

Polynomial bounds for chromatic number
VIII. Excluding a path and a complete multipartite graph

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January 9, 2023; revised December 26, 2023

¹Supported by AFOSR grant FA9550-22-1-0234 and NSF grant DMS-2154169.

²Research supported by EPSRC grant EP/X013642/1.

³Supported by AFOSR grant FA9550-22-1-0234 and NSF grant DMS-2154169.

Abstract

We prove that for every path H , and every integer d , there is a polynomial f such that every graph G with chromatic number greater than $f(t)$ either contains H as an induced subgraph, or contains as a subgraph the complete d -partite graph with parts of cardinality t . For $t = 1$ and general d this is a classical theorem of Gyárfás, and for $d = 2$ and general t this is a theorem of Bonamy et al.

1 Introduction

A graph is H -free if it contains no induced copy of H . The Gyárfás-Sumner conjecture [6, 20] says:

1.1 Conjecture: *For every forest H , and every k , every H -free graph that does not contain a clique on k vertices has bounded chromatic number.*

This is known only for a few special kinds of forest (see [19] for a survey). It is known when H is a path: this was proved by Gyárfás [6] in the original paper proposing the Gyárfás-Sumner conjecture, with an elegant proof technique now known as the “Gyárfás path” method. But even when H is a path, and indeed even when H is the five-vertex path P_5 , it is not known whether the bound in 1.1 is polynomial in k (the best current bound is that every P_5 -free graph G which does not contain a clique on k vertices has chromatic number $\chi(G) \leq k^{\log_2 k}$ [16]).

There is a stronger conjecture, which has a complicated parentage (it resulted from the joining of 1.1 and an open special case of a false conjecture of Esperet [4]):

1.2 Conjecture: *For every forest H , there is a polynomial $f(k)$ such that for every $k \geq 0$, every H -free graph that does not contain a clique on k vertices has chromatic number at most $f(k)$.*

So far the conjecture is only known for a few forests H , including all trees that do not contain P_5 as a subgraph (see [14, 15, 3, 12]).

If we exclude a complete bipartite graph $K_{t,t}$ rather than a clique, we can do better: the conjectures 1.1 and 1.2 become theorems. Rödl [9] proved the analogue of 1.1, that:

1.3 Theorem: *For every forest H , and every t , every H -free graph that does not contain $K_{t,t}$ as a subgraph has bounded chromatic number.*

The analogue of 1.2 (that is, 1.3 with polynomial bounds) is also now known. It was proved first for the case when H is a path by Bonamy et al. [1]. Two of us, with Spirkl [13], subsequently proved it for all forests:

1.4 Theorem: *For every forest H there is a polynomial $f(t)$ such that for every $t \geq 1$, every H -free graph that does not contain $K_{t,t}$ as a subgraph has chromatic number at most $f(t)$.*

What about excluding the *tripartite* graph $K_{t,t,t}$? Or, more generally, excluding as a subgraph the complete d -partite graph with parts of cardinality t , which we denote by $K_d(t)$? We will show:

1.5 Theorem: *For every path H and all $d \geq 1$, there is a polynomial f such that, for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f(t)$.*

For $t = 1$ and general d this is Gyárfás’ result for paths, and for $d = 2$ and general t this is the theorem of Bonamy et al.

Let us say a graph H is *multibounding* if for every $d \geq 1$, there is a polynomial $f(t)$ such that, for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f(t)$. It is easy to see that every multibounding graph H must be a forest and must satisfy the Gyárfás-Sumner conjecture (take $t = 1$), and any forest satisfying 1.2 is multibounding. We propose:

1.6 Conjecture: *Every forest is multibounding.*

Among all the forests that are known to satisfy 1.1, which can we show are multibounding?

- Two of us [17] showed that every tree of radius two is multibounding, strengthening the theorem of Kierstead and Penrice [8] that every tree H of radius two satisfies 1.1.
- In section 2 of this paper we will show that paths are multibounding, strengthening Gyárfás' theorem that paths satisfy 1.1.
- A *broom* is obtained from a path with one end v by adding leaves adjacent to v , and in section 3 we will show that brooms are also multibounding, strengthening a result of Gyárfás that brooms satisfy 1.1.
- Finally, in section 4 we will show that if H is a forest and each of its components is multibounding then H is multibounding.

2 The proof of 1.5

The proof uses a new version of the “Gyárfás path” method. The usual Gyárfás path technique goes by induction on the clique number: the idea (very roughly) is that the inductive hypothesis allows us to assume that neighbourhoods have small chromatic number, and it is then possible to walk step-by-step into the graph, always heading towards an unexplored portion of the graph with large chromatic number. At each step, we have visited only a bounded number of vertices, and so the portion of the graph in the neighbourhood of the explored set has bounded chromatic number.

