Polynomial bounds for chromatic number. I. Excluding a biclique and an induced tree

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Abstract

Let H be a tree. It was proved by Rödl that graphs that do not contain H as an induced subgraph, and do not contain the complete bipartite graph $K_{t,t}$ as a subgraph, have bounded chromatic number. Kierstead and Penrice strengthened this, showing that such graphs have bounded degeneracy. Here we give a further strengthening, proving that for every tree H, the degeneracy is at most polynomial in t. This answers a question of Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak.

1 Introduction

The Gyárfás-Sumner conjecture [6, 15] asserts:

1.1 Conjecture: *For every forest H, there is a function f such that $\chi(G) \le f(\omega(G))$ for every H-free graph G.*

(We use $\chi(G)$ and $\omega(G)$ to denote the chromatic number and the clique number of a graph G, and a graph is *H-free* if it has no induced subgraph isomorphic to H.) One attractive feature of this conjecture is that it is best possible in a sense: for every graph H that is not a forest, there is no function f as in 1.1 (because, as shown by Erdős [4], there are graphs with arbitrarily large chromatic number and girth). The conjecture has been proved for some special families of trees (see, for example, [3, 7, 8, 9, 11, 12, 13]) but remains open in general.

A class $\mathcal{C}$ of graphs is *χ-bounded* if there is a function f such that $\chi(G) \le f(\omega(G))$ for every graph G that is an induced subgraph of a member of $\mathcal{C}$ (see [14] for a survey). Thus the Gyárfás-Sumner conjecture asserts that the class of all H-free graphs is χ-bounded, for every forest H. For some χ-bounded classes, the function f can be taken to be polynomial, and it remains open whether for every forest H, there is a polynomial f that satisfies 1.1. (Indeed, Esperet [5] made the even stronger conjecture that, for every χ-bounded class, f can always be chosen to be a polynomial, but this has recently been shown to be false [2].)

The complete bipartite graph with parts of cardinality s, t is denoted by $K_{s,t}$. Let us define $\tau(G)$ to be the largest t such that G contains $K_{t,t}$ as a subgraph (not necessarily induced). It was proved by Rödl (mentioned in [10], and see also [8]) that the analogue of the Gyárfás-Sumner conjecture is true if we replace $\omega(G)$ by $\tau(G)$. Explicitly:

1.2 *For every forest H, there is a function f such that $\chi(G) \le f(\tau(G))$ for every H-free graph G.*

This has the same attractive feature that the result is best possible (in the same sense).

This result was strengthened by Kierstead and Penrice. Let us say a graph G is *d-degenerate* (where $d \ge 0$ is an integer) if every nonnull subgraph has a vertex of degree at most d; and the *degeneracy* $\partial(G)$ of G is the smallest d such that G is d-degenerate. Then $\chi(G) \le \partial(G) + 1$, and so the following result of Kierstead and Penrice [9] is a strengthening of 1.2:

1.3 *For every forest H, there is a function f such that $\partial(G) \le f(\tau(G))$ for every H-free graph G.*

What about the analogue of Esperet's question: do 1.2 and 1.3 remain true if we require f to be a polynomial in $\tau(G)$? This question was raised by Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak in [1], and they proved it when H is a path, that is:

1.4 *For every path H, there exists $c > 0$ such that $\partial(G) \le \tau(G)^c$ for every H-free graph G.*

In this paper we answer the question completely. Our main result is:

1.5 *For every forest H, there exists $c > 0$ such that $\partial(G) \le \tau(G)^c$ for every H-free graph G.*

We also look at a related question: what can we say about $\chi(G)$ and $\partial(G)$ if G is H-free and does not contain $K_{s,t}$ as a subgraph? More exactly, if H, s are fixed, how do $\chi(G)$ and $\partial(G)$ depend on t? We will show that the dependence is in fact linear in t:

1.6 *For every forest H and every integer $s > 0$, there exists $c > 0$ such that for every graph G and every integer $t > 0$, if G is H-free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) \leq ct$.*

We also prove a weaker result, that under the same hypotheses, $\chi(G) \leq ct$, and for this the bound on c is a small function of s, H.

Finally, there is a second pretty theorem in the paper [1] of Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak:

1.7 *Let ℓ be an integer; then there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every graph G with no induced cycle of length at least ℓ.*

We give a new proof of this, simpler than that in [1].

In this paper, all graphs are finite and have no loops or parallel edges. We denote by $|H|$ the number of vertices of a graph H. If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G[X]$. We use "G-adjacent" to mean adjacent in G, and "G-neighbour" to mean a neighbour in G, and so on.

2 Producing a path-induced rooted tree.

We will prove 1.5 in this section and the next. We need to show that if a graph G has degeneracy at least some very large polynomial in t (independent of G), and does not contain $K_{t,t}$ as a subgraph, then it contains any desired tree as an induced subgraph. We will show this in two stages: in this section we will show that G contains a large (with degrees a somewhat smaller polynomial in t) "path-induced" tree, and in the next section we will convert this to the desired induced tree. "Path-induced" means that each path of the tree starting at the root is an induced path of G; so we should be talking about rooted trees. Let us say this carefully.

