Pure pairs. VI. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix

Alex Scott
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour¹
Princeton University, Princeton, NJ 08544

November 15, 2019; revised January 20, 2020

¹Supported by AFOSR grant A9550-19-1-0187 and NSF grant DMS-1800053.
Abstract

For integer $n > 0$, let $f(n)$ be the number of rows of the largest all-0 or all-1 square submatrix of $M$, minimized over all $n \times n$ 0/1-matrices $M$. Thus $f(n) = O(\log n)$. But let us fix a matrix $H$, and define $f_H(n)$ to be the same, minimized over over all $n \times n$ 0/1-matrices $M$ such that neither $M$ nor its complement (that is, change all 0’s to 1’s and vice versa) contains $H$ as a submatrix. It is known that $f_H(n) \geq cn^\varepsilon$, where $c, \varepsilon > 0$ are constants depending on $H$.

When can we take $\varepsilon = 1$? If so, then one of $H$ and its complement must be an acyclic matrix (that is, the corresponding bipartite graph is a forest). Körándi, Pach, and Tomon [4] conjectured the converse, that $f_H(n)$ is linear in $n$ for every forest matrix $H$; and they proved it for certain matrices $H$ with only two rows.

Their conjecture remains open, but we show $f_H(n) = n^{1-o(1)}$ for every forest matrix $H$. 
1 Introduction

A 0/1-matrix can be regarded as a bipartite graph, with a distinguished bipartition \( (V_1, V_2) \) say, in which there are linear orders imposed on \( V_1 \) and on \( V_2 \). Submatrix containment corresponds, in graph theory terms, to induced subgraph containment, respecting the two bipartitions and preserving the linear orders. In two earlier papers [1, 5], with Maria Chudnovsky and Sophie Spirkl, we proved some results about excluding induced subgraphs, in a general graph and in a bipartite graph respectively. Now we impose orders on the vertex sets, and only consider induced subgraph containment that respects the orders; and we ask how far our earlier theorems remain true under this much weaker hypothesis.

In this paper, all graphs are finite and with no loops or parallel edges. Two disjoint sets are complete to each other if every vertex of the first is adjacent to every vertex of the second, and anticomplete if there are no edges between them. A pair \( (Z_1, Z_2) \) of subsets of \( V(G) \) is pure if \( Z_1 \) is either complete or anticomplete to \( Z_2 \). Let us state the earlier theorems that we want to extend to ordered graphs. First, we proved the following, with Chudnovsky and Spirkl [1]:

1.1 For every forest \( T \), there exists \( \varepsilon > 0 \) such that if \( G \) is a graph with \( n \geq 2 \) vertices, and no induced subgraph is isomorphic to \( T \) or its complement, then there is a pure pair \( (Z_1, Z_2) \) of subsets of \( V(G) \) with \( |Z_1|, |Z_2| \geq \varepsilon n \).

This theorem characterizes forests: if \( T \) is a graph that is not a forest or the complement of one, then there is no \( \varepsilon > 0 \) as in 1.1.

Second, we proved a similar theorem about bipartite graphs, but we need some more definitions. A bigraph is a graph together with a bipartition \( (V_1(G), V_2(G)) \) of \( G \). A bigraph \( G \) contains a bigraph \( H \) if there is an isomorphism from \( H \) to an induced subgraph of \( G \) that maps \( V_i(H) \) into \( V_i(G) \) for \( i = 1, 2 \). The bicomplement of a bigraph \( H \) is the bigraph obtained by reversing the adjacency of \( v_1, v_2 \) for all \( v_i \in V_i(G) \) (\( i = 1, 2 \)). With Spirkl [5] we proved the following:

1.2 For every forest bigraph \( T \), there exists \( \varepsilon > 0 \) such that if \( G \) is a bigraph with \( |V_i(G)| \geq n \) for \( i = 1, 2 \), and \( G \) does not contain \( T \) or its bicomplement, then there is a pure pair \( (Z_1, Z_2) \) with \( Z_i \subseteq V_i(G) \) and \( |Z_i| \geq \varepsilon n \) for \( i = 1, 2 \).

Again, this characterizes forests, in that if \( H \) is a bigraph that is not a forest or the bicomplement of a forest, there is no \( \varepsilon > 0 \) as in 1.2.