This argument does not give polynomial bounds in our setting, as the induction pushes up the bounds too quickly. Here, we are concerned with complete d -partite graphs $K_d(t)$, and argue by induction on d . Rather than a simple induction using neighbourhoods, we instead use the fact that for any t vertices their *common* neighbourhood does not contain $K_{d-1}(t)$ and so by induction has small chromatic number (at most polynomial in t). It follows that, if H is a subgraph with large chromatic number, then there are at most $t - 1$ vertices adjacent to almost all (in an appropriate sense) of H . We now make an argument analogous to the Gyárfás path argument, but rather than one step at a time, we move in a sequence of short dashes, and need to be more careful about the parts of the graph that we explore. At each stage, we make sure we are attached to a subgraph H that is both highly connected and has large chromatic number; we identify the next subgraph $H' \subseteq H$ (again, highly connected and with large chromatic number), and then take a path through H to H' , using the connectivity to avoid any (at most $t - 1$) “dangerous” vertices that can see almost all of H' . Iterating the argument gives the required path.

We need a lemma:

2.1 Lemma *For all integers $t > 0$, if G has chromatic number at least $4t$, then G has a t -connected induced subgraph with chromatic number at least $\chi(G) - 2t$.*

This is proved in [10], but weaker statements have been proved in several other papers: for instance, by Girão and Narayanan [5], and by Penev, Thomassé and Trotignon [11], who proved:

2.2 Theorem: *For all integers $t > 0$, if G has chromatic number at least $2(t - 1)^2 + 1$, then G has a t -connected induced subgraph with chromatic number at least $\chi(G) - 2t + 3$.*

(Any statement of this form is good enough for the application here, with appropriate adjustment of the constants.)

If $X \subseteq V(G)$, we define $\chi(X) = \chi(G[X])$. Let X, Y be disjoint subsets of $V(G)$, where $|X| = t$ and every vertex in Y is adjacent to every vertex in X . We call (X, Y) a t -biclique, and its *value* is $\chi(Y)$. For integers $p, t \geq 1$, let us say a (p, t) -balloon (P, Y) in G consists of an induced path P of G with vertices $v_1-v_2-\dots-v_p$ in order, and a subset $Y \subseteq V(G)$, with the following properties:

- $v_1, \dots, v_{p-1} \notin Y$, $v_p \in Y$, none of v_1, \dots, v_{p-2} have a neighbour in Y , and if $p \geq 2$ then v_p is the unique neighbour of v_{p-1} in Y ; and
- $G[Y]$ is t -connected.

The *value* of the (p, t) -balloon (P, Y) is $\chi(Z)$, where Z is the set of vertices in Y nonadjacent to v_p (and so $v_p \in Z$).

2.3 Lemma: *For every graph G , and all integers $p, q, s, t \geq 1$, if G contains no (p, t) -balloon of value at least q , and G contains no t -biclique of value s , then*

$$\chi(G) \leq (1 + t + \dots + t^{p-1})(s + t(2t + 9)) + t^p q.$$

Proof. For integers $p, q, s, t \geq 1$, define

$$k(p, q, s, t) = (1 + t + \dots + t^{p-1})(s + t(2t + 9)) + t^p q.$$

We proceed by induction on p . Suppose first that $p = 1$, and $\chi(G) > k(1, q, s, t)$. We want to find a $(1, t)$ -balloon of value at least q ; that is, a set Y and a vertex $v \in Y$ such that $G[Y]$ is t -connected and the non-neighbours of v in Y induce a subgraph with chromatic number at least q . By 2.1, since $\chi(G) > k(1, q, s, t) \geq 4t$, G has a t -connected induced subgraph $G[Z]$ say with chromatic number at least $\chi(G) - 2t$. Since $\chi(G) - 2t \geq k(1, q, s, t) - 2t \geq t$, there exist t distinct vertices in Z , say z_1, \dots, z_t . For each i , let X_i be the set of vertices in $Z \setminus \{z_i\}$ nonadjacent to z_i . Since $G[Z]$ is t -connected, we may assume that $G[X_i]$ has chromatic number at most q for $1 \leq i \leq t$, and hence so does $G[X_i \cup \{z_i\}]$ since $q \geq 1$. Thus the union of the sets X_1, \dots, X_t and $\{z_1, \dots, z_t\}$ has chromatic number at most tq . Moreover, since G contains no t -biclique of value s , the set of vertices in $Z \setminus \{z_1, \dots, z_t\}$ adjacent to all of z_1, \dots, z_t has chromatic number less than s . Consequently $\chi(Z) \leq s + tq$, and so $\chi(G) \leq s + t(q + 2) \leq k(1, q, s, t)$ and the theorem holds.

Thus we may assume that $p \geq 2$, and theorem holds for $p - 1$ (and all values of q, s, t). Let $q' = s + t(2t + 9) + tq$. In particular, we may assume that G contains a $(p - 1, t)$ -balloon of value q' , since $k(p - 1, q', s, t) = k(p, q, s, t)$. We assume for a contradiction that $\chi(G) > k(p, q, s, t)$. Let (P', Y') be a $(p - 1, t)$ -balloon of value q' , where P' has vertices $v_1 - \dots - v_{p-1}$. Let Z' be the set of vertices in Y' nonadjacent to v_{p-1} ; thus, $\chi(Z') \geq q'$. Since $q' \geq 12t$, 2.1 implies that there exists $Y \subseteq Z'$ such that $G[Y]$ is $3t$ -connected and $\chi(G[Y]) \geq q' - 6t$. Let X be the set of all vertices $v \in Y'$ such that the set of vertices in Y different from and nonadjacent to v has chromatic number less than $q + 2t$.

