Polynomial bounds for chromatic number. V. Excluding a tree of radius two and a complete multipartite graph

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Abstract

The Gyárfás-Sumner conjecture says that for every forest H and every integer k, if G is H-free and does not contain a clique on k vertices then it has bounded chromatic number. (A graph is H*-free* if it does not contain an induced copy of H.) Kierstead and Penrice proved it for trees of radius at most two, but otherwise the conjecture is known only for a few simple types of forest. More is known if we exclude a complete bipartite subgraph instead of a clique: Rödl showed that, for every forest H, if G is H-free and does not contain $K_{t,t}$ as a subgraph then it has bounded chromatic number. In an earlier paper with Sophie Spirkl, we strengthened Rödl's result, showing that for every forest H, the bound on chromatic number can be taken to be polynomial in t. In this paper, we prove a related strengthening of the Kierstead-Penrice theorem, showing that for every tree H of radius two and integer $d \geq 2$, if G is H-free and does not contain as a subgraph the complete d-partite graph with parts of cardinality t, then its chromatic number is at most polynomial in t.

1 Introduction

The Gyárfás-Sumner conjecture [8, 21] says:

1.1 Conjecture: *For every forest H there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

(G is *H-free* if no induced subgraph of G is isomorphic to H; and $\chi(G), \omega(G)$ denote the chromatic number and the size of the largest clique of G, respectively.)

This is open in general, although it is known to hold [14] for graphs that do not contain any induced *subdivision* of H, and has been proved for a few special kinds of forest. Notably, Kierstead and Penrice [11] proved:

1.2 *For every tree H of radius two, there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

These statements can also be phrased in terms of χ-bounded classes. A class of graphs is *hereditary* if it is closed under taking induced subgraphs; and a hereditary class $\mathcal{G}$ of graphs is *χ-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$. Thus conjecture 1.1 says that, for every forest H, the class of H-free graphs is χ-bounded; and 1.2 says that the class of H-free graphs is χ-bounded when H is a tree of radius two.

There has been a great deal of recent progress on χ-bounded classes (see [15] for a survey). In most cases, the proofs give bounds on the chromatic number that grow relatively quickly (often superexponentially) in the clique number. However, a striking conjecture of Esperet [7] asserts that this is not necessary, and that for every χ-bounded class, the function f can be taken to be polynomial. Esperet's conjecture has been shown to be false in its full generality [2], but remains open for classes of graphs excluding a forest; and in that case, the Gyárfás-Sumner conjecture and Esperet's conjecture would together give the following:

1.3 Conjecture: *For every forest H, there is a polynomial f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

So far, this is only known for a few classes of trees (see [3, 13, 17, 18, 20]).

While the conjectures 1.1 and 1.3 remain open, more is known if we exclude a complete bipartite graph rather than a clique. Rödl (see [10, 12]) proved that:

1.4 *For every forest H and integer $t \geq 2$, there exists k such that if G is H-free and does not contain $K_{t,t}$ as a subgraph then $\chi(G) \leq k$.*

It will be helpful to define one piece of notation. For a graph G, and integer $d \geq 1$, let $\tau_d(G)$ denote the largest t such that G has a subgraph (not necessarily induced) isomorphic to the complete d-partite graph with each part of cardinality t. Thus $\tau_1(G) = |G|$, and $\tau_2(G)$ is the largest t such that G contains $K_{t,t}$ as a subgraph, and 1.4 says that for every forest H there is a function f such that every H-free graph G satisfies $\chi(G) \leq f(\tau_2(G))$. It is natural to ask whether f can be taken to be a polynomial in Rödl's result. When H is a path, this was proved by Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak [1]. We proved the general case with Sophie Spirkl in [16]:

1.5 *For every forest H, there exists $c > 0$ such that $\chi(G) \leq \tau_2(G)^c$ for every H-free graph G.*

Note that this is a special case of 1.3, as $\tau_2(G) \geq \lfloor \omega(G)/2 \rfloor$. 1.3 would also imply that the same is true for $\tau_d(G)$ instead of $\tau_2(G)$ for any fixed value of $d \geq 2$ (except if $\tau_d(G) \leq 1$) since $\tau_d(G) \geq \lfloor \omega(G)/d \rfloor$. This has not been proved in general – indeed, proving it for a forest H would show that H also satisfies the Gyárfás-Sumner conjecture – but in this paper we prove it when H is a tree of radius two. Our main result is the following extension of 1.2:

1.6 *For every tree H of radius two, and every integer $d \geq 1$, there is a polynomial f such that $\chi(G) \leq f(\tau_d(G))$ for every H-free graph G.*

A referee suggests two further open questions on these lines:

- (Extending 1.6 to other trees H that we know satisfy 1.1.) Is it true that if H is a path, then for every integer $d \geq 1$, there is a polynomial f such that $\chi(G) \leq f(\tau_d(G))$ for every H-free graph G?
- (An analogue of Esperet's (false) conjecture for $\tau_d(G)$.) Let $\mathcal{C}$ be a hereditary class of graphs, and $d \geq 1$. Suppose that there is a function f such that $\chi(G) \leq f(\tau_d(G))$ for each $G \in \mathcal{C}$. Can we always choose f to be a polynomial? What if $d = 2$?

We note that for the five-vertex path P_5, the best current upper bound on chromatic number is $\omega^{\log_2 \omega}$ [19]. A polynomial upper bound, as asserted by conjecture 1.3, would imply that P_5 satisfies the Erdős-Hajnal conjecture [5, 6] (P_5 is currently the smallest open case of the Erdős-Hajnal conjecture, after C_5 was recently proved in [4]). Since P_5 is a tree of radius two, it would be very nice if the function f in 1.6 had polynomial dependence on d. But the function f we prove in this paper has doubly-exponential dependence on d. Incidentally, if we take $t = 1$ in 1.6, we have proved that $\chi(G) \leq f(\omega(G))$ for every H-free graph G, where f is doubly-exponential in $\omega(G)$. While this is admittedly fast-growing, the bound is much smaller than that of Kierstead and Penrice [11].

We use standard notation. For a graph G, we denote the number of vertices by $|G|$. When $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X. We write $\chi(X)$ for $\chi(G[X])$ when there is no ambiguity. If $v \in V(G)$, a *non-neighbour* of v in G means a vertex u of G different from v and nonadjacent to v.

2 Some Ramsey-type lemmas

We will use the following well-known version of Ramsey's theorem, proved (for instance) in [17]:

2.1 *Let $x \geq 2$ and $y \geq 1$ be integers. For a graph G, if $|G| \geq x^y$, then G has either a clique of cardinality $x + 1$, or a stable set of cardinality y.*

We also need the next result:

2.2 *Let $s \geq 2$ and $t \geq 1$ be integers, and let G be a graph with $\tau_{d+1}(G) < t$. Let $L_1, \ldots, L_{s^{2d+2}}$ be pairwise disjoint subsets of $V(G)$, each of cardinality at least $2^{s^{2d+2}} t^{ds+s^2+s}$. Then there exist $I \subseteq \{1, \ldots, s^{2d+2}\}$ with $|I| = s$, and a subset $X_i \subseteq L_i$ for each $i \in I$, where $\bigcup_{i \in I} X_i$ is a stable set, and $|X_i| \geq s$ for each $i \in I$.*

For inductive purposes, we will prove the following stronger (but messier) form: 2.2 follows from it by substituting $a = b = c = s$.

