Tournaments with near-linear transitive subsets

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Abstract

Let H be a tournament, and let $\epsilon \geq 0$ be a real number. We call ϵ an "Erdős-Hajnal coefficient" for H if there exists $c > 0$ such that in every tournament G not containing H as a subtournament, there is a transitive subset of cardinality at least $c|V(G)|^{\epsilon}$. The Erdős-Hajnal conjecture asserts, in one form, that every tournament H has a positive Erdős-Hajnal coefficient. This remains open, but recently the tournaments with Erdős-Hajnal coefficient 1 were completely characterized. In this paper we provide an analogous theorem for tournaments that have an Erdős-Hajnal coefficient larger than 5/6; we give a construction for them all, and we prove that for any such tournament H there are numbers c, d such that, if a tournament G with $|V(G)| > 1$ does not contain H as a subtournament, then $V(G)$ can be partitioned into at most $c(\log(|V(G)|))^d$ transitive subsets.

Keywords: The Erdős-Hajnal conjecture, tournaments.

1 Introduction

A *tournament* is a loopless digraph such that for every pair of distinct vertices u, v, exactly one of uv, vu is an edge. A *transitive set* is a subset of $V(G)$ that can be ordered $\{x_1, \dots, x_k\}$ such that $x_i x_j$ is an edge for $1 \leq i < j \leq k$. A *colouring* of a tournament G is a partition of $V(G)$ into transitive sets, and the *chromatic number* $\chi(G)$ is the minimum number of transitive sets in a colouring. If G, H are tournaments, we say that G is *H-free* if no subtournament of G is isomorphic to H.

There are some tournaments H with the property that every H-free tournament has chromatic number at most a constant (depending on H). These are called *heroes*, and they were all explicitly found in an earlier paper [3]. In this paper, we turn to the question: which are the most heroic non-heroes? It turns out that for some non-heroes H, the chromatic number of every H-free tournament G is at most a polylog function of the number of vertices of G, and all the others give nothing better than a polynomial bound. More exactly, we will show the following (we will often write $|G|$ instead of $|V(G)|$, when G is a graph or tournament):

1.1 *Every tournament H has exactly one of the following properties:*

- *for some c, every H-free tournament has chromatic number at most c (that is, H is a hero)*
- *for some c, d, every H-free tournament G with $|G| > 1$ has chromatic number at most $c(\log(|G|))^d$, and for all c, there are H-free tournaments G with $|G| > 1$ and with chromatic number at least $c(\log(|G|))^{1/3}$*
- *for all c, there are H-free tournaments G with $|G| > 1$ and with chromatic number at least $c|G|^{1/6}$.*

This is one of our main results. The other is an explicit construction for all tournaments of the second type, which we call *pseudo-heroes*.

This research is closely connected with, and motivated by, the Erdős-Hajnal conjecture. P. Erdős and A. Hajnal [7] made the following conjecture in 1989 (it is still open):

1.2 (The Erdős-Hajnal conjecture.) *For every graph H there exists a number $\epsilon > 0$ such that every graph G that does not contain H as an induced subgraph contains a clique or a stable set of size at least $|G|^\epsilon$.*

If G is a tournament, $\alpha(G)$ denotes the cardinality of the largest transitive subset of $V(G)$. It was shown in [1] that the conjecture 1.2 is equivalent to the following:

1.3 (Conjecture.) *For every tournament H there exists a number $\epsilon > 0$ such that every H-free tournament G satisfies $\alpha(G) \geq |G|^\epsilon$.*

Let us say that $\epsilon \geq 0$ is an *EH-coefficient* for a tournament H if there exists $c > 0$ such that every H-free tournament G satisfies $\alpha(G) \geq c|G|^\epsilon$. Thus, the Erdős-Hajnal conjecture is equivalent to the conjecture that every tournament has a positive EH-coefficient. (We introduce c in the definition of the Erdős-Hajnal coefficient to eliminate the effect of tournaments G of bounded order; now, whether ϵ is an EH-coefficient for H depends only on arbitrarily large tournaments not containing H.) If ϵ is an EH-coefficient for H, then so is every smaller non-negative number; and thus a natural invariant is the supremum of the set of all EH-coefficients for H. We call this the *EH-supremum* for H, and denote it by $\xi(H)$. The EH-supremum for H is *not* necessarily itself an EH-coefficient for H; indeed, most of this paper concerns finding the tournaments H with $\xi(H) = 1$ for which 1 is not an EH-coefficient.

While we have nothing to say about the truth of 1.3 in general, a more tractable problem is: for which tournaments is some given $\epsilon > 0$ an EH-coefficient? In an earlier paper [3], we completely answered this for $\epsilon = 1$; and in this paper one goal is a similar result for $\epsilon > 5/6$.

Before we go on, let us state the result of [3] properly; and to do so we need some more definitions. We denote by T_k the transitive tournament with k vertices. If G is a tournament and X, Y are disjoint subsets of $V(G)$, and every vertex in X is adjacent to every vertex in Y, we write $X \Rightarrow Y$. We write $v \Rightarrow Y$ for $\{v\} \Rightarrow Y$, and $X \Rightarrow v$ for $X \Rightarrow \{v\}$. If G is a tournament and (X, Y, Z) is a partition of $V(G)$ into nonempty sets satisfying $X \Rightarrow Y$, $Y \Rightarrow Z$, and $Z \Rightarrow X$, we call (X, Y, Z) a *trisection* of G. If A, B, C, G are tournaments, and there is a trisection (X, Y, Z) of G such that $G|X, G|Y, G|Z$ are isomorphic to A, B, C respectively, we write $G = \Delta(A, B, C)$. It is convenient to write k for T_k here, so for instance $\Delta(1, 1, 1)$ means $\Delta(T_1, T_1, T_1)$, and $\Delta(H, 1, k)$ means $\Delta(H, T_1, T_k)$.

A tournament is a *celebrity* if 1 is an EH-coefficient for it; that is, for some $c > 0$, every H-free tournament G satisfies $\alpha(G) \geq c|G|$. The main result of [3] is:

1.4 *The following hold:*

- *A tournament is a hero if and only if it is a celebrity.*
- *A tournament is a hero if and only if all its strong components are heroes.*
- *A strongly-connected tournament with more than one vertex is a hero if and only if it equals $\Delta(1, H, k)$ or $\Delta(1, k, H)$ for some hero H and some integer $k > 0$.*

In this paper, we study the tournaments H which are "almost" heroes, in the sense that all H-free tournaments have chromatic number at most a polylog function of their order. More precisely, we say a tournament H is

- a *pseudo-hero* if there exist $c, d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\chi(G) \leq c(\log(|G|))^d$
- a *pseudo-celebrity* if there exist $c > 0$ and $d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq c\frac{|G|}{(\log(|G|))^d}$.

Logarithms are to base two, throughout the paper. (The conditions $|G| > 1$ are included just to ensure that $\log(|G|) > 0$.) The next result is an analogue of 1.4:

1.5 *The following hold:*

- *A tournament is a pseudo-hero if and only if it is a pseudo-celebrity.*
- *A tournament is a pseudo-hero if and only if all its strong components are pseudo-heroes.*
- *A strongly-connected tournament with more than one vertex is a pseudo-hero if and only if either*

– *it equals* $\Delta(2,k,l)$ *for some* $k,l \geq 2$*, or*
– *it equals* $\Delta(1,H,k)$ *or* $\Delta(1,k,H)$ *for some pseudo-hero* H *and some integer* $k > 0$.

More generally, let $0 \leq \epsilon \leq 1$; we say that a tournament H is

- an *ϵ-hero* if there exist $c, d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\chi(G) \leq c|G|^{1-\epsilon} \log(|G|)^d$; and
- an *ϵ-celebrity* if there exist $c > 0$ and $d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq c^{-1}|G|^{\epsilon} \log(|G|)^{-d}$.

