Cycle-touching graphs

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Abstract

Let us say a graph is cycle-touching if for every two vertex-disjoint cycles $C,C'$ of $G$ there is an edge of $G$ between $V(C)$ and $V(C')$. The structure of such graphs is not well-understood. For instance, there is an open conjecture, due to Ngoc Khang Le, that cycle-touching graphs have only a polynomial number of induced paths; that is, there exists $c > 0$ such that every cycle-touching graph $G$ has at most $|G|^c$ induced paths.

Let $t \geq 0$ be an integer, and let $G$ be a cycle-touching graph with no subgraph isomorphic to the complete bipartite graph $K_{t,t}$. In this paper we prove two things about such graphs $G$:

- there is a set $X \subseteq V(G)$ that intersects every cycle of $G$, with $|X| = O(\log |G|)$; and
- $G$ has only a polynomial number of induced paths.

The first result was proved independently by Bonamy, Bonnet, Déprés, Esperet, Geniet, Hilaire, Thomassé and Wesolek.
1 Introduction

Two subsets $X, Y$ of the vertex set of a graph $G$ are anticomplete if they are disjoint and there is no edge of $G$ between $X$ and $Y$; and we say two subgraphs of $G$ are anticomplete if their vertex sets are anticomplete. A graph $G$ is cycle-touching if there are no two anticomplete cycles in $G$. We do not understand such graphs very well; for instance, we do not know a polynomial-time algorithm to recognize cycle-touching graphs. In an attempt to find such an algorithm, several years ago Ngoc Khang Le proposed the conjecture (unpublished) that there exists $c > 0$ such that every cycle-touching graph $G$ has only $|G|^c$ induced cycles; and the stronger conjecture that the same is true for paths, that is:

1.1 Conjecture: There exists $c > 0$ such that every cycle-touching graph $G$ has only $|G|^c$ induced paths.

Stéphan Thomassé showed (private communication, also several years ago, and we give a proof below) that in order to prove 1.1, it suffices to prove it for two special kinds of cycle-touching graphs: those with girth at least 100, and those that contain the complete bipartite graph $K_{100, 100}$ as a subgraph. Being cycle-touching seems to be much more restrictive in graphs of large girth than for general graphs, and in this paper we focus on cycle-touching graphs of large girth.

Both conjectures of Le remain open, but in this paper we will prove them for cycle-touching graphs of with large girth. As Thomassé showed, we can weaken the hypothesis that $G$ has large girth, replacing it by the hypothesis that $G$ does not contain $K_{t, t}$ as a subgraph, where $t \geq 0$ is some constant. Thus, one result of this paper is:

1.2 For all integers $t \geq 0$ there exists $c > 0$ such that, if $G$ is cycle-touching and does not contain $K_{t, t}$ as a subgraph, then $G$ has only at most $|G|^c$ induced paths.

At a recent workshop in Barbados, Thomassé spoke about cycle-touching graphs in an open problem session, and proposed a new question about them: can they have large tree-width? Yes, because a complete bipartite graph $G$ is cycle-touching, and its tree-width is linear in $|G|$; but what if we just consider cycle-touching graphs of large girth? Thomassé asked whether the tree-width of such graphs is at most logarithmic in the number of vertices. He also proposed the stronger conjecture that in every such graph, there is a set of vertices of at most logarithmic size, that meets every cycle.

This was proved independently by two groups: a group of participants of the workshop (Marthe Bonamy, Édouard Bonnet, Hugues Déprés, Louis Esperet, Colin Geniet, Claire Hilaire, Stéphan Thomassé and Alexandra Wesolek), and the authors of this paper. Both proofs were found a few days after the end of the workshop. And we can replace the hypothesis that $G$ has large girth by the hypothesis that $G$ does not contain $K_{t, t}$ as a subgraph. Let us say $X \subseteq V(G)$ is a cycle-hitting set if $G \setminus X$ has no cycle; then our second theorem (found independently by Bonamy et al.) is:

1.3 For all integers $t \geq 2$, there exists $c > 0$ such that if $G$ is cycle-touching and does not contain $K_{t, t}$ as a subgraph, there is a cycle-hitting set of cardinality at most $c \log |G|$.

The proof of 1.2 uses 1.3, but it is easier, so we give it first.
2 Large girth

First, let us see why the hypotheses that $G$ has large girth, and $G$ does not contain $K_{t,t}$ as a subgraph, are interchangeable. This is a consequence of the following, proved by S. Thomassé several years ago, but unpublished:

2.1 For all integers $t,g \geq 2$ there is an integer $k \geq 0$ with the following property. Let $G$ be a cycle-touching graph that does not contain $K_{t,t}$ as a subgraph; then there exists $X \subseteq V(G)$ with $|X| \leq k$ such that $G \setminus X$ has girth more than $g$.

Proof. By Ramsey’s theorem, there exists $n \geq 0$ such that for every graph $H$ with at least $n$ vertices, and every partition of $E(H)$ into at most $2^n$ subsets, there is a subset $I \subseteq V(H)$ with $|I| \geq 2t$ such that all edges of $G$ with both ends in $I$ belong to the same subset of the partition. Let $k = ng^2$; we claim that $k$ satisfies 2.1.

Let $G$ be as in the theorem. We claim:

(1) There do not exist $ng$ cycles of $G$, pairwise vertex-disjoint and each with length at most $g$.

Suppose that there exist $n$ cycles as in (1). Let $C_1, \ldots, C_n$ be pairwise vertex-disjoint cycles of $G$, all of length $\ell$. For $1 \leq i \leq n$, let $V(C_i) = \{a^i_1, \ldots, a^i_\ell\}$, enumerated in some order. For $1 \leq i < j \leq n$, the type of the pair $(i,j)$ is the function

$$f : \{1, \ldots, \ell\} \times \{1, \ldots, \ell\} \to \{0, 1\}$$

where for $1 \leq p,q \leq \ell$, $f(p,q) = 1$ if $a^i_p, a^j_q$ are adjacent, and $f(p,q) = 0$ otherwise. There are only at most $2^{\ell^2}$ distinct types, and so from the choice of $n$, there is a subset $I \subseteq \{1, \ldots, n\}$ with $|I| \geq \max 2t, s$ such that the pairs $(i,j)$ all have the same type $f$ say, for all distinct $i,j \in I$ with $i < j$. Now $f$ is not identically zero, since $|I| \geq 2t$ and $G$ is cycle-touching; so there exist $p,q \in \{1, \ldots, \ell\}$ such that $f(p,q) = 1$. Since $|I| \geq 2t$, there exist disjoint $I_1, I_2 \subseteq I$, both of cardinality $t$ and such that $i < j$ for all $i \in I_1$ and $j \in I_2$. But then each of the vertices $a^i_p$ ($i \in I_1$) is adjacent to each of the vertices $a^j_q$ ($j \in I_2$), contradicting that $G$ does not contain $K_{t,t}$ as a subgraph. This proves (1).

Choose $r$ maximum such that there are cycles $C_1, \ldots, C_r$ of $G$, pairwise vertex-disjoint and each of length at most $r$. Let $X = V(C_1) \cup \cdots \cup V(C_r)$. By (1), $r \leq ng$, and so $|X| \leq ng^2 = k$. But $G \setminus X$ has no cycle of length at most $g$, from the maximality of $r$. This proves 2.1.

3 Plantations and transitions

Let $G$ be a cycle-touching graph, and let $Z \subseteq V(G)$ be a cycle-hitting set. We call $(G, Z)$ a plantation. Our goal is to show that the number of induced paths $P$ of $G$ with $Z \subseteq V(P)$ is at most the product of an exponential function of $|Z|$ and a polynomial in $|G|$. Then it is easy to derive 1.2 from 1.3, since we can choose $Z$ of size $O(\log |G|)$. But proving the first statement takes a number of steps.

Let $F$ be the forest $G \setminus Z$, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. We say $(G, Z)$ is monic if $Z$ is stable and each vertex in $N$ has a unique neighbour in $Z$. Let us say a transition of $(G, Z)$ is a path of $F$ of length at least one, with both ends in $N$ and with no
internal vertex in $N$. Let $P$ be a transition. If $z \in Z$ is adjacent to an end of $P$, we say $z$ is a foot of $P$. If $(G, Z)$ is monic, every transition $P$ has one or two feet, and these are the only vertices in $Z$ that have a neighbour in $V(P)$. We remark that distinct transitions cannot have the same pair of ends, but they may have the same pair of feet. If $P$ only has one foot, $P$ is a self-transition. We say $(G, Z)$ is selfless if there is no self-transition. Starting with a monic plantation, our first objective is to eliminate self-transitions.

We will need the following two lemmas:

3.1 Let $F$ be a forest, and let $T_1, \ldots, T_k$ be trees of $F$ with $k > 0$.

- If no two of $T_1, \ldots, T_k$ are vertex-disjoint, then some vertex of $F$ belongs to $V(T_1 \cap \cdots \cap T_k)$.
- If no two of $T_1, \ldots, T_k$ are anticomplete, there is a clique $X$ of $F$ with $|K| \leq 2$ such that each of $T_1, \ldots, T_k$ contains a vertex of $X$.

The first claim is well-known and easy, and we assume it without proof. For the second, let $F'$ be the tree obtained from $F$ by subdividing once each edge $e$ of $F$ (let $v_e$ be the new vertex that replaces $e$). For $1 \leq i \leq k$, let $T'_i$ be the tree of $F'$ induced on the union of $V(T_i)$ and the set of all $v_e$ such that $e \in E(F)$ has an end in $V(T_i)$. The hypothesis implies that no two of $T'_1, \ldots, T'_k$ are vertex-disjoint, and so the result follows by applying the first bullet of the theorem to $F'$ and $T'_1, \ldots, T'_k$. This proves 3.1.

3.2 Let $F$ be a forest, let $T_1, \ldots, T_k$ be pairwise anticomplete trees of $F$, and let $T_{k+1}, \ldots, T_\ell$ be pairwise anticomplete trees of $F$. Let $H$ be a graph with bipartition $\{(1, \ldots, k), (k+1, \ldots, \ell)\}$, in which $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, \ell\}$ are adjacent if $T_i, T_j$ are not anticomplete. Then $H$ is a forest.

Proof. Suppose that $H$ has a cycle $C$. Since $C$ has length at least four, and $H$ is bipartite, we may assume that $1, 2 \in V(C)$. Let $P_1, P_2$ be the two paths of $C$ between 1, 2. For $h = 1, 2$ let $I_h = \{k+1, \ldots, \ell\} \cap V(P_h)$. Let $v_i \in V(T_i)$ for $i = 1, 2$. For $h = 1, 2$, there is a path $Q_h$ of $F$ between $v_1, v_2$ with interior included in the union of the sets $V(T_i)$ ($i \in V(P_h)$), and hence included in

$$V(T_1 \cup \cdots \cup T_k) \cup \bigcup_{i \in I_h} V(T_i).$$

Since $F$ is a forest, it follows that $Q_1 = Q_2$, and so every vertex of $Q_1$ not in $V(T_1 \cup \cdots \cup T_k)$ belongs to both $\bigcup_{i \in I_1} V(T_i)$ and to $\bigcup_{i \in I_2} V(T_i)$, which is impossible since these two sets are disjoint. Consequently $V(Q_1) \subseteq V(T_1 \cup \cdots \cup T_k)$, which is also impossible since $T_1, \ldots, T_k$ are anticomplete, and $Q_1$ has an end in $T_1$ and an end in $T_2$. This proves 3.2.

