (k-1)\kappa \geq 0$. This proves the claim that $\mu(D) \geq 3\epsilon$. Note that D is anticomplete to $A_i \cup B_i$ for $1 \leq i < k$.

For $X \subseteq D$, let $B(X)$ denote the set of vertices in $C' \setminus X$ with a neighbour in X. By 5.1, there exists a connected subset $X \subseteq D$ with $\mu(X) \geq \mu(D) - \epsilon$; and hence there is a connected subset $X \subseteq D$ with $\mu(X \cup B(X)) \geq \epsilon$. Choose such a set X minimal, and let $A_k = X$ and $B_k = B(X)$. Then A_k is anticomplete to $A_i \cup B_i$ for $1 \leq i < k$, since $A_k = X \subseteq D$; and B_k is anticomplete to A_i for $1 \leq i < k$, since $B_k = B(X) \subseteq C'$.

Choose $x \in A_k$ such that $A_k \setminus \{x\}$ is connected (or empty). Since $\mu(x) < \epsilon$ and $\mu(N(x)) < \epsilon$, the minimality of X implies that $\mu(A_k \cup B_k) \leq 3\epsilon$. Let $C = C' \setminus (A_k \cup B_k)$. Since $\mu(C') \geq 1 - 3(k-1)\epsilon$, it follows that $\mu(C) \geq 1 - 3k\epsilon$, and since the set of vertices in C with no neighbour in $A_k \cup B_k$ has mass less than ϵ, it follows that the set C_k say of vertices in C with a neighbour in $A_k \cup B_k$ satisfies $\mu(C_k) \geq 1 - (3k+1)\epsilon \geq \kappa - 3k\epsilon$. Also C_k is anticomplete to A_k, since $B_k = B(X)$, and so B_k covers C_k. For $1 \leq i < k$, $C_i \subseteq B_k \cup C$, and $\mu(C_i \cap B_k) \leq 3\epsilon$, and so

$$\mu(C_i \cap C) \geq \mu(C_i) - 3\epsilon \geq \kappa - 3(k-1)\epsilon - 3\epsilon = \kappa - 3k\epsilon.$$

Since B_i covers $C_i \cap C$, this proves 6.1. ∎

This is used to prove the following. Let us say a *k-ladder* in a graph G is a family of $3k$ subsets

$$A_1, \ldots, A_k, B_1, \ldots, B_k, C_1, \ldots, C_k$$

of $V(G)$, pairwise disjoint and such that

- for $1 \leq i \leq k$, A_i is connected and covers B_i, and B_i covers C_i;

- for $1 \le i \le k$, A_i, C_i are anticomplete; and
- for all distinct $i, j \in \{1, \dots, k\}$, A_i is anticomplete to $A_j \cup B_j \cup C_j$.

If in addition we have

- for $1 \le i < j \le k$, B_i is anticomplete to C_j

we say the ladder is *half-cleaned*. Let us say the *union* of the k-ladder is the triple (A, B, C) where $A = \bigcup_{1 \le i \le k} A_i$, $B = \bigcup_{1 \le i \le k} B_i$, and $C = \bigcup_{1 \le i \le k} C_i$.

6.2 *Let $\epsilon, \kappa > 0$, and let $k \ge 0$ be an integer such that $(k-1)k(\kappa+\epsilon) \le 1$ and $(k-1)(\kappa+\epsilon)+4\epsilon \le 1$. Let (G, μ) be an ϵ-coherent massed graph. Then there is a half-cleaned k-ladder*

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

in G such that $\mu(C_i) \ge \kappa$ for $1 \le i \le k$.

Proof. Let $\kappa' = k(\kappa + \epsilon)$. Since $\kappa' + 4\epsilon \le 1$ and $(k-1)\kappa' \le 1$, it follows from 6.1 that there are $2k+1$ subsets $A_1, \dots, A_k, B'_1, \dots, B'_k, C$ of $V(G)$, all disjoint, with the following properties:

- for $1 \le i \le k$, A_i is connected and covers B'_i;
- for $1 \le i \le k$, A_i, C are anticomplete;
- for all distinct $i, j \in \{1, \dots, k\}$, A_i is anticomplete to $A_j \cup B'_j$; and
- for $1 \le i \le k$, the set of vertices in C covered by B'_i has mass at least κ'.

Inductively, suppose that $0 \le j < k$ and we have defined $B_1, \dots, B_j$ with $B_i \subseteq B'_i$ for $1 \le i \le j$, and we have defined disjoint subsets $C_1, \dots, C_j$ of C such that for $1 \le i \le j$, B_i covers C_i and is anticomplete to $C \setminus (C_1 \cup \dots \cup C_i)$, with $\kappa \le \mu(C_i) \le \kappa + \epsilon$. Thus $\mu(C_1 \cup \dots \cup C_j) \le (k-1)(\kappa + \epsilon)$, and since the set of vertices in C covered by B'_{j+1} has mass at least $\kappa' = k(\kappa + \epsilon)$, we may choose $B_{j+1} \subseteq B'_{j+1}$ minimal such that the set, C_{j+1} say, of vertices in $C \setminus (C_1 \cup \dots \cup C_j)$ covered by B_{j+1} has mass at least κ. From the minimality of B_{j+1}, it follows that $\mu(C_{j+1}) \le \kappa + \epsilon$. This completes the inductive definition and so proves 6.2. ∎

6.3 *Let $\epsilon > 0$, and let $k \ge 0$ be an integer. Let (G, μ) be an $(\epsilon, 2)$-coherent massed graph, and let*

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

be a k-ladder in G such that $\mu(C_i) \ge 3k\epsilon$ for $1 \le i \le k$. Let (A, B, C) be its union. Suppose that $b_i \in B_i$ for $1 \le i \le k$, such that $b_1, \dots, b_k$ are pairwise nonadjacent. Then every pairing of $\{b_1, \dots, b_k\}$ is feasible in $G[A \cup B \cup C]$.