Thus, a 1-hero is the same thing as a pseudo-hero, and a 1-celebrity is the same as a pseudo-celebrity. We will prove:

1.6 *For all* ϵ *with* $0 \leq \epsilon \leq 1$*:*

- *a tournament is an* ϵ*-hero if and only if it is an* ϵ*-celebrity*
- *a tournament is an* ϵ*-celebrity if and only if its strong components are* ϵ*-celebrities*
- *if* H *is an* ϵ*-celebrity and* $k \geq 1$*, then* $\Delta(1,H,k)$ *and* $\Delta(1,k,H)$ *are* ϵ*-celebrities.*

(Much of 1.5 is implied by setting $\epsilon = 1$ in 1.6.) In addition, we will prove:

1.7 *Every tournament* H *with* $\xi(H) > 5/6$ *is a pseudo-hero and hence satisfies* $\xi(H) = 1$.

Thus, if $\xi(H) > 5/6$ then every H-free tournament has chromatic number at most a polylog function of its order. We do not know if 5/6 is best possible; but the polylog behaviour is best possible, in the following sense:

1.8 *For every real* d *with* $0 \leq d < \frac{1}{3}$ *and all sufficiently large integers* n *(depending on* d*), there is a tournament* G *with* n *vertices such that*

- $\alpha(G) \leq n(\log(n))^{-d}$*, and*
- *every pseudo-hero contained in* G *is a hero.*

This last is a corollary of a result of [3]; let us see that now. Since every pseudo-hero that is not a hero contains $\Delta(2,2,2)$, by 1.4 and 1.5, it follows that 1.8 is implied by the following result of [3]:

1.9 *For every real* d *with* $0 \leq d < \frac{1}{3}$*, and all sufficiently large integers* n *(depending on* d*), there is a tournament* G *with* n *vertices, not containing* $\Delta(2,2,2)$*, such that*

$$\alpha(G) \leq \frac{n}{(\log(n))^d}.$$

(More precisely, the result of [3] asserts this with $\log(n)$ replaced by $\ln(n)$; we leave the reader to check the equivalence.) The paper is organized as follows:

- in sections 2,3 and 4 we prove the first, second and third assertion of 1.6 respectively;
- in section 5 we prove that for all $k, l \geq 2$, $\Delta(2,k,l)$ is a pseudo-celebrity, and indeed there exists $c > 0$ such that every $\Delta(2,k,l)$-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq c|G|/\log(|G|)$;
- in section 6 we prove the "only if" part of the third statement of 1.5, and thereby finish the proof of 1.5; and we also prove 1.7.

2 ϵ-celebrities are ϵ-heroes

In this section we prove the first statement of 1.6. Let us say a function ϕ is *round* if for each integer $n \geq 2$, $\phi(n)$ is a real number, at least 1 and (non-strictly) increasing with n. We need:

2.1 *Let ϕ be round. Suppose that G is a tournament with $|G| > 1$, and for all $n > 1$, every n-vertex subtournament of G has a transitive set of cardinality at least $n/\phi(n)$. Then $\chi(G) \leq \phi(|G|)\log(|G|)$.*

Proof. We proceed by induction on $|G|$. Let $n = |G|$. Now $\phi(n)\log(n) \geq 1$ (since $\phi(n) \geq 1$, and logarithms are to base 2), and so we may assume that $\chi(G) \geq 2$. By hypothesis, G has a transitive set X of cardinality x say, where $x \geq n/\phi(n) > 0$. In particular, $x \leq n-1$, and so $n-1 \geq n/\phi(n)$. Consequently $\phi(n) \geq n/(n-1) \geq 2/\log(n)$, and so $2 \leq \phi(n)\log(n)$. Hence we may assume that $\chi(G) \geq 3$. In particular, $G \setminus X$ has at least two vertices, and therefore we may apply the inductive hypothesis to $G \setminus X$. Since $\chi(G) \leq 1 + \chi(G \setminus X)$, we deduce that

$$\chi(G) \leq 1 + \phi(n-x)\log(n-x) \leq 1 + \phi(n)\log(n-x).$$

Now

$$\log(1 - x/n) \leq \ln(1 - x/n) \leq -x/n \leq -(\phi(n))^{-1},$$

and so $1 + \phi(n)\log(1 - x/n) \leq 0$. Consequently

$$\chi(G) \leq 1 + \phi(n)\log(n-x) = 1 + \phi(n)\log(1 - x/n) + \phi(n)\log(n) \leq \phi(n)\log(n).$$

This proves 2.1. ∎

Sometimes the previous result can be improved:

2.2 *Let G be a tournament with $|G| > 0$, and for each integer n with $1 \leq n \leq |G|$, let $\phi(n)$ be a positive real number, and let ϵ be a real number with $0 < \epsilon \leq 1$, such that*

- *every subtournament H of G with $|H| > 0$ has a transitive set of cardinality at least $|H|/\phi(|H|)$, and*
- *$\phi(n)/\phi(m) \geq (n/m)^\epsilon$ for all m, n with $1 \leq m \leq n \leq |G|$.*

Let $c = 2^\epsilon - 1$. Then $\chi(G) \leq c^{-1}\phi(|G|)$.

Proof. We proceed by induction on $|G|$. Let $n = |G|$. From the hypothesis, there is a transitive subset with cardinality at least $n/\phi(n) \geq 2^{\epsilon-1}n/\phi(n)$. Let us choose $X_1, \ldots, X_k \subseteq V(G)$, pairwise disjoint and each transitive with cardinality at least $2^{\epsilon-1}n/\phi(n)$, with k maximal; it follows that $k \geq 1$. Let $X_1 \cup \cdots \cup X_k = W$, and let $G \setminus W = G'$, and $|G'| = n'$. Let $x = n'/n$. Now W includes k disjoint subsets of cardinality at least $2^{\epsilon-1}n/\phi(n)$, and so

$$n - n' = |W| \geq k2^{\epsilon-1}n/\phi(n),$$

that is, $k \leq (1-x)\phi(n)2^{1-\epsilon}$. If $n' = 0$, then

$$\chi(G) \leq k \leq \phi(n)2^{1-\epsilon} \leq c^{-1}\phi(|G|),$$

as required. Thus we may assume that $n' > 0$. Now G' has no transitive set of cardinality at least $2^{\epsilon-1}n/\phi(n)$ by the maximality of k, and yet by hypothesis, it has a transitive set of cardinality at least $n'/\phi(n')$. It follows that $n'/\phi(n') < 2^{\epsilon-1}n/\phi(n)$, that is,

$$\phi(n')/\phi(n) > 2^{1-\epsilon}x.$$

By hypothesis, $\phi(n')/\phi(n) \leq x^\epsilon$, and so $2^{1-\epsilon}x < x^\epsilon$, that is, $x < 1/2$. From the inductive hypothesis, $\chi(G') \leq c^{-1}\phi(n')$. Since $\chi(G) \leq \chi(G') + k$, and $k \leq (1-x)\phi(n)2^{1-\epsilon}$, we deduce that

$$\chi(G) \leq c^{-1}\phi(n') + (1-x)\phi(n)2^{1-\epsilon}.$$

Since $\phi(n') \leq \phi(n)x^\epsilon$, it follows that

$$c\chi(G)/\phi(G) \leq x^\epsilon + (1-x)2^{1-\epsilon}c.$$

Now the function $(1-x^\epsilon)/(1-x)$ is minimized for $0 \leq x \leq 1/2$ when $x = 1/2$, and its value then is $2^{1-\epsilon}c$; and so $(1-x^\epsilon)/(1-x) \geq 2^{1-\epsilon}c$, that is,

$$x^\epsilon + (1-x)2^{1-\epsilon}c \leq 1.$$

It follws that $c\chi(G)/\phi(G) \leq 1$. as required. This proves 2.2. ∎

Thus if ϕ grows sufficiently quickly then we can avoid the extra log factor introduced by 2.1. Curiously, it was proved in [3] that the same is true when ϕ is constant. We do not know whether it is also true in the cases in between, when ϕ is not constant but only grows slowly. Unfortunately, these are the cases of most interest to us in this paper, and for them we have to make do with 2.1.

We deduce the first statement of 1.6, namely:

2.3 *For $0 \leq \epsilon \leq 1$, a tournament is an ϵ-hero if and only if it is an ϵ-celebrity.*

Proof. Let H be an ϵ-celebrity, and choose $c > 0$ and $d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq c^{-1}|G|^\epsilon \log(|G|)^{-d}$. We may assume that $c \geq 1$. Define $\phi(n) = cn^{1-\epsilon}(\log(n))^d$ for $n \geq 2$. Thus ϕ is round, and every H-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq |G|/\phi(|G|)$. Then if G is H-free and $|G| > 1$, the hypotheses of 2.1 are satisfied, and so

$$\chi(G) \leq \phi(|G|)\log(|G|) \leq c|G|^{1-\epsilon}(\log(|G|))^{d+1},$$

and therefore H is an ϵ-hero. (Note that, if $\epsilon < 1$, we could apply 2.2 here instead, and avoid the extra log factor.)