2.3 *Let $a \geq 2$ and $t \geq 1$ be integers. For all integers $b, c, d \geq 0$ with $b \leq a$ and $c \geq 1$, define*

$$k_{b,c,d} = \begin{cases} (ac)^{d+1} & \text{if } b = a \\ b(ac)^d + (a(c-1))^{d+1} + 1 & \text{if } b < a \text{ and } d > 0 \\ 1 & \text{if } b < a \text{ and } d = 0 \end{cases}$$

Define $p_{b,c,d} = 2^{k_{b,c,d}} t^{a(c+d)+b}$. Now let $b, c, d \geq 0$ be integers with $b \leq a$, and let G be a graph with $\tau_{d+1}(G) < t$. Let $L_1, \ldots, L_{k_{b,c,d}}$ be pairwise disjoint subsets of $V(G)$, each of cardinality at least $p_{b,c,d}$. Then there exist $I \subseteq \{1, \ldots, k_{b,c,d}\}$ with $|I| = c$, and a stable subset $X \subseteq L_1 \cup \cdots \cup L_{k_{b,c,d}}$, where $|X \cap L_i| \geq a$ for each $i \in I \setminus \{1\}$, and $|X \cap L_1| \geq b$ if $1 \in I$.

Proof. We proceed by induction on $(a+1)(c+d)+b$ (the numbers a, t are fixed throughout the proof). If $d = 0$ then $\tau_{d+1}(G) = |G|$, so $|G| < t \leq p_{b,c,d}$, and there is no choice of L_1 satisfying the hypothesis, and therefore the theorem holds. So we may assume that $d \geq 1$.

Suppose that $c = 1$. Since G has no clique of cardinality $(d+1)t$ (because $t > \tau_{d+1}(G)$), 2.1 implies that every set of $((d+1)t)^b$ vertices of G includes a stable set of cardinality b. Thus it suffices to show that $p_{b,c,d} \geq ((d+1)t)^b$ when $c = 1$, that is, we must show that

$$2^{k_{b,1,d}} t^{a(1+d)+b} \geq ((d+1)t)^b.$$

Since $t^{a(1+d)+b} \geq t^b$, it is enough to show that $2^{k_{b,1,d}} \geq (d+1)^b$. But $k_{b,1,d} \geq ba^d$, so it suffices to show that $2^{ba^d} \geq (d+1)^b$, that is, $a^d \geq \log_2(d+1)$. Since $a \geq 2$, and $2^d \geq \log_2(d+1)$, this is true, so we may assume that $c \geq 2$.

Suppose that $b = 0$, and therefore $k_{b,c,d} = k_{a,c-1,d} + 1$. By applying the inductive hypothesis to $L_2, \ldots, L_{k_{b,c,d}}$, with b, c, d replaced by $a, c-1, d$ respectively, we deduce that there exist $I' \subseteq \{2, \ldots, k_{b,c,d}\}$ with $|I'| = c - 1$, and a stable subset $X \subseteq L_2 \cup \cdots \cup L_{k_{b,c,d}}$, where $|X \cap L_i| \geq a$ for each $i \in I'$. Then setting $I = I' \cup \{1\}$ satisfies the theorem. Thus we may assume that $b \geq 1$.

(1) *The following inequalities hold:*

$$\begin{aligned} k_{b,c,d} - k_{b-1,c,d} &\geq k_{a,c,d-1} \\ p_{b,c,d} &\geq 2(d+1)^2 t^2 \\ p_{b,c,d} &\geq 2(d+1)t p_{b-1,c,d} \\ p_{b,c,d} &\geq 2^{k_{b,c,d}} t \\ p_{b,c,d} &\geq t p_{b-1,c,d} + p_{a,c,d-1}. \end{aligned}$$

The first is clear (and holds with equality) if $b < a$, so we assume that $b = a$. Since

$$1 - 1/c > (1 - 1/c)^{d+1}$$

(because $c \geq 2$ and $d \geq 1$), we have

$$(ac)^{d+1} > a(ac)^d + (a(c-1))^{d+1},$$

and so

$$k_{a,c,d} \geq a(ac)^d + (a(c-1))^{d+1} + 1 = k_{a-1,c,d} + k_{a,c,d-1},$$

and the first inequality follows.

For the second, we must show that $2^{k_{b,c,d}}t^{a(c+d)+b} \geq 2(d+1)^2t^2$. Since $t^{a(c+d)+b} \geq t^2$, it suffices to show that $2^{k_{b,c,d}} \geq 2(d+1)^2$, and this is true since $k_{b,c,d} \geq (ac)^d + 1 \geq 2^d + 1$, and $2^{2^d+1} \geq 2(d+1)^2$. This proves the second inequality.

For the third, we must show that

$$2^{k_{b,c,d}}t^{a(c+d)+b} \geq 2(d+1)t2^{k_{b-1,c,d}}t^{a(c+d)+b-1},$$

that is,

$$k_{b,c,d} - k_{b-1,c,d} \geq 1 + \log_2(d+1).$$

But (using the first inequality if $b = a$), $k_{b,c,d} - k_{b-1,c,d} \geq (ac)^d \geq 2^d \geq 1 + \log_2(d+1)$ as required.

For the fourth, we must show that $2^{k_{b,c,d}}t^{a(c+d)+b} \geq 2^{k_{b,c,d}}t$, which is clear. Finally, for the fifth, we must show that

$$2^{k_{b,c,d}}t^{a(c+d)+b} \geq t2^{k_{b-1,c,d}}t^{a(c+d)+b-1} + 2^{k_{b,c,d-1}}t^{a(c+d-1)+b},$$

that is,

$$2^{k_{b,c,d}} \geq 2^{k_{b-1,c,d}} + 2^{k_{b,c,d-1}}t^{-a}.$$

Since $t \geq 1$, it suffices to show that $2^{k_{b,c,d}} \geq 2 \cdot 2^{k_{b-1,c,d}}$ and $2^{k_{b,c,d}} \geq 2 \cdot 2^{k_{b,c,d-1}}$, that is, $k_{b,c,d} > k_{b-1,c,d}$ and $k_{b,c,d} > k_{b,c,d-1}$, which are both true (since $ac \geq 2$). This proves (1).