We will use two operations to eliminate self-transitions: deletion and explosion. If $(G, Z)$ is a plantation, and $v \in V(G) \setminus Z$, then $(G \setminus \{v\}, Z)$ is a plantation, monic if $(G, Z)$ is monic. Moreover, each transition of $(G \setminus \{v\}, Z)$ is a transition of $(G, Z)$, so deleting vertices in $V(G) \setminus Z$ may be used to eliminate some self-transitions, without introducing new ones. Second, if $v \in Z$, let $G'$ be obtained from $G$ by deleting $v$ and all its neighbours in $V(G) \setminus Z$. Then again $(G', Z \setminus \{z\})$ is a plantation, monic if $(G, Z)$ is monic, and each of its transitions is a transition of $(G, Z)$. This operation is called exploding $v$. We will show:
3.3 Let \((G, Z)\) be a monic plantation. Then there exist \(X \subseteq Z\) and \(Y \subseteq V(G) \setminus Z\), with \(|X| \leq 1\) and \(|Y| \leq 2\), such that exploding the vertices in \(X\) and deleting the vertices in \(Y\) yields a selfless plantation.

Proof. As before, let \(F = G \setminus Z\), and let \(N\) be the set of vertices in \(V(G) \setminus Z\) with a neighbour in \(Z\).

(1) If there do not exist two self-transitions \(P_1, P_2\) with the same foot such that \(P_1, P_2\) are anticomplete, then there exists \(Y \subseteq V(F)\) with \(|Y| \leq 2\) such that deleting the vertices in \(Y\) yields a selfless plantation.

We claim that no two self-transitions are anticomplete; for suppose that \(P, P'\) are anticomplete self-transitions. Let \(P, P'\) be self-transitions with feet \(z, z'\) respectively. From the hypothesis of (1), \(z \neq z'\); and since \((G, Z)\) is monic and \(V(P), V(P')\) are disjoint and anticomplete, the cycles induced on \(V(P) \cup \{z\}\) and \(V(P') \cup \{z'\}\) are also disjoint and anticomplete, contradicting that \(G\) is cycle-touching. Thus, no two self-transitions are anticomplete. From 3.1, there is a clique \(Y\) of \(F\) with \(|Y| \leq 2\) such that every self-transition contains a vertex in \(Y\); and so deleting the vertices in \(Y\) yields a selfless plantation. This proves (1).

From (1) we may assume that there exist \(z \in Z\) and two anticomplete self-transitions \(P_1, P_2\) both with foot \(z\).

(2) There does not exist \(z' \in Z\) with \(z' \neq z\) such that there are two anticomplete self-transitions \(Q_1, Q_2\) both with foot \(z'\).

Suppose such \(z', Q_1, Q_2\) exist. For all \(i, j \in \{1, 2\}\), \(P_i\) is not anticomplete to \(Q_j\), as before, but this is contrary to 3.2, and so proves (2).

Consequently, if we explode \(z\), we obtain a plantation satisfying (1); and the result follows. This proves 3.3.

If \(P\) is a path, we denote by \(P^*\) the interior of \(P\), that is, the set of vertices that have degree two in \(P\). A clearing of \((G, Z)\) is a maximal tree \(T\) of \(F\) such that every vertex of \(T\) in \(N\) is a leaf (that is, a vertex of degree one) of \(T\). Every edge of \(F\) belongs to a unique clearing; every transition is a path of some clearing; and if \(T\) is a clearing, every path of \(T\) with distinct ends both in \(N\) is a transition.

Let \((G, Z)\) be a monic selfless plantation. If \(z, z' \in Z\), the multiplicity of the pair \((z, z')\) is the number of transitions with feet \(z, z'\). Thus the multiplicity of \((z, z)\) is zero, since \((G, Z)\) is selfless. We say that \((G, Z)\) has thickness \(k\) if \(k\) is the maximum of the multiplicity of pairs of elements of \(Z\). Our next objective is, again by deleting and exploding, to obtain a plantation with bounded thickness. We will show the following.

3.4 Let \((G, Z)\) be a monic selfless plantation. Then there exists \(X \subseteq Z\) with \(|X| \leq 6\) such that exploding the vertices in \(X\) yields a plantation with thickness at most 24.

Proof. Again, let \(N\) be the set of vertices in \(V(G) \setminus Z\) with a neighbour in \(Z\), and let \(F\) be the forest \(G \setminus Z\). We observe first:
(1) Let $z, z' \in Z$. If $P_1, P_2$ are distinct transitions both with feet $z, z'$, then $P_1^*, P_2^*$ are anticomplete, and either

- $P_1, P_2$ are anticomplete; or
- $P_1, P_2$ have a common end and $P_1 \cup P_2$ is an induced path; or
- $V(P_1), V(P_2)$ are disjoint and there is a unique edge between them, joining an end of $P_1$ and an end of $P_2$.

Let $P_i$ have ends $a_i, b_i$ for $i = 1, 2$, where $a_1, a_2$ are adjacent to $z$, and $b_1, b_2$ to $z'$. Since $P_1, P_2$ are distinct, and they are both paths in the forest $F$, they do not have the same pairs of ends; and so we may assume that $a_1 \neq a_2$. Let $T_i$ be the clearing that contains $P_i$. Since $(G, Z)$ is selfless, $a_1$ is the only neighbour of $z$ in $V(T_1)$, and so $a_2 \notin V(T_1)$; and consequently $P_1^* \cap V(T_1) = \emptyset$. The vertices of $P_1^*$ are not leaves of $T_1$, and so every vertex of $G$ with a neighbour in $P_1^*$ belongs to $V(T_1)$. Consequently $P_1^*, P_2^*$ are anticomplete.

If $V(P_1) \cap V(P_2) \neq \emptyset$, then $P_1, P_2$ have a common end, and so $b_1 = b_2$; but then the second outcome holds. Thus we may assume that $V(P_1), V(P_2)$ are disjoint. If they are anticomplete, then the first outcome holds; and if not, the edge between $V(P_1), V(P_2)$ is unique (since $F$ is a forest) and the third outcome holds. This proves (1).

Let $z, z' \in Z$, and let $(z, z')$ have multiplicity at least 25. Then there is either a $(z, z')$-linkage of cardinality four, or a $(z, z')$-star of cardinality five.

(2) Let $z, z' \in Z$, and let $(z, z')$ have multiplicity at least 25. Then there is either a $(z, z')$-linkage of cardinality four, or a $(z, z')$-star of cardinality five.

Let $P_i$ ($i \in I$) all be distinct transitions, with the same feet $z, z'$, where $|I| = 25$. For each $i \in I$ let $P_i$ have ends $a_i, b_i$, where $a_i$ is adjacent to $z$ and $b_i$ to $z'$. Every bipartite graph with 25 edges has a matching of size 7 or a vertex of degree 5, from König’s theorem; and because of this, applied to the bipartite graph with bipartition $\{(a_i : i \in I), (b_i : i \in I)\}$ and edge set $\{(a_i, b_i : i \in I)\}$, we may assume that either $a_1, \ldots, a_7, b_1, \ldots, b_7$ are all distinct, or $a_1 = \ldots = a_5$. In the second case, $\{P_1, \ldots, P_5\}$ is a $(z, z')$-star by (1), so we assume the first holds. Let $H$ be the graph with vertex set $\{1, \ldots, 7\}$, in which $i, j$ are adjacent if $P_i, P_j$ are not anticomplete (and hence they are vertex-disjoint and there is a unique edge between them, by (1)). A graph isomorphic to $H$ can be obtained from $F$ by deleting all vertices not in $P_1, \ldots, P_7$ and contracting the edges of $P_1, \ldots, P_7$; and so $H$ is a forest. Hence it has a stable set of cardinality 4, say $\{1, \ldots, 4\}$; and then $\{P_1, \ldots, P_4\}$ is a $(z, z')$-linkage. This proves (2).

(3) Let $z_1, z_2, z_3, z_4 \in Z$ be distinct. There is not both a $(z_1, z_2)$-linkage of cardinality four and a $(z_3, z_4)$-linkage of cardinality four.

Suppose that $\{P_1, \ldots, P_4\}$ is a $(z_1, z_2)$-linkage, and $\{P_5, \ldots, P_8\}$ is a $(z_3, z_4)$-linkage. Let $H$ be the bipartite graph with bipartition $\{(1, \ldots, 4), (5, \ldots, 8)\}$ in which $i \in \{1, \ldots, 4\}$ and $j \in \{5, \ldots, 8\}$. 

are adjacent if $P_i, Q_j$ are not anticomplete. By 3.2, $H$ is a forest, and so (this is an easy exercise) there exist distinct $a, b \in \{1, \ldots, 4\}$ and $c, d \in \{5, \ldots, 8\}$ such that $\{a, b, c, d\}$ is a stable set of $H$. Thus we may assume that $P_1, P_2, P_5, P_6$ are pairwise anticomplete. Consequently the cycles induced on $V(P_1 \cup P_2) \cup \{z_1, z_2\}$ and $V(P_5 \cup P_6) \cup \{z_3, z_4\}$ are anticomplete, contradicting that $G$ is cycle-touching. This proves (3).

(4) Let $z_1, z_2, z_3, z_4 \in Z$ be distinct. There is not both a $(z_1, z_2)$-star of cardinality five and a $(z_3, z_4)$-star of cardinality five that have nonadjacent centres.

Suppose that $\{P_1, \ldots, P_3\}$ is a $(z_1, z_2)$-star with centre $a_1$, and $\{P_6, \ldots, P_{10}\}$ is a $(z_3, z_4)$-star with centre $a_2$, and $a_1, a_2$ are nonadjacent. We may assume that $z_1$ is adjacent to $a_1$, and $z_3$ to $a_2$. Since $P_1, \ldots, P_3$ all have feet $z_1, z_2$, it follows that $a_2 \notin V(P_1 \cup \cdots \cup P_3)$; and since $F$ is a forest, $a_2$ has at most one neighbour in $V(P_1 \cup \cdots \cup P_3)$. If this neighbour exists, it is not $a_1$ since $a_1, a_2$ are nonadjacent; and so we may assume that $a_2$ has no neighbour in $P_1, \ldots, P_4$. Similarly we may assume that $a_1$ has no neighbour in $P_6, \ldots, P_9$. For $1 \leq i \leq 4$ let $Q_i$ be the path $P_i \setminus \{a_1\}$, and define $Q_6, \ldots, Q_9$ similarly. Thus $Q_1, \ldots, Q_4$ are pairwise anticomplete, and so are $Q_6, \ldots, Q_9$. Using 3.2 as in the proof of (3), we may assume that $Q_1, Q_2, Q_6, Q_7$ are pairwise anticomplete; but then the cycles induced on $V(P_1 \cup P_2) \cup \{z_2\}$ and $V(P_6 \cup P_7) \cup \{z_4\}$ are anticomplete, contradicting that $G$ is cycle-touching. This proves (4).

