A shorter proof of the path-width theorem

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Abstract

A graph has *path-width* at most w if it can be built from a sequence of graphs each with at most $w+1$ vertices, by overlapping consecutive terms. Every graph with path-width at least $w-1$ contains every w-vertex forest as a minor: this was originally proved by Bienstock, Robertson, Thomas and the author, and was given a short proof by Diestel. Here we give a proof even shorter and simpler than that of Diestel.

1 The proof

All graphs in this paper are finite, and may have loops or parallel edges. If G is a graph, $|G|$ denotes its number of vertices, and for $A \subseteq V(G)$, $G[A]$ denotes the subgraph induced on A. A *path-decomposition* of a graph G is a sequence $(W_1, \ldots, W_n)$ of subsets of $V(G)$ (called *bags*), with union $V(G)$, such that for every edge uv of G there exists i such that $u, v \in W_i$, and such that $W_i \cap W_k \subseteq W_j$ for $1 \leq i < j < k \leq n$; and it has *width* at most w if $|W_i| \leq w + 1$ for each i. A graph has *path-width* at most w if it admits a path-decomposition with width at most w. Robertson and the author [3] proved that for every forest F, all graphs that do not contain F as a minor have bounded path-width (and the conclusion is false for all graphs F that are not forests); and later Bienstock, Robertson, Thomas and author [1] proved:

1.1 *For every forest F, every graph that does not contain F as a minor has path-width at most $|F| - 2$.*

This is tight, since a complete graph on $|F| - 1$ vertices has path-width $|F| - 2$ and does not contain F as a minor. It was given a short proof by Diestel [2], but there is an even shorter proof, that we present here.

A *model* of a loopless graph H in a graph G is a map ϕ with domain $V(H) \cup E(H)$, such that

- $\phi(h)$ is a non-null connected subgraph of G for each $h \in V(H)$, and $\phi(h), \phi(h')$ are vertex-disjoint for all distinct $h, h' \in V(H)$;
- $\phi(f) \in E(G)$ for each $f \in E(H)$, and $\phi(f) \neq \phi(f')$ for all distinct $f, f' \in E(H)$;
- if $f \in E(H)$ is incident in H with $h \in V(H)$, then $\phi(f)$ is incident in G with a vertex of $\phi(h)$.

Thus there is a model of H in G if and only if G contains H as a minor.

A *separation* of G is a pair (A, B) of subsets of $V(G)$ with union $V(G)$, such that there are no edges between $A \setminus B$ and $B \setminus A$, and its *order* is $|A \cap B|$. If (A, B) and (A', B') are separations of G, we write $(A, B) \leq (A', B')$ if $A \subseteq A'$ and $B' \subseteq B$. For each integer $w \geq 0$, we say a separation (A, B) of a graph G is *w-good* if there is a path-decomposition of $G[A]$ with width at most w and with last bag $A \cap B$. We need the following observation, which is the heart of the proof:

1.2 *If (A', B') and (P, Q) are separations of G, where (A', B') is w-good and $(P, Q) \leq (A', B')$, and there are $|P \cap Q|$ vertex-disjoint paths of G between P and B', then (P, Q) is w-good.*

Proof. Let $t = |P \cap Q|$, and let $R_1, \ldots, R_t$ be disjoint paths between P and B'. We may assume that each has only one vertex in B', and hence in $A' \cap B'$. Each of these paths has only its first vertex in P, and so if we contract the edges of $R_1, \ldots, R_t$, we preserve the subgraph $G[P]$. Let H be the union of $G[P]$ and the paths $R_1, \ldots, R_t$. Since (A', B') is w-good, there is a path-decomposition of H of width at most w, such that its last bag consists of the t ends in B' of the paths $R_1, \ldots, R_t$. But contracting the edges of $R_1, \ldots, R_t$ brings this to a path-decomposition of $G[P]$ with last bag $P \cap Q$ (since each edge to be contracted has both ends inside a bag). This proves 1.2. ■