For the converse, let H be an ϵ-hero. Thus there exist $c, d \geq 0$ such that every H-free tournament G with $|G| > 1$ satisfies $\chi(G) \leq c|G|^{1-\epsilon}(\log(|G|))^d$. But every non-null tournament G has a transitive set of cardinality at least $|G|/\chi(G)$ (take the largest set of the partition given by the colouring). Consequently, every H-free tournament G with $|G| > 1$ has a transitive set of cardinality at least $c^{-1}|G|^\epsilon(\log(|G|))^{-d}$. It follows that H is an ϵ-celebrity. This proves 2.3. ∎

3 ϵ-celebrities that are not strongly connected

In this section we prove the second statement of 1.6, the following.

3.1 *For $0 \leq \epsilon \leq 1$, a tournament is an ϵ-celebrity if and only if all its strong components are ϵ-celebrities.*

Let T be a tournament and let $X, Y \subseteq V(T)$ be disjoint. We denote by $e_{X,Y}$ the number of edges xy where $x \in X$ and $y \in Y$. If $X, Y \neq \emptyset$, the *density from X to Y* is

$$d(X, Y) = \frac{e_{X,Y}}{|X||Y|}.$$

Note that $d(X, Y) = 1 - d(Y, X)$, since T is a tournament. We need the following, which follows easily by a standard application of the regularity lemma (see for instance [5] for an analogous argument).

3.2 *For every tournament H and every real $\lambda > 0$ there exists a real $c > 0$ with the following property. For every H-free tournament G there exist disjoint subsets $X, Y \subseteq V(G)$ with $|X|, |Y| = \lceil c|V(G)|\rceil$, such that $d(X, Y) < \lambda$.*

Let H_1, H_2 be tournaments. Let G be a tournament such that there is a partition (V_1, V_2) of $V(G)$ with $V_1 \Rightarrow V_2$, where for $i = 1, 2$, the subtournament of G with vertex set V_i is isomorphic to H_i. We denote such a tournament G by $H_1 \Rightarrow H_2$. For two sets of tournaments $\mathcal{F}_1$ and $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ the set consisting of all tournaments (up to isomorphism) of the form $H_1 \Rightarrow H_2$ for some $H_1 \in \mathcal{F}_1$ and $H_2 \in \mathcal{F}_2$. For a set $\mathcal{F}$ of tournaments, we say that a tournament T is *$\mathcal{F}$-free* if no subtournament of T is isomorphic to a member of $\mathcal{F}$. We need the following lemma.

3.3 *Let $h \geq 1$ be an integer, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sets of tournaments, where each tournament in $\mathcal{F}_1 \cup \mathcal{F}_2$ has at most h vertices. Then there exists $C > 0$ with the following property. Let ϕ be round, such that for $i = 1, 2$, every $\mathcal{F}_i$-free tournament T of order $n > 1$ satisfies $\alpha(T) \geq n/\phi(n)$. Then every ($\mathcal{F}_1 \Rightarrow \mathcal{F}_2$)-free tournament T of order $n > 1$ satisfies $\alpha(T) \geq Cn/\phi(n)$.*

Proof. If one of $\mathcal{F}_1$ and $\mathcal{F}_2$ is empty, the result is trivial, so we assume both are non-empty, and hence $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ is nonempty. Choose one of its members, H_0 say. Choose $c > 0$ satisfying 3.2, taking $H = H_0$ and $\lambda = (4h)^{-1}$. Let $C = c/2$. We will show that C satisfies the theorem.

Let T be an ($\mathcal{F}_1 \Rightarrow \mathcal{F}_2$)-free tournament with $n > 1$ vertices. By 3.2, there exist disjoint $V_1, V_2 \subseteq V(G)$ with $|V_1|, |V_2| \geq c|V(T)|$ such that $d(V_2, V_1) < (4h)^{-1}$. Let X be the set of all vertices in V_1 with at least $(1 - (2h)^{-1})|V_2|$ out-neighbours in V_2. Every vertex in $V_1 \setminus X$ is adjacent from at least $(2h)^{-1}|V_2|$ members of V_2, and so

$$|V_1 \setminus X|(2h)^{-1}|V_2| \leq (4h)^{-1}|V_1||V_2|,$$

that is, $|X| \geq |V_1|/2$.

Now $|V_1| \geq cn$. Suppose that $T|X$ is $\mathcal{F}_1$-free. From the hypothesis, X includes a transitive subset of cardinality at least $|X|/\phi(|X|)$; but $\phi(|X|) \leq \phi(n)$, and $|X| \geq cn/2$, and so $\alpha(T) \geq Cn/\phi(n)$ as required. Thus we may assume that there exists $X' \subseteq X$ such that $T|X'$ is isomorphic to some member H_1 of $\mathcal{F}_1$. For each $x \in X'$, at most $(2h)^{-1}|V_2|$ vertices in V_2 are adjacent to x, since $x \in X$;

and since $|X'| \leq h$, it follows that at most $|V_2|/2$ vertices in V_2 are adjacent to a vertex in X'. Let Y be the set of all $y \in V_2$ that are adjacent from every vertex in X'; then $|Y| \geq |V_2|/2$. Since T is $(\mathcal{F}_1 \Rightarrow \mathcal{F}_2)$-free, it follows that $T|Y$ is $\mathcal{F}_2$-free; and so from the hypothesis, Y includes a transitive subset of cardinality at least $|Y|/\phi(|Y|)$. But $\phi(|Y|) \leq \phi(n)$, and

$$|Y| \geq |V_2|/2 \geq cn/2 = Cn,$$

and so $\alpha(G) \geq Cn/\phi(n)$. This proves 3.3. ∎

Proof of 3.1. Since every subtournament of an ϵ-celebrity is an ϵ-celebrity, the "only if" part of 3.1 is clear. The "if" part is implied by 3.3, taking $\phi(n) = cn^{1-\epsilon}(\log(n))^d$ for appropriate c, d. This proves 3.1. ∎

4 Adding handles

To complete the proof of 1.6, we need to show the following, which is proved in this section:

4.1 *For $0 \leq \epsilon \leq 1$, let H be an ϵ-hero, and let $k \geq 1$ be an integer. Then $\Delta(H, 1, k)$ and $\Delta(k, 1, H)$ are ϵ-heroes.*

We prove, more generally:

4.2 *Let H be a tournament, and let ϕ be round, such that every H-free tournament G satisfies $\chi(G) \leq \phi(|G|)$. Let $k \geq 1$ be an integer. Then there exists $c \geq 0$ such that every $\Delta(H, 1, k)$-free tournament G satisfies $\chi(G) \leq c\phi(G)\log(|G|)$, and the same for $\Delta(k, 1, H)$.*

We remark that if ϕ grows sufficiently quickly to satisfy the hypotheses of 2.2 we could use the latter to avoid the extra log factor.

Let H, K be tournaments, and let $a \geq 1$ be an integer. An *(a, H, K)-jewel* in a tournament G is a subset $X \subseteq V(G)$ such that $|X| = a$, and for every partition (A, B) of X, either $G|A$ contains H or $G|B$ contains K. An *(a, H, K)-jewel-chain* of length t is a sequence $Y_1, \ldots, Y_t$ of (a, H, K)-jewels, pairwise disjoint, such that $Y_i \Rightarrow Y_{i+1}$ for $1 \leq i < t$. We need the following lemma, proved in [3]:

4.3 *Let H, K be tournaments, such that one of them is transitive, and let $a \geq 1$ be an integer. Then there are integers $\lambda_1, \lambda_2 \geq 0$ with the following property. For every $\Delta(H, 1, K)$-free tournament G, if*

- *c_1 is such that every H-free subtournament of G has chromatic number at most c_1, and every K-free subtournament of G has chromatic number at most c_1, and*
- *c_2 is such that every subtournament of G containing no (a, H, K)-jewel-chain of length four has chromatic number at most c_2,*

then G has chromatic number at most $\lambda_1 c_1 + \lambda_2 c_2$.

