Induced subgraphs of graphs with large chromatic number. I. Odd holes

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Abstract

An *odd hole* in a graph is an induced subgraph which is a cycle of odd length at least five. In 1985, A. Gyárfás made the conjecture that for all t there exists n such that every graph with no K_t subgraph and no odd hole is n-colourable. We prove this conjecture.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. A *hole* in a graph is an induced subgraph which is a cycle of length at least four, and a hole is *odd* if its length is odd. An *odd antihole* is an induced subgraph that is the complement of an odd hole. We denote the chromatic number of a graph G by $\chi(G)$. A *clique* of G is a subset of $V(G)$ such that all its members are pairwise adjacent, and the cardinality of the largest clique of G is denoted by $\omega(G)$. A clique of cardinality W is called a *W-clique*.

Clearly $\chi(G) \geq \omega(G)$, but $\chi(G)$ cannot in general be bounded above by a function of $\omega(G)$; indeed, there are graphs with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large [8, 9]. On the other hand, the "strong perfect graph theorem" [5] says:

1.1 *Let G be a graph that does not contain an odd hole and does not contain an odd antihole. Then $\chi(G) = \omega(G)$.*

This is best possible in a sense, since odd holes and odd antiholes themselves have $\chi > \omega$. But what happens if we only exclude odd holes? A. Gyárfás [12] proposed the following conjecture in 1985, and it is the main result of this paper:

1.2 (Conjecture) *There is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G with no odd hole.*

Let G have no odd hole. It is trivial that $\chi(G) = \omega(G)$ if $\omega(G) \leq 2$, and it was proved in [4] that $\chi(G) \leq 4$ if $\omega(G) = 3$; but for larger values of $\omega(G)$, no bound on $\chi(G)$ in terms of $\omega(G)$ was known. Here we prove conjecture 1.2, and give an explicit function f.

1.3 *Let G be a graph with no odd hole. Then $\chi(G) \leq 2^{2^{\omega(G)+2}}$.*

We imagine that this bound is nowhere near best possible. There is an example that satisfies $\chi(G) \geq \omega(G)^\alpha$, where $\alpha = \log(7/2)/\log(3) \approx 1.14$, but this is the best we have found. To see this, let G_0 have one vertex, and for $k \geq 1$ let G_k be obtained from G_{k-1} by substituting a seven-vertex antihole for each vertex. Then G_k has no odd hole, $\omega(G_k) = 3^k$, and $\chi(G_k) \geq (7/2)^k$ (because the largest stable set in G_k has cardinality 2^k).

2 Background

What can we say about the structure of a graph G with large chromatic number, if it does not contain a large clique? In particular, what can we say about its induced subgraphs? If $\chi(G) > \omega(G)$, then the strong perfect graph theorem tells us that some induced subgraph is an odd hole or the complement of one, and that is all. But what if $\chi(G)$ is much larger than $\omega(G)$? For instance, there are graphs G with $\omega(G) = 2$ and with $\chi(G)$ as large as we want; what induced subgraphs must such a graph contain?

Let $\mathcal{F}$ be a set of graphs. We say that $\mathcal{F}$ is *χ-bounding* if for every integer W, every graph G with $\omega(G) \leq W$ and $\chi(G)$ sufficiently large (depending on W) has an induced subgraph isomorphic to a member of $\mathcal{F}$. Which sets of graphs $\mathcal{F}$ have this property?

This is not an easy question; the answer is not known even when $|\mathcal{F}| = 1$. Let $\mathcal{F} = \{F\}$ say. Erdős [10] showed that for every integer k, there is a graph G with $\omega(G) = 2$, with $\chi(G) \geq k$ and with girth at least k (the *girth* is the length of the shortest cycle). So by taking $k > |V(F)|$, it follows that if $\{F\}$ is χ-bounding then F must be a forest, and a famous conjecture (proposed independently by Gyárfás [11] and Sumner [19]) asserts that this is sufficient, that $\{F\}$ is indeed χ-bounding for every forest F. But even this is not solved, although it has been proved for several special types of forests (see for instance the papers by Gyárfás, Szemerédi and Tuza [13], Kierstead and Penrice [14], Kierstead and Zhu [15], and the first author [16]).

The same argument shows that if $\mathcal{F}$ is finite then it must contain a forest, and if the Gyárfás-Sumner conjecture is true then that is all we can deduce; every set containing a forest is χ-bounding. But what can we say about χ-bounding sets $\mathcal{F}$ that do not contain a forest? $\mathcal{F}$ must be infinite, and the result of Erdős shows that $\mathcal{F}$ must contain graphs with arbitrarily large girth; so an obvious question is, what if all the graphs in $\mathcal{F}$ are cycles? Which sets of cycles are χ-bounding?

Gyárfás [12] conjectured that each of three conditions might be sufficient:

2.1 (Conjecture) *The set of all odd cycles of length at least five is χ-bounding.*

2.2 (Conjecture) *For each integer ℓ, the set of all cycles of length at least ℓ is χ-bounding.*

2.3 (Conjecture) *For each integer ℓ, the set of all odd cycles of length at least ℓ is χ-bounding.*

The third conjecture contains the first two; but in view of the strong perfect graph theorem, the first (which is a rephrasing of conjecture 1.2) seems the most natural and has attracted the most attention, and that is what we prove in this paper. The third conjecture remains open, although with Maria Chudnovsky, we have recently proved the second conjecture [7]. (See also [6] for related results and [17] for earlier work.)