From (3), every two pairs of vertices in $Z$ that have a linkage of cardinality four have a common end, and so there is a set $X_1 \subseteq Z$ with $|X_1| \leq 2$ such that for all $z, z' \in Z$ such that there is a $(z, z')$-linkage of cardinality four, one of $z, z' \in X_1$.

From (4), if $z_1, \ldots, z_6 \in Z$ are distinct, there is not both a $(z_1, z_2)$-star of cardinality five and a $(z_3, z_4)$-star of cardinality five and a $(z_5, z_6)$-star of cardinality five, since not all three centres can be adjacent because $F$ is a forest. Thus, if $H$ is the graph with vertex set $Z$ in which $z, z'$ are adjacent if there is a $(z, z')$-star of cardinality five, then $H$ has no matching of size three, and therefore there is a subset $X_2 \subseteq Z$ with $|X_2| \leq 4$ such that $X_2$ meets every edge of $H$. Consequently for all $z, z' \in Z$, if there is a $(z, z')$-star of cardinality five, then one of $z, z'$ belongs to $X_2$.

It follows that if we explode all the vertices in $X_1 \cup X_2$, producing a plantation $(G', Z')$ say, then for all $z, z' \in Z$, there is no $(z, z')$-linkage in $(G', Z')$ of cardinality four and there is no $(z, z')$-star in $(G', Z')$ of cardinality five. By (2), $(G', Z')$ has thickness at most 24. This proves 3.4.

4 Applying the Erdős-Pósa theorem

Let $(G, Z)$ be a plantation; we say a set $S$ of transitions in $(G, Z)$ is normal if

- for all $P, Q \in S$, either $P, Q$ are anticomplete or $P, Q$ have a common end; and
- for each $P \in S$, there is an edge $e$ of $P$ that does not belong to any other member of $S$.

We need first:

4.1 Let $(G, Z)$ be a plantation, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. Suppose that every component of $F$ contains at least two vertices of $N$. Then there is a normal set $S$ of transitions with $|S| \geq |N|/4$.  

6
Proof. Let $F$ be the forest $G \setminus Z$. By choosing transitions from each component of $F$ separately, we may assume that $F$ is a tree, and $|N| \geq 2$. If $|N| \leq 3$ the result is clear, so we may assume that $|N| \geq 4$. Choose some vertex $r \in N$, call it the root of $F$, and direct every edge of $F$ towards $r$. Let $\mathcal{R}$ be the set of all transitions of $(G, Z)$ that are directed paths. Thus $|\mathcal{R}| = |N| - 1$, since every vertex in $N$ different from $r$ is the first vertex of a unique directed transition. Moreover, for the same reason, every member of $\mathcal{R}$ has an edge that does not belong to any other member of $\mathcal{R}$. We will show that there is a normal subset of $\mathcal{R}$ with cardinality at least $|\mathcal{R}|/3$.

Let $P$ be a directed transition, and let $Q$ be the directed path of $F$ from the first vertex of $P$ to the root of $F$. It follows that $P$ is an initial subpath of $Q$. We define the height of $P$ to be the number of vertices of $Q$ that belong to $N$.

(1) Let $P_1, P_2$ be directed transitions, with heights $h_1, h_2$ where $h_1 - h_2$ is a multiple of three. Then either $P, Q$ are anticomplete, or they have the same last vertex and therefore the same height.

Let $P_i$ have first vertex $a_i$ and last vertex $b_i$ for $i = 1, 2$. We may assume that $b_1 \neq b_2$, and so $V(P_1) \cap V(P_2) = \emptyset$. Hence we may assume that there is an edge of $F$ with one end in $V(P_1)$ and the other in $V(P_2)$, and we may assume this edge is directed from its end $c_1 \in V(P_1)$ to its end $c_2 \in V(P_2)$, by exchanging $P_1, P_2$ if necessary. Since $c_1$ has at most one out-neighbour in $F$, and $c_2 \notin V(P_1)$, it follows that $c_1 = b_1$. For $i = 1, 2$, let $Q_i$ be the directed path of $F$ from $a_i$ to the root of $F$. It follows that the edge $c_1c_2$ belongs to $Q_1$, and so $Q_1$ contains all the vertices of $N \cap V(Q_2)$ except possibly $a_2$, and in addition contains $a_1, b_1$. Thus $h_1 - h_2 \in \{1, 2\}$, contradicting that $h_1 - h_2$ is a multiple of three. This proves (1).

For $i = 1, 2, 3$, let $S_i$ be the set of all directed transitions with height congruent to $i$ modulo three. By (1), each of these sets is normal, and every directed transition belongs to one of them, so one of them has cardinality at least $|\mathcal{R}|/3 = (|N| - 1)/3$, and hence at least $|N|/4$, since $|N| \geq 4$. This proves 4.1. □

We need the following result, a special case of a theorem of P. Erdős and L. Pósa [1] or of L. Lovász [2]:

4.2 If no two cycles of a multigraph $G$ are vertex-disjoint, then there is a subset $X \subseteq V(G)$ with $|X| \leq 3$ such that every cycle of $G$ contains a vertex in $X$.

We need anticomplete cycles, not just disjoint cycles: but by selecting some transitions carefully, we can make a derived graph, disjoint cycles in which would yield anticomplete cycles in the original graph. The Erdős-Pósa result 4.2 is used to show the following:

4.3 Let $(G, Z)$ be a monic plantation, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. Let $S$ be a normal set of transitions. Then there exists $X \subseteq Z$ with $|X| \leq 3$ such that at most $|Z|$ members of $S$ have no foot in $X$.

Proof. Let $H$ be the multigraph with vertex set $Z$, edge set $S$, and incidence relation defined as follows: for each $P \in S$, and each $z \in Z$, $P$ is incident with $z$ in $H$ if $z$ is a foot of $P$. We observe:

(1) If $C$ is a cycle of $H$, there is a cycle $C'$ of $G$ with $V(C') \cap Z \subseteq V(C)$, and $V(C') \setminus Z$ is a
subset of the union of the vertex sets of the transitions in $E(C)$.

Let the vertices and edges of $C$ in order be $u_1, P_1, u_2, P_2, \ldots, u_m, P_m, u_{m+1} = u_1$. Thus $u_1, \ldots, u_m \in Z$ are distinct, and for $1 \leq i \leq m$, $P_i \in S$ is a transition with feet $u_i, u_{i+1}$, and $P_1, \ldots, P_m$ are all distinct. Suppose that $m = 1$; then $H$ has a loop $P_1$, incident with $u_1$ in $H$. Let $p, q$ be the ends of the path $P_1$ in $G$; then the union of $P_1$ with the path $p-u_1-q$ is the desired cycle. Thus we may assume that $m \geq 2$.

For $1 \leq i \leq m$, let $P_i^+$ be the path between $u_i, u_{i+1}$ with interior $V(P_i)$. Since $S$ is normal, there is an edge $e$ of $P_1$ that belongs to none of $P_2, \ldots, P_m$. But the union of $P_1^+ \setminus \{e\}$ and $P_2^+ \cup \cdots \cup P_m^+$ is a connected graph, containing both ends of $e$; and so contains a path joining the ends of $e$. Adding $e$ to this path gives the desired cycle $C'$. This proves (1).

(2) No two cycles of $H$ are vertex-disjoint.

Suppose that $C, D$ are two cycles of $H$ that are vertex-disjoint. By (1), there is a cycle $C'$ of $G$ with $V(C') \cap Z \subseteq V(C)$, and $V(C') \setminus Z$ is a subset of the union of the vertex sets of the transitions in $E(C)$. Define $D'$ similarly. Since $C, D$ are vertex-disjoint, and $Z$ is stable, it follows that $V(C') \cap Z$ is anticomplete to $V(D') \cap Z$. Let the vertices and edges of $C$ in order be

$$u_1, P_1, u_2, P_2, \ldots, u_m, P_m, u_{m+1} = u_1,$$

and define $v_1, Q_1, v_2, Q_2, \ldots, v_n, Q_n, u_{n+1} = v_1$ similarly for $D$. For $1 \leq i \leq m$, two vertices in $V(C') \cap Z$ are adjacent to ends of $P_i$, and since $(G, Z)$ is monic, no other vertices in $Z$ has neighbours in $V(P_i)$. Consequently $V(D') \cap Z$ is anticomplete to $V(C')$ and similarly $V(C') \cap Z$ is anticomplete to $V(D')$. Since $G$ is cycle-touching, it follows that some $P_i$ is not anticomplete to some $Q_j$, and therefore we may assume that $P_1$ is not anticomplete to $Q_1$. Since $S$ is normal, it follows that $P_1, Q_1$ have a common end $a$; but then the unique neighbour $z \in Z$ of $a$ belongs to both $V(C'), V(D)$, a contradiction. This proves (2).

From 4.2, there exists $X \subseteq Z$ with $|X| \leq 3$ such that $H \setminus X$ is a forest, and therefore has at most $|Z \setminus X| - 1 \leq |Z|$ edges; and so at most $|Z|$ members of $S$ have no neighbour in $X$. This proves 4.3.

We use this to show:

4.4 Let $(G, Z)$ be a monic selfless plantation, with thickness $k$, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. Suppose that every component of $G \setminus Z$ contains at least two vertices in $N$. Then $|N| \leq (12k + 4)|Z|$.

**Proof.** By 4.1, there is a normal set $S$ of transitions with $|S| \geq |N|/4$. From 4.3, there exists $X \subseteq Z$ with $|X| \leq 3$ such that at most $|Z|$ members of $S$ have no neighbour in $X$. But since $(G, Z)$ has thickness $k$, for each $x \in X$ and $z \in Z$, there are at most $k$ transitions with feet $x, z$, and therefore for each $x \in X$, at most $k|Z|$ transitions in $S$ contain a neighbour of $x$. Since $|X| \leq 3$, it follows that $|S| \leq 3k|Z| + |Z|$. But $|S| \geq |N|/4$, and so $|N| \leq (12k + 4)|Z|$. This proves 4.4.
5 Non-monic plantations

The result 4.4 brings us close to what we want, but only for monic plantations. In this section we extend it to more general plantations. Let us say a plantation \((G, Z)\) is dyadic if \(Z\) is stable and every vertex in \(V(G) \setminus Z\) has at most two neighbours in \(Z\). We say \(v \in V(G) \setminus Z\) is double if it has two neighbours in \(Z\); and \((G, Z)\) is dyadically simple if it is dyadic and no two double vertices in \(Z\) have the same two neighbours in \(Z\). We claim first:

5.1 Let \((G, Z)\) be a dyadic plantation. Then there exists \(X \subseteq Z\) with \(|X| \leq 6\) such that exploding \(X\) yields a dyadic plantation with at most \(2|Z|\) double vertices.

**Proof.** We claim first:

(1) Let \(Y\) be a stable set of double vertices. Then there exists \(X \subseteq Z\) with \(|X| \leq 3\) such that at most \(|Z|\) vertices in \(Y\) have no neighbour in \(X\).