Let us point out that there is a serious error in [3]; the statement of 4.3 published there (theorem 4.5 of that paper) mistakenly omitted the hypothesis that "one of them is transitive", which is needed for the application of theorem 4.4 of that paper, three lines from the end of its proof. Fortunately, in the application of 4.3 in [3], the missing hypothesis is satisfied.

Proof of 4.2, 4.1 and 1.6. Let K be a transitive tournament with k vertices; from the symmetry, it suffices to show the result for $\Delta(H,1,K)$. Let ϕ be as in the hypothesis of the theorem. We may assume that $\phi(2) \geq 2^k$, by scaling ϕ. Let $a = 2^k|V(H)|$, and let $\lambda_1, \lambda_2 \geq 0$ be as in 4.3.

(1) *If G is a tournament with $|G| > 1$, not containing an (a,H,K)-jewel, then $\chi(G) \leq a\phi(|G|)$.*

Choose pairwise vertex-disjoint subtournaments $H_1, \ldots, H_t$ of G, each isomorphic to H, with t maximum, and let the union of their vertex sets be W. If $t \geq 2^k$, then since every tournament with at least 2^k vertices has a transitive subset of cardinality k, it follows that $V(H_1) \cup \cdots \cup V(H_{2^k})$ is an (a,H,K)-jewel, a contradiction. Thus $t < 2^k$. Consequently $\chi(G|W) \leq |W| \leq a$, and $\chi(G \setminus W) \leq \phi(|G| - |W|) \leq \phi(|G|)$ since $G \setminus W$ is H-free. It follows that $\chi(G) \leq a + \phi(|G|) \leq a\phi(|G|)$ since $a, \phi(|G|) \geq 2$. This proves (1).

(2) *There exists $C \geq 0$ such that if G is a tournament with $|G| > 1$, not containing an (a,H,K)-jewel-chain of length four, then $\chi(G) \leq C\phi(G)\log(|G|)$.*

By (1), if G is a tournament with $n > 1$ vertices, not containing an (a,H,K)-jewel, then $\alpha(G) \geq a^{-1}n/\phi(n)$. By 3.3 applied twice, there exists $C > 0$ such that every tournament G of order $n > 1$ containing no (a,H,K)-jewel-chain of length four satisfies $\alpha(G) \geq C^{-1}n/\phi(n)$. By 2.1, every such G satisfies $\chi(G) \leq C\phi(n)\log(n)$. This proves (2).

Let $c = \lambda_1 + \lambda_2 C$; we claim that c satisfies the theorem. For let G be a $\Delta(H,1,K)$-free tournament, with $n > 1$ vertices. Let $c_1 = \phi(n)$. Then every H-free subtournament of G has chromatic number at most c_1; and so does every K-free subtournament of G, since every K-free tournament has at most 2^k vertices and hence has chromatic number at most $2^k \leq \phi(2) \leq \phi(n) = c_1$. Let $c_2 = C\phi(n)\log(n)$; then every subtournament of G not containing an (a,H,K)-jewel-chain of length four has chromatic number at most c_2, by (2). By 4.3,

$$\chi(G) \leq \lambda_1 c_1 + \lambda_2 c_2 = \lambda_1\phi(n) + \lambda_2 C\phi(n)\log(n) \leq (\lambda_1 + \lambda_2 C)\phi(n)\log(n).$$

This proves 4.2, and hence 4.1, and therefore finishes the proof of 1.6. ∎

That completes all we have to say about ϵ-heroes in general.

5 Excluding $\Delta(2,k,l)$

Now we return to the case $\epsilon = 1$ and the proof of 1.5. So far we have proved the first two statements of 1.5, and part of the "if" half of the third statement, all as corollaries of 1.6. In this section we complete the proof of the "if" half of the third statement of 1.5, by proving the following.

5.1 *For all $k,l \geq 2$, there exists $c > 0$ such that every $\Delta(2,k,l)$-free tournament G with $|G| > 1$ satisfies $\alpha(G) \geq c|G|/\log(|G|)$.*

This follows immediately from 5.3 and 5.4, proved below. We need the "bipartite Ramsey theorem", proved by Beineke and Schwenk [2], the following. If X, Y are disjoint subsets of the vertex set of a graph G, we say X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and X is *anticomplete to* Y if there are no edges between X and Y.

5.2 *For all integers $l \geq 0$ there exists $K \geq 0$, such that for every graph with bipartition (A, B) where $|A|, |B| \geq K$, there exist $X \subseteq A$ and $Y \subseteq B$ with $|X| = |Y| = l$, such that either X is complete to Y or X is anticomplete to Y.*

The smallest K satisfying the statement of 5.2 will be denoted by $K(l)$.

If G is a tournament and uv is an edge, we say that u is *adjacent to* v and v is *adjacent from* u. Let $(v_1, ..., v_n)$ be an enumeration of the vertex set of a tournament G (thus, with $n = |V(G)|$). We say that an edge v_iv_j of G is a *backedge* under this enumeration if $i > j$. If $t \geq 0$ is an integer, an enumeration $(v_1, \dots, v_n)$ of $V(G)$ is said to be *t-forward* if for every two sets $X, Y \subseteq V(G)$ with $|X| = |Y| = t$, there exist $v_i \in X$ and $v_j \in Y$ such that either $i \geq j$, or v_iv_j is an edge of G.

5.3 *For all integers $k \geq 2$, there exists $c > 0$ such that, if G is a $\Delta(2, k, k)$-free tournament with $|G| > 1$ that admits a 2^k-forward enumeration, then $\alpha(G) \geq c|G|/\log(|G|)$.*

Proof. Let $M = 2^kK(2^k)$ and $c = 1/(4M)$. We will show that c satisfies the theorem. For let G be a $\Delta(2, k, k)$-free tournament with $|G| > 1$, and let $(v_1, \dots, v_n)$ be a 2^k-forward enumeration of $V(G)$. For $1 \leq i \leq n$, we define $\phi(v_i) = i$. A backedge vu of G is *left-active* if there is no set $A \subseteq V(G)$ such that:

- $|A| = K(2^k)$
- for each $a \in A$, $\phi(u) < \phi(a) < (\phi(u) + \phi(v))/2$
- each $a \in A$ is adjacent from u and from v.

Similarly, a backedge vu is *right-active* if there is no set $B \subseteq V(G)$ such that:

- $|B| = K(2^k)$
- for each $b \in B$, $(\phi(u) + \phi(v))/2 < \phi(b) < \phi(v)$
- each $b \in B$ is adjacent to u and to v.

(1) *Every backedge vu is either left-active or right-active.*

For suppose that vu is a backedge that is neither left-active nor right-active. Thus there exist sets A and B as above. Let J be the graph with bipartition (A, B), in which $a \in A$ and $b \in B$ are adjacent if ba is an edge (and hence a backedge) of G. By 5.2, there exist $X \subseteq A$ and $Y \subseteq B$ such that $|X| = |Y| = 2^k$, and X is either complete or anticomplete to Y in J. Since the enumeration is 2^k-forward, and $\phi(x) < (\phi(u) + \phi(v))/2 < \phi(y)$ for all $x \in X$ and $y \in Y$, it follows that there exist $x \in X$ and $y \in Y$ such that yx is not a backedge of G, and thus x, y are not adjacent in J; and consequently X is anticomplete to Y in J, and so every vertex in y is adjacent in G from every vertex in X. Since $|X| = |Y| = 2^k$, there are transitive subsets X' of X and Y' of Y, both of cardinality k (by a theorem of [8], that every tournament with 2^k vertices has a transitive set of cardinality

$k+1$). But then the subtournament of G with vertex set $X' \cup Y' \cup \{u, v\}$ is isomorphic to $\Delta(2,k,k)$, a contradiction. This proves (1).

For a backedge vu, we call $\phi(v) - \phi(u)$ its *length*.