(1) $|X| < t$.

Suppose that $|X| \geq t$, and choose $X' \subseteq X$ with $|X'| = t$. Since G contains no t -biclique of value s , the set of vertices in $Y \setminus X'$ that are adjacent to every vertex in X' has chromatic number less than s . But for each $v \in X'$, the set of vertices in Y different from and nonadjacent to v has chromatic

number less than $q + 2t$, and hence so does the union of this set and $\{v\}$. But Y is the union of these subsets; so $\chi(Y) < s + t(q + 2t) \leq q' - 6t \leq \chi(Y)$, a contradiction. This proves (1).

Since $G[Y']$ is t -connected, and $v_{p-1} \notin X$, and $Y \not\subseteq X$ (because $G[Y]$ is $3t$ -connected and hence $|Y| \geq 3t + 1 \geq t$), there is an induced path Q of $G[Y']$ between v_{p-1} and Y with no vertex in X . Choose Q minimal, and let its vertices be $v_{p-1} = u_1 - \dots - u_k$ in order, where $u_k \in Y$. Hence $k \geq 3$, since $u_1 = v_{p-1}$ has no neighbour in Y . From the minimality of Q , none of u_1, \dots, u_{k-1} belong to Y , and none of u_1, \dots, u_{k-2} have a neighbour in $Y \setminus X$. Let R be the path with vertices $v_1 - \dots - v_{p-1} - u_2 - \dots - u_{k-1} - u_k$. Now there are two cases.

First, suppose that u_{k-1} has at least t neighbours in $Y \setminus X$. Let R' be a subpath of $R \setminus \{u_k\}$ with p vertices and one end u_{k-1} . Since $G[(Y \setminus X) \cup \{u_{k-1}\}]$ is t -connected (because $G[Y]$ is $3t$ -connected, and $|X| < t$, and u_{k-1} has at least t neighbours in $Y \setminus X$), and the set of non-neighbours of u_{k-1} in $Y \setminus X$ has chromatic number at least $q + t \geq q$ (because $u_{k-1} \notin X$ and $|X| < t$) it follows that $(R', (Y \setminus X) \cup \{u_{k-1}\})$ is a (p, t) -balloon of value at least q , a contradiction.

Now suppose that u_{k-1} has fewer than t neighbours in $Y \setminus X$. The vertex u_k is one such neighbour; let N be the set of the neighbours of u_{k-1} in $Y \setminus X$ different from u_k . Let R' be a subpath of R with p vertices and one end u_k . Since $G[Y \setminus (X \cup N)]$ is t -connected (because $G[Y]$ is $3t$ -connected and $|X \cup N| \leq 2t$), and the set of non-neighbours of u_k in $Y \setminus (X \cup N)$ has chromatic number at least q (because $u_k \notin X$ and $|X \cup N| < 2t$) it follows that $(R', Y \setminus (X \cup N))$ is a (p, t) -balloon of value at least q , a contradiction. This proves 2.3. ■

Let us deduce that all paths are multibounding, that is, 1.5, which we restate:

2.4 Theorem: *For all $d \geq 1$ and every path H , there is a polynomial f such that, for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f(t)$.*

Proof. Let H have p vertices. H -free graphs contain no (p, t) -balloon of value 1, so by 2.3, there exist integers $b, c \geq 1$ such that for all integers $s, t \geq 1$, and every graph G , if G is H -free and contains no t -biclique of value s , then $\chi(G) \leq bt^c s$.

We prove by induction on d that if G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq (bt^c)^d$. If $d = 1$, graphs not containing $K_d(t)$ as a subgraph have fewer than t vertices, and so have chromatic number less than $t \leq bt^c$ (since $b, c \geq 1$) and the claim holds. Thus we assume that $d > 1$ and the claim holds for $d - 1$. Let G be H -free, with chromatic number more than $(bt^c)^d$; we will show that G contains $K_d(t)$ as a subgraph. Since $\chi(G) > (bt^c)^d$, we deduce from the choice of b, c (taking $s = (bc^t)^{d-1}$) that G contains a t -biclique (X, Y) of value $(bc^t)^{d-1}$. From the inductive hypothesis applied to $G[Y]$, $G[Y]$ contains $K_{d-1}(t)$ as a subgraph. But then, adding X , we find that G contains $K_d(t)$ as a subgraph. This proves 1.5. ■

The proof we just gave is redundant, because 1.5 follows from the analogous theorem 3.2 for brooms which we will prove later, but the proof above is a little simpler than the proof of 3.2.