Proof. For each $v \in B$, let $i(v) \in \{1, \dots, k\}$ such that $v \in B_{i(v)}$, and for $1 \le i \le k$ let $D_i = A_i \cup B_i \cup C_i$. Let Π be a pairing of $\{b_1, \dots, b_k\}$, and let $\{s_1, t_1\}, \dots, \{s_n, t_n\}$ be the members of Π with cardinality two. For $1 \le m \le n$ we will construct inductively a path P_m between s_m, t_m with the following properties:

- $V(P_m) \subseteq D_{i(s_m)} \cup D_{i(t_m)}$;
- for $1 \le \ell < m$, the sets $V(P_\ell), V(P_m)$ are anticomplete;
- $V(P_m)$ is anticomplete to $\{b_1, \dots, b_k\} \setminus \{s_m, t_m\}$; and
- at most two vertices of $V(P_m)$ belong to C, and at most two to $B \setminus \{s_m, t_m\}$.

Let $1 \le m \le n$, and suppose we have constructed $P_1, \dots, P_{m-1}$; we construct P_m as follows. Let

$$Z = \{b_1, \dots, b_k\} \cup ((V(P_1) \cup \dots \cup V(P_{m-1})) \cap (B \cup C)).$$

Thus $|Z| \le k + 4(m-1) \le 3k - 4$, since $m \le n \le k/2$. Let X be the set of vertices in $C_{i(s_m)}$ that have G-distance at least three from every vertex of Z. Since (G, μ) is $(\epsilon, 2)$-coherent, it follows that $\mu(X) \ge \mu(C_{i(s_m)}) - (3k-4)\epsilon \ge \epsilon$. Let Y be the set of vertices in $C_{i(t_m)}$ that have G-distance at least three from every vertex of Z; then similarly $\mu(Y) \ge \epsilon$. Since $X \cap Y = \emptyset$ and (G, μ) is ϵ-coherent, there exist $x \in X$ and $y \in Y$, adjacent. Since $B_{i(s_m)}$ covers $C_{i(s_m)}$, there exists $x' \in B_{i(s_m)}$ adjacent to x; and since the distance between x and Z is at least three, it follows that x' has no neighbour in Z. Similarly there exists $y' \in B_{i(t_m)}$ adjacent to y. Since $A_{i(s_m)}$ is connected and covers $B_{i(s_m)}$, and similarly for t_m, it follows that the subgraph of G induced on $A_{i(s_m)} \cup A_{i(t_m)} \cup \{s_m, t_m, x, y, x', y'\}$ is connected. Choose an induced path P_m joining s_m, t_m in this subgraph. Then P_m satisfies the first and fourth bullets above.

We claim that for $1 \le \ell < m$, $V(P_\ell)$ and $V(P_m)$ are anticomplete. Suppose not, and let $u \in V(P_\ell)$ and $v \in V(P_m)$ be adjacent or equal. Now either u is one of s_ℓ, t_ℓ, or u belongs to one of $A_{i(s_\ell)}, A_{i(t_\ell)}$, or $u \in Z$; and either v is one of s_m, t_m, or v belongs to one of $A_{i(s_m)}, A_{i(t_m)}$, or $v \in \{x, y, x', y'\}$. If $u \in A_{i(s_\ell)}$, then all its neighbours in $A \cup B \cup C$ belong to $A_{i(s_\ell)} \cup B_{i(s_\ell)}$ from the definition of a k-ladder; and since v is not in the latter set, it follows that $u \notin A_{i(s_\ell)}$. Similarly $u \notin A_{i(t_\ell)}$, and $v \notin A_{i(s_m)} \cup A_{i(t_m)}$. Consequently $u, v \in B \cup C$. Thus $u \in Z$, and so $v \notin \{x, y, x', y'\}$ from the choice of x, y, x', y'. Hence v is one of s_m, t_m. But from the choice of P_ℓ, $V(P_\ell)$ is anticomplete to $\{b_1, \dots, b_k\} \setminus \{s_\ell, t_\ell\}$, a contradiction. Thus P_m satisfies the second bullet.

For the third bullet, suppose that $u \in \{b_1, \dots, b_k\} \setminus \{s_m, t_m\}$ is adjacent to $v \in V(P_m)$. Since $u \in Z$, it follows that $v \notin \{x, y, x', y'\}$; and as before $v \notin A_{i(s_m)} \cup A_{i(t_m)}$, and so v is one of s_m, t_m, a contradiction since $b_1, \dots, b_k$ are pairwise nonadjacent. Thus P_m satisfies the third bullet.

This completes the inductive definition, and hence proves 6.3. ∎

In turn, 6.3 is used to prove the following, the main result of this section.

6.4 *For every caterpillar T, there exist $\epsilon > 0$ and an integer $r \ge 1$, such that for every (ϵ, r)-coherent massed graph (G, μ), there is a versatile copy of T in G.*

Proof. We may assume that $|T| \ge 3$. Assign a head to T to make it rooted, not one of the leaves. Let $k = 2^{|T|}$. Let $r = 5k$, and choose $\epsilon > 0$ such that $(k-1)k(2^k(3k+2)+1)\epsilon \le 1$. Let (G, μ) be an (ϵ, r)-coherent massed graph. By 6.2, there is a half-cleaned k-ladder

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

in G such that $\mu(C_i) \ge 2^k(3k+2)\epsilon$ for $1 \le i \le k$. Then the unfocussed case of 5.2 (taking $\delta = (3k+1)\epsilon$) implies that there is a $\{C_1, \dots, C_k\}$-spread $(3k+1)\epsilon$-realization $(X_v : v \in V(T))$ of T in G.