(2) *There do not exist $M\log(n)$ left-active edges in G with the same tail v.*

Suppose there do exist such edges. Since their lengths are all between 1 and $n-1$, it follows that for some integer t with $0 \le t \le \log(n)$, there are M left-active edges all with tail v and all with length between 2^t and $2^{t+1}-1$. Let them be vu_i ($1 \le i \le M$), numbered such that $\phi(u_i) < \phi(u_j)$ for $1 \le i < j \le M$. For $1 \le i < j \le M$, since

$$\phi(v) - \phi(u_j) \ge 2^t > (\phi(v) - \phi(u_i))/2,$$

it follows that $\phi(u_i) < \phi(u_j) < (\phi(u_i) + \phi(v))/2$. Let $X = \{u_i : 1 \le i \le 2^k\}$, and $Y = \{u_i : 2^k < i \le M\}$. For each $u_i \in X$, vu_i is left-active, and so u_i is adjacent in G to at most $(K(2^k)-1)$ members of Y. Consequently there are at least $|Y| - |X|(K(2^k)-1) \ge 2^k$ members of Y that are adjacent in G to each member of X, contradicting that the enumeration is 2^k-forward. This proves (2).

By (2) there are at most $Mn\log(n)$ left-active edges in G, and similarly at most $Mn\log(n)$ right-active. By (1), it follows that there are at most $2Mn\log(n) = (2c)^{-1}n\log(n)$ backedges. Let J be the graph with vertex set $V(G)$ in which u, v are adjacent for each backedge vu. Thus $|E(J)| \le (2c)^{-1}n\log(n)$. By Turan's theorem [4], applied to J, we deduce that J has a stable set of cardinality at least $cn/\log(n)$, and so $\alpha(G) \ge cn/\log(n)$. This proves 5.3. ∎

5.4 *For all integers $k \ge 2$ there exists $c > 0$ such that every $\Delta(2,k,k)$-free tournament G has a subtournament with at least $c|G|$ vertices that admits a 2^k-forward enumeration.*

Proof. Let $b = 2k+1$, and $d = (12k-1)b$. Let $c > 0$ be the real number satisfying

$$\log(c) = -240b^2 2^{7bd}.$$

We will show that c satisfies the theorem.

Let G be a $\Delta(2,k,k)$-free tournament. Let us say a *chain* is a sequence $A_1, \dots, A_m$ of subsets of $V(G)$ with the following properties:

- $A_1, \dots, A_m$ are pairwise disjoint
- for $1 \le i \le m$, $|A_i| = bd$ and A_i is transitive
- for $1 \le i < j \le m$, each vertex in A_j is adjacent to at most d vertices in A_i, and each vertex in A_i is adjacent from at most d vertices in A_j.

(1) *We may assume that G admits a chain $A_1, \dots, A_m$ with $m \ge 4$.*

For if $n < 2^{4bd}$ then the theorem holds, since $c < 2^{-4bd}$ and so any one-vertex subtournament of G satisfies the theorem (and if G is null then G itself satisfies the theorem). Thus we assume that $n \geq 2^{4bd}$, and so G contains a transitive set of cardinality $4bd$. But then there is a chain A_1, A_2, A_3, A_4. This proves (1).

Let $A_1, \dots, A_m$ be a chain with m maximum. Define $A = A_1 \cup \dots \cup A_m$. For $1 \leq i < m$, let B_i be the set of all $v \in V(G) \setminus A$ such that there exist $Y \subseteq A_i$ and $Z \subseteq A_{i+1}$ with $|Y| = |Z| = k$ and $\{v\} \Rightarrow Y \Rightarrow Z \Rightarrow \{v\}$. Let $B = B_1 \cup \dots \cup B_{m-1}$, and $C = V(G) \setminus (A \cup B)$.

(2) $|B| \leq m(bd)^{2k}$.

For suppose not. Then $|B_i| > (bd)^{2k}$ for some i with $1 \leq i < m$. For each $v \in B_i$, choose $Y_v \subseteq A_i$ and $Z_v \subseteq A_{i+1}$ such that $|Y_v| = |Z_v| = k$ and $\{v\} \Rightarrow Y_v \Rightarrow Z_v \Rightarrow \{v\}$. Since there are at most $(bd)^{2k}$ possibilities for the pair (Y_v, Z_v), there exist distinct u, v with $Y_u = Y_v$ and $Z_u = Z_v$. But then the subtournament of G with vertex set $\{u, v\} \cup Y_v \cup Z_v$ is isomorphic to $\Delta(2, k, k)$, a contradiction.

(3) *For each $v \in C$, there is no i with $1 \leq i < m$ such that v has at least k out-neighbours in A_i and at least $(d+1)k$ in-neighbours in A_{i+1}. Also, there is no i with $1 \leq i < m$ such that v has at least $(d+1)k$ out-neighbours in A_i and at least k in-neighbours in A_{i+1}. In particular, there is no i with $1 \leq i < m$ such that v has at least $bd/2$ out-neighbours in A_i and at least $bd/2$ in-neighbours in A_{i+1}.*

For the first claim, suppose that $Y \subseteq A_i$ and $Z \subseteq A_{i+1}$ with $|Y| = k$ and $|Z| \geq (d+1)k$, and v is adjacent to every vertex in Y and adjacent from every vertex in Z. Now each vertex in Y has at most d in-neighbours in Z, and so at most dk vertices in Z have an out-neighbour in Y. Consequently, there exists $Z' \subseteq Z$ with $|Z'| = k$, such that $Y \Rightarrow Z'$. But then Y, Z' show that $v \in B_i \subseteq B$, a contradiction. This proves the first claim, and the second follows from the symmetry. The third follows since $bd/2 \geq k$ and $bd/2 \geq (d+1)k$. This proves (3).

For $1 \leq i < m$ let C_i be the set of all vertices $v \in C$ such that v has at least $bd/2$ in-neighbours in A_i and at least $bd/2$ out-neighbours in A_{i+1}. (Note that bd is odd, so equality is not possible here.) Let C_0 be the set of all $v \in C$ with at least $bd/2$ out-neighbours in A_1, and let C_m be the set of all $v \in C$ with at least $bd/2$ in-neighbours in A_m. By (3), it follows that $C_0, C_1, \dots, C_m$ are pairwise disjoint and have union C.

(4) *Let $0 \leq i \leq m$ and let $v \in C_i$. Then for $1 \leq h < i$, v has at most $k-1$ out-neighbours in A_h; and for $i+1 < j \leq m$, v has at most $k-1$ in-neighbours in A_j.*

For v has at least $bd/2$ in-neighbours in A_i, and since $v \notin B$, it follows from (3) that v has at least $bd/2$ in-neighbours in each of $A_1, \dots, A_i$. In particular, v has at least $bd/2$ in-neighbours in A_{h+1}. By (3), v has at most $k-1$ out-neighbours in A_h. This proves the first assertion. The second follows by the symmetry. This proves (4).

For $2 \leq i \leq m$ let $L_i = A_1 \cup \dots \cup A_{i-2}$, and for $0 \leq i \leq m-2$ let $R_i = A_{i+3} \cup \dots \cup A_m$. Let L_0, L_1, R_{m-1}, R_m all be the null set. (It follows that L_2, R_{m-2} are also empty.)

(5) *Let $0 \leq i \leq m$, and let $u, v \in L_i$ be distinct. Then there is no transitive set $Z \subseteq C_i$ with*

$|Z| = k$ *such that* $Z \Rightarrow \{u, v\}$*, and consequently there are at most* 2^k *vertices in* C_i *that are adjacent to both* u *and* v*. Similarly, for* $0 \leq i \leq m$*, if* $u, v \in R_i$ *then there is no transitive set* $Z \subseteq C_i$ *with* $|Z| = k$ *such that* $\{u, v\} \Rightarrow Z$*, and hence there are at most* 2^k *vertices in* C_i *that are adjacent from both* u *and* v*.*

For let $0 \leq i \leq m$, and let $u, v \in L_i$ (thus $i \geq 3$), and suppose that there exists a transitive set $Z \subseteq C_i$ with $|Z| = k$ such that every vertex in Z is adjacent to both u, v. By (4), each member of Z has at most $k - 1$ out-neighbours in A_{i-1}. Also, u, v each have at most at in-neighbours in A_{i-1}. Consequently there is a subset Y of A_{i-1} with $|Y| = k$ such that $\{u, v\} \Rightarrow Y \Rightarrow Z$, since $bd - (k-1)k - 2d \geq k$. But then the subtournament of G with vertex set $\{u, v\} \cup Y \cup Z$ is isomorphic to $\Delta(2, k, k)$, a contradiction. This proves the first assertion, and the second follows by symmetry. This proves (5).