On a similar theme, Bonamy, Charbit and Thomassé [1] recently proved the following:

2.4 *There exists n such that every graph with no induced odd cycle of length divisible by three (and in particular, with no triangles) has chromatic number at most n.*

What could we hope for as a more general condition? Let us say that a set $F \subseteq \{3, 4, \dots\}$ is *χ-bounding* if the set of all cycles with length in F is χ-bounding. It is not true that every infinite set F is χ-bounding; not even if we restrict to sets with upper density 1 in the positive integers. For instance, consider the following sequence of graphs: let G_1 be the null graph, and for each $i > 1$, let G_i be a graph with girth at least $2^{|V(G_{i-1})|}$ and chromatic number at least i. Letting F be the set of cycle lengths that do not occur in any G_i, we see that F has upper density 1, while the family $\{G_i : i \geq 1\}$ has unbounded chromatic number.

In the positive direction, perhaps it is true that every set F with strictly positive *lower* density is χ-bounding? As a step towards resolving this, we propose the following strengthening of the Gyárfás conjectures:

2.5 (Conjecture) *Let $F \subseteq \{3, 4, \dots\}$ be an infinite set of positive integers with bounded gaps (that is, there is some $K > 0$ such that every set of K consecutive positive integers contains an element of F). Then F is χ-bounding.*

We have proved this for triangle-free graphs, in a strengthened form. The following is proved in [18]:

2.6 *For every integer $\ell \geq 0$ there exists n such that in every graph G with $\omega(G) < 3$ and $\chi(G) > n$ there are ℓ holes with lengths ℓ consecutive integers.*

In the other direction, it would be interesting to know whether there is a χ-bounding set with lower density 0.

3 Using a cograph

The proof of 1.3 needs a lemma which we prove in this section, and before that we need the following.

3.1 *Let C be an $\omega(G)$-clique in a graph G with $C \neq \emptyset$, and let $A \subseteq V(G) \setminus C$, such that every vertex in C has a neighbour in A. Then there exist $a_1, a_2 \in A$, and $c_1, c_2 \in C$, all distinct, such that a_1 is adjacent to c_1 and not to c_2, and a_2 is adjacent to c_2 and not to c_1.*

Proof. Choose a vertex $a_1 \in A$ with as many neighbours in C as possible. Since $C \cup \{a_1\}$ is not a clique (because C is an $\omega(G)$-clique), there exists $c_2 \in C$ nonadjacent to a_1. Choose $a_2 \in A$ adjacent to c_2. From the choice of a_1, a_2 does not have more neighbours in C than a_1, and so there exists $c_1 \in C$ adjacent to a_1 and not to a_2. But then a_1, a_2, c_1, c_2 satisfy the theorem. ∎

If X, Y are disjoint subsets of $V(G)$, we say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y, and X is *anticomplete* to Y if every vertex in X is nonadjacent to every vertex in Y. A graph H is a *cograph* if no induced subgraph is isomorphic to a three-edge path. We need the following, which has been discovered independently many times (see [2]):

3.2 *If H is a cograph with more than one vertex, then there is a partition of $V(H)$ into two nonempty sets A_1, A_2, such that either A_1 is complete to A_2, or A_1 is anticomplete to A_2.*

The *length* of a path or cycle is the number of edges in it, and its *parity* is $0, 1$ depending whether its length is even or odd. The *interior* of a path P is the set of vertices of P incident with two edges of P. If $A \subseteq V(G)$, an *A-path* means an induced path in G with distinct ends both in A and with interior in $V(G) \setminus A$. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$. We write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. We apply 3.2 to prove the following.

3.3 *Let $W > 0$, and let G be a graph with $\omega(G) \leq W$. Let A, B be a partition of $V(G)$, such that:*

- *A is stable;*
- *every vertex in B has a neighbour in A; and*
- *there is a cograph H with vertex set A, with the property that for every A-path P of G, its ends are adjacent in H if and only if P has odd length.*

Then there is a partition X, Y of B such that every W-clique in B intersects both X and Y.

Proof. We remark that the last bullet above is a very strong statement; it implies immediately that for every two vertices $u, v \in A$, either every A-path of G joining u, v has odd length or every such A-path has even length. We prove by induction on $|A|$ that if Z denotes the union of all W-cliques in B, then there is a partition X, Y of Z such that every W-clique in B intersects both X and Y.

(1) *If $C \subseteq B$ is a W-clique, and $M \subseteq A$, and every vertex in C has a neighbour in M, then no vertex in C is complete to M.*

Since M is stable, 3.1 implies that there is a four-vertex induced path m-c-c'-m' of G, with $m, m' \in M$ and $c, c' \in C$. Since this is an A-path of odd length, m, m' are adjacent in H. If $v \in C$ is complete to M, then m-v-m' is an A-path of even length between m, m', which is impossible. This proves (1).

We may assume there is a W-clique included in B, for otherwise we can take $X = Y = \emptyset$; and so $|A| \geq 2$ (since $W > 0$ and every vertex in this W-clique has a neighbour in A, and they do not all have the same neighbour since $\omega(G) \leq W$). By 3.2, there is a partition A_1, A_2 of A into two nonempty sets A_1, A_2, such that A_1 is either complete or anticomplete to A_2 in H. For $i = 1, 2$ let $H_i = H[A_i]$.

Suppose first that A_1 is complete to A_2 in H. For $i = 1, 2$, let B_i be the set of vertices in B with a neighbour in A_i. Thus $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$. For $i = 1, 2$, let Z_i be the union of all W-cliques included in B_i, and let Z_0 be the union of all W-cliques in B that intersect both B_1, B_2.

(2) $Z_0 \cap Z_1, Z_0 \cap Z_2 = \emptyset$.

