Polynomial bounds for chromatic number
VII. Disjoint holes

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Abstract

A hole in a graph G is an induced cycle of length at least four, and a k-multihole in G is a set of pairwise disjoint and nonadjacent holes. It is well known that if G does not contain any holes then its chromatic number is equal to its clique number. In this paper we show that, for any k, if G does not contain a k-multihole, then its chromatic number is at most a polynomial function of its clique number. We show that the same result holds if we ask for all the holes to be odd or of length four; and if we ask for the holes to be longer than any fixed constant or of length four. This is part of a broader study of graph classes that are polynomially χ-bounded.

1 Introduction

A function $\phi : \mathbb{N} \to \mathbb{N}$ is a *binding function* for a graph G if $\chi(G) \leq \phi(\omega(G))$, where $\chi(G), \omega(G)$ denote the chromatic number of G and the size of the largest clique of G, respectively. A class $\mathcal{C}$ of graphs is *hereditary* if for every $G \in \mathcal{C}$, every graph isomorphic to an induced subgraph of G also belongs to $\mathcal{C}$. A hereditary class $\mathcal{C}$ is *χ-bounded* if there is a function ϕ that is a binding function for each $G \in \mathcal{C}$, and if so, we call ϕ a *binding function* for the class; if there exists a polynomial binding function, we say that $\mathcal{C}$ is *poly-χ-bounded* (see [11] for a survey on χ-bounded classes, and [8] on poly-χ-bounded classes). While many classes are known to be χ-bounded, the proofs frequently give quite fast-growing functions, and it is natural to ask whether this is necessary. A remarkable conjecture of Louis Esperet [5] asserted that every χ-bounded hereditary class is poly-χ-bounded. But this was recently disproved by Briański, Davies and Walczak [2]. So the question now is: which hereditary classes are poly-χ-bounded?

A hereditary graph class is defined by excluding some induced subgraphs. A graph is *H-free* if it has no induced subgraph isomorphic to H, and *$\{H_1, H_2\}$-free* means both H_1-free and H_2-free. There is a mass of results on χ-bounded classes where one of the excluded graphs is a forest, but in this paper we consider some classes where every excluded graph has a cycle. A *hole* is an induced cycle of length at least four, and *odd-hole-free* means containing no odd hole. A *four-hole* means a hole of length four. Let us say a *k-multihole* of a graph G is an induced subgraph with k components, each a cycle of length at least four. We denote the k-vertex path by P_k and the k-vertex cycle by C_k.

Graphs with no 1-multihole are chordal and hence perfect. The class of graphs with no k-multihole in which all the cycles have odd length, is shown in [9] to be χ-bounded, but it contains the class of $\{P_5, C_5\}$-free graphs, and we cannot yet prove it is poly-χ-bounded (see [15] for the best current bounds). If we replace "odd" by "long", the same applies: it is shown in [10] that for every $\ell \geq 0$, the class of graphs with no k-multihole in which all the cycles have length at least ℓ is χ-bounded (and we cannot yet prove it is poly-χ-bounded, for the same reason). But we can if we permit cycles of length four to be components of the multiholes we are excluding. We will show:

1.1 *For each integer $k \geq 0$, let $\mathcal{C}$ be the class of all graphs G with no k-multihole in which every component either has length four or odd length. Then $\mathcal{C}$ is poly-χ-bounded.*

If we change "odd" to "long", it also works:

1.2 *For all integer $k \geq 0$ and $\ell \geq 4$, let $\mathcal{C}$ be the class of all graphs G with no k-multihole in which every component either has length four or length at least ℓ. Then $\mathcal{C}$ is poly-χ-bounded.*

This second one we can make stronger (we could not prove the corresponding strengthening of the first):

1.3 *For all integers $k, s \geq 0$, and $\ell \geq 4$, let $\mathcal{C}$ be the class of all graphs G such that no induced subgraph of G has exactly k components, each of which is either isomorphic to $K_{s,s}$ or a cycle of length at least ℓ. Then $\mathcal{C}$ is poly-χ-bounded.*

(In general, $K_{s,t}$ denotes the complete bipartite graph with parts of cardinality s and t.) Both these results derive from a theorem about $K_{s,s}$, which we will explain in the next section.

2 Excluding a disjoint union, and self-isolation

If $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced on A; and we write $\chi(A)$ for $\chi(G[A])$ and $\omega(A)$ for $\omega(G[A])$. Two disjoint subsets of $V(G)$ are *anticomplete* if there are no edges between them, and *complete* if every vertex of the first subset is adjacent to every vertex of the second. A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H, and such a subgraph is a *copy* of H. A function $\phi : \mathbb{N} \to \mathbb{N}$ is *non-decreasing* if $\phi(x) \leq \phi(y)$ for all $x, y \in \mathbb{N}$ with $x \leq y$.

Let us say a graph H is *self-isolating* if for every non-decreasing polynomial $\psi : \mathbb{N} \to \mathbb{N}$, there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For every graph G with $\chi(G) > \phi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$, such that either

- $G[A]$ is H-free, or
- G contains a copy H' of H such that $V(H')$ is disjoint from and anticomplete to A.

Self-isolation is of interest in considering polynomial χ-boundedness for the class of H-free graphs, where H is a forest. Say a forest H is *good* if the class of H-free graphs is polynomially χ-bounded. It might be true that every forest is good (strengthening the Gyárfás-Sumner conjecture [6, 16] from χ-boundedness to polynomial χ-boundedness), but this has only been proved for a few simple kinds of tree H, and some (not all) of the forests that are disjoint unions of these trees. It is not known that if trees H_1, H_2 are good, then the disjoint union of H_1 and H_2 is good. For instance, trees of diameter three are good [14], but disjoint unions of them might not be as far as we know. But self-isolation helps here: if H_1 and H_2 are good forests, and one of them is self-isolating, then the disjoint union of H_1 and H_2 is good. Some good trees are known to be self-isolating (namely, stars and four-vertex paths), so we can happily take disjoint unions with them and preserve goodness.