Let \(H\) be the graph with vertex set \(Z\) and edge set \(Y\), where \(y \in Y\) is incident in \(H\) with \(z \in Z\) if \(y\) is adjacent to \(z\) in \(G\). For every cycle \(C\) of \(H\), there is a cycle \(C'\) of \(G\) induced on the vertices of \(C\) that are vertices or edges of \(C\); and if \(C, D\) are vertex-disjoint cycles of \(H\), the corresponding cycles \(C', D'\) of \(G\) are anticomplete (since \(Y\) is stable, \(Z\) is stable, and each vertex in \(Y\) has exactly two neighbours in \(Z\)). Consequently there do not exist such \(C, D\), and so by 4.2, there exists \(X \subseteq Z\) with \(|Z| \leq 3\) such that \(H \setminus X\) is a forest. Hence at most \(Z\) vertices in \(Y\) have no neighbour in \(X\). This proves (1).

Let \(N_2\) be the set of all double vertices. Since \(G \setminus Z\) is a forest and hence bipartite, it follows that \(N_2\) is the union of two stable sets; and so by (1) applied to each of these sets, we deduce that there exists \(X \subseteq Z\) with \(|X| \leq 6\) such that at most \(2|Z|\) vertices in \(N_2\) have no neighbour in \(X\). But then \(X\) satisfies the theorem. This proves 5.1.

For \(z \in Z\), \(N(z)\) denotes the set of neighbours of \(z\), and for \(Z' \subseteq Z\), \(N(Z')\) denotes the union of the sets \(N(z)\) \((z \in Z')\). We deduce:

5.2 Let \((G, Z)\) be a dyadic plantation. Then there exist \(X \subseteq Z\) with \(|X| \leq 13\) and \(Y \subseteq V(G) \setminus Z\) with \(|Y| \leq 2\) and with the following property. Let \(F = G \setminus Z\). For \(i = 1, 2\), let \(N_i\) be the set of all \(v \in V(F) \setminus (Y \cup N(X))\) that have exactly \(i\) neighbours in \(Z\); and let \(N_0\) be the set of all \(v \in N_1\) such that the component of \(F \setminus (Y \cup N(X) \cup N_2)\) containing \(v\) contains no other vertex in \(N_1\). Then there are at most \(296|Z| + 4\) edges between \(Z \setminus X\) and \(V(F) \setminus (N(X) \cup N_0)\).

**Proof.** By 5.1, there exists \(X_1 \subseteq Z\) with \(|X_1| \leq 6\) such that exploding \(X_1\) yields a dyadic plantation \((G_1, Z_1)\) with at most \(2|Z|\) double vertices. Let \(Y_1\) be the set of double vertices of \((G_1, Z_1)\). It follows that \((G_1 \setminus Y_1, Z_1)\) is monic and \(|Y_1| \leq 2|Z|\). By 3.3 applied to \((G_1 \setminus Y_1, Z_1)\), there exists \(X_2 \subseteq Z_1\) and \(Y \subseteq V(G_1) \setminus (Y_1 \cup Z_1)\), with \(|X_2| \leq 1\) and \(|Y| \leq 2\), such that starting with \((G_1 \setminus Y_1, Z_1)\), and exploding the vertices in \(X_2\) and deleting the vertices in \(Y\), yields a selfless plantation \((G_2, Z_2)\) say. By 3.4, there exists \(X_3 \subseteq Z_2\) with \(|X_3| \leq 6\) such that starting with \((G_2, Z_2)\) and exploding the vertices in \(X_3\) yields a monic selfless plantation \((G_3, Z_3)\) with thickness at most 24. Let \(Y_3\) be the union of the vertex sets of all components of \(G_3 \setminus Z_3\) that have at most one vertex with a neighbour
in $Z_3$. The plantation $(G_3 \setminus Y_3, Z_3)$ satisfies the hypothesis of 4.4, and its thickness is at most 24, and so by 4.4, there are at most $(12 \cdot 24 + 4)|Z_3| \leq 292|Z|$ edges between $Z_3$ and $V(G) \setminus (Y_3 \cup Z_3)$.

Let $X = X_1 \cup X_2 \cup X_3$; we will show that $X, Y$ satisfy the theorem. We recall that $(G_3, Z_3)$ is obtained from $(G, Z)$ by exploding the vertices in $X$ and deleting the vertices in $Y_1 \cup Y$. Let $(G', Z_3)$ be obtained from $(G, Z)$ by exploding the vertices in $X$ and deleting the vertices in $Y$. There are only $2|Y_1| \leq 4|Z|$ edges of $G'$ between $Y_1$ and $|Z|$ since $|Y_1| \leq 2|Z|$ and each of its members has only two neighbours in $Z$. Thus there are at most $296|Z|$ edges $yz$ of $G'$ such that $y \in V(G') \setminus (Y_3 \cup Z')$ and $z \in Z_3$; that is, there are at most $296|Z|$ edges $yz$ of $G$ such that $y \in V(F) \setminus (N(X) \cup Y \cup N_0)$ and $z \in Z_3$. Since $|Y| \leq 2$, there are only four edges between $Y$ and $Z$. This proves 5.2.

\section{Counting paths}

Let $(G, Z)$ be a plantation. We denote by $n(G, Z)$ the number of induced paths $P$ of $G$ with $Z \subseteq V(P)$ such that both ends of $P$ belong to $Z$. Let us call such a path $P$ a $Z$-covering path. Our objective is to show that $n(G, Z)$ is at most the product of a polynomial in $|G|$ and an exponential in $|Z|$.

It is enough to work with dyadic plantations, because of the following.

\begin{proposition}
Let $(G, Z)$ be a plantation. Then there is a dyadic plantation $(G', Z')$ with $|G'| \leq |G|$ and $|Z'| \leq |Z|$ such that $n(G, Z) \leq n(G', Z')$.
\end{proposition}

\begin{proof}
We prove this by induction on $|G|$. We observe first:

\begin{itemize}
  \item If some vertex $v \in V(G) \setminus Z$ has more than two neighbours in $Z$, this vertex does not belong to any $Z$-covering path, and so we may delete it without changing the number of $Z$-covering paths. Hence in this case we can win by induction on $|G|$; so we may assume there is no such vertex.
  \item If some vertex $v \in V(G) \setminus Z$ has two neighbours $z, z' \in Z$, and $z, z'$ are adjacent, then again $v$ does not belong to any $Z$-covering path, and we can delete it and win as before. So we may assume that there is no such vertex.
  \item If some three vertices in $Z$ are pairwise adjacent, then $n(G, Z) = 0$, so we may assume there is no such triangle.
\end{itemize}

If some two vertices $z, z'$ are adjacent, then they have no common neighbour, by the assumptions of the second and third bullets above; so contracting $zz' = e$ (say) will not make any parallel edges. Let $G'$ be the graph obtained from $G$ by contracting $e$ into a new vertex $z''$ say, and let $Z' = (Z \setminus \{z, z'\}) \cup \{z''\}$. Then it is easy to see that

\begin{itemize}
  \item $(G', Z')$ is a plantation;
  \item every $Z$-covering path of $(G, Z)$ contains $e$; so for every $Z$-covering path $P$ of $(G, Z)$, there is a $Z'$-covering path $P'$ of $(G', Z')$ with $E(P') = E(P) \cup \{e\}$; and
  \item for every $Z'$-covering path $P'$ of $(G', Z')$, there is at most one $Z$-covering path $P$ of $(G, Z)$ with $E(P') = E(P) \cup \{e\}$.
\end{itemize}

Consequently, in this case $n(G, Z) \leq n(G', Z')$ and we can again win by induction on $|G|$. This proves 6.1.
A multiset is a set together with a positive integer assigned to each member of the set, called its multiplicity. The next result implies that if \((G, Z)\) is dyadic, every \(Z\)-covering path \(P\) is determined by the set of edges of \(P\) with an end in \(Z\). A linear forest is a forest in which every component is a path; and the end multiset of a linear forest \(H\) is the multiset of ends of the components of \(H\), where an end of a component \(P\) of \(H\) has multiplicity one if \(E(P) \neq \emptyset\), and multiplicity two if \(E(P) = \emptyset\).

6.2 Let \(F\) be a forest, and let \(X\) be a multiset of vertices of \(V(F)\). Then there is at most one linear forest that is a subgraph of \(F\) with end-multiset equal to \(X\).

**Proof.** We proceed by induction on \(|V(F)|\). If some vertex in \(X\) has multiplicity at least three in \(X\), then there is no linear forest with end-multiset \(X\). If some vertex \(v\) in \(X\) has multiplicity two in \(X\), then \(v\) is a component of every linear forest in \(F\) with end-multiset \(X\), so the result follows by deleting \(v\). Hence we may assume that every vertex in \(X\) has multiplicity one. Also, from the inductive hypothesis applied to each component, we may assume that \(F\) is connected. If some leaf of \(F\) is not in \(X\), we may delete it and apply the inductive hypothesis, so we assume all leaves of \(F\) belong to \(X\). If \(F\) is a path, the result is clear, so we assume \(F\) is not a path. Let us say a shoot of \(F\) is a path of \(F\) with one end a leaf of \(F\), such that all its internal vertices have degree two in \(F\), and maximal with both these properties. Every shoot has length at least one, one of its ends is a leaf of \(F\), and the other has degree at least three in \(F\), from the maximality of the shoot and since \(F\) is not a path. (Let us call the end of degree at least three the inner end.) Let \(F'\) be obtained from \(F\) by deleting all vertices of \(F\) that belong to shoots and have degree at most two in \(F\). Then \(F'\) is non-null, and therefore it has degree at least two in \(F'\). Since \(u\) is not a leaf of \(F\), it is the inner end of some shoot of \(F\); and therefore it has degree at least three in \(F\); and so is the inner end of at least two shoots of \(F\), say \(P, P'\). But then \(P \cup P'\) is a component of every linear forest in \(F\) with end-multiset \(X\), and the result follows from the inductive hypothesis by deleting \(V(P \cup P')\). This proves 6.2.

We will show:

6.3 If \((G, Z)\) is a plantation, then \(n(G, Z) \leq |G|^{82}2^{296}|Z|\).

**Proof.** By 6.1 we may assume that \((G, Z)\) is dyadic. Let \(\delta_G(Z)\) be the set of edges of \(G\) between \(Z\) and \(V(G) \setminus Z\).

1. For each subset \(D\) of \(\delta_G(Z)\), there is at most one \(Z\)-covering path \(P\) with \(E(P) \cap \delta_G(Z) = D\).

To see this, let \(X\) be a multiset of ends in \(V(G) \setminus Z\) of the edges in \(D\), where the multiplicity of a vertex \(v\) in \(X\) is the number of edges in \(D\) incident with \(v\). If \(P\) is a \(Z\)-covering path with \(E(P) \cap \delta_G(Z) = D\), then \(P \setminus Z\) is a linear forest with end-multiset \(X\), and so \(P\) is unique by 6.2. This proves (1).