Choose a clique $Y \subseteq L_1$, maximal such that at most $|Y|p_{b,c,d}/(2(d+1)t)$ vertices in L_1 have a non-neighbour in Y. (Possibly $Y = \emptyset$.) Let N be the set of vertices in $L_1 \setminus Y$ that are adjacent to every vertex in Y. Then:

(2) *$|N| \geq p_{b,c,d}/2$, and every vertex $v \in N$ has more than $p_{b,c,d}/(2(d+1)t)$ non-neighbours in N.*

Since $t > \tau_{d+1}(G)$, it follows that G has no clique of cardinality $(d+1)t$, and so $|Y| < (d+1)t$. Let $M = L_1 \setminus (N \cup Y)$. Thus $|M| \leq |Y|p_{b,c,d}/(2(d+1)t)$ from the choice of Y, and so

$$\begin{aligned}|Y \cup M| &\leq |Y|\left(1 + \frac{p_{b,c,d}}{2(d+1)t}\right) \leq ((d+1)t - 1)\left(1 + \frac{p_{b,c,d}}{2(d+1)t}\right)\\ &= (d+1)t - 1 + \frac{p_{b,c,d}}{2} - \frac{p_{b,c,d}}{2(d+1)t} \leq \frac{p_{b,c,d}}{2}\end{aligned}$$

since $2(d+1)^2t^2 \leq p_{b,c,d}$ by (1). Consequently $|N| \geq p_{b,c,d}/2$. This proves the first assertion. For the second, if some vertex $v \in N$ has at most $p_{b,c,d}/(2(d+1)t)$ non-neighbours in N, then adding v to Y gives a set Y' such that at most $|Y'|p_{b,c,d}/(2(d+1)t)$ vertices in L_1 have a non-neighbour in Y', contrary to the maximality of Y. This proves (2).

We may assume that

(3) *For each $v \in N$, there are fewer than $k_{b-1,c,d}$ values of $i \in \{2, \ldots, k_{b,c,d}\}$ such that v has at least $p_{b-1,c,d}$ non-neighbours in L_i.*

Suppose that there exists $I' \subseteq \{2, \dots, k_{b,c,d}\}$ with $|I'| = k_{b-1,c,d}$, such that for each $i \in I'$ there is a set $L'_i \subseteq L_i$ of non-neighbours of v with $|L'_i| \geq p_{b-1,c,d}$. Let L'_1 be the set of non-neighbours of v in N; then $|L'_1| \geq p_{b-1,c,d}$ by (2), since $p_{b,c,d}/(2(d+1)t) \geq p_{b-1,c,d}$ by (1). From the inductive hypothesis applied to L'_i ($i \in I' \cup \{1\}$), with b, c, d replaced by $b-1, c, d$ respectively, it follows that there exist $I \subseteq I' \cup \{1\}$ with $|I| = c$, and a stable subset $X \subseteq \bigcup_{i \in I} L'_i$, where $|X \cap L'_i| \geq a$ for each $i \in I \setminus \{1\}$, and $|X \cap L'_1| \geq b-1$ if $1 \in I$. If $1 \notin I$ then the theorem holds. If $1 \in I$, then by adding v to X we see that again the theorem holds. This proves (3).

For each $v \in N$, let I_v be the set of values of $i \in \{2, \dots, k_{b,c,d}\}$ such that v has fewer than $p_{b-1,c,d}$ non-neighbours in L_i. Thus $|I_v| \geq k_{b,c,d} - k_{b-1,c,d}$ for each v, by (3). Since there are at most $2^{k_{b,c,d}-1}$ choices of the set I_v, and $|N| \geq p_{b,c,d}/2 \geq 2^{k_{b,c,d}-1}t$ by (1), there exists $T \subseteq N$ with $|T| = t$ such that the sets I_v ($v \in T$) are all equal, and equal to some I' say. For each $i \in I'$, since each $v \in T$ has at most $p_{b-1,c,d}$ non-neighbours in L_i, it follows that there are at most $tp_{b-1,c,d}$ vertices in L_i that have a non-neighbour in T. Since $|L_i| \geq p_{b,c,d} \geq tp_{b-1,c,d} + p_{a,c,d-1}$ by (1), there is a subset $L'_i \subseteq L_i$ with $|L'_i| \geq p_{a,c,d-1}$, such that every vertex in T is adjacent to every vertex of L'_i. Since $\tau_{d+1}(G) < t$, it follows that $\tau_d(G') < t$, where $G' = G[\bigcup_{i \in I'} L'_i]$. Since $|I| \geq k_{b,c,d} - k_{b-1,c,d} \geq k_{a,c,d-1}$, the inductive hypothesis (on $c+d$) applied to L'_i ($i \in I'$), with b, c, d replaced by $a, c, d-1$ respectively, implies that there exist $I \subseteq I'$ with $|I| = c$, and a stable subset $X \subseteq \bigcup_{i \in I} L'_i$, where $|X \cap L'_i| \geq a$ for each $i \in I$. This proves 2.3. ∎

If $v \in V(G)$ and $B \subseteq V(G)$ with $v \notin B$, we say v is *complete* to B if v is adjacent to every vertex in B, and v is *anticomplete* to B if v has no neighbours in B. If A, B are disjoint subsets of $V(G)$, we say A is *complete* to B if every vertex in A is complete to B, and A is *anticomplete* to B if there are no edges between A and B. The result just proved will be used in combination with the following:

2.4 *Let G be a graph, let A, B be disjoint subsets of $V(G)$, and let $k, \ell, t \geq 1$ be integers. Suppose that*

- *for each $T \subseteq A$ with $|T| = t$, the set of vertices in B complete to T has chromatic number at most ℓ;*
- *every vertex in B has at least t^{k-1} neighbours in A; and*
- *$\chi(B) > k|A|^{2k-1}\ell$.*

Then there exist distinct $a_1, \dots, a_k \in A$, and disjoint subsets $L_1, \dots, L_k \subseteq B$, each with chromatic number more than ℓ, such that for all $i, j \in \{1, \dots, k\}$ and all $v \in L_j$, a_i is adjacent to v if and only if $i = j$.

Proof. We proceed by induction on k. Suppose first that $k = 1$. Since each vertex in B has a neighbour in A, and $\chi(B) > |A|\ell$, there is a vertex $a_1 \in A$ such that the set L_1 of neighbours of a_1 in B has chromatic number more than ℓ, and so the theorem holds. Thus we may assume that $k \geq 2$ and the theorem holds for $k-1$. Since $B \neq \emptyset$, and each vertex in B has at least t^{k-1} neighbours in A, it follows that $|A| \geq t^{k-1}$.

For each $v \in B$, let N_v be the set of neighbours of v in A. Since $|N_v| > 0$, there are at most $|A|$ possibilities for $|N_v|$; and so there exist $C \subseteq B$ with

$$\chi(C) \geq \chi(B)/|A| > k|A|^{2k-2}\ell$$

such that the sets $N_v (v \in C)$ all have the same cardinality. For each $v \in C$, let S_v be the set of all $u \in C$ with $|N_u \setminus N_v| \geq t^{k-2}$.