For suppose that $Z_0 \cap Z_1 \neq \emptyset$ say. Then there is a W-clique $C \subseteq Z_1$ and a vertex $b_2 \in B_2$ with a neighbour in C. Choose $a_2 \in A_2$ adjacent to b_2. Let N be the set of vertices in C that are adjacent to b_2. Since $\omega(G) \leq W$, it follows that $N \neq C$. Let M be the set of vertices in A_1 with a neighbour in $C \setminus N$. Thus $M \neq \emptyset$. We claim that M is complete to N in G. For let $m \in M$, and let $v \in N$. Choose $b_1 \in C \setminus N$ adjacent to m. Since every A-path between m, a_2 has odd length, it follows that m-b_1-v-b_2-a_2 is not an A-path, and so m is adjacent to v. This proves that M is complete to N. But every vertex in C has a neighbour in M, contrary to (1). This proves (2).

From the inductive hypothesis (applied to H_1, A_1 and Z_1), there is a partition X_1, Y_1 of Z_1 such that every W-clique in Z_1 intersects them both. Choose X_2, Y_2 similarly for Z_2. Then $X_1 \cup X_2 \cup (B_1 \cap Z_0)$ and $Y_1 \cup Y_2 \cup (B_2 \cap Z_0)$ are disjoint subsets of B, from (2); and we claim that every W-clique C in B meets both of them. For if $C \subseteq B_1$ then C meets both X_1 and Y_1 from the choice of X_1, Y_1; and similarly if $C \subseteq B_2$. If C intersects both B_1, B_2 then $C \subseteq Z_0$, and so meets both $B_1 \cap Z_0$ and $B_2 \cap Z_0$. This completes the inductive proof in this case.

Thus we may assume that A_1 is anticomplete to A_2 in H. Let B_0 be the set of all vertices in B with a neighbour in A_1 and a neighbour in A_2; and for $i = 1, 2$, let B_i be the set of vertices in B such that all its neighbours in A belong to A_i. Thus B_0, B_1, B_2 are pairwise disjoint and have union B. We claim:

(3) *Every W-clique in B is a subset of one of B_1, B_2.*

There is no edge between B_1 and B_2, since such an edge would give a three-edge A-path from

A_1 to A_2, which is impossible. Thus every clique in B is a subset of one of $B_0 \cup B_1, B_0 \cup B_2$. Suppose that C is a W-clique contained in $B_0 \cup B_1$, with at least one vertex in B_0. Choose $a_2 \in A_2$ with a neighbour in C, and let N be the set of neighbours of a_2 in C. Thus $N \subseteq B_0$. Since $\omega(G) = |C|$, it follows that $C \setminus N \neq \emptyset$. Let M be the set of vertices in A_1 with a neighbour in $C \setminus N$. We claim that M is complete to N. For let $m \in M$ and $v \in N$. Choose $b \in C \setminus N$ adjacent to m. Since m-b-v-a_2 is not an odd A-path (because m, a_2 are nonadjacent in H), it follows that m, v are adjacent. This proves our claim that M is complete to N. But every vertex in C has a neighbour in M, contrary to (1). This proves (3).

From the inductive hypothesis (applied to H_1, A_1 and B_1), the union of all W-cliques included in B_1 can be partitioned into two sets X_1, Y_1, such that every W-clique included in B_1 intersects them both. Choose X_2, Y_2 similarly for B_2. Then by (3), every W-clique in B intersects both of $X_1 \cup X_2, Y_1 \cup Y_2$. This completes the inductive proof, and hence completes the proof of 3.3. ∎

4 Obtaining the cograph

In a graph G, a sequence $(L_0, L_1, \dots, L_k)$ of disjoint subsets of $V(G)$ is called a *levelling* in G if $|L_0| = 1$, and for $1 \leq i \leq k$, every vertex in L_i has a neighbour in L_{i-1}, and has no neighbour in L_h for h with $0 \leq h < i - 1$.

For a fixed levelling $(L_0, \dots, L_k)$, and for $0 \leq h \leq j \leq k$, we say that a vertex $u \in L_h$ is an *ancestor* of a vertex $v \in L_j$ if there is a path between u, v of length $j - h$ (which therefore has exactly one vertex in each L_i for $h \leq i \leq j$, and has no other vertices); and if $j = h + 1$ we say that u is a *parent* of v. For $0 \leq i \leq k$, if $u, v \in L_i$ are adjacent, we say they are *siblings*.

If u, v are vertices of a path P, we denote the subpath of P joining them by u-P-v. If P is a path between u, v and $e = vw$ is an edge with $w \notin V(P)$, we denote the path with edge-set $E(P) \cup \{e\}$ by u-P-v-w or just P-v-w. If P is a path between u, v and Q is a path between v, w, and $P \cup Q$ is a path (that is, $V(P) \cap V(Q) = \{v\}$), we sometimes denote their union by u-P-v-Q-w, and use a similar notation for longer strings of concatenations of paths and edges. In this section we prove the following:

4.1 *Let $(L_0, \dots, L_k)$ be a levelling in a graph G. Suppose that:*

- *L_k is stable;*
- *for every two vertices $u, v \in L_k$, all induced paths in G with ends u, v and with interior in $L_0 \cup L_1 \cup \dots \cup L_{k-1}$ have the same parity; and*
- *for $1 \leq i \leq k-1$, if $u, v \in L_i$ are siblings, then they have the same set of parents (we call this the "parent rule").*

Let H be the graph with vertex set L_k, in which distinct vertices u, v are adjacent if there is an induced path of odd length between them with interior in $L_0 \cup L_1 \cup \dots \cup L_{k-1}$. Then H is a cograph.