A *rooted tree* (H, r) consists of a tree H and a vertex r of H called the *root*. A rooted subtree of (H, r) means a rooted tree (J, r) where J is a subtree of H and $r \in V(J)$. The *height* of (H, r) is the length (number of edges) of the longest path of H with one end r. If $u, v \in V(H)$ are adjacent and u lies on the path of H between v, r, we say v is a *child* of u and u is the *parent* of v. The *spread* of H is the maximum over all vertices $u \in V(H)$ of the number of children of u. (Thus the spread is usually one less than the maximum degree.) Let H be a subgraph of G (not necessarily induced), where (H, r) is a rooted tree. We say that (H, r) is a *path-induced rooted subgraph* of G if every path of H with one end r is an induced subgraph of G.

Let $\zeta, \eta \geq 1$. The rooted tree (H, r) is (ζ, η)-*uniform* if

- every vertex with a child has exactly ζ children;
- every vertex with no child is joined to r by a path of H of length exactly η.

We need two lemmas:

2.1 *Let $k, \zeta, \eta \geq 1$ with $\zeta \geq 2$, and let $(H_1, r_1), \ldots, (H_k, r_k)$ be $(k\zeta^{\eta+1}, \eta)$-uniform rooted trees, each a subgraph of a graph G, such that $r_i \notin V(H_j)$ for all distinct $i, j \in \{1, \ldots, k\}$. Then for $1 \leq i \leq k$ there is a (ζ, η)-uniform rooted subtree (H'_i, r_i) of (H_i, r_i), such that the trees $H'_1, \ldots, H'_k$ are pairwise vertex-disjoint.*

Proof. Choose $j \leq k$ maximum such that there are (ζ, η)-uniform rooted subtrees (H'_i, r_i) of (H_i, r_i) for $1 \leq i \leq j$, such that the trees $H'_1, \ldots, H'_j$ are pairwise vertex-disjoint. Let $X = V(H'_1) \cup \cdots \cup V(H'_j)$. Thus $|X| \leq j\zeta^{\eta+1}$, since each H'_i has

$$1 + \zeta + \zeta^2 + \cdots + \zeta^\eta \leq \zeta^{\eta+1}$$

vertices (here we use that $\zeta \geq 2$). Suppose that $j < k$. Then each vertex of (H_{j+1}, r_{j+1}) with a child has at least $(k-j)\zeta^{\eta+1} \geq \zeta^{\eta+1} \geq \zeta$ children not in X; and since $r_{j+1} \notin X$, it follows that there is a (ζ, η)-uniform rooted subtree (H'_{j+1}, r_{j+1}) of (H_{j+1}, r_{j+1}) vertex-disjoint from X, contrary to the maximality of j. Thus $j = k$, and this proves 2.1. ∎

Let (T, r) be a rooted tree, where T is a subgraph of G. For $t > 0$, a vertex $u \in V(G)$ is *t-bad* for (T, r) if there is a vertex $w \in V(T)$ such that u is distinct from and G-adjacent to more than $d(1 - 1/t)$ children of w, where d is the number of children of w. We will often use the following:

2.2 *Let $t, \eta \geq 1$ and $\zeta \geq 2$ be integers. Let (T, r) be a $(t\zeta, \eta)$-uniform rooted tree, where T is a subgraph of G; and let $u \in V(G) \setminus V(T)$. If u is not t-bad for (T, r), then there is a (ζ, η)-uniform rooted subtree (S, r) of (T, r) such that u has no G-neighbour in $V(S)$ except possibly r.*

We omit the proof, which is clear. The second lemma is:

2.3 *Let $t, \eta \geq 1$ and $\zeta \geq 2$ be integers, where t divides ζ. Let G be a graph that does not contain $K_{t,t}$ as a subgraph, and let (T, r) be a (ζ, η)-uniform rooted tree, where T is a subgraph of G. Then fewer than ζ^η vertices in $V(G)$ are t-bad for (T, r).*

Proof. There are $\zeta^\eta/(\zeta - 1)$ vertices in $V(T)$ that have children (since $\zeta \geq 2$). Let $w \in V(T)$ with ζ children, and let C_w be the set of its children in (T, r). Suppose that there are t distinct vertices $u_1, \ldots, u_t$ in $V(G)$ such that each is G-nonadjacent to more than $|C_w|(1 - 1/t)$ vertices in C_w, and hence to at least $|C_w|(1 - 1/t) + 1$ such vertices, since t divides $|C_w|$.

It follows that each u_i is equal or $|C_w|/t - 1$ G-nonadjacent to at most $|C_w|/t - 1$ vertices of C_w, and so at most $t(|C_w|/t - 1)$ vertices in C_w belong to or have a G-nonneighbour in $\{u_1, \ldots, u_t\}$. Consequently at least t vertices in C_w are G-adjacent to all of $u_1, \ldots, u_t$, contradicting that G does not contain $K_{t,t}$ as a subgraph. Thus there are at most $t - 1 \leq \zeta - 1$ vertices in $V(G)$ with more than $|C_w|(t-1)/t$ G-neighbours in C_w. So the number of vertices in $V(G)$ that are t-bad for (T, r) is at most $\zeta - 1$ times the number of vertices of T that have children, and so smaller than ζ^η. This proves 2.3. ∎

The main result of this section is the following:

2.4 *Let $\eta > 0$ be an integer and let $c = (\eta + 1)!$. Let $\zeta \geq 2$, and let (H, r) be a rooted tree of height at most η, and spread at most ζ. Let $t \geq 1$ be an integer, and suppose that the graph G does not contain $K_{t,t}$ as a subgraph, and does not contain a rooted tree isomorphic to (H, r) as a path-induced rooted subgraph. Then $\partial(G) \leq (\zeta t)^c$.*

Proof. We may assume that $t \geq 2$. We proceed by induction on η. If $\eta = 1$, it follows that G has maximum degree at most $\zeta - 1$, since it does not contain (H, r) as a path-induced rooted subgraph; and so $\partial(G) \leq \zeta - 1 \leq (\zeta t)^c$ as required. So we may assume that $\eta \geq 2$, and the result holds for all rooted trees with height less than η. Let $c' = \eta!$ and $\zeta' = t\zeta^{\eta+1}$. Let us say a *limb* is a $(\zeta', \eta - 1)$-uniform rooted tree that is a path-induced rooted subgraph of G.

(1) *For each vertex u, there are at most $\zeta - 1$ G-neighbours v of u with the property that there is a limb (J, v) of G such that $u \notin V(J)$ and u is not t-bad for (J, v).*

Suppose there are ζ such vertices $v_1, \dots, v_\zeta$, and let the corresponding limbs be (J_i, v_i) for $1 \le i \le \zeta$. By 2.2, for $1 \le i \le \zeta$, there is a $(\zeta^{\eta+1}, \eta - 1)$-uniform rooted subtree (J_i', v_i) of (J_i, v_i), such that u has no neighbour in $V(J_i')$ except v_i. By 2.1, there is a $(\zeta, \eta - 1)$-uniform rooted subtree (H_i', r_i) of (J_i', r_i) for $1 \le i \le \zeta$, such that the trees $H_1', \dots, H_k'$ are pairwise vertex-disjoint. But then adding u to the union of these trees gives a (ζ, η)-uniform rooted tree, and it is path-induced in G, and contains a rooted induced subgraph isomorphic to (H, r), a contradiction. This proves (1).

Let P be the set of vertices v of G such that there is a limb with root v, and let $Q = V(G) \setminus P$. For each $v \in P$, there is at least one limb with root v; select one, and call it (J_v, v). For each edge e with at least one end in P, select one such end, and call it the *head* of e.

- Let A be the set of all edges with both ends in Q;
- Let B be the set of all edges uv of G with head v, such that $u \notin V(J_v)$, and u is not t-bad for (J_v, v);
- Let C be the set of all edges uv of G with head v, such that $u \notin V(J_v)$, and u is t-bad for (J_v, v);
- Let D be the set of all edges uv of G with head v, such that $u \in V(J_v)$.

Thus every edge of G belongs to exactly one of A, B, C, D. Since $G[Q]$ does not contain a limb, the inductive hypothesis implies that $\partial(G[Q]) \le (\zeta' t)^{c'}$. Consequently

$$|A| \le (\zeta' t)^{c'} |Q| \le (\zeta' t)^{c'} |G|.$$

By (1), for each vertex $u \in V(G)$, there are at most $\zeta - 1$ edges $uv \in B$ with head v; and so

$$|B| \le (\zeta - 1)|G|.$$

For each $v \in P$, there are at most $\zeta'^{\eta-1}$ edges $uv \in C$ with head v by 2.3, and so

$$|C| \le \zeta'^{\eta-1}|P| \le \zeta'^{\eta-1}|G|.$$

For each $v \in P$, since (J_v, v) is path-induced, there are at most ζ' edges $uv \in D$ with head v, and so

$$|D| \le \zeta'|P| \le \zeta'|G|.$$

Summing, we deduce that

$$|E(G)| \le \left((\zeta' t)^{c'} + (\zeta - 1) + \zeta'^{\eta-1} + \zeta' \right) |G|,$$

and so some vertex of G has degree at most $2\left((\zeta' t)^{c'} + (\zeta - 1) + \zeta'^{\eta-1} + \zeta'\right)$. Since this also holds for every non-null induced subgraph of G, we deduce that

$$\partial(G) \le 2\left((\zeta' t)^{c'} + (\zeta - 1) + \zeta'^{\eta-1} + \zeta'\right).$$

We recall that $\zeta' = t\zeta^{\eta+1}$ and $c = (\eta+1)c'$; and so

$$\begin{aligned}
\partial(G) &\le 2\left(\zeta^{c'(\eta+1)} t^{c'} + (\zeta - 1) + \zeta^{\eta^2-1} t^{\eta-1} + \zeta^{\eta+1} t\right) && (1)\\
&\le 2\zeta^{c}\left(t^{c'} + 1 + t^{\eta-1} + t\right) && (2)\\
&\le 8\zeta^{c} t^{c'} \le \zeta^{c} t^{c} && (3)
\end{aligned}$$

(since $c \ge c' + 3$ and $t \ge 2$). This proves 2.4. ∎

We remark that 2.4 implies 1.4, and a strengthening:

2.5 *If H is a path, and $t \ge 1$ is an integer, and G is H-free and does not contain $K_{t,t}$ as a subgraph, then $\partial(G) \le (2t)^{|H|!}$.*

Proof. Let $\zeta = 2$, and $\eta = |E(H)| = |H| - 1$. Let r be one end of H. Then G does not contain (H, r) as a path-induced rooted subgraph, and so $\partial(G) \le (2t)^{|H|!}$ by 2.4. This proves 2.5. ∎

3 Growing a tree

If (T, r) is a rooted tree and $v \in V(T)$, the *height* of v in (T, r) is the number of edges in the path between v, r; and so the height of (T, r) is the largest of the heights of its vertices. Let (T, r) be a rooted tree, and let (S, r) be a rooted subtree. The graph obtained from T by deleting all the edges of S is disconnected, and each of its components contains a unique vertex of S; for each $v \in V(S)$, let T_v be the component that contains $v \in V(S)$. We call the rooted tree (T_v, v) the *decoration of S at v in T*.