Which graphs are self-isolating? We know very little at the moment: there are very few graphs that we know to have the property, and none that we know not to have the property. (Could it be that all graphs are self-isolating? Certainly, if we change the definition of self-isolating, replacing the polynomials ϕ, ψ by general functions, it is easy to show that all graphs have the property, by induction on $\omega(G)$.) A graph is self-isolating if all its components are self-isolating, but the only connected graphs that we know are self-isolating are complete graphs (proved below), paths of arbitrary length (proved in [4]), and complete bipartite graphs (proved in the next section). The main result of [13] was that stars are self-isolating, so our result that complete bipartite graphs are self-isolating generalizes this. The last takes up the main part of this paper, and is most of what we need to prove 1.1 and 1.3.

First, complete isolation:

2.1 *Every complete graph is self-isolating.*

Proof. (This proof was derived from a similar proof in [7].) Let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial, and let H be a k-vertex complete graph. Let ϕ be the polynomial $\phi(x) = (x+1)^k \psi(x) + x$ for $x \in \mathbb{N}$. Now let G be a graph with chromatic number more than $\phi(\omega(G))$, and let K be a clique of G with cardinality $\omega(G)$. If $\omega(G) < k$, then the first bullet in the definition of self-isolating holds, so we assume that $\omega(G) \geq k$. For each $X \subseteq K$ with $|X| = k$, let A_X be the set of vertices in $V(G) \setminus K$ that are nonadjacent to every vertex in X; and for every $Y \subseteq K$ with $|Y| = k-1$, let B_Y be the set of vertices in $V(G) \setminus K$ that are adjacent to every vertex in $K \setminus Y$. Thus $V(G) \setminus K$ is the union of the $\binom{\omega(G)}{k}$ sets A_X and the $\binom{\omega(G)}{k-1}$ sets B_Y; and since

$$\binom{\omega(G)}{k} + \binom{\omega(G)}{k-1} = \binom{\omega(G)+1}{k} \leq (\omega(G)+1)^k,$$

and $\chi(G \setminus K) > (\omega(G)+1)^k \psi(\omega(G))$, one of the sets A_X or B_Y has chromatic number more than $\psi(\omega(G))$. If $\chi(A_X) > \psi(\omega(G))$ for some X, then $G[X]$ is a copy of H anticomplete to A_X, and since $\psi(\omega(G)) \geq \psi(\omega(A_X))$, the second bullet in the definition of self-isolating holds. If $\chi(B_Y) > \psi(\omega(G))$ for some Y, then since $|K \setminus Y| = \omega(G) - k + 1$ and B_Y is complete to $K \setminus Y$, it follows that $\omega(B_Y) < k$ and so $G[B_Y]$ is H-free, and the first bullet in the definition of self-isolating holds. This proves 2.1. ∎

3 Complete bipartite isolation

We turn to the proof that

3.1 *Every complete bipartite graph is self-isolating.*

We will in fact prove something a little stronger. Let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing function. An induced subgraph H of a graph G is *ψ-nondominating* if there exists a set $A \subseteq V(G)$ disjoint from and anticomplete to $V(H)$, with $\chi(A)) \geq \psi(\omega(A))$. If $\psi : \mathbb{N} \to \mathbb{N}$ is a non-decreasing function and $q \geq 0$ is an integer, a *(ψ, q)-sprinkling* in a graph G is a pair (P, Q) of disjoint subsets of $V(G)$, such that

- $\chi(P) > \psi(\omega(P))$; and
- $\chi(Q) > \psi(\omega(Q)) + qr$, where r is the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q.

(This is closely related to what was called a "(ψ, q)-scattering" in [4].) We will prove:

3.2 *Let $s, q \geq 0$ be integers, and let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial. Then there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For every graphs G with with $\chi(G) > \phi(\omega(G))$, either:*

- *there is a ψ-nondominating copy of $K_{s,s}$ in G, or*
- *there is a (ψ, q)-sprinkling in G.*

Proof of 3.1, assuming 3.2. Let $s, s' \geq 0$ be integers, where $s' \leq s$. We will show that $K_{s,s'}$ is self-isolating. (It is not enough to show this when $s = s'$, because we do not know that every induced subgraph of a self-isolating graph is self-isolating.) Let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial, let $q = s + s'$, and let ϕ satisfy 3.2. Let G be a graph with $\chi(G) > \phi(\omega(G))$. We claim that either there is a ψ-nondominating copy of $K_{s,s'}$ in G, or there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$ such that $G[A]$ is $K_{s,s'}$-free. If there is a ψ-nondominating copy of $K_{s,s}$ in G, then there is also one of $K_{s,s'}$, so by 3.2, we may assume that there is a (ψ, q)-sprinkling (P, Q) in G. If $G[P]$ is $K_{s,s'}$-free, the claim holds, so we assume that there is a copy H of $K_{s,s'}$ in $G[P]$. Thus $|H| = q$. Let r be the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q. The set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $|H|r = qr$; and $\chi(Q) > \psi(\omega(Q)) + qr$ from the definition of a (ψ, q)-sprinkling. Consequently H is ψ-nondominating, and hence $K_{s,s'}$ is self-isolating. ∎

To prove 3.2 we will need the following lemma:

3.3 *For every graph G that is not a complete graph, there is a vertex v such that the set of vertices different from and nonadjacent to v has chromatic number at least $\chi(G)/\omega(G)$.*

Proof. Let X be a maximum clique of G, and for each $x \in X$, let D_x be the set of vertices of G different from and nonadjacent to x. Since G is nonnull, it follows that $X \neq \emptyset$. But $V(G)$ is the union of the sets $D_x \cup \{x\}$ over $x \in X$, because of the maximality of X; and so there exists $v \in X$ such that $\chi(D_v \cup \{v\}) \geq \chi(G)/\omega(G)$. Choose such a vertex v with $D_v \neq \emptyset$ if possible. If $D_v \neq \emptyset$, then $\chi(D_v \cup \{v\}) = \chi(D_v)$, since there are no edges between v and D_v, and so the theorem holds. Thus we may assume (for a contradiction) that $D_v = \emptyset$, and so $1 = \chi(D_v \cup \{v\}) \geq \chi(G)/\omega(G)$. Since $\chi(G)/\omega(G) \geq 1$, equality holds, and so $\chi(D_x \cup \{x\}) \geq \chi(G)/\omega(G)$ for every $x \in X$; and so $D_x = \emptyset$ for all $x \in X$, from the choice of v. Consequently $V(G) = X$, and G is a complete graph, a contradiction. This proves 3.3. ∎