(6) *For* $0 \leq i \leq m$*, and all* $u \in L_i$ *and* $v \in R_i$*, there are fewer than* 2^{7bd} *vertices in* C_i *that are adjacent to* u *and from* v*.*

For since $L_i, R_i \neq \emptyset$, it follows that $3 \leq i \leq m - 3$. Suppose that there are at least 2^{7bd} vertices in C_i adjacent to u and from v; then they include a transitive set Y of cardinality $7bd$. Choose a chain $Y_1, \ldots, Y_7$ of subsets of Y such that $Y_h \Rightarrow Y_j$ for all h, j with $1 \leq h < j \leq 7$. By (5), every vertex in $L_i \setminus \{u\}$ has at most $k - 1 \leq d$ in-neighbours in Y, and every vertex in $R_i \setminus \{v\}$ has at most d out-neighbours in Y. Also, each vertex in Y has at most $k - 1 \leq d$ out-neighbours in A_h for $1 \leq h \leq i - 2$, and at most d in-neighbours in A_j for $i + 2 \leq j \leq m$, by (4). Choose h, j with $u \in A_h$ and $v \in A_j$. Then

$$A_1, \ldots, A_{h-1}, A_{h+1}, \ldots, A_{i-2}, Y_1, Y_2, \ldots, Y_7, A_{i+3}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m$$

is a chain with $m + 1$ terms, contrary to the maximality of m. This proves (6).

(7) *Let* $0 \leq i \leq m$*, and let* $Z \subseteq C_i$ *be transitive. Let* p *be an integer such that* $|Z| \leq bdp$ *and* $2b(k-1)p < d$*. Then there are fewer than* $2bp$ *vertices in* L_i *that are adjacent from at least* d *members of* Z*.*

For suppose that there exists $W \subseteq L_i$ with $|W| = 2bp$ such that each member of W is adjacent from at least d members of Z. Each member of W has at least d in-neighbours in Z, and yet every two distinct members of W have at most $k - 1$ common in-neighbours in Z, by (5). Hence $|Z| \geq d|W| - (k-1)|W|^2/2$. Since $|Z| \leq bdp$ and $|W| = 2bp$, it follows that $2(k-1)bp \geq d$, a contradiction. Thus there is no such W. This proves (7).

(8) *For* $0 \leq i \leq m$ *and all* $v \in R_i$*, if* $Y \subseteq C_i$ *is transitive and* $v \Rightarrow Y$ *then* $|Y| < 12b \cdot 2^{7bd}$*.*

We may assume that $R_i \neq \emptyset$, and so $i \leq m - 3$. Choose a maximal subset Z of Y such that every vertex in L_i is adjacent from at most d members of Z. Suppose that $|Z| \geq 6bd$, and choose a chain $Z_1, \ldots, Z_6$ of subsets of Z such that $Z_h \Rightarrow Z_j$ for $1 \leq h < j \leq 6$. By (2), every vertex of R_i different from v is adjacent to at most $k - 1 \leq at$ members of Y. Let $v \in A_j$. By (4), if $i \geq 2$ then

$$A_1, \ldots, A_{i-2}, Z_1, \ldots, Z_6, A_{i+3}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m$$

is a chain with $m+1$ terms, contrary to the maximality of m; while if $i \leq 1$ then the chain

$$Z_1, \ldots, Z_6, A_{i+3}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m$$

gives a contradiction similarly. Thus $|Z| < 6bd$.

We say $u \in L_i$ is *saturated* if u is adjacent from exactly d members of Z. Since $|Z| < 6bd$ and $12(k-1)b < d$, it follows from (7) with $p = 6$ that there are fewer than $12b$ saturated vertices in L_i. But every vertex in $Y \setminus Z$ is adjacent to a saturated vertex in L_i, from the maximality of Z. Since every saturated vertex in L_i is adjacent from at most 2^{7bd} members of Y, by (6), and hence from at most $2^{7bd} - d$ members of $Y \setminus Z$, it follows that $|Y \setminus Z| \leq 12b(2^{7bd} - d)$, and so

$$|Y| \leq 12b(2^{7bd} - d) + 6bd < 12b \cdot 2^{7bd}.$$

This proves (8).

(9) *For $0 \leq i \leq m$, there is no transitive subset Y of C_i with $|Y| \geq 240b^2 2^{7bd}$.*

Let $Y \subseteq C_i$ be transitive. Choose a maximal subset Z of Y such that every vertex of L_i is adjacent from at most d members of Z, and every vertex in R_i is adjacent to at most d members of Z. Suppose that $|Z| \geq 5bd$, and choose a chain $Z_1, \ldots, Z_5$ of subsets of Z such that $Z_h \Rightarrow Z_j$ for $1 \leq h < j \leq 5$. If $2 \leq i \leq m-2$ then by (4),

$$A_1, \ldots, A_{i-2}, Z_1, \ldots, Z_5, A_{i+3}, \ldots, A_m$$

is a chain with $m+1$ terms, a contradiction; while if $i \leq 1$ then

$$Z_1, \ldots, Z_5, A_{i+3}, \ldots, A_m$$

gives a contradiction, and if $i \geq m-1$ then

$$A_1, \ldots, A_{i-2}, Z_1, \ldots, Z_5$$

gives a contradiction. Thus $|Z| < 5bd$.

We say $u \in L_i$ is *saturated* if it is adjacent from exactly d members of Z; and $v \in R_i$ is *saturated* if it is adjacent to exactly d members of Z. Since $|Z| \leq 5t$, and $10(k-1)b < d$, it follows from (7) with $p = 5$ that there are at most $10b$ saturated vertices in L_i, and similarly at most $10b$ saturated vertices in R_i. From the maximality of Z, every vertex of $Y \backslash Z$ is adjacent to at least one of the saturated vertices in L_i or from at least one of the saturated vertices in R_i. But by (8), each saturated vertex in L_i is adjacent from at most $12b2^{7bd}$ members of Y and hence from at most $12b2^{7bd} - d$ members of $Y \setminus Z$, and similarly every saturated vertex in R_i is adjacent to at most $12b2^{7bd} - d$ members of $Y \setminus Z$. We deduce that

$$|Y| < 20b(12b2^{7bd} - d) + 5bd \leq 240b^2 2^{7bd}.$$

This proves (9).

(10) $|A| \geq 2c|G|$ *where c is as defined in the statement of the theorem.*

From (9), each C_i has cardinality at most $2^{240b^2 2^{7bd}-1}$, and so $|C| \le (m+1)2^{240b^2 2^{7bd}-1}$. Since $m \ge 2$ (and hence $m+1 \le 2m$), and $|B| \le m(bd)^{2k}$ by (2), and $|A| = mbd$, we deduce that

$$|G| \le (2^{240b^2 2^{7bd}} + (bd)^{2k} + bd)m \le (2^{240b^2 2^{7bd}} + (bd)^{2k} + bd)|A|/(bd).$$

It follows that $|A| \ge 2c|G|$ where c is as defined in the statement of the theorem. This proves (10).