(1) *$\chi(S_v) > (k-1)|A|^{2k-2}\ell$ for each $v \in C$.*

Since $|N_v| \geq t^{k-1}$, there exist pairwise disjoint subsets $T_1, \ldots, T_{t^{k-2}}$ of N_v, each of cardinality t. If $u \in C \setminus S_v$, then since N_u, N_v have the same cardinality, it follows that $|N_v \setminus N_u| < t^{k-2}$, and so u is complete to one of the sets $T_1, \ldots, T_{t^{k-2}}$. For $1 \leq i \leq t^{k-2}$, the set of vertices in C complete to T_i has chromatic number at most ℓ by hypothesis; and since $C \setminus S_v$ is the union of t^{k-2} of these sets, it follows that $\chi(C \setminus S_v) \leq t^{k-2}\ell$. Since $\chi(C) > k|A|^{2k-2}\ell$, and $|A| \geq t^{k-1} \geq t$, we deduce that

$$\chi(S_v) > k|A|^{2k-2}\ell - t^{k-2}\ell \geq (k-1)|A|^{2k-2}\ell.$$

This proves (1).

Let $a_1, \ldots, a_k \in A$ be distinct. A *feathering* for the sequence $(a_1, \ldots, a_k)$ is a sequence $(L_2, \ldots, L_k)$ of pairwise disjoint subsets of C, each with chromatic number more than ℓ, such that for all $i \in \{1, \ldots, k\}$, all $j \in \{2, \ldots, k\}$, and all $u \in B_j$, a_i is adjacent to u if and only if $i = j$. For each $v \in B$, let us say a *tail* for v is a sequence $(a_1, \ldots, a_k)$ of distinct vertices in A, such that v is adjacent to a_1 and nonadjacent to $a_2, \ldots, a_k$, and there is a feathering for $(a_1, \ldots, a_k)$.

(2) *For each $v \in B$, there is a tail for v.*

Since every $u \in S_v$ has a non-neighbour in N_v, and $\chi(S_v) > (k-1)|A|^{2k-2}\ell$, there exists $a_1 \in N_v$ such that the set B' of vertices in S_v nonadjacent to a_1 has chromatic number more than $(k-1)|A|^{2k-3}\ell$. From the inductive hypothesis on k, applied with A, B, k replaced by $A \setminus N_v$, B', $k-1$ respectively, there exist distinct $a_2, \ldots, a_k \in A \setminus N_v$, and disjoint subsets $L_2, \ldots, L_k \subseteq B'$, each with chromatic number more than ℓ, such that for all $i \in \{2, \ldots, k\}$, a_i is complete to L_i and anticomplete to all the other L_j. Since a_1 has no neighbours in B', it follows that $(a_1, \ldots, a_k)$ is a tail for v. This proves (2).

For every sequence $(a_1, \ldots, a_k)$ of distinct vertices in A, let $M(a_1, \ldots, a_k)$ be the set of $v \in C$ such that $(a_1, \ldots, a_k)$ is a tail for v. By (2), C is the union of the sets $M(a_1, \ldots, a_k)$ over all choices of $(a_1, \ldots, a_k)$; and since there are at most $|A|^k$ such sequences $(a_1, \ldots, a_k)$, it follows that $\chi(M(a_1, \ldots, a_k)) \geq \chi(C)|A|^{-k}$ for some choice of $(a_1, \ldots, a_k)$. Let $L_1 = M(a_1, \ldots, a_k)$; then $\chi(L_1) > \ell$, since $\chi(C)|A|^{-k} > (k-1)|A|^{k-2}\ell \geq \ell$. Let $(L_2, \ldots, L_k)$ be a feathering for $a_1, \ldots, a_k$. (The latter exists since $M(a_1, \ldots, a_k) \neq \emptyset$ and so $a_1, \ldots, a_k$ is a tail for some $v \in C$.) Since every vertex in L_1 is adjacent to a_1, and the vertices in $L_2, \ldots, L_k$ are all nonadjacent to a_1, it follows that $L_1, \ldots, L_k$ are pairwise disjoint. Then $a_1, \ldots, a_k$ and $L_1, \ldots, L_k$ satisfy the theorem. This proves 2.4. ∎

3 Cores and their neighbourhoods

For each integer $s \geq 1$, let H_s be the tree with $1 + s + s^2$ vertices, in which some vertex has degree s and all its neighbours have degree $s+1$. Every tree of radius two is an induced subgraph of H_s for some choice of $s \geq 2$, and so to prove 1.6 in general, it suffices to prove it in the case that $H = H_s$

for some $s \geq 2$. Thus, we need to show that for every integer $s \geq 2$ and every integer $d \geq 2$, there is a polynomial f such that $\chi(G) \leq f(\tau_d(G))$ for every H_s-free graph G. We might as well assume that f is *increasing*, that is, $f(y) \geq f(x)$ for all $y \geq x \geq 0$, and *integral*, that is, all its coefficients are integers. (It is tempting to assume further that f is of the form $f(x) = x^c$ for some c, but we cannot, because $\tau_d(G)$ might be zero or one.)

To prove 1.6, we will proceed by induction on d, with s fixed. When $d = 1$, the result is trivial, and when $d = 2$ it follows from 1.5, so we assume that $d \geq 2$ and the result holds for d, and we will prove that it holds for $d + 1$. In summary, then, we have:

3.1 Hypothesis A:

- *$s, d \geq 2$ are integers.*
- *f is an increasing integral polynomial such that $\chi(G) \leq f(\tau_d(G))$ for every H_s-free graph G, and $f(t) \geq t$ for all integers $t \geq 1$.*

(We may assume the statement $f(t) \geq t$ without loss of generality, and it will be convenient later.) We must show that there is a polynomial (and therefore there is an increasing integral polynomial) f' such that if G is H_s-free then $\chi(G) \leq f'(\tau_{d+1}(G))$; or equivalently, that there is a polynomial f'' such that if G is H_s-free and $t > \tau_{d+1}(G)$ is an integer, then $\chi(G) \leq f''(t)$. Thus, to complete the proof of 1.6, we will prove:

3.2 *Assuming Hypothesis A, there is a polynomial f_1 such if G is H_s-free and $t > \tau_{d+1}(G)$ is an integer, then $\chi(G) \leq f_1(t)$.*

We need the following, a special case of theorem 3.2 of [16]:

3.3 *Let $s \geq 2$ be an integer. If G is an H_s-free graph, and $t > \tau_2(G)$ is an integer, then*

$$\chi(G) \leq (s(s^2 + s + 1)t)^{120(s^2+s+1)}.$$

For integers $w, d \geq 1$, a subgraph C of G is a (w, d)-*core* if $V(C)$ is the disjoint union of d sets that are stable in C (but not necessarily stable in G), each of cardinality w and pairwise complete in C. We call these sets the *parts* of the core. A core is *stable* if each of its parts is stable in G.