Let G be a graph, let (S, r) be a rooted tree, and let $\zeta \ge 2$ and $\eta \ge 1$. We say that (S, r) is (ζ, η)-*decorated* in G if S is an induced subgraph of G with height at most $\eta - 1$, and there is a rooted tree (T, r) with the following properties:

- (S, r) is a rooted subtree of (T, r), and (T, r) is a path-induced rooted subgraph of G;
- for each $u \in V(S)$ and $v \in V(T) \setminus V(S)$, if u, v are G-adjacent then they are T-adjacent;
- for each $v \in V(S)$, the decoration of S at v in T is $(\zeta, \eta - h)$-uniform, where h is the height of v in (S, r).

Thus, informally, T is obtained from S by attaching to S uniform trees rooted at each vertex of S. Note that T is only required to be path-induced: the various uniform trees that are attached to S might have edges between them.

In view of 2.4, if we have a graph G with huge degeneracy that does not contain $K_{t,t}$, then it contains a (ζ, η)-uniform rooted tree (T, r) as a path-induced rooted subgraph; and consequently there is a one-vertex rooted tree (S, r) that is (ζ, η)-decorated in G. The next result shows that if we start with ζ large enough, then by reducing ζ we can grow S into any larger tree that we wish, and that will prove 1.5.

3.1 *Let $\eta, t \geq 1$ and $\zeta \geq 2$ be integers, let G be a graph that does not contain $K_{t,t}$ as a subgraph, and let (S', r) be a (ζ', η)-decorated rooted tree in G, where $\zeta' \geq (\zeta t)^\eta |S'| + \zeta t$. Let $p \in V(S')$ with height in (S', r) less than η. Then there is a G-neighbour q of p, with $q \in V(G) \setminus V(S')$, and with no other G-neighbour in $V(S')$, such that, if S denotes the tree obtained from S' by adding q and the edge pq, then (S, r) is a (ζ, η)-decorated rooted tree in G.*

Proof. For each $v \in V(S')$, let $h(v)$ denote the height of v in (S', r). Since (S', r) is (ζ', η)-decorated in G, it follows that S' is an induced subgraph of G, and there is a rooted tree (T', r) such that

- (S', r) is a rooted subtree of (T', r), and (T', r) is a path-induced rooted subgraph of G;
- for each $u \in V(S')$ and $v \in V(T') \setminus V(S')$, if u, v are G-adjacent then they are T'-adjacent;
- for each $v \in V(S')$, the decoration of S' at v in T' is $(\zeta', \eta - h(v))$-uniform.

For each $v \in V(S')$, let (T_v, v) be the decoration of S' at v in T'. Since T_p is $(\zeta', \eta - h(p))$-uniform, and $h(p) < \eta$, it follows that p has ζ' children in (T_p, p). We need to select one of these children, say q, to add to S', forming S. Any one of them would make a larger induced tree when added to S', since (S', r) is (ζ, η)-decorated. But in order to make the new rooted tree (ζ, η)-decorated, we will delete from T' all vertices of T' that are G-adjacent and not T'-adjacent to q; and doing so must not destroy too much of T'.

For each $v \in V(S')$, let (S_v, v) be a $(t\zeta, \eta - h(v))$-uniform rooted subtree of (T_v, v). By 2.3, there are fewer than $(t\zeta)^{\eta - h(v)} \leq\leq (t\zeta)^\eta$ vertices not in $V(S_v)$ that are t-bad for (S_v, v), and so there are fewer than $(t\zeta)^\eta |S'|$ children of p in (T_p, p) that are t-bad for one of the rooted trees (S_v, v) $(v \in V(S'))$. Also, since (S_p, p) is path-induced, every G-neighbour of p in $V(S_p)$ is an S_p-neighbour of p; so there are only $t\zeta$ children of p in (T_p, p) that belong to $V(S_p)$. Since $\zeta' \geq (\zeta t)^\eta |S'| + \zeta t$, there is a child q of p in (T_p, p) that is t-bad for none of the trees (S_v, v) $(v \in V(S'))$ and does not belong to $V(S_p)$.