Proof. We proceed by induction on $|V(G)|$. Suppose there is a four-vertex induced path in H, say with vertices a_k-b_k-c_k-d_k in order. From the inductive hypothesis, we may assume that

- $V(G) = L_0 \cup L_1 \cup \dots \cup L_k$;
- $L_k = \{a_k, b_k, c_k, d_k\}$; and
- for $0 \le i < k$, and each $v \in L_i$, there exists $u \in L_{i+1}$ such that v is the only parent of u (we call such a vertex u a *dependent* of v).

It follows that $|L_i| \le 4$ for $0 \le i \le k$. For distinct $u, v \in L_k$, we say $(u, v) \in L_k$ is an *odd pair* if u, v are adjacent in H, that is, if every L_k-path between them has odd length; and (u, v) is an *even pair* otherwise, that is, all such paths have even length. Thus $(a_k, b_k), (b_k, c_k), (c_k, d_k)$ are odd pairs, and $(a_k, c_k), (b_k, d_k), (a_k, d_k)$ are even pairs. We will have to check the parity of several L_k-paths, and here is a quick rule: the parity of an L_k-path is the same as the parity of the number of edges between siblings that it contains.

For $i = k-1, k-2, \dots, 0$, let a_i be a parent of a_{i+1}, and define b_i, c_i, d_i similarly. Thus $a_i, b_i, c_i, d_i \in L_i$ for each i, and and $a_0 = b_0 = c_0 = d_0$ (since $|L_0| = 1$). Let A be the path

$$a_k\text{-}a_{k-1}\text{-}\cdots\text{-}a_1\text{-}a_0,$$

and define B, C, D similarly. For each i with $0 \le i \le k$, let A_i be the subpath a_k-A-a_i. Since b_1 is adjacent to $b_0 = c_0$, there is a maximum value of i with $i \le k$ such that b_i has a neighbour in C; and so B_i is disjoint from C. Since (b_k, c_k) is an odd pair, and since the only possible neighbours of b_i in C are c_{i-1}, c_i, c_{i+1} from the definition of a levelling, it follows that b_i is adjacent to c_i, and there are no other edges between the paths B_i, C_i. Since b_i, c_i are distinct it follows that $i \ge 1$; and since L_k is stable it follows that $i < k$. By the parent rule, we may assume that $b_{i-1} = c_{i-1}$.

(1) *Some vertex of A_i has a neighbour in $B_i \cup C_i$.*

For suppose not. Then in particular, A_i is disjoint from B_i, C_i. Choose an induced path P between a_i and b_i with interior a subset of $L_0 \cup \dots \cup L_{i-1}$. Thus the neighbour of b_i in P is the unique parent $b_{i-1} = c_{i-1}$ of b_i. But the paths A_i-a_i-P-b_i-B_i and A_i-a_i-P-c_{i-1}-c_i-C_i have the same length, and both are L_k-paths, and one is between the odd pair (a_k, b_k), and the other between the even pair (a_k, c_k), a contradiction. This proves (1).

By the same argument, it follows that some vertex of D_i has a neighbour in $B_i \cup C_i$. Choose $h \ge i$ maximum such that a_h has a neighbour in $B_i \cup C_i$, and similarly choose $j \ge i$ maximum such that d_j has a neighbour in $B_i \cup C_i$.

(2) *A_h, B_i, C_i, D_j are pairwise disjoint, and there is no edge between A_h, D_j except possibly $a_h d_j$.*

For we have seen that B_i, C_i are disjoint, and no vertex of A_h belongs to $B_i \cup C_i$ from the maximality of h, and the same for D_j. Suppose that A_h, D_j share a vertex, and choose $g \ge \max(h, j)$ maximum such that $a_g = d_g$. There is a path from a_g to b_g with interior in $L_0 \cup \dots \cup L_{g-1}$, and so there is an induced path from a_g to b_k with interior contained in $L_0 \cup \dots \cup L_{g-1} \cup V(B_g)$. The union of this path with A_g and with D_g are induced paths of the same length, joining an even pair and an odd pair, which is impossible. This proves that A_h, D_j are disjoint.

Now suppose that there is an edge between A_h and D_j, not between a_h and d_j; say between a_f and d_g, where $f > h$ and $g \ge j$. Choose such an edge with $f + g$ maximum. Since (a_k, d_k) is an even pair it follows that $g = f+1$ or $g = f-1$. If $g = f-1$, then a_{f-1} has no dependent, a contradiction. So $g = f+1$. If $g \neq j$ then d_{g-1} has no dependent, again a contradiction; so $g = j$. Hence d_g has a neighbour in one of B_i, C_i, and there is an induced path P from d_g to one of b_k, c_k, with vertex set included in one of $\{d_g\} \cup B_i, \{d_g\} \cup C_i$ respectively. If $g < k$, then the L_k-paths $D_g \cup P$ and A_f-a_f-d_g-P have lengths differing by two, and so have the same parity, yet one joins an even pair and the other an odd pair, a contradiction. So $g = k$, and therefore $f = k-1$, and $i \leq h \leq k-2$. Since $g = j = k$, d_k has a neighbour in one of B_i, C_i. Since L_k is stable, d_k is not adjacent to either of b_k, c_k, and so it is adjacent to at least one of b_{k-1}, c_{k-1}. The second is impossible since (c_k, d_k) is an odd pair, so d_k is adjacent to b_{k-1} and to no other vertex of $B \cup C$. There is an induced path between a_{k-1}, c_{k-1} with interior in $L_0 \cup \cdots \cup L_{k-2}$. Adding the edge $c_k c_{k-1}$ to it gives an induced path P between a_{k-1}, c_k such that neither of a_k, d_k have neighbours in its interior; but then a_k-a_{k-1}-P-c_k and d_k-a_{k-1}-P-c_k are induced paths of the same length, one joining an odd pair and one joining an even pair, which is impossible. This proves (2).