Let V be the union of all A_i with $1 \le i \le m$ and i odd. Then $|V| \ge |A|/2 \ge c|G|$. Number the members of V as $\{v_1, \dots, v_t\}$ say, where for $1 \le r < s \le t$, if $x_r \in A_i$ and $x_s \in A_j$ then $i \le j$, and either $i < j$ or x_r is adjacent to x_s. (This is possible since each A_i is transitive.) We claim that this order is 2^k-forward. For let Y, Z be disjoint subsets of V with $|Y| = |Z| = 2^k$, such that for $1 \le r, s \le t$, if $x_r \in Y$ and $x_s \in Z$ then $r < s$. We must show that there exist $y \in Y$ and $z \in Z$ such that y is adjacent to z. Suppose not. Choose i with $1 \le i \le m$ and i odd, maximum such that $A_i \cap Y \neq \emptyset$. It follows that $A_h \cap Z = \emptyset$ for all $h < i$. If $Z \cap A_i \neq \emptyset$, let $v_r \in A_i \cap Y$ and $v_s \in A_i \cap Z$; it follows that $r < s$ from the choice of the numbering, and so v_r is adjacent to v_s, a contradiction. Thus $Z \cap A_i = \emptyset$. It follows that $j \ge i+2$ for each j with $1 \le j \le m$ such that $z \cap A_j \neq \emptyset$. Since $|Y| = 2^k$, there exists $Y' \subseteq Y$ with $|Y| = k$ such that Y is transitive, and similarly there exists a transitive $Z' \subseteq Z$ with $|Z'| = k$. Now each member of Y' is adjacent from at most d members of A_{i+1}, and so there are at most dk vertices in A_{i+1} adjacent to some member of Y'; and similarly at most dk are adjacent from some member of Z'. Since $bd \ge 2dk+2$, there are two vertices $u, v \in A_{i+1}$ such that $Y' \Rightarrow \{u, v\}$ and $\{u, v\} \Rightarrow Z'$. But then the subtournament of G with vertex set $\{u, v\} \cup Y' \cup Z'$ is isomorphic to $\Delta(2, k, k)$, a contradiction. This proves that the order is 2^k-forward, and so completes the proof of 5.4. ∎

Proof of 5.1. This follows immediately from 5.3 and 5.4. ∎

6 Strongly-connected pseudo-heroes

In this section we complete the proof of 1.5, and also prove 1.7. As a biproduct of the remainder of the proof of 1.5, we are able to identify all the minimal tournaments that are not pseudo-heroes (there are six). Here they are:

- Let H_1 be the tournament with five vertices $v_1, \dots, v_5$, in which v_i is adjacent to v_{i+1} and v_{i+2} for $1 \le i \le 5$ (reading subscripts modulo 5).
- Let H_2 be the tournament obtained from H_1 by replacing the edge v_5v_1 by an edge v_1v_5.
- Let H_3 be the tournament with five vertices $v_1, \dots, v_5$ in which v_i is adjacent to v_j for all i, j with $1 \le i < j \le 4$, and v_5 is adjacent to v_1, v_3 and adjacent from v_2, v_4.
- Let H_4 be the tournament $\Delta(1, \Delta(1,1,1), \Delta(1,1,1))$
- Let H_5 be the tournament $\Delta(2, 2, \Delta(1,1,1))$
- Let H_6 be the tournament $\Delta(3,3,3)$.

First, we prove they are not pseudo-heroes, but also it is helpful to give the best upper bounds on their ξ-values that we can. We begin with:

6.1 *If H is a strongly-connected tournament with more than one vertex that does not admit a trisection, then $\xi(H) \leq 1/\log(3)$. In particular, $\xi(H_i) \leq 1/\log(3)$ for $i = 1,2,3$, and so H_1, H_2, H_3 are not pseudo-heroes.*

Proof. Let D_0 be the one-vertex tournament, and for $i \geq 1$ let $D_i = \Delta(D_{i-1}, D_{i-1}, D_{i-1})$. Thus $|D_i| = 3^i$. For $i > 0$, no transitive subtournament of D_i intersects all three parts of the trisection of D_i, so $\alpha(D_i) = 2\alpha(D_{i-1})$; and consequently $\alpha(D_i) = 2^i = |D_i|^{1/\log(3)}$. We claim that for all $i \geq 0$, D_i does not contain H; for suppose D_i contains H for some value of i, and choose the smallest. Then $i \geq 1$ since $|V(H)| \geq 2$, and so D_i admits a trisection (A, B, C) where $D_i|A, D_i|B, D_i|C$ are all isomorphic to D_{i-1}. Choose a subtournament T of D_i isomorphic to H. From the minimality of i, $V(T)$ is not a subset of any of A, B, C, and therefore has nonempty intersection with at least two of them; and since H is strongly-connected, $V(T)$ has nonempty intersection with all three of A, B, C. But then T admits a trisection, a contradiction.

This proves that no D_i contains H. Let ϵ be an EH-coefficient for H, and choose $c > 0$ such that every H-free tournament G satisfies $\alpha(G) \geq c|G|^\epsilon$. In particular, taking $G = D_i$ implies that

$$|D_i|^{1/\log(3)} = \alpha(D_i) \geq c|D_i|^\epsilon,$$

for all $i \geq 0$. It follows that $1/\log(3) \geq \epsilon$. Since this holds for all EH-coefficients ϵ, it follows that $\xi(H) \leq 1/\log(3)$. This proves 6.1. ∎

6.2 $\xi(H_4) \leq 1/2$, *and hence H_4 is not a pseudo-hero.*

Proof. For $k \geq 1$, let D_k be the tournament with k^2 vertices $v_1, \ldots, v_{k^2}$, in which for $1 \leq i < j \leq k^2$, v_i is adjacent to v_j if k does not divide $j - i$, and otherwise v_j is adjacent to v_i. (This construction is due to Gaku Liu, in private communication.) For $1 \leq i \leq k$, let $C_i = \{v_i, v_{i+k}, v_{i+2k}, \ldots, v_{i+(k-1)k}\}$. Then $C_1, \ldots, C_k$ are disjoint and have union $V(D_k)$.

(1) $\alpha(D_k) \leq 2k - 1$.

Let $X \subseteq V(D_k)$ induce a transitive tournament. For $1 \leq i \leq k$, if $X \cap C_i \neq \emptyset$, let p_i be the smallest value of j such that $v_j \in X \cap C_i$, and q_i the largest; and let I_i be $\{v_j : p_i \leq j \leq q_i\}$. If $X \cap C_i = \emptyset$, let $I_i = \emptyset$. Note that if $v_j \in X \cap I_i$ then $j \in C_i$; because otherwise $\{v_{p_i}, v_{q_i}, v_j\}$ would induce a cyclic triangle, contradicting that X is transitive. This has two consequences:

- For each $i \in \{1, \ldots, k\}$, $|X \cap I_i| \leq 1 + (|I_i| - 1)/k$, since between any two members of X in I_i there are $k - 1$ members of $C_i \setminus X$. Summing over i, we deduce that $|X| \leq k - 1 + \sum_i |I_i|/k$.
- The sets I_i $(1 \leq i \leq k)$ are pairwise disjoint, and so $\sum_i |I_i| \leq k^2$.

Combining these, we deduce that $|X| \leq 2k - 1$. This proves (1).

(2) *D_k does not contain H_4.*

For $1 \leq j \leq k^2$, let $\phi(v_j)$ be the value of $i \in \{1, \ldots, k\}$ with $v_j \in C_i$. Thus, let $a, b, c \in V(D_k)$ be distinct with the following properties:

(P) if $\{a, b, c\}$ induces a cyclic triangle in D_k then $|\{\phi(a), \phi(b), \phi(c)\}| = 2$; and

(Q) if ab, ac, bc are edges and $\phi(a) = \phi(c)$ then $\phi(b) = \phi(a)$.

(R) if $\{a, b, c\}$ induces a cyclic triangle and d is some other vertex such that $d \Rightarrow \{a, b, c\}$ or $d \Leftarrow \{a, b, c\}$ then $\phi(d) \neq \phi(a), \phi(b), \phi(c)$.

(The third condition above follows easily from the other two, but we use it enough to give it a separate name.) For $X \subseteq V(D_k)$, $\phi(X)$ denotes $\{\phi(v) : v \in X\}$. Suppose that D_k contains H_4, and let A, B, C be the trisection of H_4 with $|A| = |B| = 3$; let $A = \{a_1, a_2, a_3\}$, and $B = \{b_1, b_2, b_3\}$, and $C = \{c\}$. Thus from property P applied to A, $|\phi(A)| = 2$, and similarly $|\phi(B)| = 2$; by property R applied to A and each member of B, $\phi(A)$ and $\phi(B)$ are disjoint; and by property R applied to A and c, $\phi(c) \notin \phi(A)$ and similarly $\phi(c) \notin \phi(B)$. Choose $a \in A$ and $b \in B$; then $\phi(a), \phi(b), \phi(c)$ are all distinct, contrary to property P. This proves (2).