3.4 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer and let $w \geq 1$ be an integer. If G' is an induced subgraph of G with*

$$\chi(G') > f\left(\left(s(s^2 + s + 1)t\right)^{120(s^2+s+1)} w\right)$$

then G' has a stable (w, d)-core.

Proof. Let $T = \left(s(s^2 + s + 1)t\right)^{120(s^2+s+1)} w$. Since $\chi(G') > f(T)$, it follows that G' has a (T, d)-core C. Let $A_1, \ldots, A_d$ be the parts of C. For $1 \leq i \leq d$, since $t > \tau_{d+1}(G)$, and the parts of C all have cardinality at least t, it follows that $\tau_2(G[A_i]) < t$, and so by 3.3,

$$\chi(A_i) \leq (s(s^2 + s + 1)t)^{120(s^2+s+1)}.$$

Hence there is a stable subset B_i of A_i of cardinality w, since $T \geq (s(s^2 + s + 1)t)^{120(s^2+s+1)} w$. But then $B_1 \cup \cdots \cup B_d$ induces a stable (w, d)-core. This proves 3.4. ∎

3.5 *Assuming Hypothesis A, let G be an H_s-free graph, let $w, d \geq 1$ be integers, and let $t > \tau_{d+1}(G)$ be an integer. Let C be a stable (w,d)-core, and let P be the set of all vertices in G that have at least t^{s-1} neighbours in some part of C and have a non-neighbour in $V(C)$. (Thus $V(C) \subseteq P$.) Then*

$$\chi(P) \leq sd^2 w^{2s}(f(t) + 2^{s^{2d+2}} t^{ds+s^2+s}).$$

Proof. Define $q(t) = f(t) + 2^{s^{2d+2}} t^{ds+s^2+s}$. Now let G, C, P, t be as in the theorem. Every vertex in P has at least t^{s-1} neighbours in some part of C, and a non-neighbour in some part of C, and we may choose these two parts to be different. Since there are only $d^2 w$ choices of a part of C and a vertex in a different part, there is a part A of C, and a vertex v_0 of a different part of C, such that $\chi(B) \geq \chi(P)/(d^2 w)$, where B is the set of vertices in P that are nonadjacent to v_0 and have at least t^{s-1} neighbours in A. (Thus $A \cap B = \emptyset$, since A is stable.) For each $T \subseteq A$ with $|T| = t$, the set of vertices in B that are complete to T contains no (t,d)-core (since G contains no $(t, d+1)$-core), and so has chromatic number at most $f(t) \leq q(t)$.

Suppose for a contradiction that

$$\chi(B) > s|A|^{2s-1} q(t).$$

Then by 2.4, taking $\ell = q(t)$, there exist distinct $a_1, \dots, a_k \in A$, and disjoint subsets $L_1, \dots, L_k \subseteq B$, each with chromatic number more than $q(t)$ and hence with cardinality at least $2^{s^{2d+2}} t^{ds+s^2+s}$, such that for all $i, j \in \{1, \dots, k\}$ and all $v \in L_j$, a_i is adjacent to v if and only if $i = j$. By 2.2, there exist $I \subseteq \{1, \dots, k\}$ with $|I| = s$, and a subset $X_i \subseteq L_i$ for each $i \in I$, where $\bigcup_{i \in I} X_i$ is a stable set, and $|X_i| = s$ for each $i \in I$. But then the subgraph induced on $\{v_0\} \cup \bigcup_{i \in I}(\{a_i\} \cup X_i)$ is isomorphic to H_s, a contradiction.

This proves that $\chi(B) \leq s|A|^{2s-1} q(t)$, and since $|A| = w$ and $\chi(B) \geq \chi(P)/(d^2 w)$, it follows that $\chi(P) \leq sd^2 w^{2s} q(t)$. This proves 3.5. ∎

4 Templates

Our proof of 1.6 follows the "template" approach used by Kierstead and Penrice in [11], but modified to make the numbers polynomial. Assuming Hypothesis A, let $w, t \geq 1$ be integers, and let us define a (w,t)-*template* in a graph G to be a pair (C, P) such that

- C is a stable (w,d)-core in G;
- $P \subseteq V(G)$ with $V(C) \subseteq P$;
- for every vertex $v \in P \setminus V(C)$ there is a part A of C such that v has at least st^{s-1} neighbours in A; and
- for every vertex $v \in P \setminus V(C)$ there is a part A of C such that v has at least $\lfloor w/t \rfloor$ non-neighbours in A.

A (w,t)-*template sequence* in G is a sequence (C_i, P_i) $(i \in I)$ of (w,t)-templates, where I is a set of integers, such that the sets P_i $(i \in I)$ are pairwise disjoint, and for all $i, j \in I$ with $i < j$, every vertex of P_j either has fewer than st^{s-1} neighbours in each part of C_i, or has fewer than $\lfloor w/t \rfloor$

non-neighbours in each part of C_i (that is, adding this vertex to P_i would violate the definition of a template).

The method of proof is, we will let w be some appropriately large polynomial, and assume that G is H_s-free and $t > \tau_{d+1}(G)$; and greedily choose a $(w = w(t), d)$-template sequence (C_i, P_i) $(1 \le i \le n)$ in G, where each P_i is as large as possible among the set of vertices that have so far not been used, and with n maximum. It follows that the set of vertices not in any of the templates of the sequence has bounded chromatic number, and so it remains to bound the chromatic number of the union of the templates. Each template has bounded chromatic number by 3.5, but we need to control the edges between templates in the sequence, to bound the chromatic number of their union. Let us assume that G has very large chromatic number. If we partition the template sequence into a bounded number of other sequences, one of them will induce a subgraph that still has very large (not quite so large) chromatic number. By this process we can make successively nicer template sequences, still inducing large chromatic number, until eventually we will obtain a contradiction (we will obtain a template sequence in which for each template, each of its vertices has only a bounded number of neighbours in other templates of the sequence.)

If (C_i, P_i) $(i \in I)$ is a (w, t)-template sequence in a graph G, its *vertex set* is $\bigcup_{i \in I} P_i$; its *support* is the subgraph of G induced on its vertex set; and its *chromatic number* is the chromatic number of its support.