Let $t_1, \dots, t_q$ be the vertices of T that are not leaves, where t_1 is the head of T and t_it_{i+1} is an edge of T for $1 \le i < q$. Choose $x_1 \in X_{t_1}$. For $2 \le i \le q$ in turn, since X_{t_i} covers $X_{t_{i-1}}$ by definition of a realization, we may choose $x_i \in X_{t_i}$ adjacent to x_{i-1}. Since each x_i belongs to one of $C_1, \dots, C_k$, it follows that $\{x_1, \dots, x_q\}$ is anticomplete to $A_1 \cup \dots \cup A_k$. Also, since there are no edges between X_u, X_v for nonadjacent $u, v \in V(T)$, it follows that $x_1, \dots, x_q$ are the vertices in order of an induced path of G. For the same reason we have the following two statements:

(1) *For each leaf v of T with neighbour t_j say, x_j has a neighbour in X_v, and X_v is anticomplete to $\{x_1, \dots, x_q\} \setminus \{x_j\}$.*

(2) *For all distinct leaves u, v of T, X_u is anticomplete to X_v.*

We recall that $C_1, \dots, C_k$ are pairwise disjoint, and X_v ($v \in V(T)$) is $\{C_1, \dots, C_k\}$-spread; let I be the set of $i \in \{1, \dots, k\}$ such that $X_v \subseteq C_i$ for some leaf v of T. For each $i \in I$, let v_i be the (unique) leaf of T with $X_{v_i} \subseteq C_i$. Let $i \in I$, and let $j \in \{1, \dots, q\}$ such that v_i is adjacent to t_j in T. We define $x^i = x_j$. Thus there may be distinct values $i, i' \in I$ with $x^i = x^{i'}$.

Let $i \in I$. Since X_{v_i} is $(2k+1)\epsilon$-dominant, and (G, μ) is (ϵ, r)-coherent, there exists $u \in X_{v_i}$ with a neighbour v such that the G-distance between v and x_1 is at least $r+1$. Consequently the G-distance between u and x_1 is at least r. By (1), $X_{v_i} \cup \{x^i\}$ is connected, and so there is a path of $G[X_{v_i} \cup \{x^i\}]$ between x^i and u; and hence there is a minimal path P_i of $G[X_{v_i} \cup \{x^i\}]$ with one end x^i and the other u_i say, such that the G-distance between x_1 and u_i is at least $q + 4i$. It follows that the G-distance between x_1 and u_i is exactly $q + 4i$. Choose a vertex $b_i \in B_i$ adjacent to u_i. Let c_i be the second vertex of P_i, that is, the vertex adjacent to x^i, and let $Q_i = P_i \setminus \{x^i\}$. Thus Q_i is a path of $G[X_{v_i}]$. The subgraph T' of G induced on $\{x_1, \dots, x_q\} \cup \{c_i : i \in I\}$ is a copy of T, by (1) and (2), and we will show that it is versatile.

(3) *For all distinct $i, j \in I$, the sets $V(Q_i) \cup \{b_i\}$ and $V(Q_j) \cup \{b_j\}$ are anticomplete.*

We may assume that $i < j$. Certainly $V(Q_i)$ and $V(Q_j)$ are anticomplete, by (2). Also b_i has no neighbour in $V(Q_j)$ since the k-ladder is half-cleaned; so it remains to check that b_j has no neighbour in $V(Q_i) \cup \{b_i\}$. Let $v \in V(Q_i)$; then from the minimality of Q_i, the G-distance between v, x_1 is at most $q + 4i$. But the G-distance between u_j and x_1 is $q + 4j$, and so the G-distance between v, u_j is at least four. Consequently the G-distance between v, b_j is at least three, and in particular b_j has no neighbour in $V(Q_i)$; and setting $v = u_i$, since the G-distance between v, b_j is at least three it follows that b_i, b_j are nonadjacent. This proves (3).

(4) *For all $i \in I$, the sets $V(Q_i) \cup \{b_i\}$ and $\{x_1, \dots, x_q\} \setminus \{x^i\}$ are anticomplete.*

Let $j \in \{1, \dots, q\}$ and suppose that x_j is adjacent to some $v \in V(Q_i) \cup \{b_i\}$ and $x_j \neq x^i$. Since the G-distance between x_1, b_i is $q + 4i$, and the G-distance between x_1, x_j is at most $q - 1$, it follows that b_i, x_j are nonadjacent, and so $v \in Q_i$, contrary to (1).

Let Z be the set of vertices in G with G-distance from x_1 at most $4k + q + 4$, and for $1 \le i \le k$ let $C'_i = C_i \setminus Z$. Since $k \ge 2^{q+2}$, it follows that $r = 5k \ge 4k + q + 4$, and so (G, μ) is $(\epsilon, 4k + q + 4)$-coherent. Hence $\mu(Z) \le \epsilon$. Thus for $1 \le i \le k$, $\mu(C'_i) \ge \mu(C_i) - \epsilon \ge 3k\epsilon$. Let $Q = \bigcup_{i \in I} V(Q_i) \cup \{b_i\}$.