The proof of 3.2 will be by examining the largest "template" in G. With s fixed, let us say that, for all integers $t, k \geq 0$, a (t,k)*-template* in G is a sequence $(A_1, \ldots, A_k)$ of pairwise disjoint subsets of $V(G)$, each of cardinality t, such that for $1 \leq i < j \leq k$, and for every stable set $S \subseteq A_j$ with $|S| = s$, every vertex in A_i has a neighbour in S. The next result will enable us to find a $(t,2)$-template. If $v \in V(G)$, we denote the set of neighbours of a vertex v by $N(v)$ or $N_G(v)$.

3.4 *Let $s, q, t \geq 0$ be integers, and let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial. Let G be a graph with*

$$\chi(G) > \omega(G)^s \left((s + t^s)\, \psi(\omega(G)) + t\right) \text{ and}$$
$$\chi(G) \geq q^s t + \left(2 + q + q^2 + \cdots + q^{s-1}\right) \psi(\omega(G)) + 2.$$

Then either

- *there is a ψ-nondominating copy of $K_{s,s}$ in G, or*
- *there is a (ψ, q)-sprinkling in G, or*
- *G contains a $(t,2)$-template.*

Proof. We may assume that $s, t \geq 1$. Define $p = \psi(\omega(G))$. For $0 \leq i \leq s$, define

$$m_i = \omega(G)^{s-i}\left(t^s p + t\right) + \left(1 + \omega(G) + \cdots + \omega(G)^{s-i-1}\right) p$$
$$n_i = q^{s-i} t + \left(1 + q + q^2 + \cdots + q^{s-i-1}\right) p.$$

Thus $m_s = t^s p + t$, and $m_i = \omega(G) m_{i+1} + p$ for $0 \leq i < s$; and $n_s = t$ and $n_i = q n_{i+1} + p$ for $0 \leq i < s$. By hypothesis, $\chi(G) > m_0$ and $\chi(G) > n_0 + p + 1$.

(1) *There is a vertex v_1 such that $\chi(N(v_1)) > n_1$ and $\chi(M(v_1)) > m_1$, where $M(v_1) = V(G) \setminus (N(v_1) \cup \{v_1\})$.*

Let S be the set of all vertices v with $\chi(N(v)) \le n_1$. If $\chi(S) > p$, choose a subset $P \subseteq S$ with $\chi(P) = p+1$, and let $Q = V(G) \setminus P$. Then

$$\chi(Q) \ge \chi(G) - (p+1) > n_0 = p + qn_1,$$

and so (P, Q) is a (ψ, q)-sprinkling. We therefore assume that $\chi(S) \le p$. Let $R = V(G) \setminus S$. Thus

$$\chi(R) \ge \chi(G) - p > m_0 - p = \omega(G)m_1 \ge \omega(G),$$

and so R is not a clique. By 3.3, there exists $v_1 \in R$ such that the set of vertices in R different from and nonadjacent to v_1 has chromatic number at least $\chi(R)/\omega(G) > m_1$, and so $\chi(M(v_1)) > m_1$. This proves (1).

Choose a stable set $S \subseteq V(G)$ with $|S| \le s$, maximal such that $\chi(N(S)) > n_{|S|}$ and $\chi(M(S)) > m_{|S|}$, where $N(S)$ denotes the set of all vertices in $V(G) \setminus S$ that are adjacent to every vertex in S, and $M(S)$ denotes the set of all vertices in $V(G) \setminus S$ that are nonadjacent to every vertex in S. From (1), $|S| \ge 1$. Now there are two cases, $|S| < s$ and $|S| = s$.

Suppose first that $|S| < s$. Let A be the set of all vertices $v \in M(S)$ such that the set of neighbours of v in $N(S)$ has chromatic number at most $n_{|S|+1}$. Since $\chi(N(S)) > n_{|S|} = qn_{|S|+1} + p$, we may assume that $\chi(A) \le p$, because otherwise $(A, N(S))$ is a (ψ, q)-sprinkling. Hence

$$\chi(B) \ge \chi(M(S)) - p > m_{|S|} - p = \omega(G)m_{|S|+1},$$

where $B = M(S) \setminus A$. Since $m_{|S|+1} \ge 1$ (because $t \ge 1$), it follows that B is not a clique, and so from 3.3, there is a vertex $v \in B$ such that the set of vertices in B, different from and nonadjacent to v, has chromatic number at least $\chi(B)/\omega(G) > m_{|S|+1}$. But then adding v to S contradicts the maximality of S.

Now suppose that $|S| = s$. Since $\chi(N(S)) > n_s = t$, we may choose $T \subseteq N(S)$ with $|T| = t$. Let A be the set of vertices in $M(S)$ that have s non-neighbours in T that are pairwise nonadjacent, and let $B = M(S) \setminus A$. For each stable set $S' \subseteq T$ with $|S'| = s$, we may assume that the set of vertices in $M(S)$ with no neighbour in S' has chromatic number at most p, because otherwise $G[S \cup S']$ is a ψ-nondominating copy of $K_{s,s}$. The number of such sets S' is at most t^s, and so $\chi(A) \le t^s p$. Hence

$$\chi(B) \ge \chi(M(S)) - t^s p > m_s - t^s p = t,$$

and so there exists $M \subseteq B$ with $|M| = t$. But then (M, T) is a $(t, 2)$-template. This proves 3.4. ∎

We also need the following version of Ramsey's theorem (proved for instance in [13]).