We will often use the following lemma:

4.1 *Let D be a directed graph in which every vertex has out-degree at most d. Then $V(D)$ can be partitioned into $2d + 1$ stable sets.*

Proof. Every non-null subgraph H has at most $d|H|$ edges, and so has a vertex v such that the sum of its indegree and outdegree is at most $2d$. Hence the undirected graph underlying D has degeneracy at most $2d$, and so is $(2d + 1)$-colourable. This proves 4.1. ∎

Let us say a (w, t)-template sequence (C_i, P_i) $(i \in I)$ with vertex set U in a graph G is

- **1-nice** if for each $i \in I$, there is no vertex $v \in U \setminus V(C_i)$ that has fewer than $\lfloor w/t \rfloor$ non-neighbours in each part of C_i.
- **2-nice** if it is 1-nice and for each $v \in U$, there are fewer than s values of $i \in I$ such that $v \notin V(C_i)$ and v has at least $s^3 t^{s-1}$ neighbours in some part of C_i.
- **3-nice** if it is 2-nice and for all distinct $i, j \in I$, every vertex in C_i has fewer than $s^3 t^{s-1}$ neighbours in each part of C_j.
- **4-nice** if it is 3-nice and for each $v \in U$, there are fewer than $(dt)^s$ values of $i \in I$ such that $v \notin P_i$ and v has a neighbour in $V(C_i)$.
- **5-nice** if it is 4-nice and for all distinct $i, j \in I$, there are no edges between $V(C_i)$ and $V(C_j)$.
- **6-nice** if it is 5-nice and for all $v \in U$, there are fewer than $2(dt)^{2s} + (dt)^s$ values of $i \in I$ such that $v \notin P_i$ and v has a neighbour in P_i.
- **7-nice** if it is 6-nice and for all $i \in I$, there are fewer than $60dws(dt)^{5s}$ values of $j \in I \setminus \{i\}$ such that some $v \in P_i$ has at least $(dt)^s$ neighbours in P_j.

- **8-nice** if it is 7-nice and for all $i \in I$ and $v \in P_i$, v has fewer than $3(dt)^{3s}$ neighbours in $U \setminus P_i$.

We will show that if G has large chromatic number, then so does some 1-nice template sequence. Then for $i = 2, \ldots, 8$ in turn, we will deduce that some i-nice template sequence has large chromatic number; and finally, we will show by applying 3.5 that when $i = 8$, this is impossible. It will follow that G has bounded chromatic number. We begin with:

4.2 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \geq 1$ be an integer. If every 1-nice (w, d)-template sequence in G has chromatic number at most k, then*

$$\chi(G) \leq f\left(\left(s(s^2+s+1)t\right)^{120(s^2+s+1)} w\right) + 2kt.$$

Proof. We observe first:

(1) *Let C be a (w, d)-core in G, and let Q be the set of vertices $v \in V(G) \setminus V(C)$ that have fewer than $\lfloor w/t \rfloor$ non-neighbours in each part of C. Then $|Q| < t$.*

Suppose not; then there is a set T of t vertices of G, with $T \cap V(C) = \emptyset$, such that every vertex in T has fewer than $\lfloor w/t \rfloor$ non-neighbours in each part of C. Let A be a part of C; then at most $(w/t - 1)t$ vertices of A have a non-neighbour in T, and since $|A| = w$, there are t vertices in A that are complete to T. Since this is true for each part of C, it follows that G contains a $(t, d+1)$-core, contradicting that $t > \tau_{d+1}(G)$. This proves (1).

Choose an integer $n \geq 0$, maximum such that there is a sequence $C_1, P_1, C_2, P_2, \ldots, C_n, P_n$ with the following properties:

- $C_1, C_2, \ldots, C_n$ are stable (w, d)-cores of G, and $P_1, \ldots, P_n$ are subsets of $V(G)$;
- the sets $P_1, P_2, \ldots, P_n$ are pairwise disjoint, and $V(C_i) \subseteq P_i$ for $1 \leq i \leq n$;
- for $1 \leq i \leq n$, P_i consists of $V(C_i)$ together with all vertices v of $G \setminus (P_1 \cup \cdots \cup P_{i-1})$ such that v has at least st^{s-1} neighbours in some part of C_i and v has at least $\lfloor w/t \rfloor$ non-neighbours in some part of C_i.

It follows that (C_i, P_i) $(i \in \{1, \ldots, n\})$ is a (w, t)-template sequence in G. Let its vertex set be U. From the maximality of n, there is no stable (w, d)-core in $G \setminus U$, and so

$$\chi(G \setminus U) \leq f\left(\left(s(s^2+s+1)t\right)^{120(s^2+s+1)} w\right)$$

by 3.4. Let D be the digraph with vertex set $\{1, \ldots, n\}$, in which for distinct $i, j \in \{1, \ldots, n\}$, j is adjacent from i if some vertex of P_j has fewer than $\lfloor w/t \rfloor$ non-neighbours in each part of C_i. By (1), every vertex of D has outdegree at most $t - 1$, and so by 4.1, $V(D)$ can be partitioned into $2t$ stable sets $I_1, \ldots, I_{2t}$. For $1 \leq j \leq 2t$, (C_i, P_i) $(i \in I_j)$ is a (w, t)-template sequence in G, and it is 1-nice from the definition of D. Consequently each such template sequence has chromatic number at most k, from the hypothesis; and so $\chi(U) \leq 2kt$. Hence

$$\chi(G) \leq \chi(U) + \chi(G \setminus U) \leq 2kt + f\left(\left(s(s^2+s+1)t\right)^{120(s^2+s+1)} w\right).$$

This proves 4.2. ∎

We observe:

4.3 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \geq 1$ be an integer. Let (C_i, P_i) $(i \in I)$ be a 1-nice (w,t)-template sequence in a graph G. Then*

- *for all $i, j \in I$ with $i < j$, every vertex of P_j has fewer than st^{s-1} neighbours in each part of C_i.*
- *for all distinct $i, j \in I$, every vertex of P_j has at least $\lfloor w/t \rfloor$ non-neighbours in some part of C_i.*

Proof. Let $i, j \in I$ be distinct, and let $v \in P_j$. Since the sequence is 1-nice, v has at least $\lfloor w/t \rfloor$ non-neighbours in some part of C_i; and if $i < j$, then from the definition of a template sequence, v has fewer than st^{s-1} neighbours in each part of C_i. This proves 4.3. ∎

4.4 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \geq (s+1)t + s^3t^s$ be an integer. Then every 1-nice (w,d)-template sequence in G is 2-nice.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 1-nice (w,d)-template sequence in G that is not 2-nice, with vertex set U, suppose that $v \in U$ and $I' \subseteq I$ such that $|I'| = s$, and for each $i \in I'$, $v \notin V(C_i)$ and v has at least s^3t^{s-1} neighbours in some part of C_i.

For each $i \in I'$, there is a part A_i of C_i such that v has at least s^3t^{s-1} neighbours in A_i, and a part B_i of C_i such that v has at least $\lfloor w/t \rfloor$ non-neighbours in B_i; and since C_i has at least two parts, all with the same cardinality, we may choose A_i, B_i distinct.