What if we impose an order on the vertex set, and ask for the induced subgraph containment to respect the order? Let us say an ordered graph is a graph with a linear order on its vertex set. Every induced subgraph inherits an order on its vertex set in the natural way: let us say an ordered graph \( G \) contains an ordered graph \( H \) if \( H \) is isomorphic to an induced subgraph \( H' \) of \( G \), where the isomorphism carries the order on \( V(H) \) to the inherited order on \( V(H') \). One could ask for an analogue of 1.1 for ordered graphs, but it is false. Fox [3] showed:

1.3 Let \( H \) be the three-vertex path with vertices \( h_1, h_2, h_3 \) in order, and make \( H \) an ordered graph using the same order. For all sufficiently large \( n \), there is an ordered graph \( G \) with \( n \) vertices, that does not contain \( H \), and such that there do not exist two disjoint subsets of \( V(G) \), both of size at least \( n/\log(n) \), and complete or anticomplete.
To deduce that 1.1 does not extend to ordered graphs, let $T$ be an ordered tree such that both $T$ and its bicomplement contain $H$.

On the other hand, there are some positive results about ordered bipartite graphs, proved recently by Korándi, Pach, and Tomon [4]. Let us say an ordered bigraph is a bigraph with linear orders on $V_1(G)$ and on $V_2(G)$. This is just a 0/1 matrix in disguise, but graph theory language is convenient for us. (Note that we are not giving a linear order of $V(G)$: that is much too strong and trivially does not work.) An ordered bigraph $G$ contains an ordered bigraph $H$ if there is an induced subgraph $H'$ of $G$ and an isomorphism from $H$ to $H'$ mapping $V_i(H)$ to $V_i(G)$ and mapping the order on $V_i(H)$ to the inherited order on $V_i(H')$, for $i = 1, 2$. (In matrix language, this is just submatrix containment.) Korándi, Pach, and Tomon [4] showed:

1.4 Let $H$ be an ordered bigraph with $|V_1(H)| \leq 2$, such that either

- $|V_2(H)| \leq 2$ and both $H$ and its bicomplement are forests, or
- every vertex in $V_2(H)$ has degree exactly one.

Then there exists $\varepsilon > 0$ with the following property. Let $G$ be an ordered bigraph that does not contain $H$, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair $(Z_1, Z_2)$ with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.

In both cases of 1.4, the bigraph $H$ is a forest and so is its bicomplement. Korándi, Pach, and Tomon asked which other ordered bigraphs $H$ satisfy the conclusion of 1.4. They observed that every such bigraph must be a forest and the bicomplement of a forest, and conjectured that this was sufficient as well as necessary, that is:

1.5 Conjecture: Let $H$ be an ordered bigraph such that both $H$ and its bicomplement are forests. Then there exists $\varepsilon > 0$ with the following property. Let $G$ be an ordered bigraph that does not contain $H$, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair $(Z_1, Z_2)$ with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.

We have not been able to decide this conjecture, but we will prove in 1.9 that, for a much more general class of bigraphs, it is possible to find pairs of almost linear size. Korándi, Pach, and Tomon also proposed a stronger conjecture:

1.6 Conjecture: For every ordered forest bigraph $H$, there exists $\varepsilon > 0$ with the following property. Let $G$ be an ordered bigraph that does not contain $H$ or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair $(Z_1, Z_2)$ with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.

This seems a natural extension of 1.5, in analogy with 1.1 and 1.2; but, for what it is worth, our guess, in view of 1.3, is that 1.6 is false. (Perhaps $n/\text{polylog}(n)$ might be true?) We have not even been able to prove it for the tree with five vertices and three leaves.

There is another result of Korándi, Pach, and Tomon, in the same paper [4]:

1.7 Let $H$ be an ordered forest bigraph such that $|V_1(H)| = 2$ and $|V(H_2)| = k$. For every $\varepsilon > 0$, there exists $\delta > 0$ with the following property. Let $G$ be an ordered bigraph that does not contain $H$, with $|V_1(G)| = |V_2(G)| = n$, and such that its bicomplement has at least $\varepsilon n^2$ edges. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$ with $|Z_1| \geq \delta n 2^{-(1+o(1))(\log \log (\delta n))^k}$ and $|Z_2| \geq \delta n$, such that $Z_1, Z_2$ are anticomplete.
So, $|Z_1|$ is not quite linear, but there is more of significance. There is nothing here about forbidding $G$ to contain the bicomplement of a forest, since the bicomplement of $H$ need not be a forest; and the “$Z_1$ complete to $Z_2$” outcome is gone. In compensation they have the assumption that the bicomplement of $G$ is not too sparse.