3.5 *For all integers $s \ge 1$ and $r \ge 2$, if a graph G has no stable subset of size s and no clique of size more than r, then $|V(G)| < r^s$.*

Now we use 3.4 to prove 3.2, which we restate in a strengthened form:

3.6 *Let $s, q \ge 0$ be integers, and let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial. Let $\phi, \phi' : \mathbb{N} \to \mathbb{N}$ be the polynomials defined by*

$$\begin{aligned}\phi'(x) &= x^s \left(s\psi(x) + (s+1)^s x^{s(s+1)}\psi(x) + (s+1)x^{s+1}\right)\\ &\qquad + q^s(s+1)x^{s+1} + \left(2 + q + q^2 + \cdots + q^{s-1}\right)\psi(x) + 2\\ \phi(x) &= (s+1)^{2s} x^{2+2s(s+1)}\psi(x) + (s+1)^s x^{1+s(s+1)}\phi'(x) + (x+1)(s+1)x^{s+1}.\end{aligned}$$

for all $x \in \mathbb{N}$. Let G be a graph with $\chi(G) > \phi(\omega(G))$. Then either:

- *there is a ψ-nondominating copy of $K_{s,s}$ in G, or*
- *there is a (ψ, q)-sprinkling in G.*

Proof. Let $t = (s+1)\omega(G)^{s+1}$. Thus

$$\chi(G) > \omega(G)^2 t^{2s}\psi(\omega(G) + \omega(G)t^s\phi'(\omega(G)) + (\omega(G)+1)t.$$

We claim we may assume that:

(1) *If $A \subseteq V(G)$ with $\chi(A) > \phi'(\omega(G))$ then $G[A]$ contains a $(t, 2)$-template.*

Suppose not. Let $G' = G[A]$. Since $\chi(A) > \phi'(\omega(G))$ and ψ is nondecreasing, it follows that

$$\chi(G') > \omega(G')^s \left(t^s\psi(\omega(G')) + t\right) + s\omega(G')^s\psi(\omega(G'))$$

and $\chi(G') \geq q^s t + \left(2 + q + q^2 + \cdots + q^{s-1}\right)\psi(\omega(G')) + 2$. By 3.4 applied to G', either

- there is a ψ-nondominating copy of $K_{s,s}$ in G' (and hence in G), or
- there is a (ψ, q)-sprinkling in G' (and hence in G), or
- G' contains a $(t, 2)$-template.

We may assume that neither of the first two bullets hold, so the third holds. This proves (1).

For $2 \leq k \leq \omega(G)+1$, define $t_k = (s+1)\omega(G)^{s+1} - s(k-2)\omega(G)^s$. Thus $t_2 = t$, and $0 \leq t_k \leq t$ for $2 \leq k \leq \omega(G)+1$. By (1) applied to G, there is a $(t_2, 2)$-template in G. Choose an integer k with $2 \leq k \leq \omega(G)+1$, maximum such that there is a (t_k, k)-template in G, and let $(A_1, \ldots, A_k)$ be such a template.

(2) *$k \leq \omega(G)$.*

Suppose that $k = \omega(G)+1$. Inductively for $i = 1, \ldots, k$, suppose that vertices $a_1, \ldots, a_{i-1}$ are defined, and define a_i as follows. For $1 \leq h < i$, the non-neighbours of a_h in A_i do not include a stable set of cardinality s, from the definition of a (t_k, k)-template. Hence by 3.5 (taking $r = \omega(G)$), there are at most $\omega(G)^s$ vertices in A_i nonadjacent to a_h, and hence at most $\omega(G)^{s+1}$ vertices in A_i that are nonadjacent to at least one of $a_1, \ldots, a_{i-1}$. Since

$$|A_i| = t_k \geq (s+1)\omega(G)^{s+1} - s(\omega(G)-1)\omega(G)^s > \omega(G)^{s+1},$$

some vertex $a_i \in A_i$ is adjacent to all of $a_1, \ldots, a_{i-1}$. This completes the inductive definition. But then $\{a_1, \ldots, a_{\omega(G)+1}\}$ is a clique in G, a contradiction. This proves (2).

Let $Z = V(G) \setminus (A_1 \cup \cdots \cup A_k)$. For $1 \leq i \leq k$, let $\mathcal{S}_i$ be the set of all stable sets contained in A_i with cardinality s. For each $S \in \mathcal{S}_i$, let D_S be the set of vertices in Z with no neighbour in S, and let Y_i be the union of the sets D_S over $S \in \mathcal{S}_i$.

(3) *$|Z \setminus (Y_1 \cup \cdots \cup Y_k)| < t_{k+1}$.*

Suppose not, and choose $A \subseteq Z \setminus (Y_1 \cup \cdots \cup Y_k)$ with $|A| = t_{k+1}$. For $1 \le i \le k$, choose $B_i \subseteq A_i$ with $|B_i| = t_{k+1}$. Then $(A, B_1, B_2, \ldots, B_k)$ is a $(t_{k+1}, k+1)$-template, contrary to the maximality of k. This proves (3).

For each $v \in Y_1 \cup \cdots \cup Y_k$, choose $i \in \{1, \ldots, k\}$ minimum such that $v \in Y_i$, and choose $S \in \mathcal{S}_i$ such that $v \in D_S$. We call S the *home* of v.

(4) *Let $1 \le i \le k$, and let $S \in \mathcal{S}_i$. The set of vertices in D_S with home S has chromatic number at most $\omega(G)t^s\psi(\omega(G)) + \phi'(\omega(G))$.*

Let F be the set of vertices in D_S with home S. By 3.5, as in the proof of (2), for $i+1 \le j \le k$ there are at most $s\omega(G)^s$ vertices in A_j with a non-neighbour in S, and since $|A_j| = t_k = t_{k+1} + s\omega(G)^s$, there exists $B_j \subseteq A_j$ with $|B_j| = t_{k+1}$ complete to S. For $1 \le h < i$, choose $B_h \subseteq A_h$ with $|B_h| = t_{k+1}$ arbitrarily. Let F' be the set of vertices $v \in F$ such that v has no neighbour in S' for some $j \in \{i+1, \ldots, k\}$ and some $S' \in \mathcal{S}_j$. For $i+1 \le j \le k$, and each $S' \in \mathcal{S}_j$, the chromatic number of the set of vertices in F with no neighbour in S' is at most $\psi(\omega(G))$, since the copy of $K_{s,s}$ induced on $S \cup S'$ is not ψ-nondominating; and so $\chi(F') \le \omega(G)t^s\psi(\omega(G))$, since there are at most $\omega(G)t^s$ choices for the pair (j, S'). Let $F'' = F \setminus F'$. If $G[F'']$ contains a $(t,2)$-template, then it contains a $(t_{k+1}, 2)$-template (C_1, C_2) say; and then

$$(C_1, C_2, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k)$$

is a $(t_{k+1}, k+1)$-template in G, from the definition of a home, a contradiction. Thus $G[F'']$ contains no such template, and so $\chi(F'') \le \phi'(\omega(G))$ by (1). Hence $\chi(F) \le \omega(G)t^s\psi(\omega(G)) + \phi'(\omega(G))$. This proves (4).