We define $a_i \in A_i$ and $Y_i \subseteq B_i$ with $|Y_i| = s$ for $i \in I'$ inductively as follows. Assume that $i \in I'$, and a_j and Y_j are defined for all $j \in I'$ with $j > i$. Let $X = \bigcup_{j \in I',\, j>i} \{a_j\} \cup Y_j$. Thus $|X| \leq (s-1)(s+1)$. For all $j \in I'$ with $j > i$, every vertex in $V(C_j)$ has fewer than st^{s-1} neighbours in A_i, and since v has at least $s^3t^{s-1} > (s-1)s(s+1)t^{s-1}$ neighbours in A_i, there is a neighbour a_i of v in A_i that is nonadjacent to every vertex in X. Similarly, since v has at least $\lfloor w/t \rfloor \geq s + (s-1)s(s+1)t^{s-1}$ non-neighbours in B_i, there is a set $Y_i \subseteq B_i$ of s vertices each nonadjacent to v and each with no neighbours in X. This completes the inductive definition. But then the subgraph induced on v together with all the sets $\{a_i\} \cup Y_i$ $(i \in I')$ is isomorphic to H_s, a contradiction. This proves 4.4. ∎

4.5 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \geq 1$ be an integer. If every 3-nice (w,d)-template sequence in G has chromatic number at most k, then every 2-nice (w,d)-template sequence in G has chromatic number at most $2sdwk$.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 2-nice (w,d)-template sequence in G. Let D be the digraph with vertex set I, in which for distinct $i, j \in I$, there is an edge of D from i to j if some vertex of $V(C_i)$ has at least s^3t^{s-1} neighbours in some part of C_j. Since $|C_i| = dw$, and the sequence is 2-nice, it follows that every vertex of D has outdegree at most $dw(s-1)$, and so $V(D)$ can be partitioned into $2dw(s-1)+1 \leq 2sdw$ stable sets. Each gives a 3-nice (w,d)-template sequence in G, and therefore has chromatic number at most k, and so (C_i, P_i) $(i \in I)$ has chromatic number at most $2sdwk$. This proves 4.5. ∎

4.6 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge (s-1)s^2t^{s-1} + s^4t^{s-1} + s$ be an integer. Then every 3-nice (w,d)-template sequence in G is 4-nice.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 3-nice (w,d)-template sequence in G, with vertex set U, let $v \in U$, and suppose that there exists $I' \subseteq I$ with $|I'| \ge (dt)^s$ such that for each $i \in I'$, $v \notin P_i$ and v has a neighbour in $V(C_i)$. Since $\omega(G) \le dt$, 2.1 implies that there exists $I'' \subseteq I'$ with $|I''| = s$, such that for each $i \in I'$, v has a neighbour $a_i \in V(C_i)$, where the vertices a_i $(i \in I'')$ are pairwise nonadjacent. For $1 \le i \le s$ let B_i be a part of C_i that does not contain a_i. Inductively for each $i \in I''$ we choose $Y_i \subseteq B_i$ of cardinality s as follows. Assume that $i \in I''$ and Y_j has been defined for all $j \in I''$ with $j > i$. Let $X = \bigcup_{j \in I'', j>i} Y_j$. Thus $|X| \le (s-1)s$. Each vertex in X has at most st^{s-1} neighbours in B_i, and v and each vertex a_j $(j \in I'' \setminus \{i\})$ has at most s^3t^{s-1} neighbours in B_i. Since $w - (s-1)s^2t^{s-1} - s^4t^{s-1} \ge s$, there exist a set Y_i of s distinct vertices in B_i that are nonadjacent to every vertex in X, and nonadjacent to v, and nonadjacent to each vertex a_j $(j \in I'' \setminus \{i\})$. This completes the inductive definition. But then the subgraph induced on $\{v\} \cup \bigcup_{i \in I''}(\{a_i\} \cup Y_i)$ is isomorphic to H_s, a contradiction. This proves 4.6. ∎

4.7 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge 1$ be an integer. If every 5-nice (w,d)-template sequence in G has chromatic number at most k, then every 4-nice (w,d)-template sequence in G has chromatic number at most $2dw(dt)^sk$.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 4-nice (w,d)-template sequence in G, with vertex set U, and let D be the digraph with vertex set I, in which for distinct $i, j \in I$, j is adjacent from i if some vertex in $V(C_i)$ has a neighbour in $V(C_j)$. Since $|C_i| = wd$ and the sequence is 4-nice, it follows that D has maximum outdegree less than $wd(dt)^s$. By 4.1, $V(D)$ is the union of $2wd(dt)^s$ stable sets, each forming a 5-nice sequence, and therefore with chromatic number at most k. This proves 4.7. ∎

4.8 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge 1$ be an integer. Then every 5-nice (w,d)-template sequence in G is 6-nice.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 5-nice (w,d)-template sequence in G, with vertex set U, and let $v \in U$, and suppose that there are at least $2(dt)^{2s} + (dt)^s$ values of $i \in I$ such that $v \notin P_i$ and v has a neighbour in P_i. Since the sequence is 4-nice, there are fewer than $(dt)^s$ values of $i \in I$ such that $v \notin P_i$ and v has a neighbour in $V(C_i)$; so there exists $I_1 \subseteq I$ with $|I_1| = 2(dt)^{2s}$ such that for each $i \in I_1$, $v \notin P_i$, and v has a neighbour a_i in P_i, and v has no neighbour in $V(C_i)$. Let D be the digraph with vertex set I_1, in which for distinct $i, j \in I_1$, j is adjacent from i if a_i has a neighbour in $V(C_j)$. Since the sequence is 4-nice, D has maximum outdegree at most $(dt)^s - 1$, and so by 4.1, I_1 can be partitioned into $2(dt)^s$ sets that are stable in D. One of these stable sets has cardinality at least $(dt)^s$, since $|I_1| = 2(dt)^{2s}$; so there exists exists $I_2 \subseteq I_1$ with $|I_2| \ge (dt)^s$, such that for all $i \in I_2$, $v \notin P_i$, and v has a neighbour a_i in P_i, and v has no neighbour in $V(C_i)$; and for all distinct $i, j \in I_2$, a_i has no neighbour in $V(C_j)$. By 2.1, there exists $I_3 \subseteq I_2$ with $|I_3| = s$, such that the vertices a_i $(i \in I_3)$ are pairwise nonadjacent. For each $i \in I_3$, since $a_i \in P_i$, there is a part of C_i such that a_i has at least $st^{s-1} \ge s$ neighbours in this part, and so there exist $b_i^1, \dots, b_i^s \in V(C_i)$, distinct, pairwise nonadjacent, and all adjacent to a_i. But then the subgraph induced on $\{v\} \cup \bigcup_{i \in I_3}\{a_i, b_i^1, \dots, b_i^s\}$ is isomorphic to H_s, a contradiction. This proves 4.8. ∎