Let ϵ be an EH-coefficient for H_4, and choose $c > 0$ such that every H_4-free tournament G satisfies $\alpha(G) \geq c|G|^\epsilon$. In particular, for each $k \geq 1$, $\alpha(D_k) \geq c|D_k|^\epsilon$, and so from (1), $2k - 1 \geq ck^{2\epsilon}$. Since this holds for all $k \geq 1$, we deduce that $\epsilon \leq 1/2$, and so $\xi(H_4) \leq 1/2$. This proves 6.2. ∎

The above is not the easiest way to prove that H_4 is not a pseudo-hero, but it gives a better bound on $\xi(H_4)$.

Next we need a lemma proved in [6], the following:

6.3 *The vertex set of every tournament H can be ordered such that the set of backward edges of every non-null subtournament S of H has cardinality at most $(|S| - 1)(\xi(H))^{-1}$.*

We deduce

6.4 *$\xi(H_5) \leq 5/6$, and so H_5 is not a pseudo-hero.*

Proof. Let $H = H_5$, and let $V(H) = A \cup B \cup C$, where

- $A = \{a_1, a_2\}, B = \{b_1, b_2\}$, and $C = \{c_1, c_2, c_3\}$
- $A \Rightarrow B \Rightarrow C \Rightarrow A$
- c_1-c_2-c_3-c_1 is a directed cycle.

Suppose there is an ordering of $V(H)$ such that no cycle of the backedge graph has length at most six. (By the *backedge graph* we mean the graph with vertex set $V(H)$ in which distinct vertices are adjacent if they are ends of a backedge of H.) Let X be the set of backedges in this ordering, and let $Y = E(H) \setminus X$. We have two properties:

(P) For every directed cycle of H, at least one of its edges is in X.

(Q) For every undirected cycle of H of length at most six, at least one of its edges is in Y.

Since every undirected graph with seven vertices and eight edges has a cycle of length at most six (indeed, at most five), it follows that $|X| \leq 7$. Suppose first that $a_1b_1, a_2b_2 \in Y$. From property P applied to the directed cycle c_i-a_j-b_j-c_i, at least one of c_ia_j, b_jc_i is in X, for $i = 1,2,3$ and $j = 1,2$. Thus there are at least six edges in X between $A \cup B$ and C. By property P applied to $H|C$, some edge of X has both ends in C. Since $|X| \leq 7$, it follows that all edges from A to B belong to Y; and so by property P, for $i = 1,2,3$ either $c_ia_1, c_ia_2 \in X$, or $b_1c_i, b_2c_i \in X$. Thus from the symmetry we may assume that $c_1a_1, c_1a_2, c_2a_1, c_2a_2 \in X$. But these four edges form a cycle contrary to property Q.

Thus not both $a_1b_1, a_2b_2 \in Y$, and similarly not both $a_1b_2, a_2b_1 \in Y$. Suppose next that $a_1b_1, a_1b_2 \in Y$. Thus $a_2b_1, a_2b_2 \in X$. By property Q applied to the cycle a_2-b_1-c_i-b_2-a_2, for $i = 1,2,3$ not both $b_1c_i, b_2c_i \in X$. By property P applied to the directed cycles c_i-a_1-b_1-c_i and c_i-a_1-b_2-c_i it follows that $c_ia_1 \in X$, for $i = 1,2,3$. But some edge of X has both ends in C, contrary to property Q.

It follows that not both $a_1b_1, a_2b_2 \in Y$, and so from the symmetry, at most one edge from A to B belongs to Y. By property Q, not all four of these edges are in X, so we may assume that $a_1b_1 \in Y$, and $a_2b_1, a_1b_2, a_2b_2 \in X$. From property P, some edge of $H|C$ belongs to X, say c_1c_2. Now by property P again, for $i = 1,2$ at least one of $c_ia_1, b_1c_i \in X$. But then there are six edges in X each with both ends in $V(H) \setminus \{c_3\}$, contrary to property Q.

It follows that in every ordering of $V(H)$, some cycle of the backedge graph has length at most six. From 6.3, we deduce that $\xi(H) \leq 5/6$. This proves the first assertion of the theorem, and the second follows. ∎

Finally:

6.5 $\xi(H_6) \leq 3/4$, *and so* H_6 *is not a pseudo-hero.*

Proof. Let $H = H_6$, and let $V(H) = A \cup B \cup C$, where

- $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2, c_3\}$
- $A \Rightarrow B \Rightarrow C \Rightarrow A$
- A, B, C are all transitive.

Suppose there is an ordering of $V(H)$ such that no cycle of the backedge graph has length at most four; let X be the set of backedges in this ordering, and let $Y = E(H) \setminus X$. We have two properties:

(P) For every directed cycle of H, at least one of its edges is in X.

(Q) For every undirected cycle of H of length at most four, at least one of its edges is in Y.

If there is a three-edge matching of members of Y between A, B, and also between B, C and between C, A, then the union of these three matchings includes a directed cycle of H, contrary to property P. So we may assume there is no three-edge matching of members of Y between A and B. By König's theorem, there are two vertices $x, y \in A \cup B$ such that every edge in Y between A and B is incident with one of x, y. If $x \in A$ and $y \in B$, and $x = a_3, y = b_3$ say, then $a_1b_1, a_1b_2, a_2b_1, a_2b_2$

are all in X, contrary to property Q. Thus we may assume that $x, y \in A$; say $x = a_1, y = a_2$. Hence $a_3b_1, a_3b_2, a_3b_3 \in X$. Let $1 \leq k \leq 3$. We claim that $c_ka_1, c_ka_2 \in X$. For suppose that $c_ka_1 \in Y$ say. From property Q at most one of the edges a_1b_1, a_1b_2, a_1b_3 is in X (otherwise there is a cycle of edges in X of length four passing through a_3); say $a_1b_1, a_1b_2 \in Y$. Now from property P applied to a_1-b_j-c_k-a_1, it follows that $b_jc_k \in X$ for $j = 1, 2$, contrary to property Q. This proves that $c_ka_1, c_ka_2 \in X$, for $k = 1, 2, 3$; but again this contradicts property Q. This proves 6.5. ∎

Now we complete the proof of 1.5; all that remains is to prove the "only if" half of the third statement of 1.5, which is the equivalence of the first two statements of the following.

6.6 *Let H be a strongly-connected tournament with more than one vertex. Then the following are equivalent:*

- *H is a pseudo-hero*
- *every strong component of H is isomorphic to $\Delta(2, k, l)$ for some $k, l \geq 2$, or to $\Delta(1, P, T)$ or $\Delta(1, T, P)$ for some pseudo-hero P and some nonempty transitive tournament T*
- *H contains none of $H_1, \dots, H_6$.*

Proof. The first statement implies the third, by 6.1, 6.2, 6.4 and 6.5, since every subtournament of a pseudo-hero is a pseudo-hero. By 5.1 and 4.1 with $\epsilon = 1$, and 3.1 with $\epsilon = 1$, the second statement implies the first. It remains to show that the third implies the second, and we proceed by induction on $|V(H)|$. Thus, let H contain none of $H_1, \dots, H_6$. If H is not strongly-connected, then inductively we may assume that all its strong components are pseudo-heroes, and hence so is H, by 3.1 with $\epsilon = 1$. If H is strongly-connected, then by a theorem of Gaku Liu, published as theorem 5.1 of [3], since H contains none of H_1, H_2, H_3, it admits a trisection (A, B, C). We may assume that $|C| \leq |A|, |B|$. If $|C| = 1$ then since H does not contain H_4, it follows that at least one of A, B is transitive, and so $H = \Delta(1, P, T)$ or $H = \Delta(1, T, P)$ for some pseudo-hero P and some nonempty transitive tournament T, and the theorem holds. If $|C| \geq 2$, then since H does not contain H_5 and $|A|, |B| \geq 2$ it follows that A, B, C are all transitive, and therefore $|C| = 2$ since H does not contain H_6; but then $H = \Delta(2, k, l)$ for some $k, l \geq 2$, and the theorem holds. This proves 6.6, and hence completes the proof of 1.5. ∎

Proof of 1.7. If H is not a pseudo-hero then from 6.6, H contains one of $H_1, \dots, H_6$, and so $\xi(H) \leq \max(\xi(H_1), \dots, \xi(H_6))$. But by 6.1, 6.2, 6.4 and 6.5, this maximum is at most 5/6. This proves 1.7. ∎

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