3 Brooms

A *broom* is obtained from a path with one end v by adding leaves adjacent to v . In this section we deduce from 2.3 that brooms are multibounding. Essentially the proof of 1.5 still works. For $k \geq 1$,

let S_k be the star with $k+1$ vertices: that is, a tree in which some vertex is incident with every edge. Let $\omega(G)$ denote the size of the largest clique of G . We need the following lemma, due to Gyárfás [7]:

3.1 Lemma: *If G is S_k -free, then $\chi(G) \leq \omega(G)^k$.*

We deduce that brooms are multibounding, that is:

3.2 Theorem: *For all $d \geq 1$ and every broom H , there is a polynomial f such that, for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f(t)$.*

Proof. Let H be obtained from a p -vertex path by adding r new vertices each adjacent to one (and the same) end of the path. We prove the result by induction on d .

If $d = 1$ then the result holds trivially, because graphs without $K_1(t)$ as a subgraph have fewer than t vertices. So we assume that $d > 1$ and the result holds for $d-1$. Let $g(t)$ be the corresponding polynomial. Define

$$f(t) = (1 + t + \dots + t^{p-1})(g(t) + 1 + t(2t + 9)) + t^p(dt)^{2r}.$$

Now let G be an H -free graph that does not contain $K_d(t)$ as a subgraph; we will show that $\chi(G) \leq f(t)$.

(1) G contains no (p, t) -balloon of value $(dt)^{2r}$.

Suppose that (P, Y) is such a (p, t) -balloon, where P has vertices $v_1 \dots v_p$ in order. Since G does not contain $K_d(t)$ as a subgraph, it follows that $\omega(G) < dt$, and so $\chi(Y) > \omega(G)^{2r}$, and therefore $G[Y]$ contains an induced copy of S_{2r} by 3.1. Let $a \in Y$, and let $\{b_1, \dots, b_{2r}\} \subseteq Y \setminus \{a\}$ be a stable set of neighbours of a . Since $G[Y]$ is connected, there is an induced path Q from v_p to a , say with vertices $v_p = q_1 \dots q_n = a$. Choose $i \in \{1, \dots, n\}$ minimum such that q_i belongs to or is adjacent to a vertex in $\{a, b_1, \dots, b_{2r}\}$. If $q_i \in \{a, b_1, \dots, b_{2r}\}$, then $i = 1$ from the minimality of i , and the subgraph induced on $\{v_1, \dots, v_p, a, b_1, \dots, b_{2r}\}$ contains a copy of H , a contradiction. So $q_i \notin \{a, b_1, \dots, b_{2r}\}$. Let N be the set of neighbours in $\{a, b_1, \dots, b_{2r}\}$ of q_i . If N contains at least r of $\{b_1, \dots, b_{2r}\}$, then the subgraph induced on $\{v_1, \dots, v_p, q_1, \dots, q_i\} \cup N$ contains a copy of H , a contradiction. So we may assume that $b_1, \dots, b_r \notin N$. If $a \in N$ then the subgraph induced on $\{v_1, \dots, v_p, q_1, \dots, q_i, a, b_1, \dots, b_r\}$ contains a copy of H , a contradiction. If $a \notin N$, we may assume that $b_{r+1} \in N$, and then the subgraph induced on $\{v_1, \dots, v_p, q_1, \dots, q_i, a, b_1, \dots, b_r, b_{r+1}\}$ contains a copy of H , a contradiction. This proves (1).

(2) G contains no t -biclique of value $g(t) + 1$.

If (X, Y) is such a t -biclique, from the inductive hypothesis $G[Y]$ contains $K_{d-1}(t)$ as a subgraph, and hence G contains $K_d(t)$ as a subgraph, a contradiction. This proves (2).

From (1), (2) and 2.3,

$$\chi(G) \leq (1 + t + \dots + t^{p-1})(g(t) + 1 + t(2t + 9)) + t^p(dt)^{2r} = f(t).$$

This proves 3.2. ■

For every forest H that is known to satisfy the Gyárfás-Sumner conjecture 1.1, one might hope to show that H is multibounding. So far, we know that trees of radius two, and brooms, are multibounding. What is the next simplest forest we could try? There are several candidates for this: disjoint unions of trees that are known to be multibounding? Subdivisions of a star, or of a claw? Double brooms?

We have answered the first of these, as we shall explain in the next section, and made some progress on the third, as we explain now. A *double broom* is a tree obtained from a path by adding new vertices each adjacent to an end of the path, but not necessarily all to the same end. It was proved in [18] that double brooms satisfy 1.1.

Let $B(p, r, s)$ be the double broom obtained from a p -vertex path by adding r leaves at one end and s at the other. We have not been able to show that double brooms are multibounding, but we proved the following:

3.3 Theorem: *For all $d, p, r, s, t \geq 1$, there is a polynomial $f(t)$, such that if a graph G does not contain $K_d(t)$, and its chromatic number is more than $f(t)$, then G contains a double broom $B(p', r, s)$ for some $p' \geq p$.*

The proof is a modification of the proof of 2.3, beginning with a large star induced subgraph and growing a balloon tethered to this star. Here is a sketch. We may assume that $r = s$, and that G has chromatic number at least some large polynomial in t , and G does not contain a t -biclique with large value (again, value at least a large polynomial in t , but not so large). We may assume that G is t -connected, by 2.1. For each $v \in V(G)$, $N(v)$ and $M(v)$ denote its set of neighbours and non-neighbours. Since there is no t -biclique with large value, there are fewer than t vertices v for which $\chi(M(v))$ is small. So there exists v with $|N(v)|$ and $\chi(M(v))$ large. Choose $N \subseteq N(v)$ with cardinality equal to a large polynomial in t .