Thus, in order to bound \(n(G, Z)\), it is enough to bound the number of different intersections of such paths with \(\delta_G(Z)\), and we will use 5.2 to do this. Let \(F = G \setminus Z\). By 5.2, there exist \(X \subseteq Z\) with \(|X| \leq 13\) and \(Y \subseteq V(G) \setminus Z\) with \(|Y| \leq 2\) and with the following property. Let \(N(X)\) be the set of vertices of \(F\) with a neighbour in \(X\). For \(i = 1, 2\), let \(N_i\) be the set of all \(v \in V(F) \setminus (Y \cup N(X))\)
that have exactly $i$ neighbours in $Z$; and let $N_0$ be the set of all $v \in N_1$ such that the component of $F \setminus (Y \cup N(X) \cup N_2)$ containing $v$ contains no other vertex in $N_1$. There are at most $296|Z| + 4$ edges between $Z \setminus X$ and $V(F) \setminus (N(X) \cup N_0)$.

The edges of $\delta_G(Z)$ fall into three groups that we will handle differently, as follows:

- Edges between $Z$ and $N(X)$. If $P$ is a $Z$-covering path, then every edge of $P$ between $Z$ and $N(X)$ belongs to a two-edge subpath of $P$ with an end in $X$. There are only $2|X|$ such subpaths in $P$, and for each $x \in X$ the number of two-edge paths in $G$ with one end $x$ is at most $|G|^2$. Thus the number of possibilities for the set of edges of $P$ between $Z$ and $N(X)$ is at most $|G|^{2|X|} \leq |G|^5$.

- Edges between $Z \setminus X$ and $N_0$. Let $T_1, \ldots, T_k$ be the components of $F \setminus (Y \cup N(X) \cup N_2)$ that contain a unique vertex in $N_1$. We claim that if $P$ is a $Z$-covering path, there are at most 28 values of $i \in \{1, \ldots, k\}$ such that $P$ contains the edge between $Z$ and $V(T_i)$. To see this, suppose that $P$ contains the unique edge between $Z$ and $V(T_i)$. Since both ends of $P$ are in $Z$, $P$ contains at least one edge between $V(T_i)$ and $V(F) \setminus V(T_i)$, say $uv$, where $v \in (F) \setminus V(T_i)$. Since $T_i$ is a component of $F \setminus (Y \cup N(X) \cup N_2)$, it follows that $v \in Y \cup N(X) \cup N_2$. Suppose that $v \in N_2$; then $v \in V(P)$, but the two neighbours of $v$ in $Z$ also belong to $V(P)$, and so $v$ has degree more than two in $P$, a contradiction. Thus $v \in Y \cup N(X)$. We have shown that the number of $i$ such that $P$ contains the unique edge between $Z$ and $V(T_i)$ is at most the number of edges of $P$ between $V(T_1 \cup \cdots \cup T_k)$ and $Y \cup N(X)$. For each $v \in Y$ there are at most two edges of $P$ between $V(T_1 \cup \cdots \cup T_k)$ and $v$; and for each $v \in N(X)$ there is at most one such edge, since there is an edge of $P$ between $v$ and $X$. Since at most $2|X| \leq 26$ vertices of $P$ belong to $N(X)$, and $|Y| \leq 2$, it follows that there are at most 30 edges of $P$ between $V(T_1 \cup \cdots \cup T_k)$ and $Y \cup N(X)$. Consequently $P$ contains at most 30 edges between $Z \setminus X$ and $N_0$. There are at most $|G|$ edges between $Z \setminus X$ and $N_0$, and so there are at most $|G|^{30}$ possibilities for the subset that belongs to $P$.

- Edges between $Z \setminus X$ and $V(F) \setminus (N(X) \cup N_0)$. From the choice of $X, Y$, there are only $296|Z| + 4$ such edges, so the number of possibilities for the subset that belongs to a $Z$-covering path is at most $2^{296|Z| + 4}$.

It follows that the number of possibilities for $E(P) \cap \delta_G(Z)$ is at most the product of these three; and so

$$n(G, Z) \leq |G|^5 \cdot |G|^{30} \cdot 2^{296|Z| + 4} = |G|^{52} \cdot 2^{296|Z| + 4}.$$

This proves 6.3.

We deduce:

6.4 Let $G$ be cycle-touching, and let $Z \subseteq V(G)$ be a cycle-hitting set. Then $G$ has at most $|G|^{52} \cdot 2^{297|Z| + 4}$ induced paths.

**Proof.** There are at most $|G|^2/2$ induced paths that are vertex-disjoint from $Z$, since such paths are determined by their ends. Let us count the induced paths that have a vertex in $Z$. For each such path $Q$, with ends $s, t$ say, let $a$ be the vertex of $Q$ in $Z$ that is closest to $s$ in $Q$, and define $b$ similarly for $t$. (Possibly $a = b$.) Thus $Q$ is divided into three subpaths: the part from $s$ to $a$, the part between $a$ and $b$, and the part from $b$ to $t$. There are only $|G|^2/2$ possibilities for the first
part, since it is determined by its first vertex and penultimate vertex; and similarly there are only \(|G|^2/2\) possibilities for the last part. We need to count the possibilities for the middle part \(P\) say, between \(a\) and \(b\). Let \(Z' = Z \cap V(P)\); then \(P\) is a \(Z'\)-covering path in the plantation \(G \setminus (Z \setminus Z')\), and so by 6.3, for each choice of \(Z'\), the number of choices of \(P\) is at most \(|G|^{822 \cdot 297}|Z|^{+4}\). Since there are only \(2|Z|\) choices for \(Z\), there are only \(|G|^{822 \cdot 297}|Z|^{+4}\) choices for \(P\) in total, and hence only \(|G|^{822 \cdot 297}|Z|^{+4}/4 + |G|^2/2 \leq |G|^{862 \cdot 297}|Z|^{+4}\) choices for \(Q\). This proves 6.4.

It follows from 6.4 that if \(Z\) can be chosen with \(|Z| \leq c \log |G|\) (with logarithms to base two), as in 1.3, then the number of induced paths of \(G\) is at most \(16|G|^{862 + 297c}\), and thus is indeed a polynomial in \(|G|\).

7 Finding a cycle-hitting set of logarithmic size

Now we turn to the proof of 1.3. The first step, done in this section, is reduce the problem to finding a logarithmic-sized cycle-hitting set in graphs \(G\) with a plantation \((G, Z)\) such that the members of \(Z\) are at least a large constant distance apart. Let \((G, Z)\) be a plantation, in which every two vertices in \(Z\) have distance at least \(k\); we say that \((G, Z)\) is \(k\)-spaced. (Thus “monic” means the same as “3-spaced”.)

We begin with:

7.1 Let \(G\) be a cycle-touching graph, and let \(C\) be a cycle of \(G\) with minimum length, \(g\) say. Let \(Z\) be the set of vertices in \(V(G) \setminus V(C)\) that have a neighbour in \(V(C)\). For all distinct \(p, q \in V(C)\), if \(P\) is a path with ends \(p, q\) and with no other vertex in \(V(C)\) then \(P\) has length at least \(g/2\). Consequently, every two distinct vertices in \(Z\) have distance at least \(g/2 - 2\) in \(G \setminus V(C)\).

**Proof.** Let \(p, q \in V(C)\) be distinct, and let \(P\) be a path with ends \(p, q\) and with no other vertex in \(V(C)\). There is a path \(Q\) of \(C\) between \(p, q\) of length at most \(|C|/2 = g/2\); and the union of \(P, Q\) is a cycle of length at least \(g\); so \(P\) has length at least \(g/2\). This proves the first statement. If \(z_1, z_2 \in Z\) are distinct, let them be adjacent to \(v_1, v_2 \in V(C)\) respectively. If \(v_1 = v_2\) then every path of \(G \setminus V(C)\) between \(z_1, z_2\) has length at least \(g - 2 \geq g/2 - 2\), since \(G\) has girth at least \(g\); and otherwise every such path has length at least \(g/2 - 2\) by the first statement, since adding two edges gives a path between \(v_1, v_2\). This proves 7.1.

7.2 Let \(G\) be a cycle-touching graph. Then either \(G\) has girth at most 12, or \(G\) has a vertex of degree at most one, or \(G\) has two adjacent vertices both of degree two.

**Proof.** Let \(C\) be a cycle of \(G\) with minimum length, \(g\) say, and suppose that \(g \geq 13\). Let \(Z\) be the set of vertices in \(V(G) \setminus V(C)\) that have a neighbour in \(V(C)\). If \(Z = \emptyset\), any two adjacent vertices of \(C\) satisfy the theorem, so we may assume that \(Z \neq \emptyset\). Let \(F\) be the forest \(G \setminus (V(C) \cup Z)\), and let \(N\) be the set of vertices in \(V(F)\) with a neighbour in \(Z\). From 7.1, every two vertices in \(N\) have distance at least \(g/2 - 4\) in \(F\). Also from 7.1, since \(g \geq 7\) it follows that every vertex in \(Z\) has only one neighbour in \(V(C)\), and \(Z\) is stable; so we may assume that \(F\) is non-null, since otherwise each vertex in \(Z\) is a leaf and the theorem holds. Let \(L\) be the set of leaves of \(F\). Since \(g \geq 9\), it follows that every vertex in \(V(F)\) has at most one neighbour in \(Z\) (because otherwise either 7.1 is violated

13
or there is a cycle of length four). Thus we may assume that every vertex of \( F \) has degree at least one in \( F \) (since otherwise it has degree at most one in \( G \)); and \( L \subseteq N \).

Let \( P \) be a path of \( H \) of maximum length, and let its vertices be \( p_1, \ldots, p_k \) in order. Since \( F \) has minimum degree at least one, it follows that \( k > 1 \). Moreover, the maximality of the length of \( P \) implies that \( p_1 \in L \) and hence \( p_1 \in N \); let \( p_1 \) be adjacent to \( z \in Z \). Since every two vertices in \( N \) have distance at least \( g/2 - 4 > 2 \) in \( F \), it follows that \( p_2 \not\in N \) and no neighbour of \( p_2 \) different from \( p_1 \) belongs to \( N \). On the other hand, from the maximality of the length of \( P \), all neighbours of \( p_2 \) in \( G \) different from \( p_3 \) are leaves of \( F \) and so belong to \( L \subseteq N \). This proves that \( p_2 \) has degree two in \( F \), and \( p_2 \not\in N \). Thus both \( p_1, p_2 \) have degree exactly two in \( G \) and the theorem holds. This proves 7.2.

We deduce:

7.3 Let \( k \geq 4 \) be an integer, and let \( f \) be a non-decreasing function such that if \( (G, Z) \) is a \( k \)-spaced plantation, then \( G \) admits a cycle-hitting set of cardinality at most \( f(|G|) \). Then every cycle-touching graph \( G \) of girth at least \( 2k + 4 \) admits a cycle-hitting set of cardinality at most \( f(|G|) + 2k + 4 \).