Our objective in this paper is essentially to generalize 1.7 to all ordered forest bigraphs. We had to modify it a little. First, the assumption that the bicomplement of $G$ has at least $\epsilon n^2$ edges is not strong enough for our proof method, when $H$ is a general forest; we have to assume that $G$ itself is fairly sparse. Second, we will obtain anticomplete sets $Z_1, Z_2$, but they will both be not quite linear; and their sizes are a little worse than the size bound in 1.7. Third, every forest is an induced subgraph of a tree, so we will assume $H$ is a tree, for convenience. The radius of a tree $T$ is the minimum $r$ such that for some vertex $v$, every vertex of $T$ can be joined to $v$ by a path with at most $r$ edges. Our main result is the following:

1.8 Let $T$ be an ordered tree bigraph, of radius $r$, and with $t \geq 2$ vertices. Let $G$ be a bigraph not containing $T$, with $|V_1(G)|, |V_2(G)| \geq n$, such that every vertex of $G$ has degree at most $n/(4t^2)$. Choose $K$ such that $t^{K^r} = n$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq nt^{-5K^{r-1}}$, such that $Z_1, Z_2$ are anticomplete.

As a consequence we will prove:

1.9 Let $T$ be an ordered tree bigraph, of radius $r$, and with $t \geq 2$ vertices. Let $G$ be an ordered bigraph not containing $T$ or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. Choose $K$ such that $t^{K^r} = n$. Then there is a pure pair $(Z_1, Z_2)$ with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq (16t^2)^{-t}nt^{-5K^{r-1}}$ for $i = 1, 2$.

Both of these theorems imply that $|Z_1|, |Z_2| = n^{1-o(1)}$, and in fact

$$|Z_1|, |Z_2| \geq n^{1-c(\log n)^{1-1/r}}.$$  

It is easy to show, with a random graph argument, that 1.9 characterizes forests and their bicomplements, and indeed, if $T$ is an ordered bigraph such that neither $T$ nor its bicomplement are forests, then the conclusion of 1.9 is far from true. Let both $T$ and its bicomplement have a cycle of length at most $g$. For large $n$, there are ordered bigraphs $G$, with $|V_1(G)|, |V_2(G)| = n$, and with girth more than $g$, which therefore contain neither $T$ nor its bicomplement, and such that $\min(|Z_1|, |Z_2|) \leq O(n^{1-1/(2g)})$ for every pure pair $(Z_1, Z_2)$ with $Z_i \subseteq V_i(G)$ for $i = 1, 2$. We omit the argument, which is standard.

2 Reduction to the sparse case

In this section we deduce 1.9 assuming 1.8, and will prove the latter in the next section. We need the following lemma, a version of a theorem of Erdős, Hajnal and Pach [2] adapted for ordered bipartite graphs.

2.1 Let $H$ be an ordered bigraph, let $|V(H_i)| = h_i$ for $i = 1, 2$, let $0 < \epsilon < 1/8$, let $d = \lfloor 1/(4\epsilon) \rfloor$, and let $m > 0$ be an integer. Let $G$ be an ordered bigraph not containing $H$, with $|V_1(G)| \geq h_1 d^{b_2} m$ and $|V_2(G)| \geq 2h_1 b_2 m$. Then there are subsets $Y_i \subseteq V_i(G)$ with $|Y_i| = m$ for $i = 1, 2$, such that either

- every vertex in $Y_1$ has at most $\epsilon |V_2|$ neighbours in $Y_2$, and every vertex in $Y_2$ has at most $\epsilon |V_1|$ neighbours in $Y_1$, or
every vertex in $Y_1$ has at most $\varepsilon|Y_2|$ non-neighbours in $Y_2$, and every vertex in $Y_2$ has at most $\varepsilon|Y_1|$ non-neighbours in $Y_1$.