Suppose there is a stable subset $S \subseteq N$ with $|S| = 3s$ such that the set X of vertices in $M(v)$ with no neighbour in S has large chromatic number. Apply 2.1 to obtain a t -connected subset Y of X , still with large χ . Fewer than t vertices in $V(G) \setminus Y$ are adjacent to almost all of Y , so there is a path from v to Y avoiding such vertices. Let v, v_1, v_2 be the first three vertices of this path (we may assume that no other vertices of the path have neighbours in S). Every vertex in S is either adjacent to neither of v_1, v_2 , or adjacent to v_1 and not to v_2 , or adjacent to v_2 ; and since $|S| = 3s$, at least s vertices in S behave the same. In each case, we have half the double broom, and now use the argument of 3.2 to obtain the other half.

If there is no such S , then since there are only polynomially many choices of stable sets in N with size $3s$ (because $|N|$ is a polynomial in t), the set of all vertices $u \in M(v)$ such that $N(v) \cap M(u)$ contains a stable set of size $3s$ has small chromatic number; and so the set Z of $u \in M(v)$ such that $N(v) \cap M(u)$ contains no such stable set has large chromatic number. But if $N(v) \cap M(u)$ contains no stable set of size $3s$, then its cardinality is at most a polynomial in t (by Ramsey's theorem, since $\omega(G) \leq td$). So every vertex in Z is adjacent to almost all of $N(v)$. Inductively, $G[Z]$ contains $K_{d-1}(t)$ as a subgraph (K say), and each vertex of K has only a few non-neighbours in $N(v)$; so K has at least t common neighbours in $N(t)$, giving a copy of $K_d(t)$, a contradiction. This ends the sketch of the proof of 3.3.

4 Disjoint unions

In this section we show that:

4.1 Theorem: *If every component of H is multibounding, then H is multibounding.*

We will deduce this by applying a strengthened version of a theorem from [2]. It turns out that a modification of the proof in [2] gives a stronger result than was noticed at the time. We say J is the *complete join* of J_1, J_2 if the complement of J is the disjoint union of the complements of J_1 and J_2 .

The theorem from [2] says that if H is the disjoint union of H_1, H_2 , and J is the complete join J_1 and J_2 , and G is both H - and J -free, then $V(G)$ can be partitioned into a bounded number of parts such that for each part X , $G[X]$ is either H_1 -free or H_2 -free or J_1 -free or J_2 -free (unless all of H_1, H_2, J_1, J_2 have only one vertex, when the statement is false). We need to improve this in two ways. First, we want to tighten the bound on the number of parts, which turns out only to depend on $|H_1|$ and $|J_1|$, and is polynomial in $|J_1|$; and second, we want to apply it to 4.1, taking J to be $K_d(t)$. This presents a difficulty, because in 4.1, we are working with containing $K_d(t)$ as a subgraph, not as an induced subgraph, and we need to work around this somehow. The easiest fix seems to be to work with excluding sets of graphs rather than single graphs: because excluding, say, $K_{t,t}$ as a subgraph is equivalent to excluding as induced subgraphs all graphs with $2t$ vertices in which half the vertices are adjacent to the other half.

Let \mathcal{H} be a nonempty class of graphs. A graph is \mathcal{H} -free if it is H -free for each $H \in \mathcal{H}$. The *max-size* of \mathcal{H} is the maximum number of vertices of members of \mathcal{H} , if this maximum exists, and ∞ otherwise; and the *min-size* is the minimum number of vertices of members of \mathcal{H} . If $\mathcal{H}_1, \mathcal{H}_2$ are classes of graphs, we denote by $\mathcal{H}_1 \uplus \mathcal{H}_2$ the class of all graphs that are the disjoint union of a member of \mathcal{H}_1 and a member of \mathcal{H}_2 ; and $\mathcal{H}_1 * \mathcal{H}_2$ is the class of all graphs that are the complete join of a member of \mathcal{H}_1 and a member of \mathcal{H}_2 .

We will show:

4.2 Lemma: *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{J}_1, \mathcal{J}_2$ be nonempty classes of non-null graphs, and let $k \geq 0$ be an integer such that \mathcal{H}_1 and \mathcal{J}_1 both have max-size at most $k-2$. Let at least one of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{J}_1, \mathcal{J}_2$ have min-size at least two. Let $S \in \mathcal{H}_1$, let $s = |S|$, and let $n = sk^s + 3$. If G is $\mathcal{H}_1 \uplus \mathcal{H}_2$ -free and $\mathcal{J}_1 * \mathcal{J}_2$ -free, then $V(G)$ can be partitioned into n sets X_1, \dots, X_n , such that for $1 \leq i \leq n$, $G[X_i]$ is either \mathcal{H}_1 -free or \mathcal{H}_2 -free or \mathcal{J}_1 -free or \mathcal{J}_2 -free.*

Proof. Let us say a subset $X \subseteq V(G)$ is *good* if $G[X]$ is either \mathcal{H}_1 -free or \mathcal{H}_2 -free or \mathcal{J}_1 -free or \mathcal{J}_2 -free. For $X \subseteq V(G)$, let $\mu(X)$ be the minimum t such that there is a partition of X into t good sets. (This exists, since at least one of the four classes has min-size at least two, and so all one-vertex sets are good.) We call $\mu(X)$ the *measure* of X . Let $V(S) = \{h_1, \dots, h_s\}$, and let us add a new vertex h_0 to S , with no neighbours, forming a graph S^+ .