4.9 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge 1$ be an integer. Then every 6-nice (w, d)-template sequence in G is 7-nice.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 6-nice (w, d)-template sequence in G, with vertex set U, and suppose that $h \in I$, and there exists $I_1 \subseteq I \setminus \{h\}$ with $|I_1| \ge 60dws(dt)^{5s}$ such that for each $i \in I_1$ there exists $a_i \in P_h$ that has at least $(dt)^s$ neighbours in P_i. Let D be the digraph with vertex set I_1, where for distinct $i, j \in I_1$, j is adjacent from i if a_i has a neighbour in P_j. Since the sequence is 6-nice, D has maximum outdegree less than $2(dt)^{2s} + (dt)^s \le 3(dt)^{2s}$, and so by 4.1, there is a subset $I_2 \subseteq I_1$ with $|I_2| \ge |I_1|/(6(dt)^{2s}) \ge 10dws(dt)^{3s}$, such that for all distinct $i, j \in I_2$, a_i has no neighbour in P_j. Since for each $i \in I_2$, a_i has a neighbour in P_i, it follows that the vertices a_i $(i \in I_2)$ are all distinct. Since $|C_h| = dw$, there exists $I_3 \subseteq I_2$ with $|I_3| \ge |I_2| - dw \ge 9dws(dt)^{3s}$ such that $a_i \in P_h \setminus V(C_h)$ for each $i \in I_3$. For each $i \in I_3$, a_i has a neighbour in $V(C_h)$, and since $|C_h| = dw$, there exists $I_4 \subseteq I_3$ and $c \in V(C_h)$ such that $|I_4| \ge |I_3|/(dw) \ge 9s(dt)^{3s}$ and a_i is adjacent to c for each $i \in I_4$.

There are at most $3(dt)^{2s}$ values of $i \in I_4$ such that c has a neighbour in $V(C_i) \cup P_i$, so there exists $I_5 \subseteq I_4$ with $|I_5| = |I_4| - 3(dt)^{2s} \ge 6s(dt)^{3s}$ such that c has no neighbour in $V(C_i) \cup P_i$ for each $i \in I_5$. For each $i \in I_5$, since a_i has $(dt)^s$ neighbours in P_i, 2.1 implies that there is a stable subset $Y_i \subseteq P_i$ with $|Y_i| = s$ such that a_i is complete to Y_i. Let D' be the digraph with vertex set I_5, in which for distinct $i, j \in I_5$, j is adjacent from i if some vertex of Y_i has a neighbour in Y_j. Since each vertex in Y_i has a neighbour in P_j for fewer than $3(dt)^{2s}$ values of j, it follows that D' has maximum outdegree less than $3s(dt)^{2s}$, and so by 4.1, there exists $I_6 \subseteq I_5$ with $|I_6| \ge |I_5|/(6s(dt)^{2s}) \ge (dt)^s$ such that for all distinct $i, j \in S_6$, there are no edges between Y_i and Y_j. Since $|I_6| \ge (dt)^s$, 2.1 implies that there exists $I_7 \subseteq I_6$ with $|I_7| = s$ such that the set of a_i with $i \in I_7$ is stable in G. But then the subgraph induced on $\{c\} \cup \bigcup_{i \in I_7} (\{a_i\} \cup Y_i)$ is isomorphic to H_s, a contradiction. This proves 4.9. ∎

4.10 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge 1$ be an integer. If every 8-nice (w, d)-template sequence in G has chromatic number at most k, then every 7-nice (w, d)-template sequence in G has chromatic number at most $120dws(dt)^{5s}k$.*

Proof. Let (C_i, P_i) $(i \in I)$ be a 7-nice (w, d)-template sequence in G, with vertex set U. Let D be the digraph with vertex set I, in which for distinct $i, j \in I$, j is adjacent from i if some vertex in P_i has more than $(dt)^s$ neighbours in P_j. Since the sequence is 7-nice, D has maximum outdegree less than $60dws(dt)^{5s}$, and so by 4.1, I is the union of $120dws(dt)^{5s}$ subsets each stable in D. We observe that if $I_1 \subseteq I$ is stable in D, then for each $i \in I_1$ and each $v \in P_i$, there is no $j \in I_1 \setminus \{i\}$ such that v has at least $(dt)^s$ neighbours in P_j; and so, since v has neighbours in P_j for at most $2(dt)^{2s} + (dt)^s \le 3(dt)^{2s}$ values of $j \ne i$, it follows that v has fewer than $3(dt)^{3s}$ neighbours in $U \setminus P_i$, and so the sequence (C_i, P_i) $(i \in I_1)$ is 8-nice. This proves 4.10. ∎

To finish this chain of reductions, we have:

4.11 *Assuming Hypothesis A, let G be an H_s-free graph, let $t > \tau_{d+1}(G)$ be an integer, and let $w \ge 1$ be an integer. Then every 8-nice (w, d)-template sequence in G has chromatic number at most*

$$3sd^{3s+2}w^{2s-1}t^{3s}(f(t) + 2^{s^{2d+2}}t^{ds+s^2+s}).$$

Proof. Let (C_i, P_i) $(i \in I)$ be an 8-nice (w, d)-template sequence in G, with vertex set U. Let G_1, G_2 be the subgraphs of G, both with vertex set U, where G_1 is the union of the subgraphs $G[P_i]$ $(1 \le i \le n)$ and G_2 contains precisely those edges of $G[U]$ that do not belong to G_1. By 3.5,

$$\chi(G_1) \le sd^2w^{2s}(f(t) + 2^{s^{2d+2}}t^{ds+s^2+s}),$$

and $\chi(G_2) \le 3(dt)^{3s}$ since G_2 has maximum degree less than $3(dt)^{3s}$ (because the sequence is 8-nice). Taking the product colouring shows that

$$\chi(U) \le 3sd^{3s+2}w^{2s-1}t^{3s}(f(t) + 2^{s^{2d+2}}t^{ds+s^2+s}).$$

This proves 4.11. ∎

Now we can complete the proof of 1.6, by proving 3.2, which we restate:

4.12 *Assuming Hypothesis A, there is a polynomial f_1 with the following property. Let G be an H_s-free graph, and let $t > \tau_{d+1}(G)$ be an integer. Then $\chi(G) \le f_1(t)$.*

Proof. Define

$$\begin{aligned}
w(t) &= s^4t^s + s \\
f_8(t) &= 3sd^{3s+2}w^{2s-1}t^{3s}(f(t) + 2^{s^{2d+2}}t^{ds+s^2+s}) \\
f_5(t) = f_6(t) = f_7(t) &= 120sd^{5s+1}wt^{5s}f_8(t) \\
f_3(t) = f_4(t) &= 2d^{s+1}wt^sf_5(t) \\
f_2(t) &= 2sdwf_3(t) \\
f_1(t) &= f\left(\left(s(s^2+s+1)t\right)^{120(s^2+s+1)}w\right) + 2tf_2(t).
\end{aligned}$$

Thus, $w, f_8, f_7, \dots, f_1$ are all polynomials in t, since d, s are constants and f is a polynomial by Hypothesis A. By 4.11, every 8-nice (w, d)-template sequence in G has chromatic number at most $f_8(t)$. For $i = 7, 6, \dots, 1$ in turn, it follows that every i-nice (w, d)-template sequence in G has chromatic number at most $f_i(t)$, by applying 4.10, 4.9, 4.8, 4.7, 4.6, 4.5, 4.4 respectively. By 4.2, $\chi(G) \le f_1(t)$. This proves 4.12. ∎

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