We need to figure out the possible edges between the four paths A_h, B_i, C_i, D_j. Except for $b_i c_i$ every such edge is incident with a_h or d_j; there is symmetry between a_h and d_j, and the only possible neighbours of a_h in B_i, C_i, D_j are

$$b_{h-1}, b_h, b_{h+1}, c_{h-1}, c_h, c_{h+1}, d_j.$$

Note that it is possible that $h = k$, in which case b_{h+1}, c_{h+1} are not defined.

(3) *a_h has no neighbour in C except possibly c_{h-1}.*

For suppose first that $h < k$ and a_h, c_{h+1} are adjacent. Since $h \geq i$, b_{h+1} is not adjacent to c_h; and since c_h has a dependent, which cannot be any of $a_{h+1}, b_{h+1}, c_{h+1}$, it follows that $j = h+1$, and c_h is the unique parent of d_{h+1}. Since (b_k, d_k) is an even pair, it follows that the path D_{h+1}-d_{h+1}-c_h-$\cdots$-c_i-b_i-B_i is not induced, and so d_{h+1} has a neighbour in B. But b_{h+1} is not a sibling of d_{h+1} by the parent rule, and b_h is not a parent of d_{h+1} since d_{h+1} is a dependent of c_h; and so $h+1 < k$ and d_{h+1} is adjacent to b_{h+2}. Consequently b_{h+1} has no dependent, which is impossible. Thus it is not the case that $h < k$ and a_h, c_{h+1} are adjacent; and so a_h has no neighbour in C except possibly c_h, c_{h-1}. Finally, if a_h, c_h are adjacent, then A_h-a_h-c_h-C_h is an odd L_k-path joining an even pair, a contradiction. This proves (3).

(4) *a_h has no neighbour in B.*

Suppose that a_h has a neighbour in B, and so a_h is adjacent to one of b_{h-1}, b_h, b_{h+1}. It is not adjacent to b_{h+1} since (a_k, b_k) is an odd pair, and for the same reason if a_h is adjacent to b_{h-1} then it is also adjacent to b_h. On the other hand, if a_h is adjacent to b_h then it is adjacent to b_{h-1} by the parent rule. Thus a_h is adjacent to both b_h, b_{h-1}. From its definition, $h \geq i$; suppose that $h > i$. Then the path A_h-a_h-b_{h-1}-$\cdots$-b_i-c_i-C_i has odd length, and therefore is not induced, since (a_k, c_k) is an even pair; and so a_h has a neighbour in C. By (3), a_h is adjacent to c_{h-1}; but then the siblings a_h, b_h do not have the same sets of parents, which is impossible. Thus $h = i$.

Suppose that d_j has a neighbour in C. Then similarly $j = i$ and d_i is adjacent to c_i, c_{i-1} and to no other vertex in $B \cup C$. Since (a_k, d_k) is an even pair it follows that a_i, d_i are not adjacent. But then

$$A_i\text{-}a_i\text{-}b_i\text{-}c_i\text{-}d_i\text{-}D_i$$

is an odd L_k-path joining an even pair, a contradiction.

So d_j has no neighbour in C; and since d_j has a neighbour in $B_i \cup C_i$, (3) (with a_h, d_j exchanged) implies that d_j is adjacent to b_{j-1} and has no other neighbour in $B \cup C$, and therefore $j > i$ (since $b_{i-1} = c_{i-1}$ and d_j has no neighbour in C). The path $D_j\text{-}d_j\text{-}b_{j-1}\text{-}\cdots\text{-}b_i$ is induced; and if a_i, d_j are nonadjacent then the union of this path with $b_i\text{-}a_i\text{-}A_i$ is an odd L_k-path joining an even pair, which is impossible. So a_i, d_j are adjacent, and since $j > i$ it follows that $j = i + 1$. But then

$$D_{i+1}\text{-}d_{i+1}\text{-}a_i\text{-}c_{i-1}\text{-}C_{i-1}$$

is an even L_k-path joining an odd pair, a contradiction. This proves (4).

From (3) and (4), a_h is adjacent to c_{h-1} and to no other vertex in $B \cup C$, and similarly d_j is adjacent to b_{j-1} and to no other vertex in $B \cup C$. Since $b_{i-1} = c_{i-1}$ and a_h has no neighbour in B, it follows that $h > i$, and similarly $j > i$. If a_h, d_j are nonadjacent, let P be the induced path between a_h and d_j with interior in $V(B_i \cup C_i)$. Since this path uses one edge between siblings, its union with A_h and D_j is an odd L_k-path joining an even pair, which is impossible. So a_h, d_j are adjacent, and so $j = h + 1$ or $h - 1$ since (a_k, d_k) is an even pair, and from the symmetry we may assume that $j = h + 1$; but then $C_{h-1}\text{-}c_{h-1}\text{-}a_h\text{-}d_{h+1}\text{-}b_h\text{-}B_h$ is an even L_k-path joining an odd pair, a contradiction. This completes the proof of 4.1. ∎

By combining this and 3.3, we deduce:

4.2 *Let $n \geq 0$ be an integer. Let G be a graph with no odd hole and let $(L_0, \ldots, L_k)$ be a levelling in G. Suppose that:*

- *L_{k-1} is stable;*
- *for $1 \leq i \leq k-2$, if u, v are adjacent vertices in L_i, then they have the same sets of parents; and*
- *every induced subgraph of $G[L_k]$ with clique number strictly less than $\omega(G)$ is n-colourable.*

Then $\chi(L_k) \leq 2n$.