Let Q be the component containing q of the graph obtained from T' by deleting $V(S)$; thus (Q, q) is $(\zeta', \eta - h(p) - 1)$-uniform, and so we may choose a $(\zeta, \eta - h(p) - 1)$-uniform rooted subtree (R_q, q) of (Q, q). Note that q has no neighbours in $V(Q)$ except its neighbours in T', since (T', r) is path-induced. Since q is not t-bad for any of the rooted trees (S_v, v) $(v \in V(S'))$, it follows by 2.2 that for each v there is a $(\zeta, \eta - h(v))$-uniform rooted subtree (R_v, v) of (S_v, v) such that q has no G-neighbour in $V(R_v)$ except possibly v, and q is G-adjacent to v if and only if they are T'-adjacent (that is, $v = p$), since $v \in V(S')$ and (S', r) is (ζ', η)-decorated. Let S be the tree induced on $V(S') \cup \{q\}$, and let T be the union of T', the trees R_v $(v \in V(S') \cup \{q\})$ and the edge pq. Then S satisfies the theorem, because the tree T exists. This proves 3.1. ∎

We deduce 1.5, which we restate in a strengthened form:

3.2 *Let $\eta, t \geq 1$ and $\zeta \geq 2$. For every rooted tree (H, s) with height at most η and spread at most ζ, let $c = (\eta + 3)!|H|$; then $\partial(G) \leq (|H|\zeta t)^c$ for every H-free graph G that does not contain $K_{t,t}$ as a subgraph.*

Proof. Choose $\eta \geq 1$ and $\zeta \geq 2$ such that (H, s) has height at most η and spread at most ζ. Let H have k vertices. Define $\zeta_k = \zeta$, and for $i = k-1, k-2, \ldots, 1$ let $\zeta_i = k(t\zeta_{i+1})^\eta$. Thus $\zeta_i \geq i(t\zeta_{i+1})^\eta + t\zeta_{i+1}$

Let G be an H-free graph that does not contain $K_{t,t}$ as a subgraph. Suppose that G contains a one-vertex rooted tree that is (ζ_1, η)-decorated in G. Choose a maximal rooted subtree (F, s) of (H, s) such that there is a rooted subtree (S, r) of G, isomorphic to (F, s), such that (S, r) is (ζ_i, η)-decorated in G, where $i = |F|$. By 3.1, $i = k$; and so G contains an induced subgraph isomorphic to H, a contradiction.

Thus G contains no one-vertex rooted tree that is (ζ_1, η)-decorated in G. Hence G contains no (ζ_1, η)-uniform rooted tree as a path-induced rooted subgraph, and so by 2.4 (applied with (H, r) replaced by a (ζ_1, η)-uniform rooted tree), $\partial(G) \leq (\zeta_1 t)^d$ where $d = (\eta+1)!$.

Now $\zeta_k = \zeta$, and $\zeta_{k-1} = k(t\zeta)^\eta$. For all i with $1 \leq i \leq k-2$, $\zeta_{i+1} \geq kt^\eta$, and so $\zeta_i = k(t\zeta_{i+1})^\eta \leq \zeta_{i+1}^{\eta+1}$. Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq (k(t\zeta)^\eta)^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So $\partial(G) \leq (k\zeta t)^c$ where $c = (k-2)(\eta+1)^2(\eta+1)! + (\eta+1)! \leq (\eta+3)!k$. This proves 3.2. ∎

Now $\zeta_k = \zeta$, and $\zeta_{k-1} = (k-1)\zeta^\eta t^\eta + \zeta t$. For all i with $1 \leq i \leq k-2$, $\zeta_{i+1} \geq it^{\eta+1}$, and so $\zeta_i = i\zeta_{i+1}^\eta t^{\eta+1} \leq \zeta_{i+1}^{\eta+1}$. Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq \left(k\zeta^\eta t^{\eta+1}\right)^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So $\partial(G) \leq (k\zeta t)^c$ where $c = (k-2)(\eta+1)^2(\eta+1)! + (\eta+1)! \leq (\eta+3)!k$. This proves 3.2. ∎

4 Excluding $K_{s,t}$

In this section we prove 1.6, and before that we prove a weaker statement, with $\partial(G)$ replaced by $\chi(G)$. For the latter we need the following lemma:

4.1 *Let J be a digraph such that every vertex has outdegree at most k. Then the undirected graph underlying J has chromatic number at most $2k+1$.*

Proof. Let G be the undirected graph underlying J. Since every subgraph of G has the property that its edges can be directed so that it has outdegree at most k, it follows that every such subgraph H has at most $k|H|$ edges; and therefore (if it is non-null) has a vertex of degree at most $2k$. Consequently G is $2k$-degenerate, and so is $(2k+1)$-colourable. This proves 4.1. ∎

We use 4.1 to prove the following (which we include here because the proof gives a relatively small constant c, although the fact that some c exists follows from 1.6):

4.2 *Let H be a tree and $s \geq 1$ an integer, and let $c = (2s|H|)^{s+|H|}$. Then for every H-free graph G and every integer $t \geq 1$, if G does not contain $K_{s,t}$ as a subgraph then $\chi(G) \leq ct$.*

Proof. We will prove this by induction on $|H|$ (for the same value of s). Let H be a tree and $s \geq 0$ an integer, and suppose the theorem holds for all smaller trees and the same value of s. We may assume that $|H| \geq 3$, since the theorem is true for trees with at most two vertices; let $p \in V(H)$ have degree one, and let q be its H-neighbour. Let H' be obtained by deleting p from H. Let $c' = (2s|H'|)^{s+|H'|}$. We observe that

(1) $c \geq \max\left((|H|-2)^{s-1}, (s-1)(|H|-2), (2(s-2)(|H|-2)+1)\, c' + 1\right)$.