Since every vertex in Q has G-distance at most $4k+q+1$ from x_1, it follows that every vertex in C_i' has G-distance at least three from Q. Let B_i' be the set of vertices in $B_i \setminus \{b_i\}$ with no neighbours in $V(Q)$. Since every vertex in C_i' has a neighbour in B_i and has G-distance at least three from Q, it follows that B_i' covers C_i'. Hence the sets

$$A_i\ (i \in I), B_i' \cup \{b_i\}\ (i \in I), C_i'\ (i \in I)$$

form an $|I|$-ladder, with union (A', B', C') say. Since $\mu(C_i') \geq 3k\epsilon$ and $r \geq 2$, every pairing of $\{b_i : i \in I\}$ is feasible in $G[A' \cup B' \cup C']$ by 6.3.

Since $A' \cup B' \cup C' \setminus \{b_i\ : i \in I\}$ is anticomplete to

$$(Q \setminus \{b_i\ : i \in I\}) \cup \{x_1, \ldots, x_q\},$$

it follows from (3) and (4) that every pairing of $\{c_i : i \in I\}$ is feasible in the subgraph induced on $A' \cup B' \cup C' \cup \bigcup_{i \in I} V(Q_i)$. But $\{c_i : i \in I\}$ is the set of leaves of T', and since

$$A' \cup B' \cup C' \cup \bigcup_{i \in I} (V(Q_i) \setminus \{c_i\})$$

is anticomplete to $\{x_1, \ldots, x_q\}$, it follows that T' is versatile. This proves 6.4. ∎

7 The general proof

Now we turn to the proof of 4.1 in general. Fix the caterpillar T, and choose ϵ_r and r to satisfy 6.4 with ϵ replaced by ϵ_r. Now we will choose ϵ much smaller than ϵ_r, and try to prove that in every ϵ-coherent massed graph (G, μ), some copy of T is versatile. We can therefore assume that for every $Z \subseteq V(G)$, there is no mass μ' on $G[Z]$ that is (ϵ_r, r)-coherent; and in particular (assuming $\mu(Z) > 0$), the mass μ' on $G[Z]$ defined by $\mu'(X) = \mu(X)/\mu(Z)$ for each $X \subseteq Z$ is not (ϵ_r, r)-coherent. Consequently, either there is a vertex $v \in Z$ with $\mu(N^r_{G[Z]}[v]) \geq \epsilon_0 \mu(Z)$, or there are two anticomplete sets $A, B \subseteq Z$ with $\mu(A), \mu(B) \geq \epsilon_r \mu(Z)$. The latter is only helpful if $\epsilon_r \mu(Z) \geq \epsilon$, but in that case we can assume the latter never occurs. Thus, for every $Z \subseteq V(G)$ with $\mu(Z) \geq \epsilon/\epsilon_r$, there is a vertex $v \in Z$ with $\mu(N^r_{G[Z]}[v]) \geq \epsilon_r \mu(Z)$. Since $N^r_{G[Z]}[v]$ is anticomplete to $Z \setminus N^{r+1}_{G[Z]}[v]$, and $\mu'(N^r_{G[Z]}[v]) \geq \epsilon_r$ (and because of the "latter never occurs" assumption above), it follows that $\mu'(Z \setminus N^{r+1}_{G[Z]}[v]) < \epsilon_r$, and so $\mu'(N^{r+1}_{G[Z]}[v]) \geq 1 - \epsilon_r$, that is, $\mu(N^{r+1}_{G[Z]}[v]) \geq (1-\epsilon_r)\mu(Z)$. Initially we could have chosen ϵ_r as small as we want, and in particular we may assume that $\epsilon_r \leq 1/2$; and so $\mu(N^{r+1}_{G[Z]}[v]) \geq \mu(Z)/2$. Because of this we will be able to apply the focussed case of 5.2.

For $X \subseteq V(G)$ and $v \in V(G)$, we say v *touches* X if either $v \in X$ or v has a neighbour in X; and otherwise v is *anticomplete* to X. We need the following.

7.1 *Let $t, r \geq 1$ be integers and $\epsilon > 0$, and let (G, μ) be an ϵ-coherent massed graph. Let T be a caterpillar in G with t vertices, and let $x_1, \ldots, x_q$ be the vertices of T with degree more than one. For each leaf v of T let x^v be its neighbour in $\{x_1, \ldots, x_q\}$, and let $X_v \subseteq V(G)$, such that*

- *for each leaf v, $v \in X_v$, and $X_v \cap \{x_1, \ldots, x_q\} = \emptyset$, and X_v is anticomplete to $\{x_1, \ldots, x_q\} \setminus \{x^v\}$;*
- *for all distinct leaves u, v, X_u is anticomplete to X_v;*

- *for each leaf v, x^v is an r-centre for $X_v \cup \{x^v\}$;*
- *for each leaf v, v is the unique neighbour of x^v in X_v; and*
- *for each leaf v, $X_v \cup \{x^v\}$ is $(r+2)t^{t+1}\epsilon$-dominant.*

Then T is versatile.

Proof. Define $\kappa_i = (r+2)t^{t-i+1}\epsilon$ for $0 \le i \le t$. Let Π be a pairing of the set of leaves L of T. Let $L = \{v_1, \dots, v_\ell\}$, where Π consists of the sets $\{v_{2i-1}, v_{2i}\}$ for $1 \le i \le k$ for some $k \le \ell/2$, together with the singleton sets $\{v_i\}$ for $2k+1 \le i \le \ell$. Let $X_v^0 = X_v \cup \{x^v\}$ for each $v \in L$. For $1 \le i \le k$, we define X_v^i ($v \in \{v_{2i+1}, \dots, v_\ell\}$) and P_i inductively as follows. We assume $P_1, \dots, P_{i-1}$ and X_v^{i-1} ($v \in \{v_{2i-1}, \dots, v_\ell\}$) have been defined, such that