**Proof.** Divide $V_i(G)$ into $h_1$ disjoint intervals, each of cardinality at least $md^{h_2}$, numbered $B_u (u \in V_i(H))$ in order. Divide $V_2(G)$ into disjoint intervals $B_u (u \in V_2(H))$ of cardinality at least $mh_1$. Choose $W \subseteq V_2(H)$ maximal such that for each $v \in W$ there exists $x_v \in B_v$, and for each $u \in V_1(H)$ there exist $Q_u \subseteq B_u$, with the following properties:

- $|Q_u| \geq md^{h_2}|W|$ for each $u \in V_1(H)$;
- for each $u \in V_1(H)$ and each $v \in W$, if $u, v$ are $H$-adjacent then $x_v$ is complete to $Q_u$, and if $u, v$ are not $H$-adjacent then $x_v$ is anticomplete to $Q_u$.

This is possible since we may take $W = \emptyset$ and $Q_u = B_u$ for each $u \in V_1(H)$. Since $G$ does not contain $H$, it follows that $W \neq V_2(H)$. Say $u \in V_1(H)$ is a problem for $x \in B_v$ if either $u, v$ are $H$-adjacent and $x$ has fewer than $|Q_u|/d$ non-neighbours in $Q_u$, or $u, v$ are not $H$-adjacent and $x$ has fewer than $|Q_u|/d$ non-neighbours in $Q_u$. From the maximality of $W$, for each $x \in B_v$ there exists $u \in V_1(H)$ that is a problem for $x$. Since there are only $h_1$ possible problems, there exist $u \in V_1(H)$, and $C \subseteq B_v$ with $|C| \geq |B_v|/h_1$, such that for every $x \in C$, $u$ is a problem for $x$. By moving to the bicomplement if necessary, we may assume that $u, v$ are $H$-adjacent; and so every vertex in $C$ has fewer than $|Q_u|/d$ neighbours in $Q_u$. Since $|Q_u| \geq md^{h_2}|W| \geq md \geq 2m$ and $|C| \geq |B_v|/h_1 \geq 2m$, it follows by averaging that there are subsets $X_1 \subseteq Q_u$ and $X_2 \subseteq C$, both of cardinality exactly $2m$, such that there are at most $|X_1| \cdot |X_2|/d = 4m^2/d$ edges joining them.

Let $Y_1$ be the set of the $m$ vertices in $X_1$ that have fewest neighbours in $X_2$; then each have at most $4m/d$ neighbours in $X_2$. Define $Y_2$ similarly; then $|Y_1| = |Y_2| = m$, and every vertex in $Y_1$ has at most $4m/d \leq \varepsilon m$ neighbours in $Y_2$ and vice versa. This proves 2.1.

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**Proof of 1.9, assuming 1.8.** Let $T$ be an ordered tree bigraph, of radius $r$, and with $t$ vertices. Let $\varepsilon = 1/(4t^2)$, and let $d = \lceil 1/(4\varepsilon) \rceil$. Let $c = (16t^2)^{-1}$. Let $G$ be a bigraph not containing $T$ or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. We may assume that $\text{cnt}^{-5Kr^{-1}} > 1$, for otherwise the result is true, taking $|Z_1| = |Z_2| = 1$. Hence $2cn \geq 2^{5Kr^{-1}} \geq 1$. Let $m$ be the largest integer such that $m \leq 2cn$. Thus $m \geq cn$, since $2cn \geq 1$.

Let $|V_i(T)| = h_i$ for $i = 1, 2$. Now $n \geq \max(h_1d^{h_2}m, 2h_1h_2m)$. By 2.1, and moving to the bicomplement if necessary, we may assume that there exist $Y_i \subseteq V_i(G)$ with $|Y_i| = m$ for $i = 1, 2$, such that every vertex in $Y_1$ has at most $\varepsilon|Y_2|$ neighbours in $Y_2$, and every vertex in $Y_2$ has at most $\varepsilon|Y_1|$ neighbours in $Y_1$. Choose $J$ with $m = t^J$. By 1.8 applied to the ordered bigraph induced on $Y_1 \cup Y_2$, there exist $Z_i \subseteq Y_i$ with $|Z_i|, |Z_2| \geq m^{-5Kr^{-1}}$, such that $Z_1, Z_2$ are anticomplete, where $m = t^J$. Since $J \leq K$, it follows that $|Z_1|, |Z_2| \geq \text{cnt}^{-5Kr^{-1}}$. This proves 1.9.