Proof. We proceed by induction on $|L_0 \cup \cdots \cup L_k|$. If $G[L_k]$ is not connected, the result follows from the inductive hypothesis applied to the levellings $(L_0, \ldots, L_{k-1}, V(C))$, for each component C of $G[L_k]$ in turn. Thus we assume that $G[L_k]$ is connected. If some vertex $v \in L_{k-1}$ has no neighbour in L_k, the result follows by the inductive hypothesis applied to the levelling $(L_0, L_1, \ldots, L_{k-1} \setminus \{v\}, L_k)$. Thus we assume that every vertex in L_{k-1} has a neighbour in L_k.

(1) *For all distinct $u, v \in L_{k-1}$, all induced paths between u, v with interior in $L_0 \cup \cdots \cup L_{k-2}$ have the same parity.*

For u, v are nonadjacent since L_{k-1} is stable. Since $G[L_k]$ is connected and u, v both have neighbours in L_k, there is an induced path Q joining u, v with interior in L_k. For every induced path P joining

u, v with interior in $L_0 \cup \cdots \cup L_{k-2}$, the union of P and Q forms a hole, which is necessarily even; and so P has the same parity as Q. This proves (1).

Let H be the graph with vertex set L_{k-1}, in which distinct u, v are adjacent if and only if some induced path joining u, v with interior in $L_0 \cup \cdots \cup L_{k-2}$ has odd length. From 4.1 applied to the levelling $(L_0, \ldots, L_{k-1})$, H is a cograph. From 3.3 applied to the sets L_{k-1}, L_k, there is a partition X, Y of L_k such that $\omega(G[X]), \omega(G[Y]) < \omega(G)$, and therefore $\chi(X), \chi(Y) \leq n$; and it follows that $\chi(L_k) \leq 2n$. This proves 4.2. ∎

At some cost in the chromatic number, we can do without the first bullet in 4.2:

4.3 *Let $n \geq 0$ be an integer. Let G be a graph with no odd hole, and let $(L_0, \ldots, L_k)$ be a levelling in G. Suppose that:*

- *for $1 \leq i \leq k-2$, if u, v are adjacent vertices in L_i, then they have the same sets of parents; and*
- *every induced subgraph of G with clique number strictly less than $\omega(G)$ is n-colourable.*

Then $\chi(L_k) \leq 4n^2\omega(G)$.

Proof. We may assume that $G[L_k]$ is connected, and for $i = k-1, k-2$ and for every vertex $v \in L_i$, there exists $u \in L_{i+1}$ such that v is its only parent (or else we could just delete v). Now we claim that $G[L_{k-2}]$ is a cograph (thanks to Bruce Reed for this improvement); for suppose that there is a three-edge path a-b-c-d that is an induced subgraph of $G[L_{k-2}]$. By the parent rule, a, b, c, d have the same parents, and in particular some vertex $z \in L_{k-3}$ is adjacent to them all. Choose $a' \in L_{k-1}$ with no neighbour in L_{k-2} except a, and choose d' similarly for d. Since a', d' have neighbours in L_k, and $G[L_k]$ is connected, there is an induced path P between a', d' with interior in L_k. But then a-a'-P-d'-d-z-a and a-a'-P-d'-d-c-b-a are holes of different parity, which is impossible. This proves that $G[L_{k-2}]$ is a cograph.

In particular, $G[L_{k-2}]$ is perfect, so L_{k-2} can be partitioned into $\omega(G)$ stable sets $X_1, \ldots, X_{\omega(G)}$. For $1 \leq j \leq \omega(G)$, let Y_j be the set of vertices in L_{k-1} with a neighbour in X_j. From 4.2 applied to the levelling $(L_0, \ldots, L_{k-3}, X_j, Y_j)$, it follows that $\chi(Y_j) \leq 2n$. Take a partition of Y_j into $2n$ stable sets $Y_j^1, \ldots, Y_j^{2n}$, and for $1 \leq h \leq 2n$ let Z_j^h be the set of vertices in L_k with a neighbour in Y_j^h. From 4.2 applied to the levelling $(L_0, \ldots, L_{k-2}, Y_j^h, Z_j^h)$, it follows that $\chi(Z_j^h) \leq 2n$. Since every vertex in L_k belongs to Z_j^h for some choice of h, j, and there are $2n\omega(G)$ possible choices of h, j, it follows that $\chi(L_k) \leq 4n^2\omega(G)$. This proves 4.3. ∎

5 The parent rule

A major hypothesis in 4.1 and 4.3 was the parent rule; and in this section we show how to arrange that it holds. In fact our proof works under the weaker hypothesis that G has no odd holes of length more than $2\ell + 1$, and so may be a valuable step in proving the strongest of the three conjectures of Gyárfás mentioned earlier, although in the present paper we only use it for $\ell = 1$. Let the *odd hole number* of G be the length of the longest induced odd cycle (or 1, if G is bipartite). First we prove the following:

5.1 *Let G be a graph with odd hole number at most $2\ell+1$. Let $(L_0, L_1, \dots, L_k)$ be a levelling in G. Then there exists a levelling $(M_0, \dots, M_k)$ in G, such that*

- $\chi(M_k) \geq \chi(L_k)/(3\ell+3)$*; and*
- *for $1 \leq i \leq k - \max(2, \ell)$, if $u, v \in M_i$ are adjacent then they have the same sets of neighbours in M_{i-1}.*

Proof. We proceed by induction on $|V(G)|$. Thus we may assume:

- $V(G) = L_0 \cup L_1 \cup \cdots \cup L_k$;
- $G[L_k]$ is connected; and
- for $1 \leq i < k$ and every vertex $u \in L_i$, there exists $v \in L_{i+1}$ such that u is its only parent.