Let $t \geq 1$ be an integer, and let G be an H-free graph not containing $K_{s,t}$ as a subgraph. We will show that $\chi(G) \leq ct$. Suppose that this is false, and choose a minimal induced subgraph G' of G with $\chi(G') > ct$. It follows that every vertex of G' has degree at least ct (since c is an integer).

Let $v \in V(G')$. We say a subset $X \subseteq V(G') \setminus \{v\}$ is a *v-bag* if there is an isomorphism from H' to $G[X \cup \{v\}]$ that maps q to v. (Thus each v-bag has cardinality $|H|-2$.)

Let $v \in V(G')$, and suppose that there are $s-1$ pairwise disjoint v-bags, say $X_1, \ldots, X_{s-1}$. Since G is H-free, every G-neighbour u of v either belongs to X_i or has a G-neighbour in X_i, for $1 \leq i \leq s-1$. In particular, every G-neighbour u of v not in $X_1 \cup \cdots \cup X_{s-1}$ has a G-neighbour in each of $X_1, \ldots, X_{s-1}$. But for each choice of $x_i \in X_i$ $(1 \leq i \leq s-1)$ there are at most $t-1$ G-neighbours of v G-adjacent to each of $x_1, \ldots, x_{s-1}$ (since they are also all adjacent to v, and G has no $K_{s,t}$ subgraph). Consequently there are at most $(t-1)(|H|-2)^{s-1}$ G-neighbours of v not in $X_1 \cup \cdots \cup X_{s-1}$; and hence

$$(s-1)(|H|-2) + (t-1)(|H|-2)^{s-1} > ct.$$

Since $ct = c + c(t-1)$, and $(s-1)(|H|-2) < c$, and $(t-1)(|H|-2)^{s-1} \leq c(t-1)$, this contradicts (1); so there is no such choice of $X_1, \ldots, X_{s-1}$.

Choose an integer r maximum such that there are r pairwise disjoint v-bags, say $X_1, \ldots, X_r$. Consequently $r \leq s-2$. Let $Y_v = X_1 \cup \cdots \cup X_r$; then from the maximality of r, $X \cap Y_v \neq \emptyset$ for every v-bag X. Moreover $|Y_v| \leq (s-2)(|H|-2)$.

Let J be the digraph with vertex set $V(G')$ in which every vertex in Y_v is J-adjacent from v, for each $v \in V(G')$. Thus J has maximum outdegree at most $(s-2)(|H|-2)$, and so by 4.1, the undirected graph J' underlying J has chromatic number at most $2(s-2)(|H|-2)+1$; and so $V(G') = V(J')$ can be partitioned into $2(s-2)(|H|-2)+1$ sets each of which is a stable set of J'. Let Z be one of these sets. Then $G[Z]$ is H'-free (because otherwise there would be a vertex $v \in Z$, and a subset $X \subseteq Z \setminus \{v\}$, and an isomorphism from H' to $G[X \cup \{v\}]$ mapping q to v, and hence with $X \cap Y_v \neq \emptyset$; but no vertex of Y_v belongs to Z, since Z is stable in J'). From the inductive hypothesis, $\chi(Z) \leq c't$, and hence

$$ct < \chi(G) = \chi(G') \leq (2(s-2)(|H|-2)+1)c't$$

contrary to (1). This proves 4.2. ∎

To prove 1.6, we will need the following strengthening of 1.3, also proved in [9]:

4.3 *For every forest H, and every integer $s > 0$, there is a tree S such that for every H-free graph G, if G contains S as a subgraph, then G contains $K_{s,s}$ as a subgraph.*

Now we prove 1.6, which we restate:

4.4 *For every forest H and every integer $s > 0$, there exists $c > 0$ such that for every graph G and every integer $t > 0$, if G is H-free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) < ct$.*

Proof. Let S be as in 4.3, and let $c = |S|^s$; we will show that c satisfies the theorem. Let $t > 0$ be an integer, and let G be an H-free graph that does not contain $K_{s,t}$ as a subgraph. Suppose that $\partial(G) \geq ct$, and choose G minimal with these properties: then every vertex of G has degree at least ct.

(1) *Let R be a tree. If every vertex of G has degree at least $t|R|^s$, then G contains a subgraph T isomorphic to R, and $V(T)$ can be ordered as $\{t_1, \dots, t_n\}$, such that for $1 \leq i \leq n$, t_i is G-adjacent to at most $s-1$ of $t_1, \dots, t_{i-1}$.*

We prove this by induction on $|R|$. We may assume that $|R| > 1$; let $p \in V(R)$ have degree one in R, and let q be its R-neighbour. Let R' be obtained from R by deleting p. From the inductive hypothesis, G contains a subgraph T' isomorphic to R', and its vertex set can be ordered as $\{t_1, \dots, t_{n-1}\}$, such that for $1 \leq i \leq n-1$, t_i is G-adjacent to at most $s-1$ of $t_1, \dots, t_{i-1}$. Choose $v \in V(T')$ such that some isomorphism from R' to T' maps q to v. If some G-neighbour u of v does not belong to $V(T')$ and has at most $s-1$ G-neighbours in $V(T')$, then we may set $t_n = u$ as required; so we may assume that every G-neighbour u of v in G either belongs to $V(T')$ or has at least s G-neighbours in $V(T')$. Let $X \subseteq V(T')$ with $|X| = s$. If there are at least t vertices in $V(G)$ that are G-adjacent to every vertex in X, then G contains $K_{s,t}$ as a subgraph, a contradiction. So for each such X, there are at most $t-1$ vertices in $V(G)$ that are G-adjacent to every vertex in X. Since there are most $|R'|^s$ choices of X, there are at most $(t-1)|R'|^s$ vertices in $V(G) \setminus V(T')$ that have at least s G-neighbours in $V(T')$. Consequently v has at most $(t-1)|R'|^s$ G-neighbours not in $V(T')$. But it has at most $|R'|$ G-neighbours in $V(T')$ and so the degree of v in G is at most $(t-1)|R'|^s + |R'| < t|R|^s$. This proves (1).