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**3 Proof of the main theorem**

In this section we prove 1.8, which we restate:

**3.1** Let $T$ be a ordered tree bigraph, of radius $r$, and with $t \geq 2$ vertices. Let $G$ be a bigraph with $|V_1(G)|, |V_2(G)| \geq n$, that does not contain $T$, and such that every vertex has degree at most $n/(4t^2)$. Choose $K$ such that $t^{Kr} = n$. Then there are two anticomplete subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq nt^{-5Kr^{-1}}$.
Proof. Let \( \varepsilon = 1/(4t^2) \). Since \( \varepsilon < 1 \), \( G \) is not complete bipartite, and so we may assume that \( t^{-5K^{r-1}-n} > 1 \) (or else the theorem holds); that is, \( t^{-5K^{r-1}} < t^{K^r} \). So \( K \geq 5 \), and \( n \geq t^{5r} \).

If \( r = 1 \), then \( T \) has a vertex of degree \( d \) say, and all other vertices of \( T \) are neighbours of \( d \). Let this vertex belongs to \( V_1(T) \) say. Since \( G \) does not contain \( T \), all vertices in \( V_1(G) \) have degree at most \( d - 1 \). Choose a set \( Z_1 \) of at most \( n/d \) vertices in \( V_1(G) \); then the set of vertices with neighbours in \( X \) has cardinality at most \( (d-1)|X| \leq (d-1)n/d \), and so there is a set of at least \( n/d \) vertices in \( V_2(G) \) anticomplete to \( Z_1 \). So to prove 1.8 in this case, we just have to check that \( n/d \geq n^{-5K^{r-1}} \). But \( t = d + 1 \) and \( n \geq t \) (because \( n \geq t^{5r} \)), so \( n/d \geq n/(2(t-1)) \); and now it remains to check that \( n/(2(t-1)) \geq n^{-5K^{r-1}} \), which is clear. Thus we may assume that \( r \geq 2 \), and so \( t \geq 4 \).

Choose a real number \( x \geq 0 \) with \( x \leq K^{r-1} \), maximum such that there exist \( A_1 \subseteq V_1(G) \) and \( A_2 \subseteq V_2(G) \) with the properties that

- \( |A_1|, |A_2| \geq nt^{-x} \); 
- every vertex in \( A_1 \) has at most \( \varepsilon nt^{-Kx} \) neighbours in \( A_2 \) and vice versa.

This is possible since we may take \( x = 0 \) and \( A_i = V_i(G) \) for \( i = 1, 2 \). Let \( A_1, A_2 \) be as above. Let \( d = \varepsilon nt^{-Kx} \). Since \( |A_1| \geq t^{-x} \geq t^{-5K^{r-1}} \), we may assume that \( A_1 \) is not anticomplete to \( A_2 \) (or else the theorem holds); so \( d \geq 1 \). For \( 1 \leq s \leq r - 1 \), let \( k_s = 4(K^{r-1} + K^{r-2} + \cdots + 1) \).

\[ (1) \ x \leq K^{r-1} - k_{r-1}. \]

Suppose not. Since \( |A_1| \geq t^{-x}n \geq t^{-K^{r-1}}n \geq 2t^{-5K^{r-1}}n \), there exists a set \( X \) of \( \lfloor t^{-5K^{r-1}}n \rfloor \leq 2t^{-5K^{r-1}}n \) vertices in \( A_1 \). The union of the neighbours in \( A_2 \) of vertices in \( X \) has cardinality at most \( (2t^{-5K^{r-1}}n)/\varepsilon nt^{-Kx} \). Since \( |A_2|/2 \geq t^{-5K^{r-1}}n \), there are fewer than \( |A_2|/2 \) vertices in \( A_2 \) anticomplete to \( X \); so \( 2t^{-5K^{r-1}}n \varepsilon nt^{-Kx} \geq |A_2|/2 \), and hence \( 4\varepsilon t^{-5K^{r-1}}nt^{-Kx} \geq t^{-x} \). Consequently \( 4\varepsilon t^{-5K^{r-1}}n \geq t(K-1)x \). But

\[ (K-1)x \geq (K-1)(K^{r-1} - k_{r-1}) = K^r - 5K^{r-1} + 4, \]

and so

\[ 4\varepsilon t^{-5K^{r-1}}n \geq t(K-1)x. \]

Hence \( n \geq t^{K^{r+6}} \), a contradiction. This proves (1).