Now every vertex in $Y_1 \cup \cdots \cup Y_k$ has a home, and there are only at most $\omega(G)t^s$ choices of a home; so by (4), $\chi(Y_1 \cup \cdots \cup Y_k) \le \omega(G)^2 t^{2s}\psi(\omega(G)) + \omega(G)t^s\phi'(\omega(G))$. Hence

$$\begin{aligned}\chi(G) &\le \omega(G)^2 t^{2s}\psi(\omega(G)) + \omega(G)t^s\phi'(\omega(G)) + |Z \setminus (Y_1 \cup \cdots \cup Y_k)| + |A_1 \cup \cdots \cup A_k| \\ &\le \omega(G)^2 t^{2s}\psi(\omega(G)) + \omega(G)t^s\phi'(\omega(G)) + (\omega(G)+1)t,\end{aligned}$$

a contradiction. This proves 3.6. ∎

4 Odd holes

Now we deduce 1.2. Let us say a hole in G is *special* if its length is either four or odd. We need a result proved in [9], the following:

4.1 *Let $x \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \le x$ for every vertex $v \in V(G)$. If C is a shortest odd hole in G, the set of vertices of G that belong to or have a neighbour in $V(C)$ has chromatic number at most $21x$.*

We deduce:

4.2 *Let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing polynomial, let $n \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \le n$ for every vertex $v \in V(G)$. If $\chi(G) > \max(\omega(G), 21n + \psi(\omega(G)))$ then G contains a ψ-nondominating special hole.*

Proof. Since $\chi(G) > \omega(G)$, G is not perfect, and so contains either a four-hole or an odd hole (by the strong perfect graph theorem [3], since odd antiholes of length at least seven contain four-holes). Let C be either a four-hole, or a shortest odd hole of G. Let A be the set of vertices in $V(G) \setminus V(C)$ that have no neighbour in $V(C)$, and $B = V(G) \setminus A$. If C has length four then $\chi(B) \le 4n$, and if C is a shortest odd hole of G, then $\chi(B) \le 21n$ by 4.1. Consequently $\chi(A) > \psi(\omega(G)) \ge \psi(\omega(A))$, and so C is a ψ-nondominating special hole. This proves 4.2. ∎

We also need:

4.3 *Let G be a graph containing no four-hole, let $n \in \mathbb{N}$, and let $X \subseteq V(G)$ be the set of all $v \in V(G)$ with $\chi(N(v)) > n$. If $\chi(X) > \omega(G)$, then there exist disjoint sets $A, B \subseteq V(G)$, anticomplete, with $\chi(A), \chi(B) > n/2 - \omega(G)$.*

Proof. Let us say an edge xy of G is *rich* if $\chi(N(x) \setminus N(y)) > n/2 - \omega(G)$ and $\chi(N(y) \setminus N(x)) > n/2 - \omega(G)$. Since there is no four-hole, it is enough to prove that there is a rich edge.

Since $\chi(X) > \omega(G)$, the graph $G[X]$ is not perfect, and so contains a four-vertex induced path with vertices v_1-v_2-v_3-v_4 in order. Let

$$\begin{aligned} A_1 &= N(v_1) \setminus (N(v_3) \cup N(v_4)) \\ A_2 &= N(v_2) \setminus (N(v_4) \cup (N(v_1) \cap N(v_3))) \\ A_3 &= N(v_3) \setminus (N(v_1) \cup (N(v_2) \cap N(v_4))) \\ A_4 &= N(v_4) \setminus (N(v_2) \cup N(v_1)). \end{aligned}$$

Since there is no four-hole, $N(v_1) \cap N(v_3)$ is a clique, and so is $N(v_1) \cap N(v_4)$, and therefore $\chi(A_1) > n - 2\omega(G)$. Since $N(v_2) \cap N(v_4)$ and $N(v_1) \cap N(v_3))$ are cliques, it also follows that $\chi(A_2) > n - 2\omega(G)$, and similarly $\chi(A_i) > n - 2\omega(G)$ for $1 \le i \le 4$.

Now v_2 is anticomplete to $A_1 \setminus A_2$, and v_1 is anticomplete to $A_2 \setminus A_1$, so if $\chi(A_1 \cap A_2) \le n/2 - \omega(G)$, then $\chi(A_1 \setminus A_2) > n/2 - \omega(G)$ and $\chi(A_2 \setminus A_1) > n/2 - \omega(G)$, and so the edge v_1v_2 is rich.

Thus we may assume that $\chi(A_1 \cap A_2) > n/2 - \omega(G)$, and similarly $\chi(A_3 \cap A_4) > n/2 - \omega(G)$. But $A_1 \cap A_2 \subseteq N(v_2) \setminus N(v_3)$, and $A_3 \cap A_4 \subseteq N(v_3) \setminus N(v_2)$, and so the edge v_2v_3 is rich. This proves 4.3. ∎

We put 4.2 and 4.3 together to prove the following:

4.4 *Let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing polynomial. If G is a C_4-free graph with*

$$\chi(G) > 85\omega(G) + 43\psi(\omega(G))$$

then G contains a ψ-nondominating odd hole.

Proof. Let G be a C_4-free graph with $\chi(G) > 85\omega(G) + 43\psi(\omega(G))$. Define $n = 4\omega(G) + 2\psi(\omega(G))$.