Now let $1 \leq q \leq s$, and assume that ϕ is an isomorphism from $S[\{h_q, \dots, h_s\}]$ to some induced subgraph F of G . We call ϕ a q -map, and we say that ϕ is t -general if for $0 \leq p < q$ there is a subset Y_p of $V(G) \setminus V(F)$ with the following properties:

- the sets Y_0, \dots, Y_{q-1} are pairwise disjoint, and all have measure at least t ;
- for $q \leq r \leq s$, and $0 \leq p < q$, if h_r, h_p are adjacent in S^+ then $\phi(h_r)$ is adjacent to every vertex of Y_p in G , and if h_r, h_p are nonadjacent in S^+ then $\phi(h_r)$ is nonadjacent to every vertex of Y_p in G .

We call the list of sets Y_0, \dots, Y_{p-1} a choice of t -minions for ϕ , and F the *image* of ϕ . We deduce:

(1) *No 1-map is 2-general.*

Suppose that ϕ is a 2-general 1-map, and let Y_0 be a choice of the (single) t -minion. Since $\mu(Y_0) > 1$, $G[Y_0]$ is not \mathcal{H}_2 -free; but there are no edges between Y_0 and $V(F)$, and F is isomorphic to S , contradicting that G is $\mathcal{H}_1 \uplus \mathcal{H}_2$ -free. This proves (1).

(2) *For $1 \leq q < s$ and $t \geq 2$, if no q -map is t -general then no $(q+1)$ -map is t' -general, where $t' = 2 + (t-1)(k-2)$. Consequently no s -map is k^{s-1} -general.*

Suppose that ϕ is a t' -general $(q+1)$ -map, let Y_0, \dots, Y_q be some choice of t -minions for ϕ , and let F be its image. Let $v \in Y_q$. If we extend ϕ by mapping h_q to v , we obtain a q -map, which therefore is not t -general. Consequently, either:

- for some $p \in \{0, \dots, q-1\}$, h_p, h_q are adjacent in S^+ , and the set of vertices in Y_q adjacent to v in G has measure at most $t-1$; or
- for some $p \in \{0, \dots, q-1\}$, h_p, h_q are nonadjacent in S^+ , and the set of vertices in Y_q nonadjacent to v in G has measure at most $t-1$.

In either case we say p is a *problem* for v . For $0 \leq p < q$, let V_p be the set of $v \in Y_q$ such that p is a problem for v . Thus $V_0 \cup \dots \cup V_{q-1} = Y_q$, and since

$$\mu(Y_q) \geq t' \geq (t-1)|S| \geq |S| = s > q,$$

it follows that $\mu(V_p) > 1$ for some $p \in \{0, \dots, q-1\}$.

Suppose that h_p, h_q are adjacent in S^+ . Since $\mu(V_p) > 1$, $G[V_p]$ contains a copy F of some member $H_1 \in \mathcal{H}_1$. For each $v \in V(F)$, the set of vertices in Y_q adjacent to v has measure at most $t-1$, and so the set of vertices in Y_q with a neighbour in $V(F)$ has measure at most $|F|(t-1) = |H_1|(t-1) \leq (k-2)(t-1)$. Since $\mu(Y_q) \geq t'$, the set of vertices X in $V(F_1)$ with no neighbour in $V(F_1)$ has measure at least $t' - (k-2)(t-1) = 2$ and so contains a copy of some member of \mathcal{H}_2 , contradicting that G is $\mathcal{H}_1 \uplus \mathcal{H}_2$ -free.

Now suppose that h_p, h_q are nonadjacent in S^+ . Since $\mu(V_p) > 1$, $G[V_p]$ contains a copy F of some member $J_1 \in \mathcal{J}_1$. For each $v \in V(F)$, the set of vertices in Y_q nonadjacent to v has measure at most $t-1$, and so the set of vertices in Y_q with a nonneighbour in $V(F_1)$ has measure at most $|F|(t-1) \leq (k-2)(t-1)$. Since $\mu(Y_q) \geq t'$, the set of vertices X in $V(F_1)$ that are adjacent to every vertex in $V(F_1)$ has measure at least $t' - (k-2)(t-1) \geq 2$ and so contains a copy of some member of \mathcal{J}_2 , contradicting that G is $\mathcal{J}_1 * \mathcal{J}_2$ -free. This proves the first assertion.