Let $s_0 \in L_0$, and for $1 \leq i \leq k$ choose $s_i \in L_i$ such that s_{i-1} is its only parent. Let S be the path s_0-s_1-$\cdots$-s_k. Let $N(S)$ be the set of vertices of G not in S but with a neighbour in S. If $v \in L_i \cap N(S)$, then v is adjacent to one or both of s_i, s_{i-1} and has no other neighbour in S; because every neighbour of v belongs to one of L_{i-1}, L_i, L_{i+1}, and v is not adjacent to s_{i+1} since s_i is the only parent of s_{i+1}. So every vertex in $L_i \cap N(S)$ is of one of three possible types. We wish to further classify them by the value of i modulo $\ell+1$. Let us say the *type* of a vertex $v \in L_i \cap N(S)$ is the pair (α, λ) where

- $\alpha = 1, 2$ or 3 depending whether v is adjacent to s_{i-1} and not to s_i, adjacent to both s_i and s_{i-1}, or adjacent to s_i and not to s_{i-1}
- $\lambda \in \{0, \dots, \ell\}$ is congruent to i modulo $\ell+1$.

Let us fix a type $(\alpha, \lambda) = \gamma$ say. Let $V(\gamma)$ be a minimal subset of $V(G) \setminus V(S)$ such that

- every vertex in $N(S)$ of type γ belongs to $V(\gamma)$; and
- for every vertex $v \in V(G) \setminus (V(S) \cup N(S))$, if some parent of v belongs to $V(\gamma)$ then $v \in V(\gamma)$.

There are $3\ell+3$ possible types γ, and every vertex in $L_k \setminus \{s_k\}$ belongs to $V(\gamma)$ for some type γ, so either there is a type $\gamma \neq (1,1)$ such that $\chi(V(\gamma) \cap L_k) \geq \chi(L_k)/(3\ell+3)$, or

$$\chi((V(\gamma) \cap L_k) \cup \{s_k\}) \geq \chi(L_k)/(3\ell+3)$$

when $\gamma = (1,1)$. In the latter case, since s_k has no neighbour in $V(\gamma) \cap L_k$, it follows that $\chi(V(\gamma) \cap L_k) \geq \chi(L_k)/(3\ell+3)$ anyway; so we may choose a type γ such that $\chi(V(\gamma) \cap L_k) \geq \chi(L_k)/(3\ell+3)$. Let C be the vertex set of a component of $G[(V(\gamma) \cap L_k)]$ with maximum chromatic number, so $\chi(C) \geq \chi(L_k)/(3\ell+3)$. Let $J_k = C$, and for $i = k-1, k-2, \dots, 1$ choose $J_i \subseteq V(\gamma) \cap L_i$ minimal such that every vertex in $J_{i+1} \setminus N(S)$ has a neighbour in J_i. Consequently, for every vertex $v \in J_i$, there is a path $v = p_i$-p_{i+1}-$\cdots$-p_k such that

- $p_j \in J_j$ for $i \leq j \leq k$
- $p_j \notin N(S)$ for $i < j \leq k$

- p_{j-1} is the only parent of p_j in J_{j-1} for $i < j \leq k$.

We call such a path a *pillar* for v.

(1) *For $1 \leq i \leq k-2$, if $v \in J_i$ and v is nonadjacent to s_i, then there is an induced path Q_v between v and s_i of length at least $2(k-i)$, with interior in $L_{i+1} \cup \cdots \cup L_k$, such that no vertex in J_i different from v has a neighbour in the interior of Q_v.*

Let P_v be a pillar for v; then none of its vertices are in $N(S)$ except possibly v, and since both P_v and the path s_i-s_{i+1}-$\cdots$-s_k end in the connected graph $G[L_k]$, there is an induced path Q_v between v, s_i with vertex set contained in the union of the vertex sets of these two paths and L_k, using all vertices of P_v and s_i-s_{i+1}-$\cdots$-s_k except possibly their ends in L_k. It follows that Q_v has length at least $2(k-i)$. If $u \in J_i \setminus \{v\}$, then since u has no neighbours in $L_{i+2} \cup \cdots \cup L_k$, and u is nonadjacent to the second vertex of Q_v (because v is its unique parent in J_i) and nonadjacent to s_{i+1} (because s_i is its only parent), it follows that u has no neighbour in the interior of Q_v. This proves (1).

For $1 \leq i \leq k$ and for every vertex $v \in J_i$, either $v \in N(S)$ or it has a parent in J_{i-1}; and so there is a path $v = r_i$-r_{i-1}-$\cdots$-r_h for some $h \leq i$, such that $r_j \in J_j$ for $h \leq j \leq i$, and $r_h \in N(S)$, and $r_j \notin N(S)$ for $h+1 \leq j \leq i$. Since r_h has a neighbour in S, one of

$$r_i\text{-}r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_{h-1}\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i,$$

$$r_i\text{-}r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i$$

is an induced path (the first if $\alpha = 1$ and the second if $\alpha = 2$ or 3). We choose some such path and call it R_v. Note that for all $v \in J_1 \cup \cdots \cup J_k$, the path R_v has even length if $\alpha = 1$, and odd length otherwise.