Let A be the set of all vertices v of G such that $\chi(N(v)) \le n$, and $B = V(G) \setminus A$. By 4.2 applied to $G[A]$, we may assume that

$$\chi(A) \le \max(\omega(A), 21n + \psi(\omega(A))) = 21n + \psi(\omega(A)) \le 84\omega(G) + 43\psi(\omega(G))$$

and so $\chi(B) \geq \chi(G) - \chi(A) > \omega(G)$. By 4.3 there exist disjoint sets $X, Y \subseteq V(G)$, anticomplete, with $\chi(X), \chi(Y) > n/2 - \omega(G) \geq \omega(G) + \psi(\omega(G))$. Since $\chi(X) > \omega(G) \geq \omega(X)$, $G[X]$ is not perfect and so contains a special hole C, and hence an odd hole since G has no four-holes. Since $V(C)$ is anticomplete to Y, and $\chi(Y) > \psi(\omega(G)) \geq \psi(\omega(Y))$, C is ψ-nondominating. This proves 4.4. ∎

This in turn is used to prove:

4.5 *Let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing polynomial. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ then G contains a ψ-nondominating special hole.*

Proof. Let $\psi'(x) = 85x + 43\psi(x)$ for $x \in \mathbb{N}$, and let ϕ satisfy 3.2 with ψ replaced by ψ', taking $s = 2$ and $q = 4$. We claim that ϕ satisfies 4.5. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$. By 3.2, either there is a ψ'-nondominating four-hole in G, or there is a $(\psi', 4)$-sprinkling in G. In the first case, this four-hole is also ψ-nondominating, since $\psi(x) \leq \psi'(x)$ for $x \in \mathbb{N}$, so we assume the second case holds. Let (P, Q) be a $(\psi', 4)$-sprinkling in G, and let r be the maximum chromatic number over $v \in P$ of the set of neighbours of v in Q. Thus $\chi(Q) > 4r + \psi'(\omega(Q))$, from the definition of a $(\psi', 4)$-sprinkling. If $G[P]$ has a four-hole H, the set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $4r$, and so there is a subset of Q with chromatic number more than $\psi'(\omega(Q)) \geq \psi(\omega(Q))$ anticomplete to H, and so H is ψ-nondominating. Thus we may assume that $G[P]$ has no four-hole. By 4.4, $G[P]$, and hence G, contains a ψ-nondominating odd hole. This proves 4.5. ∎

We deduce 1.1, which we restate:

4.6 *For each integer $k \geq 0$, let $\mathcal{C}$ be the class of all graphs G with no k-multihole in which every component is special. Then $\mathcal{C}$ is poly-χ-bounded.*

Proof. Let us say a k-multihole is *special* if each of its components is a special hole. We proceed by induction on k. The result is true when $k = 1$, because graphs containing no special hole are perfect; so we assume that $k \geq 2$, and there is a polynomial binding function $\psi : \mathbb{N} \to \mathbb{N}$ for the class of all graphs with no special $(k-1)$-multihole $\mathcal{C}_{k-1}$ (and we may assume ψ is non-decreasing). Let ϕ satisfy 4.5; we claim that ϕ is a binding function for the class of all graphs with no special k-multihole. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$; we must show that G contains a special k-multihole. By the choice of ϕ, G contains a ψ-nondominating special hole H say. Choose $A \subseteq V(G) \setminus V(H)$, anticomplete to $V(H)$, such that $\chi(A) > \psi(\omega(A))$. From the inductive hypothesis, $G[A]$ contains a special $(k-1)$-multihole, and so G contains a special k-multihole. This proves 4.6. ∎

5 Long holes

In this section we will prove 1.3. The proof is similar to that of 1.1. Fix an integer $\ell \geq 4$, and we say a hole is *long* if its length is at least ℓ. Let $\tau(G)$ denote the largest integer t such that G contains $K_{t,t}$ as a subgraph. We need a result proved in [1] (see also [12]), the following:

5.1 *There exists an integer $c > 0$ such that $\chi(G) \leq \tau(G)^c + 1$ for every graph G with no long hole.*

We deduce:

5.2 *Let $s \in \mathbb{N}$; then the class of $K_{s,s}$-free graphs with no long hole is poly-χ-bounded.*

Proof. Let $c \geq 1$ be as in 5.1, and let ϕ be the polynomial $\phi(x) = x^{cs}$ for $x \in \mathbb{N}$. Let G be a $K_{s,s}$-free graph with no long hole. We will show that ϕ is a binding function for G. Suppose that $\tau(G) \geq \omega(G)^s$, and let A, B be disjoint subsets of $V(G)$, both of cardinality at least $\omega(G)^s$ and complete to each other. By 3.5, there exist stable sets $A' \subseteq A$ and $B' \subseteq B$ both of cardinality s; but then $G[A' \cup B']$ is a copy of $K_{s,s}$, a contradiction. So $\tau(G) < \omega(G)^s$. By 5.1,

$$\chi(G) \leq (\omega(G)^s - 1)^c + 1 \leq \omega(G)^{cs} = \phi(\omega(G)),$$

and so ϕ is a binding function for G, and hence for the class of $K_{s,s}$-free graphs with no long hole. This proves 5.2. ∎

Next we need an analogue of 4.2, the following:

5.3 *Let $n \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \leq n$ for every vertex $v \in V(G)$. If C is a shortest long hole in G, the set of vertices of G that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell + 1)n$.*

Proof. Let C have vertices c_1-c_2-$\cdots$-c_k-c_1 in order. Let P be the path c_1-c_2-$\cdots$-$c_{\ell-3}$, and let Q be the path $C \setminus V(P)$.

(1) *If $v \in V(G) \setminus V(C)$ has no neighbour in $V(P)$, then all neighbours of v in $V(Q)$ belong to a three-vertex subpath of Q.*

Suppose not, and choose i, j minimum and maximum respectively such that $c_i, c_j \in V(Q)$ are neighbours of v. Thus $j - i \geq 3$, and so

$$c_1\text{-}c_2\text{-}\cdots\text{-}c_i\text{-}v\text{-}c_j\text{-}c_{j+1}\text{-}\cdots\text{-}c_k\text{-}c_1$$

is a long hole (because $j \geq \ell - 2$) that is shorter than C, a contradiction. This proves (1).