For the second assertion, (1) and the first assertion imply that no 2-map is k -general. Also the first assertion implies that for $1 \leq q < s$ and $t \geq 1$, if no q -map is t -general then no $(q+1)$ -map is kt -general. From (1), it follows that for $1 \leq q \leq s$ and $t \geq 1$, no q -map is k^{q-1} -general, and in particular, no s -map is k^{s-1} -general. This proves (2).

For each $v \in V(G)$, let $N(v)$ be the set of neighbours of v in G , and let $M(v)$ be the set of its non-neighbours; so $M(v), N(v)$ have union $V(G) \setminus \{v\}$.

(3) For every vertex $v \in V(G)$, either $\mu(N(v)) < sk^{s-1}$ or $\mu(M(v)) < sk^{s-1}$.

Let h_s have a neighbours and b non-neighbours in S^+ . Thus $a \leq s - 1$ and $b \leq s$. Let ϕ be the function mapping h_s to v . Thus ϕ is an s -map, and so is not k^{s-1} -general. Consequently, either $N(v)$ cannot be partitioned into a sets each with measure at least k^{s-1} , or $M(v)$ cannot be partitioned into b sets each with measure at least k^{s-1} . But a minimal subset of $N(v)$ with measure at least k^{s-1} has measure k^{s-1} (since singleton sets are good); and so, by successively extracting minimal sets with measure at least k^{s-1} , we deduce that either $\mu(N(v)) < ak^{s-1}$, or $\mu(M(v)) < bk^{s-1}$. In particular, one of $N(v), M(v)$ has measure less than sk^{s-1} . This proves (3).

Let A be the set of $v \in V(G)$ such that $N(v)$ has measure at most sk^{s-1} , and let B be the set of $v \in V(G)$ such that $M(v)$ has measure at most sk^{s-1} . If $\mu(A) > 1$, then $G[A]$ contains a copy F of some member of \mathcal{H}_1 , and the set of vertices of G with a neighbour in $V(F)$ has measure at most $sk^{s-1}|F| \leq sk^s$; and since the set of vertices with no neighbours in $V(F)$ has measure at most 1 (because it induces an \mathcal{H}_2 -free subgraph) and $V(F)$ has measure at most 2, it follows that $V(G)$ has measure at most $sk^s + 3$. If $\mu(B) > 1$, then $G[B]$ contains a copy of some member of \mathcal{J}_1 , and it follows similarly that $\mu(V(G)) \leq sk^s + 3$. If neither of these, then $\mu(V(G)) \leq \mu(A) + \mu(B) \leq 2$. In each case the theorem holds. This proves 4.2. \blacksquare

Let us deduce 4.1, which is implied by the following:

4.3 Theorem: *Let H be the disjoint union of H_1, H_2 , and suppose that H_1, H_2 are both multibounding. Then H is multibounding.*

Proof. We need to show that for every $d \geq 1$, there is a polynomial $f_d(t)$ such that, for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f_d(t)$. We show this by induction on d . If $d = 1$, the statement is trivial, taking $f_1(t) = t$, so we may assume that $d > 1$, and the statement holds for all $d' < d$. For $i = 1, 2$, since H_i is multibounding, there is a polynomial $g_i(t)$ such that for all $t \geq 1$, if a graph G is H_i -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq g_i(t)$. Let $g(t)$ be a polynomial such that $g_1(t), g_2(t) \leq g(t)$, and $f_{d'}(t) \leq g(t)$ for all d' with $1 \leq d' < d$.

Let $k = 2 + |H_1| + d_1t$, and $s = |H_1|$, and let $f_d(t) = (sk^s + 3)g(t)$. We will show that for all $t \geq 1$, if a graph G is H -free and does not contain $K_d(t)$ as a subgraph, then $\chi(G) \leq f_d(t)$.

We may assume that at least one of H_1, H_2 has more than one vertex. Let $\mathcal{H}_i = \{H_i\}$ for $i = 1, 2$. Thus G is $\mathcal{H}_1 \uplus \mathcal{H}_2$ -free. Choose integers $d_1, d_2 > 0$ with $d_1 + d_2 = d$. For $i = 1, 2$, let \mathcal{J}_i be the class of all graphs with exactly $d_i t$ vertices that have $K_{d_i}(t)$ as a subgraph. Since G does not contain $K_d(t)$ as a subgraph, it is $\mathcal{J}_1 * \mathcal{J}_2$ -free, and so we can apply 4.2. By 4.2, it follows that $V(G)$ can be partitioned into $n = sk^s + 3$ sets X_1, \dots, X_n , such that for $1 \leq i \leq n$, $G[X_i]$ is either \mathcal{H}_1 -free or \mathcal{H}_2 -free or \mathcal{J}_1 -free or \mathcal{J}_2 -free. If $G[X_i]$ is \mathcal{H}_1 -free, then $\chi(G[X_i]) \leq g_1(t) \leq g(t)$, and similarly if $G[X_i]$ is \mathcal{H}_2 -free, then $\chi(G[X_i]) \leq g(t)$. If $G[X_i]$ is \mathcal{J}_1 -free, then it does not contain $K_{d_1}(t)$ as a subgraph, and so $\chi(G[X_i]) \leq f_{d_1}(t) \leq g(t)$, and similarly if $G[X_i]$ is \mathcal{J}_2 -free, then $\chi(G[X_i]) \leq g(t)$. So in each case, $\chi(G[X_i]) \leq g(t)$, and hence $\chi(G) \leq ng(t) = f_d(t)$. This proves 4.1. \blacksquare