- for $1 \le h \le i-1$, P_h is an induced path between v_{2h-1}, v_{2h}, of length at most $2r+1$;
- for $1 \le h \le i-1$, $V(P_h) \setminus \{v_{2h-1}, v_{2h}\}$ is anticomplete to $V(T) \setminus \{v_{2h-1}, v_{2h}\}$;
- for $1 \le h < h' \le i-1$, $V(P_h)$ is anticomplete to $V(P_{h'})$;
- for $1 \le h \le i-1$ and $v \in \{v_{2i-1}, \dots, v_\ell\}$, $V(P_h)$ is anticomplete to X_v^{i-1};
- for $v \in \{v_{2i-1}, \dots, v_\ell\}$, X_v^{i-1} is κ_{i-1}-dominant; and
- for $v \in \{v_{2i-1}, \dots, v_\ell\}$, $\{x^v, v\} \subseteq X_v^{i-1}$ and x^v is an r-centre for X_v^{i-1}.

For each $w \in \{v_{2i+1}, \dots, v_\ell\}$, choose $X_w^i \subseteq X_w^{i-1}$, minimal such that X_w^i is κ_i-dominant and x^w is a r-centre for X_w^i. By deleting a vertex in X_w^i with maximum $G[X_w^i]$-distance from x^w, the minimality of X_w^i implies that the set of vertices that touch X_w^i has mass at most $\kappa_i + \epsilon$. Also the set of vertices that touch $\{x_1, \dots, x_q\} \cup \bigcup_{h<i} V(P_h)$ has mass at most $(q + (k-1)(2r+2))\epsilon$. Let $u = v_{2i-1}$ and $v = v_{2i}$. Let C be the set of all vertices that do not touch $\{x_1, \dots, x_q\} \cup \bigcup_{h<i} V(P_h)$ and do not touch X_w^i for $w \in \{v_{2i+1}, \dots, v_\ell\}$. Let $A, B \subseteq C$ be the sets of all vertices in C that touch X_u^{i-1} and touch X_v^{i-1} respectively.

Since X_u^{i-1} is κ_{i-1}-dominant, it follows that

$$\mu(A) \ge \kappa_{i-1} - (q + (k-1)(2r+2))\epsilon - \ell(\kappa_i + \epsilon).$$

The expression on the right side of this inequality is at least ϵ, since $q + \ell = t$ and $\ell \le t-1$ and $k \le \ell/2$ and $r \ge 1$ (we leave checking this to the reader); and so $\mu(A) \ge \epsilon$. The same holds for $\mu(B)$; and so A, B are not anticomplete. Consequently there are vertices a, b, adjacent or equal, such that a touches X_u^{i-1} and b touches X_v^{i-1}, and a, b are anticomplete to $\{x_1, \dots, x_q\} \cup \bigcup_{h<i} V(P_h)$ and to X_w^i for $w \in \{v_{2i+1}, \dots, v_\ell\}$. Since x^u is an r-centre for X_u^{i-1}, and u is the unique neighbour of x^u in X_u^{i-1}, there is a path of $G[X_u^{i-1} \cup \{a\}]$ between u and a of length at most r, and the same for v; and therefore there is an induced path P_i between u, v of length at most $2r+1$, and all its vertices belong to $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$. In particular, $V(P_i)$ is anticomplete to $V(P_h)$ for $h < i$, since $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$ is anticomplete to $V(P_h)$; and $V(P_i)$ is anticomplete to X_w^i for $w \in \{v_{2i+1}, \dots, v_\ell\}$, since $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$ is complete to X_w^i. This completes the inductive definition.

But then the paths $P_1, \dots, P_k$ together with the singletons $\{c_i\}$ ($2k+1 \le i \le \ell$) show that Π is feasible relative to T. Consequently T is versatile. This proves 7.1. ∎

This is used to prove:

7.2 *Let T be a caterpillar with t vertices, let $r \geq 1$ be an integer, and let $\epsilon, \delta > 0$ with $\epsilon \leq \delta/2$, such that $\delta \leq 2^{-(t+2^t)}t^{-t}$ and $\epsilon \leq 2^{-(t+2^t)}t^{-2t}(3r+5)^{-1}$. Let (G, μ) be a (δ, r)-focussed ϵ-coherent massed graph. Then there is a versatile copy of T in G.*

Proof. We may assume that $|T| \geq 3$, T is rooted, and its head is an internal vertex. Let $t = |T|$ and $k = 2^t$. Let $\lambda = 1/k - \epsilon$.

(1) *There exist pairwise disjoint subsets $Y_1, \ldots, Y_k$ of $V(G)$ with $\mu(Y_i) \geq \lambda$ for $1 \leq i \leq k$.*

We define $Y_1, \ldots, Y_k \subseteq V(G)$ inductively as follows. Let $1 \leq i < k$, and assume we have chosen $Y_1, \ldots, Y_i \subseteq V(G)$, pairwise disjoint and each with $\lambda \leq \mu(Y_i) \leq \lambda + \epsilon$. Thus

$$\mu(Y_1 \cup \cdots \cup Y_i) \leq (k-1)(\lambda + \epsilon),$$

and so

$$\mu(V(G) \setminus (Y_1 \cup \cdots \cup Y_i)) \geq 1 - (k-1)(\lambda + \epsilon) \geq \lambda.$$

Consequently we may choose Y_{i+1} disjoint from $Y_1 \cup \cdots \cup Y_i$ with $\mu(Y_i) \geq \lambda$. Choose Y_{i+1} minimal with this property; then $\mu(Y_i) \leq \lambda + \epsilon$. This completes the inductive definition of $Y_1, \ldots, Y_k$ and so proves (1).