Since \( x < K^{r-1} - 4 \), and \( n \geq t^{5K^{r-1}} \), it follows that \( nt^{-x-1} \geq t \). Hence \( |A| \geq nt^{-x} \geq t - 1 \)nt^{-x-1} + t. Since \( |V(T_1)| \leq t - 1 \), it follows that we may choose \( V_1(T) \) disjoint boxes \( B_u \) \( u \in V_1(T) \), all intervals of \( A_1 \), and of cardinality \( \lfloor nt^{-x-1} \rfloor \), numbered in order. Partition \( A_2 \) into \( B_v \) \( v \in V_2(T) \) similarly. For \( 1 \leq s \leq r - 1 \), let \( p_s = dt^{-Kk_s} \). Let \( p_r = 1 \). For \( 2 \leq s \leq r \), let \( f_s = dt^2/p_{s-1} = t^{Kk_{s-1}+2} \). So \( f_s \geq t^{4K+2} \).

Let \( v_0 \) be the root of \( T \), and for \( 1 \leq s \leq r \) let \( T_s \) be the subtree of \( T \) induced on the vertices with distance at most \( s \) from the root. So \( V(T_0) = \{v_0\} \), and \( T_r = T \). Let \( L_s \) be the set of vertices with distance exactly \( s \) from \( v_0 \). For \( 2 \leq s \leq r \), and each edge \( uv \) of \( T \) with \( u \in L_{s-1} \) and \( v \in L_s \), choose \( X_{uv} \subseteq B_u \) and \( Y_{uv} \subseteq B_v \) satisfying the following conditions:

- every vertex in \( X_{uv} \) has fewer than \( p_s \) neighbours in \( B_v \setminus Y_{uv} \);
- \( |Y_{uv}| \leq f_s|X_{uv}| \) and \( |X_{uv}| \leq |B_v|/2 \); and
• subject to these conditions, $Y_{uv}$ is maximal.

(2) For $2 \leq s \leq r$, and each edge $uv$ of $T$ with $u \in L_{s-1}$ and $v \in L_s$, we may assume that $|X_{uv}| = \lceil |Y_{uv}|/f_s \rceil$, and $|X_{uv}| \leq |B_v|/(2t)$, and $|Y_{uv}| \leq |B_v|/2 - td$.

We may assume that $|X_{uv}| = \lceil |Y_{uv}|/f_s \rceil$, by removing elements from $X_{uv}$ if necessary. Suppose first that $s = r$. Then $X_{uv}$ is anticomplete to $B_v \setminus Y_{uv}$ (because $p_r = 1$), and so either $|X_{uv}| < t^{-5K^{r-1}}n$ or $|B_v \setminus Y_{uv}| < t^{-5K^{r-1}}n$. The second implies that $|B_v| < 2t^{-5K^{r-1}}n$ (since $|Y_{uv}| \leq |B_v|/2$), and so $nt^{-x-1} < 2t^{-5K^{r-1}}$, that is, $4t^{-5K^{r-1}-x-1} < 2$, a contradiction. So $|X_{uv}| < t^{-5K^{r-1}}n$.

Since $t^{-5K^{r-1}}n \leq nt^{-x-1}/(2t)$, it follows that $|X_{uv}| \leq |B_u|/(2t)$. Also, $|Y_{uv}| \leq f_r t^{-5K^{r-1}}n$. We claim that $f_r t^{-5K^{r-1}}n \leq |B_v|/2 - td$. Suppose not; then either $f_r t^{-5K^{r-1}}n > |B_v|/4$ or $td > |B_v|/4$. The first implies that $t^{KKr_{r-1}+2t^{-5K^{r-1}}}n > nt^{-x-1}/4$, and so $t^{KKr_{r-1}+x+4t^{-5K^{r-1}}} > 1$. Hence $Kkr_{r-1} + x + 4 - 5K^{r-1} > 0$. But $x \leq K^{r-1} - kr_{r-1}$ by (1), so

$$Kkr_{r-1} + K^{r-1} - kr_{r-1} + 4 - 5K^{r-1} > 0,$$

a contradiction (in fact, the left side sums to zero). The second implies that $4et^{-Kx} > t^{-x-2}$ and so $K < 1$ since $4e = t^{-2}$, a contradiction. Thus when $s = r$, all three statements of (2) hold.