For $1 \leq i \leq k$, let A_i be the set of vertices in $V(G) \setminus V(C)$ that are adjacent to c_i and to none of $c_1, \ldots, c_{i-1}$.

(2) *A_i is anticomplete to A_j for $\ell - 2 \leq i < j \leq k$ with $j - i \geq 4$.*

Suppose that $u \in A_i$ and $v \in A_j$ are adjacent. Choose $j' \geq j$ maximum such that $c_{j'}$ is adjacent to v; thus $j' \geq j \geq i + 4$, and so by (1), u is non-adjacent to $c_{j'}, \ldots, c_k$. Hence

$$c_1\text{-}c_2\text{-}\cdots\text{-}c_i\text{-}u\text{-}v\text{-}c_{j'}\text{-}c_{j'+1}\text{-}\cdots\text{-}c_k\text{-}c_1$$

is a long hole shorter than C, a contradiction. This proves (2).

For $t = 1, 2, 3, 4$ let I_t be the set of all integers $i \in \{\ell - 2, \ldots, k\}$ such that $i - t$ is divisible by four. Thus I_1, I_2, I_3, I_4 form a partition of $\{\ell - 2, \ldots, k\}$. Moreover, for all $t \in \{1, \ldots, 4\}$, and all distinct $i, j \in I_t$, there is no edge between $A_i \cup \{c_{i+1}\}$ and $A_j \cup \{c_{j+1}\}$, by (2); and so $\bigcup_{i \in I_t} A_i \cup \{c_{i+1}\}$ has chromatic number at most n. Hence the set of all vertices in $V(G)$ that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell + 1)n$, since those that belong to or have a neighbour in P have chromatic number at most $(\ell - 3)n$, and the others have chromatic number at most $4n$. This proves 5.3. ∎

Now we need an analogue of 4.3, the following:

5.4 *Let $s \in \mathbb{N}$, let G be a $K_{s,s}$-free graph, with no long hole of length at most $2s\ell$. Let $n \in \mathbb{N}$, and let $B \subseteq V(G)$ be the set of vertices v of G such that $\chi(N(v)) > n$. If $G[B]$ contains a long hole, then there exist disjoint sets $X, Y \subseteq B$, anticomplete, with $\chi(X), \chi(Y) > n - (2s\ell)^s\omega(G)^s$.*

Proof. We may assume that $G[B]$ has a hole of length more than $2s\ell$, and so contains an induced path P with $2s\ell - 1$ vertices. Let the vertices of P be p_1-p_2-$\cdots$-p_r in order, where $r = 2s\ell - 1$. For each stable subset $S \subseteq V(P)$ with $|S| = s$, let D_S be the set of vertices in $V(G) \setminus V(P)$ that are adjacent to every vertex in S. Since G is $K_{s,s}$-free, it follows from 3.5 that $|D_S| \leq \omega(G)^s$. Let D be the set of vertices in $V(G) \setminus V(P)$ that have s pairwise nonadjacent neighbours in $V(P)$. Since there are at most $(2s\ell)^s$ choices of S, it follows that $\chi(D) \leq (2s\ell)^s\omega(G)^s$. Let $F = V(G) \setminus (V(P) \cup D)$.

(1) *For each $v \in F$, if i, j are minimum and maximum such that v is adjacent to p_i, p_j, then $j - i \leq (s-2)(\ell-2) + 1$.*

Let $v \in F$. Choose $t \geq 0$ maximum such that there exist $1 \leq i_1 < \cdots < i_t \leq r$ satisfying:

- i_1 is the least i such that v is adjacent to p_i;
- v is adjacent to p_{i_k} for $1 \leq k \leq t$;
- $i_{k+1} \geq i_k + 2$ for $1 \leq k \leq t-1$;
- v is nonadjacent to p_j for $1 \leq k \leq t-1$ and for each $j \in \{i_k + 2, \ldots, i_{k+1} - 1\}$.

Since $\{p_{i_1}, p_{i_2}, \ldots, p_{i_t}\}$ is a stable set, and $v \in F$, it follows that $t < s$. Moreover, for $1 \leq k < t$, v is nonadjacent to each p_j for each $j \in \{i_k + 2, \ldots, i_{k+1} - 1\}$; so one of

$$v\text{-}p_{i_k}\text{-}p_{i_k+1}\text{-}\cdots\text{-}p_{i_{k+1}}$$

$$v\text{-}p_{i_k+1}\text{-}p_{i_k+2}\text{-}\cdots\text{-}p_{i_{k+1}}$$

is an induced cycle. This cycle has length at most $2s\ell$, since P has only $r = 2s\ell - 1$ vertices; and so the cycle has length less than ℓ, since G has no long hole of length at most $2s\ell$. Consequently $i_{k+1} - i_k \leq \ell - 2$, and so $i_t - i_1 \leq (s-2)(\ell-2)$. ¿From the maximality of t, v is nonadjacent to p_j for all $j \geq i_t + 2$. This proves (1).

Let X be the set of neighbours of p_1 in $V(G) \setminus D$, and let Y be the set of neighbours of p_r in $V(G) \setminus D$.

(2) *X is disjoint from and anticomplete to Y.*

Since $r - 1 > (s-2)(\ell-2) + 1$, (1) implies that $X \cap Y = \emptyset$. Suppose that $u \in X$ and $v \in Y$ are adjacent. Choose $i \in \{1, \ldots, r\}$ maximum such that u is adjacent to p_i, and choose $j \in \{1, \ldots, r\}$ minimum such that v is adjacent to p_j. By (1), $i - 1 \leq (s-2)(\ell-2)+1$, and $r - j \leq (s-2)(\ell-2)+1$. Hence $i - 1 + r - j \leq 2((s-2)(\ell-2)+1)$, and so

$$j - i \geq (r-1) - 2((s-2)(\ell-2)+1) = 4\ell + 4s - 12.$$

But then u-p_i-p_{i+1}-$\cdots$-p_j-v-u is a hole of length at least $4\ell + 4s - 9 \geq \ell$ and at most $2s\ell$, a contradiction. This proves (2).