Let $\kappa_0 = 2^{-k}\lambda - \epsilon$, and for $1 \leq i \leq t$ let $\kappa_i = \kappa_0 t^{-i}$. Since G is (δ, r)-focussed, it is also (κ_0, r)-focussed, since $\kappa_0 \geq \delta$. From the focussed case of 5.2, there is a $\{Y_1, \ldots, Y_k\}$-spread κ_0-realization $(X_v : v \in V(T))$ of T in G such that X_v has an r-centre for each $v \in V(T)$ except the head. Let the vertices of degree more than one in T be $t_1, \ldots, t_q$, where t_1 is the head and $t_i t_{i+1}$ is an edge of T for $1 \leq i < q$. Choose $x_1 \in X_{t_1}$, and inductively for $i = 2, \ldots, q$, choose $x_i \in X_{t_i}$ adjacent to x_{i-1}. As in the proof of 6.4, $x_1, \ldots, x_t$ are the vertices in order of an induced path of G. We need to arrange that for each leaf v of T adjacent to t_i say, x_i has a unique neighbour in X_v.

Let $L = \{v_1, \ldots, v_\ell\}$ be the set of leaves of T, and for each $v \in L$ let x^v be the vertex x_j such that v is adjacent to t_j in T. Thus x^v is the unique vertex in $\{x_1, \ldots, x_q\}$ covered by X_v. Since X_v has an r-centre, and x^v has a neighbour in X_v, it follows that x^v is a $(2r+1)$-centre of $X_v \cup \{x^v\}$.

Let $X_v^0 = X_v \cup \{x^v\}$ for $v \in L$. For $0 \leq i \leq \ell$ we will inductively define X_v^i $(v \in L)$ satisfying the following: for $0 \leq i \leq \ell$,

- for each $v \in L$, $x^v \in X_v^i$ and $X_v^i \setminus \{x^v\}$ is anticomplete to $\{x_1, \ldots, x_q\} \setminus \{x^v\}$;
- for all distinct $u, v \in L$, $X_u^i \setminus \{x^u\}$ is anticomplete to $X_v^i \setminus \{x^v\}$;
- for $1 \leq j \leq i$, x^{v_j} is a $(3r+2)$-centre of $X_{v_j}^i$, and x^{v_j} has a unique neighbour in $X_{v_j}^i$;
- for $i < j \leq \ell$, x^{v_j} is a $(2r+1)$-centre of $X_{v_j}^i$; and
- for each $v \in L$, X_v^i is κ_i-dominant.

Suppose that $1 \le i \le \ell$, and we have defined X_v^{i-1} ($v \in L$) as above. For each $u \in L \setminus \{v_i\}$, let $u = v_j$ say. Let $r_j = 3r + 2$ if $j < i$, and $r_j = 2r + 1$ if $j > i$, so in either case x^u is an r_j-centre of X_u^{i-1}. Choose $X_u^i \subseteq X_u^{i-1}$, minimal such that X_u^i is κ_i-dominant and x^u is an r_j-centre for X_u^i. It follows from the minimality of each X_u^i that the set of vertices that touch X_u^i has mass at most $\kappa_i + \epsilon$.

Let $v = v_i$, and let Y'' be the set of all vertices that touch X_v^{i-1}; then $\mu(Y'') \ge \kappa_{i-1}$ since X_v^{i-1} is κ_{i-1}-dominant. Let Y' be the set of vertices in Y'' that are anticomplete to

$$\{x_1, \dots, x_q\} \cup \bigcup_{u \in L \setminus \{v\}} X_u^i.$$

Thus $\mu(Y') \ge \mu(Y'') - q\epsilon - (\ell - 1)(\kappa_i + \epsilon)$. Since $\mu(Y'') \ge \kappa_{i-1}$ and $\delta \le 2^{-(t+2^t)}t^{-t}$ by hypothesis, it follows that $\mu(Y') \ge \delta$ (we leave it to the reader to check this arithmetic). Since G is (δ, r)-focussed, there is a subset $Y \subseteq Y'$ with $\mu(Y) \ge \mu(Y')/2$, such that Y has an r-centre y say. Since x^v is a $(2r+1)$-centre for X_v^{i-1}, and y touches this set, there is an induced path P of $G[X_v^{i-1} \cup \{y\}]$ between x^v and y, of length at most $2r + 2$. Let $X_v^i = Y \cup V(P)$. Then x^v is a $(3r + 2)$-centre for X_v^i. Since x^v is anticomplete to Y and has only one neighbour in P, it follows that x^v has only one neighbour in $Y \cup V(P)$. Moreover, Y is κ_i-dominant since $\mu(Y) \ge \epsilon$. This completes the inductive definition.

For each $v \in L$, let c_v be the unique neighbour of x^v in X_v^ℓ. The subgraph T' induced on $\{x_1, \dots, x_q\} \cup \{c_v : v \in L\}$ is a copy of T. Since $\epsilon \le 2^{-(t+2^t)}t^{-2t}(3r + 5)^{-1}$ by hypothesis, it follows that $\kappa_\ell \ge (3r + 4)t^{t+1}\epsilon$, and so 7.1 (with r replaced by $3r + 2$) implies that T' is versatile. This proves 7.2. ∎

Now let us put these pieces together, to prove 4.1, which we restate:

7.3 *For every caterpillar T, there exists $\epsilon > 0$, such that for every ϵ-coherent massed graph (G, μ), there is a versatile copy of T in G.*

Proof. By 6.4 there exist $\epsilon_r > 0$ and an integer $r \ge 1$, such that for every (ϵ_r, r)-coherent massed graph (G, μ), there is a versatile copy of T in G. We may assume $\epsilon_r \le t^{-t}(3r + 5)^{-1}$. Choose ϵ such that $\epsilon \le 2^{-(t+2^t)}t^{-t}\epsilon_r$. Let $\delta = \epsilon/\epsilon_r$. Then ϵ, δ satisfy the hypotheses of 7.2.