Now we assume that $2 \leq s < r$. We have $|X_{uv}| \leq |Y_{uv}|/f_s + 1 \leq |B_v|/(2f_s) + 1 \leq |B_v|/4$, because $f_s \geq t^{4K+2} \geq 4$ and $|B_v| \geq 8$. Consequently the $|X_{uv}|$ vertices in $B_v \setminus Y_{uv}$ with fewest neighbours in $X_{uv}$ each have at most $p_s$ neighbours in $X_{uv}$. Since $p_s = dt^{-Ks}$, and $k_s \leq 4K^{r-2} + 4K^{r-3} + \cdots + 4$, the maximality of $x$ and (1) imply that $|X_{uv}| < nt^{-x-k_s}$. Hence $|X_{uv}| \leq |B_u|/(2t)$, because $nt^{-x-k_s} \leq nt^{-x-1}/(2t)$ (because $k_s \geq 3$). Thus the second claim holds. For the third claim, since $|Y_{uv}| \leq f_s |X_{uv}|$, it suffices to show that $f_s |X_{uv}| \leq |B_v|/2 - td$, and to prove this, it suffices to show that $f_s |X_{uv}| \leq |B_v|/4$ and $td \leq |B_v|/4$. To show the first, it suffices to show that $nt^{-x-k_s} f_s \leq |B_v|/t$, that is, $nt^{-x-k_s} t^{KKr_{r-1}+2} \leq nt^{-x-2}$, which simplifies to $4 + Kkr_{r-1} \leq k_s$, and this holds with equality. To show that $td \leq |B_v|/4$, it suffices to show that $te t^{2x-Kx} \leq nt^{-x-1}/4$, which simplifies to $4et^{2x-Kx} \leq 1$; and this is true since $4et^{2} \leq 1$, and $K \geq 1$. This proves (2).

For $2 \leq s \leq r$, and each $u \in L_{s-1}$, let $X_u$ be the union of the sets $X_{uv}$ over all $v \in L_s$ that are $T$-adjacent to $u$. Then:

(3) For $2 \leq s \leq r$, and each $u \in L_{s-1}$, $|X_u| \leq |B_u|/2$.

For each $v \in L_s$ that is $T$-adjacent to $u$, $|X_{uv}| \leq |B_u|/(2t)$ by (2), and the claim follows. This proves (3).

Let $P_v = B_v$. For $s = 1, \ldots, r-1$ we will choose $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, and $x_v \in P_v$ for each $v \in L_{s-1}$, satisfying the following conditions:

• for all distinct $u, v \in V(T_{s-1})$, $u, v$ are $T$-adjacent if and only if $x_u, x_v$ are $G$-adjacent;

• for all $u \in V(T_{s-1})$ and $v \in L_s$, and all $x \in P_v$, $u, v$ are $T$-adjacent if and only if $x_u, x$ are $G$-adjacent;

• for each $v \in L_s$, $|P_v| \geq p_s$. 

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First let us assume $s = 1 < r$. If there exists $x \in B_{v_0}$ with at least $p_1$ neighbours in $B_v \setminus X_v$ for each $v$ such that $v_0, v$ are $T$-adjacent, then we may set $x_{v_0} = x$; so we assume there is no such $x$. Consequently there is a $T$-neighbour $v$ of $v_0$ such that for at least $|B_{v_0}|/t \geq nt^{-x-2}$ vertices $x \in B_{v_0}$, $x$ has fewer than $p_1$ neighbours in $B_v \setminus X_v$. Choose a set $X$ of exactly $[nt^{-x-2}]$ such vertices $x$. Since $|X| \leq nt^{-x-2} + 1 \leq 2nt^{-x-2} - 2$, and $|B_{v_0}| \geq 8nt^{-x-2}$ (because $t \geq 4$), it follows that $|X| < |B_{v_0}|/4$. Since $|X_v| \leq |B_v|/2$, it follows that $|B_v \setminus X_v| \geq 2|X|$, and so at least $|X|$ vertices in $B_v \setminus X_v$ have at most $p_1$ neighbours in $X$. Since $p_1 \leq dt^{-K_ki} = dt^{-4K}$, the maximality of $x$ implies that $|X| < nt^{-x-4}$, a contradiction. So we can satisfy the three bullets above when $s = 1 \leq r - 1$.