But $\chi(N(p_1)) \geq n$, and so $\chi(X) \geq n - \chi(D) \geq n - (2s\ell)^s\omega(G)^s$, and the same for Y. This proves 5.4. ∎

Next, combining 5.3 and 5.4, we have an analogue of 4.4:

5.5 *Let $s \in \mathbb{N}$, and let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing polynomial. There is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. If G is a $K_{s,s}$-free graph with no long hole of length at most $2s\ell$, and no ψ-nondominating long hole, then $\chi(G) \leq \phi(\omega(G))$.*

Proof. By 5.2, there is a non-decreasing polynomial $\theta : \mathbb{N} \to \mathbb{N}$ that is a binding function for the class of $K_{s,s}$-free graphs with no long hole. Define ϕ by

$$\phi(x) = 2\theta(x) + \psi(x) + (\ell + 1)\left((2s\ell)^s x^s + \theta(x) + \psi(x)\right).$$

We claim that ϕ satisfies 5.5. Thus, let G be a $K_{s,s}$-free graph with no long hole of length at most $2s\ell$, and no ψ-nondominating long hole. Let

$$n = (2s\ell)^s\omega(G)^s + \theta(\omega(G)) + \psi(\omega(G)).$$

Let A be the set of vertices $v \in V(G)$ such that $\chi(N(v)) \leq n$, and $B = V(G) \setminus A$.

(1) *$\chi(A) \leq \theta(\omega(G)) + \psi(\omega(G)) + (\ell + 1)n$.*

Suppose not. Then by 5.2, $G[A]$ has a long hole; let C be a shortest long hole of $G[A]$. By 5.3 applied to $G[A]$, the set of vertices of A that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell+1)n$, and so there is a subset of $A \setminus V(C)$ anticomplete to $V(C)$ with chromatic number more than $\chi(A) - (\ell + 1)n \geq \psi(\omega(G))$. Hence C is ψ-nondominating, a contradiction. This proves (1).

(2) *$\chi(B) \leq \theta(\omega(G))$.*

Suppose not. Then $G[B]$ has a long hole by 5.2. By 5.4, there exist disjoint sets $X, Y \subseteq B$, anticomplete, with $\chi(X), \chi(Y) > n - (2s\ell)^s\omega(G)^s$. Since $\chi(X) \geq \theta(\omega(G))$, $G[X]$ has a long hole, and it is ψ-nondominating since $\chi(Y) \geq \psi(\omega(G))$, a contradiction. This proves (2).

¿From (1) and (2), it follows that

$$\chi(G) \leq 2\theta(\omega(G)) + \psi(\omega(G)) + (\ell + 1)n.$$

This proves 5.5. ∎

This implies:

5.6 *Let $s \in \mathbb{N}$, and let $\psi : \mathbb{N} \to \mathbb{N}$ be some non-decreasing polynomial. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ then G contains either a ψ-nondominating copy of $K_{s,s}$, or a ψ-nondominating long hole.*

Proof. By 5.5, there is a non-decreasing polynomial $\psi' : \mathbb{N} \to \mathbb{N}$ with the following property. If G is a $K_{s,s}$-free graph with no long hole of length at most $2s\ell$, and $\chi(G) > \psi'(\omega(G))$, then G contains a ψ-nondominating long hole.

Let ϕ satisfy 3.2 with ψ replaced by ψ', taking $q = 2s\ell$. We claim that ϕ satisfies 5.6. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$. By 3.2, either there is a ψ'-nondominating copy of $K_{s,s}$ in G, or there is a $(\psi', 2s\ell)$-sprinkling in G. In the first case, this copy of $K_{s,s}$ is also ψ-nondominating, since $\psi(x) \leq \psi'(x)$ for $x \in \mathbb{N}$, so we assume the second case holds. Let (P, Q) be a $(\psi', 2s\ell)$-sprinkling in G, and let r be the maximum chromatic number over $v \in P$ of the set of neighbours of v in Q. Thus $\chi(Q) > 2s\ell r + \psi'(\omega(Q))$, from the definition of a $(\psi', 2s\ell)$-sprinkling. If $G[P]$ contains H where H is either a copy of $K_{s,s}$ or a long hole of length at most $2s\ell$, the set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $|H|r \leq 2s\ell r$, and so there is a subset of Q with chromatic number more than $\psi'(\omega(Q)) \geq \psi(\omega(Q))$ anticomplete to H; and therefore H is ψ-nondominating. Thus we may assume that $G[P]$ is $K_{s,s}$-free and has no long hole of length at most $2s\ell$. By 5.5, $G[P]$, and hence G, contains a ψ-nondominating long hole. This proves 5.6. ∎

Finally, we prove 1.3, which we restate:

5.7 *For all integers $k, s \geq 0$ and $\ell \geq 4$, let $\mathcal{C}$ be the class of all graphs G such that no induced subgraph of G has exactly k components, each of which is either a copy of $K_{s,s}$ or a cycle of length at least ℓ. Then $\mathcal{C}$ is poly-χ-bounded.*

Proof. (The proof is just like that of 4.6.) Let us say an induced subgraph H of a graph G is a *k-object* if it has exactly k components, and each is either a copy of $K_{s,s}$ or a cycle of length at least ℓ. Thus $\mathcal{C}_k$ is the class of graphs with no k-object. We prove by induction on k that $\mathcal{C}_k$ is poly-χ-bounded. The result is true when $k = 1$, by 5.2, so we assume that $k \geq 2$, and there is a polynomial binding function $\psi : \mathbb{N} \to \mathbb{N}$ for $\mathcal{C}_{k-1}$ (and we may assume ψ is non-decreasing). Let ϕ satisfy 5.6; we claim that ϕ is a binding function for $\mathcal{C}_k$. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$; we must show that G contains a k-object. By the choice of c, G contains a ψ-nondominating induced subgraph H, where H is either a copy of $K_{s,s}$ or a long hole. Choose $A \subseteq V(G) \setminus V(H)$, anticomplete to $V(H)$, such that $\chi(A) > \psi(\omega(A))$. From the inductive hypothesis, $G[A]$ contains a $(k-1)$-object, and so G contains a k-object. This proves 5.7. ∎

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