Proof. We proceed by induction on \(|G|\). If \( G \) has a vertex of degree at most one, the result follows from the inductive hypothesis by deleting the vertex, so we assume that \( G \) has minimum degree at least two. Let \( g = 2k + 4 \), and suppose that \( G \) has girth strictly more than \( g \). By 7.2, since \( g \geq 12 \), there are two adjacent vertices \( v_1, v_2 \) of \( G \) both of degree two; let \( G' \) be obtained from \( G \) by contracting the edge between these two vertices and thereby making a new vertex \( v \). Then \( G' \) is cycle-touching and has girth at least \( g \), and therefore has a cycle-hitting set \( X \) of cardinality at most \( f(|G|) + g \). If \( v \in X \), then we may replace \( v \) by one of its two neighbours in \( G' \); so we may assume that \( v \notin X \), and so \( X \subseteq V(G) \). Moreover, \( X \) is a cycle-hitting set of \( G \) and the result follows.

Thus we may assume that \( G \) has a cycle \( C \) of length exactly \( g \). Let \( Z \) be the set of vertices in \( V(G) \setminus V(C) \) that have a neighbour in \( V(C) \). Then \( (G \setminus V(C), Z) \) is a plantation, and every two vertices in \( Z \) have distance at least \( g/2 - 2 = k \) by 7.1. From the hypothesis, \( G \setminus V(C) \) admits a cycle-hitting set of cardinality at most \( f(|G|) \); and adding \( V(C) \) to this set gives a cycle-hitting set in \( G \) of cardinality at most \( f(|G|) + g \). This proves 7.3.

8 Getting more transitions

In 4.1 we showed that if \( (G, Z) \) is a plantation and every component of \( F \) contains at least two vertices of \( N \), then there is a normal set \( S \) of transitions with \(|S| \geq |N|/4\). Now we have a \( k \)-spaced plantation where \( k \) is large, but we need to do better than 4.1; we need to show that for some \( c > 1/2 \), we can get a normal set of size at least \( c|N| \). In fact we will show that there is a normal set of cardinality at least \( 2|N|/3 \), provided that every component of \( F \) contains at least three vertices in \( N \). For the proof, we will find a normal set within each clearing (we recall that a clearing is a maximal tree \( T \) of \( F \) such that every vertex of \( N \cap V(T) \) is a leaf of \( T \)); and take the union of them all. But the union of a normal set taken from a clearing, and a normal set taken from another clearing, need not be normal, and we have to be cautious about this. The problem is, the two clearings might have a vertex \( v \) in common, and one normal set might contain a transition \( P \) with \( v \) as an end, and the other might contain a transition \( Q \) that does not contain \( v \) but contains a neighbour of \( v \); then in the
union, $P,Q$ are not anticomplete. So we will institute another rule, saying that if $v$ is a vertex of $F$ that belongs to more than one clearing, then we must not use transitions that contain a neighbour of $v$ and do not contain $v$. But we only need this rule if $v$ belongs to more than one clearing, that is, if $v ∈ N$ and $v$ has degree more than one in $F$, and such vertices $v$ also have a good side; since they belong to at least two clearings, they will be double-counted when we treat each clearing separately and add, and we can take advantage of this.

Thus, let $T$ be a tree with $E(T) \neq \emptyset$, and let $X,Y ⊆ V(T)$ be disjoint, such that every vertex in $X ∪ Y$ is a leaf of $T$. (In the application, $T$ will be a clearing, $X$ will be the set of vertices of $N ∩ V(T)$ that are leaves of $F$, and $Y$ will be the set of vertices of $T$ that belong to another clearing.) Let us say an $(X,Y)$-tuft in $T$ is a set $F$ of distinct paths of $T$, such that

- the paths in $F$ all have a common end in $Y$ (called a root of the tuft);
- each path in $F$ has another end in $X ∪ Y$ (necessarily all distinct);
- for each $y ∈ Y$, no path in $F$ contains a neighbour of $y$ and does not contain $y$.

Two $(X,Y)$-tufts are anticomplete if each member of the first is anticomplete to each path of the second. If $Y \neq \emptyset$, an $(X,Y)$-lawn is the union of a set of pairwise anticomplete $(X,Y)$-tufts. If $Y = \emptyset$, an $(X,Y)$-lawn is a set $F$ of paths of $T$, all with a common end, and each with two distinct ends both in $X$. We will show:

**8.1** Let $T,X,Y$ be as above, with $|X ∪ Y| ≥ 2$, such that either $Y \neq \emptyset$ or $|X| ≥ 3$, and such that every two distinct vertices in $X ∪ Y$ have distance at least 9. Then there is an $(X,Y)$-lawn of cardinality at least $(2|X| + |Y|)/3$.

**Proof.** We proceed by induction on $|T|$, and with $T$ fixed, by induction on $|Y|$. If some leaf of $T$ does not belong to $X ∪ Y$, we may delete it and apply the inductive hypothesis, so we assume that every such vertex is in $X ∪ Y$. If $|Y| ≤ 1$, let $Y ⊆ \{y\}$ say, where $y ∈ Y$ if $Y \neq \emptyset$, and $y ∈ X$ otherwise: then the set of all paths from $y$ to members of $X$ is an $(X,Y)$-tuft of cardinality $|X| + |Y| − 1 ≥ (2|X| + |Y|)/3$ and the theorem holds. Thus we may assume that $|Y| ≥ 2$. If there exists $y ∈ Y$ such that its neighbour has degree at most two, let $Y' = Y \setminus \{y\}$ and $X' = X ∪ \{y\}$; then every $(X',Y')$-tuft is an $(X,Y)$-tuft (because none of its paths can contain the neighbour of $y$ without containing $y$, since this neighbour has degree at most two and does not belong to $X ∪ Y$). Consequently the result follows from the second inductive hypothesis. Thus we may assume that for each $y ∈ Y$, its neighbour has degree at least three.

Choose $y, y' ∈ Y$ with maximum distance, and let $P$ be the path between them. Let its vertices be $y−p_1−⋯−p_k−y'$ in order. From the hypothesis, $P$ has length at least 9, and so $k ≥ 8$. For $1 ≤ i ≤ k$ let $T_i$ be the component that contains $p_i$ of the forest obtained from $T$ by deleting all edges of $P$.

(1) For $1 ≤ i ≤ 4$, no vertex of $T_i$ belongs to $Y$.

Suppose that $y'' ∈ V(T_i) ∩ Y$ where $1 ≤ i ≤ 4$. Let $Q$ be the path of $T_i$ between $p_i,y''$. Since the distance between $y,y''$ is at least 9, it follows that $Q$ has length at least $9 − i$. But the path between $p_i,y'$ has length $k + 1 − i$, and so the path between $y',y''$ has length at least $(9 − i) + (k + 1 − i)$. This path has length at most $k + 1$, from the choice of $P$, and therefore $(9 − i) + (k + 1 − i) ≤ k + 1$, that is, $2i ≥ 9$, a contradiction. This proves (1).
For $1 \leq i \leq 4$ let $X_i = X \cap V(T_i)$, and $x_i = |X_i|$. Since $y \in Y$, it follows that $T_1$ has an edge, and so some vertex of $T_1$ is a leaf of $T$ and hence belongs to $X \cup Y$; and so $x_1 \geq 1$. Similarly $x_k \geq 1$.

(2) For $2 \leq i \leq 4$, we may assume that $2x_i \geq x_1 + \cdots + x_{i-1}$ and $2x_i \geq x_2 + \cdots + x_i + 3$.

Since $k \geq 8 > i$, it follows that $T_j, T_k$ are vertex-disjoint. Let $S, S'$ be the two trees obtained from $T$ by deleting $V(T_i)$, where $t \in S$ and $t' \in S'$. Thus $T_k$ is a subtree of $S'$, and so $|(X \cup Y) \cap V(S')| \geq x_k + 1 \geq 2$. Let $X' = X \cap V(S')$ and $Y' = Y \cap V(S')$. From the inductive hypothesis applied to $S'$, there is an $(X', Y')$-lawn $L$ in $S'$ with cardinality at least $(2|X'| + |Y'|)/3$. The set of all paths from $y$ to $X \cap (V(T_1 \cup \cdots \cup T_{i-1}))$ is an $(X, Y)$-tuft, of cardinality $x_1 + \cdots + x_{i-1}$ and with root $y'$, and all its paths are contained in $S'$; and since $S, S'$ are anticomplete, adding this tuft to the $L$ gives an $(X, Y)$-lawn of cardinality at least $(2|X'| + |Y'|)/3 + x_1 + \cdots + x_{i-1}$. We may assume this is less than $(2|X| + |Y|)/3$, and so

$$(2|X \setminus X'| + |Y \setminus Y'|)/3 > x_1 + \cdots + x_{i-1}.$$  

But $|X \setminus X'| = x_1 + \cdots + x_i$, and $|Y \setminus Y'| = 1$ by (1), and so

$$2(x_1 + \cdots + x_i)/3 + 1/3 > x_1 + \cdots + x_{i-1},$$

that is, $2x_i + 1 > x_1 + \cdots + x_{i-1}$. This proves the first assertion.

Suppose that $p_i+1$ does not belong to any path of $L$. The set of all paths from $y$ to $X \cap (V(T_1 \cup \cdots \cup T_i))$ is an $(X, Y)$-tuft, and its the union of $L$ is an $(X, Y)$-lawn of cardinality at least $(2|X'| + |Y'|)/3 + x_1 + \cdots + x_i$. We may assume this is less than $(2|X| + |Y|)/3$, and so

$$(2|X \setminus X'| + |Y \setminus Y'|)/3 > x_1 + \cdots + x_i.$$  

But $|X \setminus X'| = x_1 + \cdots + x_i$, and $|Y \setminus Y'| = 1$ by (1), and so

$$2(x_1 + \cdots + x_i)/3 + 1/3 > x_1 + \cdots + x_i,$$

that is, $1 > x_1 + \cdots + x_i$, which is impossible since $x_1 > 0$. Thus $p_i+1$ belongs to a path of $L$.

Now $L$ is a union of pairwise anticomplete $(X', Y')$-tufts, and therefore all the paths of $L$ that contain $p_i+1$ belong to the same $(X', Y')$-tuft $F$. Let $y''$ be the root of $F$. Let $F'$ be the set of all paths in $T$ from $y''$ to a vertex in $X \cap V(T_2 \cup \cdots \cup T_i)$ together with the path from $y''$ to $y$; then $F \cup F'$ is an $(X, Y)$-tuft. (Note that we are excluding paths between $y''$ and $X \cap V(T_1)$ from $F'$, because they contain a neighbour of $y$ and do not contain $y$, contrary to the definition of an $(X, Y)$-tuft.) Hence $L \cup F'$ is an $(X, Y)$-lawn of cardinality at least $(2|X'| + |Y'|)/3 + x_2 + \cdots + x_i + 1$, and so we may assume that this is less than $(2|X| + |Y|)/3$. Consequently

$$(2|X \setminus X'| + |Y \setminus Y'|)/3 > x_2 + \cdots + x_i + 1.$$  

As before, it follows that

$$2(x_1 + \cdots + x_i + 1)/3 > x_2 + \cdots + x_i + 1,$$

that is, $2x_1 + 1 > x_2 + \cdots + x_i + 3$. This proves (2).
From the first statement of (2) with $i = 2$, it follows that $x_2 \geq x_1/2$; with $i = 3$, we deduce that $x_3 \geq (x_1 + x_2)/2 \geq 3x_1/4$; and with $i = 4$, we deduce that $x_4 \geq (x_1 + x_2 + x_3)/2 \geq (1 + 1/2 + 3/4)x_1/2 = 9x_1/8$. Consequently
\[ x_2 + x_3 + x_4 \geq (1/2 + 3/4 + 9/8)x_1 > 2x_1. \]
But this contradicts the second statement of (2) with $i = 4$. This proves 8.1.