Let (G, μ) be an ϵ-coherent massed graph; we will prove there is a versatile copy of T in G. If (G, μ) is $(\delta, r + 1)$-focussed, the result follows from 7.2, so we assume not. Hence there exists $Z \subseteq V(G)$ with $\mu(Z) \ge \delta$, such that $\mu(N_{G[Z]}^{r+1}[v]) < \mu(Z)/2$ for each $v \in Z$. Let $\mu'(X) = \mu(X)/\mu(Z)$ for all $X \subseteq Z$; then $(G[Z], \mu')$ is a massed graph. If it is (ϵ_r, r)-coherent then the result follows from 6.4, so we assume not, for a contradiction. If there exist anticomplete subsets A, B of Z with $\mu'(A), \mu'(B) > \epsilon_r$, then $\mu(A), \mu(B) \ge \epsilon$, which is impossible. Thus there exists $v \in Z$ such that $\mu'(N_{G[Z]}^r[v]) \ge \epsilon_r$, and hence such that $\mu(N_{G[Z]}^r[v]) \ge \epsilon_r\mu(Z) \ge \epsilon$. But $\mu(N_{G[Z]}^{r+1}[v]) < \mu(Z)/2$ from the choice of Z, and so $\mu(Z \setminus N_{G[Z]}^{r+1}[v]) > \mu(Z)/2 \ge \epsilon$, a contradiction since the two sets $N_{G[Z]}^r[v]$ and $Z \setminus N_{G[Z]}^{r+1}[v]$ are anticomplete. This proves 7.3. ∎

8 Parallels with χ-boundedness

An ideal is *χ-bounded* if there is a function f such that $\chi(G) \le f(\omega(G))$ for each graph G in the ideal. Such ideals have been studied intensively, and it turns out that incoherent ideals and χ-bounded ideals are in some ways very similar. Here are two instances:

- Take a graph H, and let $\mathcal{I}$ be the ideal of all H-free graphs. The Gyárfás-Sumner conjecture [17, 25] asserts that $\mathcal{I}$ is χ-bounded if and only if H is a forest, and a conjecture of [18] asserts that $\mathcal{I}$ is incoherent if and only if H is a forest. The second is now (very recently) a theorem [8], and the "only if" part of the first is true, and the "if" part has been shown for some forests.
- A *hole* in G means an induced cycle of length at least four. Let $\mathcal{I}$ be the class of all graphs with no hole of length at least k, for some fixed integer k. A theorem of [6] says that $\mathcal{I}$ is χ-bounded, and a theorem of [2] says that $\mathcal{I}$ is incoherent.

But the parallel does not always work, and in fact neither property implies the other. Here are two examples showing this, one for each direction:

- The ideal of all perfect graphs is χ-bounded, and indeed so is the ideal of all graphs with no odd hole [21], but an example of Fox [15] shows that these ideals are coherent.
- Fix a graph H, and let $\mathcal{I}$ be the ideal of all graphs such that no induced subgraph is isomorphic to a subdivision of H. Then $\mathcal{I}$ is not necessarily χ-bounded [19], but our main result proves that it is incoherent.

There are a number of hard results and longstanding open conjectures about ideals that are not χ-bounded, and it is entertaining to try their parallels for coherent ideals. (The proofs below are just sketched.)

It is proved in [24] that every ideal that is not χ-bounded contains cycles of all lengths modulo k, for every integer $k \geq 1$. The same is not true for coherent ideals, as the example of Fox [15] shows; a coherent ideal need not contain a cycle of odd length more than three, and in particular need not contain a cycle of length 1 modulo 6. But it follows from 2.3 that for all integers ℓ, every coherent ideal contains a cycle of length 2ℓ modulo k, and hence contains one of every length modulo k if k is odd. To see this, choose $\epsilon > 0$ very small, and choose an ϵ-coherent graph G from the ideal. By 2.3, G contains a P-filleting of complete graph H of some large (constant) size, where P is a Hamilton path of H. Choose many disjoint subpaths of P, each of length $2k$, with no edges joining them. Let these paths be $P_1, \ldots, P_n$ say, and let the ith vertex of P_j be v^i_j. For each i, and $1 \leq h < k \leq n$, there is a path of the P-filleting that joins v^i_h, v^i_j, say $Q^{h,j}_i$; and by Ramsey's theorem, we may choose many of the paths P_j such that all the paths $Q^{h,j}_i$ have the same length modulo k depending on i. (Redefine n, and renumber $P_1, \ldots, P_n$ so that this holds.) Let R_i be the union of

$$Q^{1,2}_i, Q^{2,3}_i, \ldots, Q^{k,k+1}_i;$$

then R_i has length divisible by k. But then for any t modulo k, the union of R_1, R_{t+1} and subpaths of P_1 and P_{k+1} (both of length t) makes a hole of length $2t$ modulo k, and this cycle belongs to the ideal.

It is conjectured in [22] that in every graph with huge chromatic number and bounded clique number, there are k holes with consecutive lengths. The same is not true in ϵ-coherent graphs G with ϵ very small, because they need not have odd holes; but perhaps there must always be k even holes with successive lengths differing by two?