Each vertex of G has degree at least $ct = t|S|^s$; let us apply (1) taking $R = S$. We deduce that G contains a subgraph T isomorphic to S, and its vertex set can be ordered as $\{t_1, \dots, t_n\}$, such that for $1 \leq i \leq n$, t_i is G-adjacent in G to at most $s-1$ of $t_1, \dots, t_{i-1}$. By 4.3, $G[V(T)]$ contains $K_{s,s}$ as a subgraph. Choose i maximum such that t_i belongs to this subgraph; then t_i is G-adjacent to at least s vertices that are earlier in the ordering, a contradiction. This proves 4.4. ∎

5 Long holes

There is another result in the paper by Bonamy et al. [1]:

5.1 *Let $\ell \geq 2$ be an integer; then there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every graph G with no induced cycle of length at least ℓ.*

In this section we give a simpler proof of this result.

Let $\eta, t \geq 1$ be integers. We say a rooted tree (H, r) is (t, η)-*tapering* if (H, r) has height η, and every vertex $v \in V(H)$ of height $i < \eta$ has exactly $t^{\eta - i}$ children. For each $v \in V(H)$, let $h(v)$ be its height in (H, r).

Let G be a graph. A map ϕ from $V(H)$ into $V(G)$ is a (t, η)-*infusion* of (H, r) into G if

- for all distinct $u, v \in V(H)$, if $u, v \in V(H)$ are H-adjacent then $\phi(u), \phi(v)$ are distinct and G-adjacent;
- for each $u \in V(H)$, if v, w are distinct children of u in (H, r), then $\phi(v) \neq \phi(w)$;
- for every path P of H with one end r, the vertices $\phi(v)$ $(v \in V(P))$ are all distinct; and
- for every path P of H with one end r, and for all distinct $u, v \in V(P)$, $\phi(u), \phi(v)$ are G-adjacent if and only if u, v are H-adjacent.

Let ϕ be a (t, η)-infusion into G. We define $V(\phi) = \{\phi(v) : v \in V(H)\}$, and we define the *root* of ϕ to be $\phi(r)$. We say $u \in V(G)$ is *t-bad* for ϕ if there exists $v \in V(H)$ with $h(v) < \eta$, such that u is distinct from and G-adjacent to $\phi(w)$ for more than $(t-1)t^{\eta - h(v) - 1}$ children w of v in (H, r). Then we have:

5.2 *Let $t, \eta \geq 1$ be integers, let (H, r) be a (t, η)-tapering rooted tree, let G be a graph not containing $K_{t,t}$ as a subgraph, and let ϕ be a (t, η)-infusion of (H, r) into G. There are at most t^{η^η} vertices in G that are t-bad for ϕ.*

The proof is like that for 2.3, using that H has at most $t^{\eta^\eta - 1}$ vertices that have children, and we omit it.

The next result strengthens 1.7:

5.3 *Let $\eta \geq 2$ be an integer, and let G be a graph with no induced cycle of length more than η. For every integer $t \geq 1$, if G does not contain $K_{t,t}$ as a subgraph then $\partial(G) \leq t^{7\eta^\eta}$.*

Proof. Let $t \geq 1$ be an integer, and let G be a graph with no induced cycle of length more than η that does not contain $K_{t,t}$. We may assume that $t \geq 2$. Let (H, r) be a (t, η)-tapering rooted tree (not necessarily contained in G).

(1) *If $u \in V(G)$ and v_i is a G-neighbour of u for $1 \leq i \leq t^\eta$, all distinct, and for each i there is a (t, η)-infusion of (H, r) into G with root v_i, such that $u \notin V(\phi_i)$, and u is not t-bad for ϕ_i, then there is a (t, η)-infusion of (H, r) into G, with root u.*

Let (H', r) be a $(t, \eta - 1)$-tapering rooted subtree or (H, r). It follows (analogously to 2.2) that for $1 \leq i \leq t^\eta$, there is a $(t, \eta - 1)$-infusion ϕ'_i of (H', r) into G such that u has no G-neighbour in $V(\phi'_i)$ except v_i. Let us number the components of $H \setminus \{r\}$ as $H_1, \ldots, H_{t^\eta}$. Let $\psi(r) = v$, and for $1 \leq i \leq t^\eta$ and each $v \in V(H_i)$, define $\psi(v) = \phi'_i(w)$ where w is the parent of v in (H, r). Then ψ is a (t, η)-infusion of (H, r) into G, with root v. This proves (1).

In these circumstances we say that ψ, constructed as in the proof of (1), is *derived from* the sequence $(\phi_i : 1 \leq i \leq t^\eta)$.