We deduce:

**8.2** Let $(G, Z)$ be a 9-spaced plantation, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. Suppose that every component of $F$ contains at least three vertices of $N$. Then there is a normal set $S$ of transitions with $|S| \geq 2|N|/3$.

**Proof.** Let $F = G \setminus Z$. If the result holds for each component of $F$ then it holds for $F$ by adding, so we may assume that $F$ is a tree. We may also assume that every leaf of $F$ belongs to $N$, since otherwise we can delete it. For each clearing $T$, let $X(T)$ be the set of vertices of $T$ that belong to $N$ and not to any other clearing; and let $Y(T)$ be the set that belong to $N$ and to another clearing (and therefore belong to $N$). Let $x(T) = |X(T)|$ and $y(T) = |Y(T)|$. Every clearing $T$ has at least two leaves, and they belong to $N$ (since every leaf $F$ belongs to $N$); and so $x(T) + y(T) \geq 2$. Moreover, either $y(T) \neq \emptyset$, or $T = F$ and hence $x(T) \geq 3$. By 8.1, for each clearing $T$ there is an $(X(T), Y(T))$-lawn of cardinality at least $(2x(T) + y(T))/3$. The union of all these lawns is a normal set of transitions, and so there is a normal set of transitions of cardinality at least the sum of $(2x(T) + y(T))/3$ over all clearings $T$. The sum of $x(T)$ over all $T$ is the number of leaves of $F$; and the sum of $y(T)$ over all $T$ is at least twice the number of vertices in $N$ that have degree more than one in $F$, and hence belong to at least two clearings. Consequently the sum of $(2x(T) + y(T))/3$ over all $T$ is at least $2|N|/3$. This proves 8.2.

We can assume that $G$ has minimum degree at least two (because if some $v$ has degree at most one, then any cycle-hitting set when $v$ is deleted is also one for $G$.) Consequently every component of $F$ contains at least two vertices in $N$. A component of $F$ is small if it contains only two vertices in $N$, and big otherwise. We will eventually need to modify 8.2 to allow small components, and the next result is a first step. Here is a kind of problem vertex: in the usual notation, let us say $z \in Z$ is tangential if

- $z$ has exactly two neighbours in $N$, say $v_1, v_2$;
- $v_1, v_2$ belong to components $P_1, P_2$ of $F$ respectively (possibly $P_1 = P_2$); and
- $P_1, P_2$ are both small.

**8.3** Let $(G, Z)$ be a 3-spaced plantation, and let $N$ be the set of vertices in $V(G) \setminus Z$ with a neighbour in $Z$. Suppose also that every vertex of $G$ has degree at least two, and there are no tangential vertices in $Z$. Let $N_b$ be the set of vertices in $N$ that belong to big components of $F$, and let $s$ be the number of small components of $F$. Then $2|N_b|/3 + s \geq (7/6)|Z|$.

**Proof.** Let $A$ be the set of edges between $Z$ and small components, and $B$ the set between $Z$ and big components. For each $z \in Z$, let $d_A(z)$ be the number of edges in $A$ incident with $z$, and define $d_B(z)$ similarly. For each $z \in A$, since $z$ has degree at least two and is not tangential, it follows that $(1/2)d_A(z) + (2/3)d_B(z) \geq 7/6$. The sum over all $z$ of $d_A(z)$ equals $2s$, and the sum of $d_B(z)$ equals $|N_b|$. Thus, by summing over all $z$, we deduce that $s + (2/3)|N_b| \geq (7/6)|Z|$. This proves 8.3.
9 Shunning popularity

In the usual notation, let $S$ be a normal set of transitions. The popularity of a vertex $z \in Z$ in $S$ is the number of transitions in $S$ for which $z$ is a foot. From 4.3, we have:

9.1 Let $(G, Z)$ be a 3-spaced plantation, and let $S$ be a normal set of transitions. Then some vertex in $Z$ has popularity at least $(|S| - |Z|)/3$ in $S$.

Proof. By 4.3, there exists $X \subseteq Z$ with $|X| \leq 3$ such that at most $|Z|$ members of $S$ have no foot in $X$; and consequently some vertex in $X$ has popularity in $S$ at least $(|S| - |Z|)/3$. This proves 9.1.

If $z \in Z$, $N^2(z)$ denotes the set of vertices in $V(G) \setminus Z$ that have distance exactly two from $z$; and for $Z' \subseteq Z$, $N^2(Z')$ is the union of the sets $N^2(z)$ for $z \in Z'$.

9.2 Let $(G, Z)$ be a 4-spaced plantation, such that every vertex of $G$ has degree at least two; then $|N^2(Z)| \leq 2|N(Z)|$.

Proof. Let $N = N(Z)$ and $F = G \setminus Z$ as usual. Since $(G, Z)$ is 4-spaced, no two vertices in $N$ are adjacent; and since every vertex has degree at least two, every vertex with degree at most one in $F$ belongs to $N$. Let $H$ be the set of all clearings, and let $J$ be the bipartite graph with vertex set $N \cup H$ in which $T \in H$ is adjacent to $v \in N$ if $v \in V(T)$. Thus $J$ is a forest (because an isomorphic graph can be obtained from $F$ by contracting all edges that have no end in $N$, since $N$ is stable in $G$); and so $|E(J)| \leq |N| + |H|$. Moreover, every vertex in $H$ has degree at least two in $J$ (since every vertex with degree at most one in $F$ belongs to $N$), and so $|E(J)| \geq 2|H|$. Consequently $|E(J)| \leq 2|N|$; but $|N^2(Z)| \leq |E(J)|$. This proves 9.2.

9.3 Let $(G, Z)$ be a 9-spaced plantation, and let $N, F$ be as before. Suppose that $G$ has minimum degree at least two, and no vertex in $Z$ is tangential. Let $\varepsilon = 5 \cdot 10^{-5}$. Then either:

- some component of $F$ contains at least $\varepsilon|N|$ vertices in $N$; or
- for some $z \in Z$, $|N(z)| \geq \varepsilon|N(Z)|$; or
- for some $z \in Z$, $|N^2(z)| \geq \varepsilon|N^2(Z)|$.

Proof. Let $c = 1/100$. Let the components of $F$ be $C_1, \ldots, C_k$. For $1 \leq i \leq k$ we will choose a normal set $S_i$ of transitions chosen from $C_i$, as follows. Let $Z_{i-1}$ be the set of vertices in $Z$ with popularity at least $c|N|$ in $S_1 \cup \cdots \cup S_{i-1}$, and let $N' = N(Z_{i-1})$ be the set of vertices in $V(C_i)$ with a neighbour in $Z_{i-1}$. For each clearing $T$ of $C_i$, let $X(T)$ be the set of leaves of $T$ that are leaves of $C_i$, and let $Y(T)$ be the set of vertices of $T$ that belong to at least two clearings. Let $X'(T) = X(T) \setminus N'$, and $Y'(T) = Y(T) \setminus N'$. For each such $T$:

- if $|X'(T)| + |Y'(T)| \geq 3$, let $L(T)$ be an $(X'(T), Y'(T))$-lawn of $T$ with cardinality at least $(2|X'(T)| + |Y'(T)|)/3$ (this is possible by 8.1 applied to $T \setminus N'$);
- if $|X'(T)| + |Y'(T)| \leq 2$ and $N' \cap V(T) \neq \emptyset$, let $L(T) = \emptyset$.
that is, \( |X'(T)| + |Y'(T)| \leq 2 \) and \( N' \cap V(T) = \emptyset \) (and hence \( |X(T)| + |Y(T)| = 2 \)) let \( \mathcal{L}(T) \) be an \( (X(T), Y(T)) \)-lawn of \( T \) with cardinality 1 (note that \( 1 \geq (2|X(T)| + |Y(T)|)/3 \) if \( Y(T) \neq \emptyset \), and otherwise \( T = C_i \)).

Note that in each case, the members of \( \mathcal{L}(T) \) contain no vertex in \( N' \), and so the union \( S_i \) of the sets \( \mathcal{L}(T) \) over all clearings \( T \) of \( C \) is normal, and none of its paths contain members of \( N' \). We claim:

1. If either \( C_i \) is a big component of \( F \), or \( N(Z_{i-1}) \cap V(C_i) \neq \emptyset \), then
   \[
   |S_i| \geq (2/3)|N \cap V(C_i)| - (2/3)|N(Z_{i-1}) \cap V(C_i)| - (4/3)|N^2(Z_{i-1}) \cap V(C_i)|.
   \]

   The result is true if \( C_i \) is small, so we assume that \( C_i \) is big. It follows that for each clearing \( T \) of \( C_i \), either \( Y(T) \neq \emptyset \) or \( |X(T)| \geq 3 \). For each clearing \( T \) of \( C_i \), let \( a(T) = 4/3 \) if \( N(Z_{i-1}) \cap V(T) \neq \emptyset \), and zero otherwise. It follows that for each clearing \( T \) of \( C_i \),
   \[
   |\mathcal{L}(T)| \geq (2|X'(T)| + |Y'(T)|)/3 - a(T).
   \]

By summing over all \( T \), we deduce that that \( |S_i| \) is at least the sum of \( (2|X'(T)| + |Y'(T)|)/3 - a(T) \) over all clearings \( T \) of \( C_i \). The sum over all \( T \) of \( 2|X'(T)|/3 \) equals \( 2/3 \) times the number of leaves of \( C_i \) that are not in \( N(Z_{i-1}) \). The sum of \( |Y'(T)|/3 \) over all \( T \) is at least \( 2/3 \) times the number of vertices of \( N \cap V(C_i) \) with degree more than one that are not in \( N(Z_{i-1}) \). The sum of \( a(T) \) over all \( T \) is at most \( (4/3)|N^2(Z_{i-1}) \cap V(C_i)| \). Thus, we deduce that
   \[
   |S_i| \geq (2/3)|N \cap V(C_i)| - (2/3)|N(Z_{i-1}) \cap V(C_i)| - (4/3)|N^2(Z_{i-1}) \cap V(C_i)|.
   \]

This proves (1).

Let \( S = S_1 \cup \cdots \cup S_k \). Then \( S \) is a normal set of transitions.

2. Every vertex in \( Z \) has popularity at most \( (c+\varepsilon)|N| \) in \( S \).

Let \( z \in Z \). If \( z \notin Z_k \) the result is clear, so we assume there exists \( i \) with \( 1 \leq i \leq k \) minimum such that \( z \notin Z_i \). Since \( z \notin Z_{i-1} \), it follows that \( z \) has popularity less than \( c|N| \) in \( S_1 \cup \cdots \cup S_{i-1} \). Its popularity in \( S_i \) is at most the number of transitions in \( S_i \) for which \( z \) is a foot; and this is at most \( |N \cap V(C_i)| \leq c|N| \), by hypothesis. Thus the popularity of \( z \) in \( S_1 \cup \cdots \cup S_i \) is at most \( (c+\varepsilon)|N| \). Since \( z \in Z_i \), none of the paths of \( S_{i+1}, \ldots, S_k \) have \( z \) as a foot. Consequently the popularity of \( z \) in \( S \) is the same as its popularity in \( S_1 \cup \cdots \cup S_i \). This proves (2).