If (A, B) and (A', B') are separations of G, the second *extends* the first if $(A, B) \leq (A', B')$ and $|A \cap B| \geq |A' \cap B'|$. A w-good separation of G is *maximal* if no different w-good separation extends it. Let $w \geq 0$ be an integer, let T be a tree or the null graph, and let (A, B) be a separation of a graph G. We say that (A, B) is *(w, T)-spanning* if

- $|A \cap B| = |T|$;
- there is a model ϕ of T in $G[A]$ such that $V(\phi(h)) \cap A \cap B \neq \emptyset$ for each $h \in V(T)$; and
- if $|T| \leq w+1$ then (A, B) is maximal w-good.

In order to prove 1.1, we may assume that F is a tree T say (by adding edges to F if necessary), and so it suffices to prove:

1.3 *Let $w \geq 0$ be an integer, let G be a graph that has path-width more than w, and let T be a tree or the null graph, with $|T| \leq w+2$. Then there is a (w, T)-spanning separation of G.*

Proof. We proceed by induction on $|T|$, keeping w fixed. If $|T| = 0$, the result holds since there is a maximal w-good separation of order zero, say (A, B) (possibly with $A = \emptyset$), which is therefore (w, T)-spanning. So we assume that $1 \leq |T| \leq w+2$ and the result holds for $|T|-1$. Choose $j \in V(T)$ with degree at most one, and if $|T| \geq 2$ let i be the neighbour of j in T.

From the inductive hypothesis, there is a $(w, T \setminus \{j\})$-spanning separation (A, B) of G, which is therefore maximal w-good, since $|T \setminus \{j\}| < w+2$. Let ϕ be a model of $T \setminus \{j\}$ in $G[A]$ such that $V(\phi(h)) \cap A \cap B \neq \emptyset$ for each $h \in V(T) \setminus \{j\}$. We choose $v \in B \setminus A$ as follows. If $|T| = 1$, then $A \cap B = \emptyset$; choose $v \in B$ arbitrarily. (This is possible since $B \neq \emptyset$, because G has path-width more than w: this is the only place where we use that the path-width is large.) If $|T| \geq 2$, let $u \in V(\phi(i)) \cap B$. Then u has a neighbour $v \in B \setminus A$, since otherwise $(A, B \setminus \{u\})$ is w-good and extends (A, B), contradicting the maximality of (A, B). This defines v.

If $|T| = w+2$, then $(A \cup \{v\}, B)$ is (w, T)-spanning, so we may assume that $|T| < w+2$, and therefore $(A \cup \{v\}, B)$ is w-good. So there is a maximal w-good separation (A', B') of G that extends $(A \cup \{v\}, B)$. Since (A', B') does not extend (A, B) (because (A, B) is maximal w-good), its order is exactly $|T|$. Suppose that there is a separation (P, Q) of G of order less than $|T|$, with $(A \cup \{v\}, B) \leq (P, Q) \leq (A', B')$. Choose (P, Q) with minimum order; then it follows from Menger's theorem that there are $|P \cap Q|$ vertex-disjoint paths from P to B', and so from 1.2, (P, Q) is w-good. But (P, Q) extends (A, B), since $|P \cap Q| \leq |T| - 1 = |A \cap B|$, and $(P, Q) \neq (A, B)$ since $v \in P$, contradicting the maximality of (A, B). Thus there is no such (P, Q), and so by Menger's theorem, there are $|T|$ disjoint paths of G between $A \cup \{v\}$ and B'. By combining these with the model ϕ, we deduce that (A', B') is (w, T)-spanning. This proves 1.3. ∎

References

[1] D. Bienstock, N. Robertson, P. Seymour and R. Thomas, "Quickly excluding a forest", *J. Combinatorial Theory, Ser. B,* **52** (1991), 274–283.

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