If P is a path of H with length η and one end r, and ϕ is a (t, η)-infusion of (H, r) into G, then ϕ maps P to an induced path $\phi(P)$ of G with length η and with one end the root of ϕ. We call $\phi(P)$ a *column* of ϕ. We observe that if ψ is derived from $(\phi_i : 1 \leq i \leq t^\eta)$ as above, then for every column Q of ψ, there is a column Q' of one of ϕ_i $(1 \leq i \leq t^\eta)$, say of ϕ', such that $Q \setminus \psi(r)$ is a subpath of Q'. Let us call (ϕ', Q') a *shift* of (ϕ, Q).

Let $\mathcal{A}_1$ be the set of all (t,η)-infusions of (H,r) into G. Inductively for $i>1$, let $\mathcal{A}_i$ be the set of all (t,η)-infusions ϕ such that for some choice of $\phi_1,\phi_2,\ldots,\phi_{t^\eta}\in\mathcal{A}_{i-1}$, ϕ is derived from the sequence $(\phi_j:\ 1\le j\le t^\eta)$. Thus $\mathcal{A}_i\subseteq\mathcal{A}_{i-1}$ for each i. There are two cases: either $\mathcal{A}_i$ is empty for some i, or it remains nonempty for all values of i. Suppose first that $\mathcal{A}_i$ is nonempty for all i, and let $\mathcal{A}$ be the intersection of all the sets $\mathcal{A}_i$ $(i\ge 1)$. Choose $\phi_1\in\mathcal{A}$, and let Q_1 be a column of ϕ_1. Since ϕ_1 is derived from some members of $\mathcal{A}$, there exists $\phi_2\in\mathcal{A}$ with root u_2, and a column Q_2 of ϕ_2, such that (ϕ_2,Q_2) is a shift of (ϕ_1,Q_1). Similarly we can choose an infinite sequence (ϕ_i,Q_i) $(i=1,2,3\ldots)$ such that each $\phi_i\in\mathcal{A}$ and each (ϕ,Q_i) is a shift of its predecessor. Let v_i be the root of ϕ_i for each i. Then $v_i,v_{i+1},\ldots,v_{i+\eta}$ are the vertices in order of Q_i for each i; and so form an induced path of G. Since G is finite, there exists $j>0$ such that v_j is adjacent to one of $v_1,\ldots,v_{j-2}$; choose a minimum such value of j, and choose $i\le j-2$ maximum such that v_i,v_j are adjacent. Then $\{v_i,\ldots,v_j\}$ induces a cycle of G of length more than η, a contradiction.

So the second case holds, that is, $\mathcal{A}_i$ is empty for some i. Choose k minimum such that $\mathcal{A}_{k+1}=\emptyset$. For $1\le i\le k$ let X_i be the set of all vertices v such that v is the root of a member of $\mathcal{A}_i$ and not the root of any member of $\mathcal{A}_{i+1}$. Thus the sets $X_1,\ldots,X_k$ are pairwise disjoint. Let X_0 be the set of vertices that are not the root of any member of $\mathcal{A}_1$; so the sets $X_0,\ldots,X_k$ form a partition of $V(G)$. For each edge e of G with an end in one of $X_1,\ldots,X_k$, choose i maximum such that e has an end in X_i, let v be an end of e in X_i, and call v the *head* of e. For each $v\in X_i$, choose $\phi_v\in\mathcal{A}_i$ with root v. (Thus $\phi_v\notin\mathcal{A}_{i+1}$ from the definition of X_i.)

- Let A be the set of all edges of G with both ends in X_0;
- Let B be the set of all edges uv with head v such that $u\notin V(\phi_v)$ and u is not bad for ϕ_v;
- Let C be the set of all edges uv with head v such that $u\notin V(\phi_v)$ and u is bad for ϕ_v;
- Let D be the set of all edges uv with head v such that $u\in V(\phi_v)$.

Since there is no (t,η)-infusion of (H,r) into $G[X_0]$, it follows that $G[X_0]$ does not contain a (ζ,η)-uniform tree as a path-induced rooted subgraph, where $\zeta=t^\eta$, and so $\partial(G[X_0])\le(\zeta t)^{(\eta+1)!}$ from 2.4. Hence

$$|A|\le(\zeta t)^{(\eta+1)!}|G|.$$

For each $u\in V(G)$, with $u\in X_i$ say, there do not exist t^η neighbours v of u such that uv has head v and belongs to B, since there is no (t,η)-infusion of (H,r) with root u that is derived from members of $\mathcal{A}_i$. Hence

$$|B|\le t^\eta|G|.$$

For each $v\in V(G)$, there are at most t^{η^η} neighbours u of v such that the edge uv has head v and belongs to C, by 5.2; so

$$|C|\le t^{\eta^\eta}|G|.$$

Finally, for each $v\in V(G)$, there are at most t^η neighbours u of v such that the edge uv has head v and belongs to D; so

$$|D|\le t^\eta|G|.$$

Summing, we obtain

$$|E(G)|\le\left(\left(t^{\eta+1}\right)^{(\eta+1)!}+t^\eta+t^{\eta^\eta}+t^\eta\right)|G|\le\left(t^{(\eta+2)!}+t^{\eta^\eta}\right)|G|\le t^{7\eta^\eta}/2.$$

Consequently $\partial(G)\le t^{7\eta^\eta}$. This proves 5.3. ∎

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