It is proved in [23] that, in any colouring of a graph with huge chromatic number and bounded clique number, some induced k-vertex path is *rainbow* (that is, all its vertices have different colours), and no other types of connected subgraph have this property. What if we colour a graph which is ϵ-coherent for ϵ very small? Then we can do better than just paths; the results of this paper show that we can get a rainbow copy of any caterpillar. Each colour class has cardinality at most $2\epsilon|G|$, so by grouping the colour classes, we can partition the vertex set into many disjoint sets each of about the same size (differing by at most $2\epsilon|G|$), and each a union of colour classes. Then 5.2 gives a copy T' of T with at most one vertex from each block of the partition; and in particular, T' is rainbow. Actually, we can do even better; results of [8] show we can get a rainbow copy of any forest.

If we direct the edges of a graph with huge chromatic number and bounded clique number, some digraphs must be present as induced subdigraphs. For instance, it is proved in [7] that every oriented star has this property, and so does a three-edge path where both ends point outwards. What if we direct the edges of a graph which is ϵ-coherent for ϵ very small? Now much less is true. We need not get a directed two-edge path, because of Fox's example from [15] (this is a comparability graph, and so can be directed so that there is no induced directed two-edge path). We also need not get an outdirected 3-star (a tree with four vertices, three of them adjacent from the fourth). To see this, fix ϵ, choose k with $k\epsilon \geq 2$, and take k disjoint sets $A_1, \dots, A_k$ each of the same size n/k say, with n large. Now take a random graph on $A_1 \cup \cdots \cup A_k$ with average degree $\log n$; with high probability the outcome is ϵ-coherent, and its maximum degree is $O(\log(n))$. For $1 \leq i \leq k$ in turn, and for every pair of vertices $u, v \in A_{i+1} \cup \cdots \cup A_k$ with a common neighbour in A_i, add an edge uv. Let the result be G'. Since this process is repeated only k times and the maximum degree at most squares at each step, the maximum degree of G' is still less than ϵn. Now add more edges so that each A_i is a clique, forming G''. Thus G'' is ϵ-coherent. Orient every edge uv of G', from u to v if $u \in A_i$ and $v \in A_j$ where $i < j$, and arbitrarily if u, v belong to the same A_i. In the resulting digraph, there is no induced outdirected 3-star. (Incidentally, because of our main theorem 2.3 we always get a subdivision of $K_{2,3}$ as an induced subgraph, and however this is oriented it contains an outdirected 2-star; so we always get an outdirected 2-star.)

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, "Crossing patterns of semi-algebraic sets", *J. Combinatorial Theory, Ser. A*, **111** (2005), 310–326.

[2] M. Bonamy, N. Bousquet, and S. Thomassé, "The Erdős-Hajnal conjecture for long holes and antiholes", *SIAM J. Discrete Math.*, **30** (2016), 1159–1164.

[3] N. Bousquet, A. Lagoutte, and S. Thomassé, "The Erdős-Hajnal conjecture for paths and antipaths", *J. Combinatorial Theory, Ser. B*, **113** (2015), 261–264.

[4] K. Choromanski, D. Falik, A. Liebenau, V. Patel and M. Pilipczuk, "Excluding hooks and their complements", `arXiv:1508.00634`.

[5] M. Chudnovsky, "The Erdős-Hajnal conjecture – a survey", *J. Graph Theory* **75** (2014), 178–190.

[6] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. III. Long holes", *Combinatorica* **37** (2017), 1057–72.

[7] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. XI. Orientations", *European Journal of Combinatorics* 76 (2019), 53–61, `arXiv:1711.07679`.

[8] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs" (manuscript March 2018), `arXiv:1809.00919`.

[9] M. Chudnovsky and P. Seymour, "Excluding paths and antipaths", *Combinatorica* **35** (2015), 389–412.

[10] M. Chudnovsky and Y. Zwols, "Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement", *J. Graph Theory*, **70** (2012), 449–472.

[11] P. Erdős, "Some remarks on the theory of graphs", *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[12] P. Erdős and A. Hajnal, "On spanned subgraphs of graphs", *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977), www.renyi.hu/~p_erdos/1977-19.pdf.

[13] P. Erdős and A. Hajnal, "Ramsey-type theorems", *Discrete Applied Mathematics* **25** (1989), 37–52.

[14] P. Erdős and G. Szekeres, "A combinatorial problem in geometry", *Compositio Mathematica* **2** (1935), 463–470.

[15] J. Fox, "A bipartite analogue of Dilworth's theorem", *Order* **23** (2006), 197–209.

[16] J. Fox and J. Pach, "Erdős-Hajnal-type results on intersection patterns of geometric objects", in *Horizon of Combinatorics* (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.

[17] A. Gyárfás, "On Ramsey covering-numbers", *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), 10.

[18] A. Liebenau, M. Pilipczuk, P. Seymour and S. Spirkl, "Caterpillars in Erdős-Hajnal", *J. Combinatorial Theory, Ser. B*, 136 (2019), 33–43, `arXiv:1810.00811`.

[19] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter and B. Walczak, "Triangle-free intersection graphs of line segments with large chromatic number", *J. Combinatorial Theory, Ser. B*, **105** (2014), 6–10.

[20] V. Rödl, "On universality of graphs with uniformly distributed edges", *Discrete Math.* **59** (1986), 125–134.

[21] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. I. Odd holes", *J. Combinatorial Theory, Ser. B*, **121** (2016), 68–84.

[22] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes", *J. Combinatorial Theory, Ser. B*, 132 (2018), 180–235, `arXiv:1509.06563`.

[23] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. IX. Rainbow paths", *Electronic J. Combinatorics*, 24.2:#P2.53, 2017.

[24] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. X. Holes with specific residue", submitted for publication, `arXiv:1705.04609`.

[25] D.P. Sumner, "Subtrees of a graph and chromatic number", in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.