Suppose that $2 \leq s \leq r$, and we have chosen $P_v \subseteq B_v \setminus X_v$ for each $v \in L_{s-1}$ and $x_v \in P_v$ for each $v \in V(T_{s-2})$. We must define $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, and $x_v \in P_v$ for each $v \in L_{s-1}$, satisfying the bullets above. From the symmetry we may assume that $L_s \subseteq V_1(T)$.

Let $C$ be the set of vertices in $A_2$ that are equal or adjacent to $x_v$ for some $v \in V(T_{s-2})$. Let $x_u \in P_u$ for each $u \in L_{s-1}$; we call $(x_u : u \in L_{s-1})$ a transversal. A transversal $(x_u : u \in L_{s-1})$ is valid if for each edge $uv$ of $T$ with $u \in L_{s-1}$ and $v \in L_s$, there are at least $p_s$ vertices in $B_v \setminus C$ that are adjacent to $x_u$ and that have no other neighbour in $\{x_{u'} : u' \in L_{s-1}\}$.

(4) There is a valid transversal.

Suppose not. Let $E$ be the set of ordered pairs $(u, v)$ such that $uv$ is an edge of $T$ with $u \in L_{s-1}$ and $v \in L_s$. Then for every transversal $(x_u : u \in L_{s-1})$, there exists $(u, v) \in E$ such that there are fewer than $p_s$ vertices in $B_v$ that are adjacent to $x_u$ and that have no other neighbour in $\{x_{u'} : u' \in L_{s-1}\}$. Call $(u, v)$ a problem for the transversal $(x_u : u \in L_{s-1})$. Since $|E| = |L_s|$, there are only $|L_s|$ possible problems, and so there exists $(u, v) \in E$ that is a problem for at least a fraction $1/|L_s|$ of all transversals. Hence there exist a subset $X \subseteq P_u$ with $|X| \geq |P_u|/|L_s| \geq p_{s-1}/t$, and a choice of $x_{u'} \in P_{u'}$ for each $u' \in L_{s-1} \setminus \{u\}$, such that for all $x_u \in X$, $(u, v)$ is a problem for the transversal $(x_{u'} : u' \in L_{s-1})$. Let $C'$ be the set of vertices in $A_2$ that are adjacent to a vertex in $(x_{u'} : u' \in L_{s-1} \setminus \{u\})$. Since every vertex in $C \cup C'$ has a neighbour $x_w$ for some $w \in V(T_{s-1}) \setminus \{u\}$, it follows that $|C \cup C'| \leq dt$. Every vertex in $X$ has fewer than $p_s$ neighbours in $B_v \setminus (C \cup C')$. If $(C \cup C') \cap B_v \subseteq Y_{uv}$ let $Y = (C \cup C') \cap B_v$, and otherwise let $Y$ be a singleton subset of $B_v \setminus Y_{uv}$. Thus $|Y| \leq dt \leq f_s|X|$, since $dt \geq 1$ by (1). Consequently $|Y \cup Y_{uv}| \leq f_s|X \cup X_{uv}|$, since $X \cap X_{uv} = \emptyset$; and since $C \cup C' \cup Y_{uv} \subseteq B_v/2$ (because $|C \cup C'| \leq dt$ and by (2)), and every vertex in $X \cup X_{uv}$ has fewer than $p_s$ neighbours in $B_v \setminus (Y \cup Y_{uv})$, this contradicts the maximality of $Y_{uv}$. This proves (4).

From (4), the inductive definition of $x_v$ $(v \in V(T_{r-1}))$ and $P_v$ $(v \in V(T))$ is complete. For each $v \in L_s$, choose $x_v \in P_v$. Then the map sending each $v \in V(T)$ to $x_v$ is an ordered parity-preserving isomorphism of $T$ to an induced subgraph of $G$, a contradiction. This proves the theorem. 

Acknowledgement

Thanks to Sophie Spirkl, who worked with Seymour on trying to settle 1.6.
References