3. \( |Z_k| \leq 6 + 2/c \).

Let \( |Z_k| = t \). Since each vertex in \( Z_k \) is a foot of at least \( c|N| \) members of \( S \), it follows that \( |S| \geq c|N|t/2 \). By 9.1, some vertex in \( Z \) has popularity at least \( (|S| - |Z|)/3 \) in \( S \), and by (2) it follows that \( (|S| - |Z|)/3 \leq (c+\varepsilon)|N| \), that is, \( |S| \leq 3(c+\varepsilon)|N| + |Z| \). Hence \( c|N|t/2 \leq 3(c+\varepsilon)|N| + |Z| \).

Since every vertex in \( Z \) has degree at least two, and therefore \( |Z| \leq |N|/2 \), it follows that
   \[
   c|N|t/2 \leq 3(c+\varepsilon)|N| + |N|/2,
   \]

that is, \( t \leq 6(c+\varepsilon)/c + 1/c \). Since \( \varepsilon \leq 1/6 \), we deduce that \( t \leq 6 + 2/c \). This proves (3).
Let $N_b$ be the set of all vertices in $N$ that belong to big components of $F$. By summing the inequality of (1) over $1 \leq i \leq k$, and since each $Z_i \subseteq Z_k$, we deduce that

$$|S| \geq (2/3)|N_b| - (2/3)|N(Z_k)| - (4/3)|N^2(Z_k)| + s,$$

where $s$ is the number of small components of $F$.

Since $|Z_k| \leq 6 + 2/c$ by (3), it follows that $|N(Z_k)| \leq (6 + 2/c)\varepsilon|N|$, and similarly $|N^2(Z_k)| \leq (6 + 2/c)\varepsilon|N^2(Z)|$. By 9.2, $|N^2(Z)| \leq 2|N|$, and so $|N^2(Z_k)| \leq (12 + 4/c)\varepsilon|N|$. Thus


Now $|N| = |N_b| + 2s$, and so

$$|S| \geq (2/3 - (10/3)(6 + 2/c)\varepsilon)|N_b| + (1 - (20/3)(6 + 2/c)\varepsilon)s.$$

By 9.1 and (2), $(|S| - |Z|)/3 \leq (c + \varepsilon)|N|$, that is, $|S| \leq 3(c + \varepsilon)(|N_b| + 2s) + |Z|$; and so

$$3(c + \varepsilon)(|N_b| + 2s) + |Z| \geq (2/3 - (10/3)(6 + 2/c)\varepsilon)|N_b| + (1 - (20/3)(6 + 2/c)\varepsilon)s.$$

We deduce that

$$|Z| \geq (2/3 - (10/3)(6 + 2/c)\varepsilon - 3(c + \varepsilon))|N_b| + (1 - (20/3)(6 + 2/c)\varepsilon - 6(c + \varepsilon))s.$$

By 8.3, $2|N_b|/3 + s \geq (7/6)|Z|$; so

$$2|N_b|/3 + s \geq (7/6)(2/3 - (10/3)(6 + 2/c)\varepsilon - 3(c + \varepsilon))|N_b| + (7/6)(1 - (20/3)(6 + 2/c)\varepsilon - 6(c + \varepsilon))s.$$

We chose $c, \varepsilon$ such that such that

$$2/3 < (7/6)(2/3 - (10/3)(6 + 2/c)\varepsilon - 3(c + \varepsilon))$$

and

$$1 < (7/6)(1 - (20/3)(6 + 2/c)\varepsilon - 6(c + \varepsilon)).$$

Then the inequality above gives a contradiction. This proves 9.3.

### 10 Completing the proof of 1.3

In view of 7.3, to prove 1.3, it remains to find a logarithmic-sized cycle-hitting set in graphs $G$ with a $k$-spaced plantation where $k$ is some large constant. For inductive purposes we will prove a stronger theorem, but we need some definitions first. Let $(G, Z)$ be a plantation, let $F$ be the forest $G \setminus Z$, and let $N = N(Z)$. We will assume that $(G, Z)$ is 3-spaced, and so $Z$ is stable and every vertex in $N$ has exactly one neighbour in $Z$. Consequently $|N|$ has the same cardinality as the number of edges between $Z$ and $V(F)$. Let us make the following definitions:

- Let $n_1(G, Z) = |N|$.
- Let $n_2(G, Z) = |N^2(Z)|$. 

20
Let \( n_3(G, Z) \) be the maximum of \(|N \cap V(C)|\), taken over all components \( C \) of \( F \).

We will prove:

10.1 Let \((G, Z)\) be a 9-spaced plantation, such that \( G \) has a cycle. Then there is a cycle-hitting set with cardinality at most

\[
10^5 (\log n_1(G, Z) + \log n_2(G, Z) + \log n_3(G, Z)).
\]

**Proof.** Note that since \( G \) has a cycle, and \( Z \) is a stable cycle-hitting set, it follows that all the quantities \( n_1(G, Z), n_2(G, Z), n_3(G, Z) \) are at least two. Let \( \varepsilon = 10^5 \). We proceed by induction on \(|G|\), and for fixed \(|G|\) by induction on \(|Z|\). Thus we may assume that every vertex of \( G \) has degree at least two, because if some \( v \) has degree zero or one, then \( G \setminus \{z\} \) has a cycle, and the result follows from the inductive hypothesis applied to \((G \setminus \{v\}, Z)\) (if \( v \notin Z \)) or to \((G \setminus \{v\}, Z \setminus \{v\})\) if \( v \in Z \).

Suppose that \( z \in Z \) is tangential, and let \( v_1, v_2 \) be its neighbours; for \( i = 1, 2 \) let \( P_i \) be the small component containing \( v_i \). Suppose first that \( P_1 = P_2 \); then the cycle formed by \( P_1 \) and the two edges to \( z \) form a component of \( G \), and since \( G \) is cycle-touching, it follows that the remainder of \( G \) is a forest, and so \( G \) has a cycle-hitting set of cardinality one. Since \( n_1(G, Z), n_2(G, Z), n_3(G, Z) \geq 2 \), and \( 1 \leq 3c \), the result holds in this case. Thus we may assume that \( P_1 \neq P_2 \). Let \( Z' = Z \setminus \{z\} \). Consequently \( G \setminus Z' \) is a forest, and so \((G, Z')\) is a 9-spaced plantation, and \( G' \) has a cycle. Moreover, \( n_1(G, Z') \leq n_1(G, Z) \), and \( n_2(G, Z') \leq n_2(G, Z) \), and \( n_3(G, Z') \leq n_3(G, Z) \) (the last because the new component of \( G \setminus Z' \) that contains \( P_1, P_2 \) and \( v \) only has two vertices with neighbours in \( Z' \)). Hence the result follows from the second inductive hypothesis.

We may therefore assume that no vertex in \( Z \) is tangential. By 9.3, either:

- some component of \( F \) contains at least \( \varepsilon |N| \) vertices in \( N \); or
- for some \( z \in Z \), \( |N(z)| \geq \varepsilon |N(Z)| \); or
- for some \( z \in Z \), \( |N^2(z)| \geq \varepsilon |N^2(Z)| \).

where \( \varepsilon = 5 \cdot 10^{-5} \). In the second and third case, let \( G' = G \setminus \{z\} \) and \( Z' = Z \setminus \{z\} \); then \((G', Z')\) is a 9-spaced plantation. If \( G' \) has no cycle, then there is a 1-vertex cycle-hitting set in \( G \) and the theorem holds, so we assume \( G' \) has a cycle. In the second case, from the inductive hypothesis, \( G' \) has a cycle-hitting set of cardinality at most

\[
c(\log n_1(G', Z') + \log n_2(G', Z') + \log n_3(G', Z')),
\]

and therefore adding \( z \) to this set gives a cycle-hitting set for \( G \). But \( n_1(G', Z') \leq (1 - \varepsilon)n_1(G, Z) \), and so \( c \log n_1(G', Z') \leq c \log n_1(G, Z) - 1 \), since \( c \log(1 - \varepsilon) \leq -1 \); and \( n_2(G', Z') \leq n_2(G, Z) \), and \( n_3(G', Z') \leq n_3(G, Z) \); and consequently the result holds in this case. Similarly in the third case, \( n_2(G', Z') \leq (1 - \varepsilon)n_2(G, Z) \), and so \( c \log n_2(G', Z') \leq c \log n_2(G, Z) - 1 \), and again the result follows. Thus we may assume that the first case holds, and some component of \( F \) contains at least \( \varepsilon |N| \) vertices in \( N \). There are at most \( 3/(2\varepsilon) \) components \( C \) of \( F \) with \(|V(C) \cap N| \geq n_3(G, Z) \), say \( C_1, \ldots, C_k \). For \( 1 \leq i \leq k \), choose \( v_i \in V(C_i) \) such that each component of \( C_i \setminus \{v_i\} \) has at most \(|V(C_i) \cap N|/2 \) vertices in \( N \). Let \( G' = G \setminus \{v_1, \ldots, v_k\} \); then \((G', Z')\) is a 9-spaced plantation. If \( G' \) has no cycle, then \( \{v_1, \ldots, v_k\} \) is a cycle-hitting set for \( G \) of cardinality at most

\[
3/(2\varepsilon) \leq 3c \leq c(\log n_1(G, Z) + \log n_2(G, Z) + \log n_3(G, Z)).
\]
as required. If $G'$ has a cycle, then from the inductive hypothesis, it has a cycle-hitting set of cardinality at most
\[ c(\log n_1(G', Z) + \log n_2(G', Z) + \log n_3(G', Z)); \]
and since $n_1(G', Z) \leq n_1(G', Z)$, and $n_2(G', Z) \leq n_2(G', Z)$, and $n_3(G', Z) \leq n_3(G', Z)/2$, and therefore $c \log n_3(G', Z) \leq c \log n_3(G, Z) - c \leq c \log n_3(G, Z) - 3/(2\epsilon)$, again the result follows. This proves 10.1.

Since each of $n_1(G, Z), n_2(G, Z), n_3(G, Z)$ is at most $|G|$, we deduce that if $G$ admits a 9-spaced plantation, then $G$ has a cycle-hitting set of cardinality at most $3 \cdot 10^5 \log |G|$. From 7.3 with $k = 9$, every cycle-touching graph $G$ of girth at least 22 admits a cycle-hitting set of cardinality at most $3 \cdot 10^5 \log |G| + 22$. Consequently, our second main theorem 1.3 follows from this because of 2.1.

What about graphs that do not contain three, or some fixed number $k$, of cycles that are pairwise anticomplete? It seems straightforward to extend the result 1.3 to such graphs, using induction on $k$, and using the theorem of Erdős and Pósa for graphs that do not have $k$ vertex-disjoint cycles, but at the moment we do not see how to extend 1.2 similarly.

References