While we are on the subject, let us give another nice application of 4.2. It is known that if every component of a forest H satisfies the Gyárfás-Sumner conjecture 1.1 then H also satisfies it. It is not

known whether, if every component of a forest H satisfies 1.2, then H necessarily satisfies 1.2 (see [3]); but this is true for a related property. Let us say a graph H is *near-Esperet* if there exists $c > 0$ such that for all $d > 0$, every H -free graph G with $\omega(G) < d$ satisfies $\chi(G) \leq d^{c \log d}$. (Logarithms are to base two throughout.) For instance, the theorem of [17] says that the five-vertex path is near-Esperet. Here, we prove that if every component of H is near-Esperet then H is near-Esperet, because of the following:

4.4 Theorem: *Let H be the disjoint union of H_1, H_2 , and let H_1, H_2 be near-Esperet. Then H is near-Esperet.*

Proof. For $i = 1, 2$ we know that H_i is near-Esperet, so there exists $c_i > 0$ such that for all $d > 0$, every $\{H_i, K_d\}$ -free graph G satisfies $\chi(G) \leq d^{c_i \log d}$. Let $c_0 = \max(c_1, c_2)$. Choose $b \geq 0$ such that $|H_1|(2 + d + |H_1|)^{|H_1|} + 3 \leq d^b$ for all $d \geq 2$. Define $c = \max(b + c_0, b/\log(3/2))$. We will show by induction on d that for all $d \geq 0$, every $\{H, K_d\}$ -free graph G satisfies $\chi(G) \leq d^{c \log d}$. If $d \leq 2$ then every K_d -free graph G satisfies $\chi(G) \leq 1$, and the claim holds since $c \geq 0$; so we may assume that $d \geq 2$, and the claim holds for all $d' < d$. (In fact we could start the induction at any convenient value of d , since we know that $H_1 \cup H_2$ satisfies 1.1.) We may assume that at least one of H_1, H_2 has more than one vertex. Choose integers $d_1, d_2 \leq 2d/3$ with $d_1 + d_2 = d$, and let J_i be the complete graph with d_i vertices for $i = 1, 2$.

Now let G be H -free, with clique number less than d . Then G does not contain the complete join of J_1, J_2 , so by 4.2, taking $\mathcal{H}_i = \{H_i\}$ and $\mathcal{J}_i = \{J_i\}$ for $i = 1, 2$, there is a partition of $V(G)$ into n sets X_1, \dots, X_n , where $k = 2 + \max(d_1, |H_1|)$ and $n = |H_1|k^{|H_1|} + 3$, such that for $1 \leq i \leq n$, $G[X_i]$ is either H_1 -free or H_2 -free or J_1 -free or J_2 -free. If $G[X_i]$ is H_1 -free then $\chi(X_i) \leq d^{c_1 \log d} \leq d^{c_0 \log d}$, and similarly if $G[X_i]$ is H_2 -free then $\chi(X_i) \leq d^{c_0 \log d}$. If $G[X_i]$ is J_1 -free then from the inductive hypothesis,

$$\chi(X_i) \leq d_1^{c \log d_1} \leq (2d/3)^{c \log(2d/3)},$$

and similarly if $G[X_i]$ is J_2 -free then $\chi(X_i) \leq (2d/3)^{c \log(2d/3)}$. Since $n \leq d^b$, it follows that

$$\chi(G) \leq d^b \max\left(d^{c_0 \log d}, (2d/3)^{c \log(2d/3)}\right).$$

Thus, to show that $\chi(G) \leq d^{c \log d}$, it suffices to check that

$$d^{c \log d} \geq d^b d^{c_0 \log d}$$

and

$$d^{c \log d} \geq d^b (2d/3)^{c \log(2d/3)}.$$

The first says (taking logarithms)

$$c(\log d)^2 \geq b \log d + c_0(\log d)^2,$$

and this is true since $b \log d \leq b(\log d)^2$ and $c \geq b + c_0$. The second says (taking logarithms)

$$c(\log d)^2 \geq b \log d + c(\log(2d/3))^2.$$

Let us write $\log d = x$ and $\log(3/2) = \varepsilon$; then we need to show that

$$cx^2 \geq bx + c(x - \varepsilon)^2,$$

that is, $2c\varepsilon x \geq bx + c\varepsilon^2$. But this is true since $c\varepsilon x \geq bx$ and $c\varepsilon x \geq c\varepsilon^2$ (because $d > 3/2$). This proves 4.4. ■

It is proved in [16] that the five-vertex path P_5 is near-Esperet; so an application of 4.4 yields that every forest that is the disjoint union of copies of P_5 is also near-Esperet.

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