(2) *For $0 \leq i \leq k-l$, there is no edge with one end in $J_i \cap N(S)$ and the other in $J_i \setminus N(S)$.*

For suppose that uv is an edge with $u \in J_i \cap N(S)$ and $v \in J_i \setminus N(S)$. Since $u \in N(S)$, the path R_u has length one or two, and in either case adding the edge uv to it gives an induced path between v, s_i, with parity different from that of R_v. But the union of either of these paths with Q_v gives a hole of length at least $2(k-i)+2 > 2\ell+1$, a contradiction. This proves (2).

(3) *For $0 \leq i \leq k-1$, if $\alpha = 2$ and $u \in J_i \cap N(S)$, then u has no parent in J_{i-1}.*

Because suppose $t \in J_{i-1}$ is a parent of u. It follows that $t \notin N(S)$ since $i, i-1$ are not congruent modulo $\ell+1$. Hence R_t has length at least $2\ell+1$ (because some vertex of R_t belongs to $N(S) \cap J_h$ where $h < i-1$, and so $i-h$ is a nonzero multiple of $\ell+1$, implying $i-1-h \geq \ell$). Also R_t has odd length, since $\alpha = 2$. But then the union of R_t with the path s_{i-1}-u-t is an odd hole of length at least $2\ell+3$, which is impossible. This proves (3).

(4) *For $0 \leq i \leq \min(k-2, k-\ell)$, if $u, v \in J_i$ are adjacent, then they have the same sets of parents in $J_{i-1} \cup \{s_{i-1}\}$.*

Let $u, v \in J_i$ be adjacent. Suppose first that $u, v \in N(S)$. If $\alpha = 2$ then the claim follows from (3), so we may assume that $\alpha = 1$ or 3. In either case u, v have the same sets of parents in $V(S)$; so suppose that there is a vertex $t \in J_{i-1}$ adjacent to v and not to u. Choose $w \in J_i \setminus N(S)$ such that t is its unique parent in J_{i-1}. From (2), w is nonadjacent to u, v. Let P_w, P_u be pillars for w, u respectively. There is an induced path T between u, w with interior in $V(P_u \cup P_w) \cup C$. If its length is even then it includes all vertices of P_u, P_w except their ends in C, from the definition of "pillar", and so has length at least $2(k-i)$; but then the union of T with the path w-t-v-u is an odd hole of length at least $2(k-i)+3 \geq 2\ell+3$, which is impossible. So T is odd. If $\alpha = 1$ then w-t-R_t-s_{i-1}-u is an even path joining u, w of length at least $2\ell+4$; and if $\alpha = 3$ then w-R_w-s_i-u is an even path joining u, w of length at least $2\ell+4$; and in either case the union of this path with T is an odd hole of length at least $2\ell+7$, which is impossible.

So from (2), we may assume that $u, v \notin N(S)$. Suppose that some vertex $t \in J_{i-1}$ is adjacent to v and not to u. The union of R_u and Q_u is a hole of length at least $2(k-i)+2$, and so is even; and hence Q_u and R_u have the same parity, which is the same as the parity of R_t. But v has no neighbour in the interior of Q_u, by (1); so u-Q_u-s_i-s_{i-1}-R_t-t-v-u is a hole of odd length and length at least $2(k-i)+4$, which is impossible. This proves (4).

If $\alpha = 1$ or 2 let $M_i = \{s_i\} \cup J_i$ for $0 \leq i \leq k$. We claim that $M_0, \ldots, M_k$ satisfies the theorem. To see this, we must check:

- Every vertex $v \in M_i$ has a neighbour in M_{i-1}. This is true if $v = s_i$ (because then it is adjacent to s_{i-1}), and if $v \in N(S)$ (because then it is adjacent to s_{i-1}, since $\alpha = 1$ or 2), and if $v \in J_i \setminus N(S)$ (because then it has a parent in J_{i-1}).
- Every vertex $v \in M_i$ has no neighbour in M_h if $h < i-1$; because $M_i \subseteq L_i$ and $M_h \subseteq L_h$.
- If $u, v \in M_i$ are adjacent and $i \leq \min(k-2, k-\ell)$ then u, v have the same sets of neighbours in M_{i-1}. Because if $u, v \neq s_i$ then the claim follows from (4), and if say $u = s_i$ then v is adjacent to s_{i-1} (since $\alpha = 1$ or 2), and $\alpha = 2$ (since v is adjacent to s_i), and so u, v have no other neighbours in M_{i-1} by (3).

We assume then that $\alpha = 3$. Let $M_0 = \{s_1\}$, $M_i = \{s_{i+1}\} \cup J_i$ for $1 \leq i < k$, and $M_k = J_k$, and we claim that $(M_0, \ldots, M_k)$ is the desired levelling. Once again we must check:

- Every vertex $v \in M_i$ has a neighbour in M_{i-1}. For if $v = s_{i+1}$ then v is adjacent to s_i; if $v \in N(S)$ then v is adjacent to s_i (because $\alpha = 3$); and if $v \in J_i \setminus N(S)$ then it has a neighbour in J_{i-1}.
- If $v \in M_i$ then it has no neighbour in M_h where $h \leq i-2$. Because $v \in L_i \cup L_{i+1}$, and $M_h \subseteq L_h \cup L_{h+1}$, so this is clear unless $h = i-2$; and in that case, $v \in L_i$ is nonadjacent to s_{i-1} since $\alpha = 3$.
- If $i \leq \min(k-2, k-\ell)$ and $u, v \in M_i$ are adjacent then u, v have the same neighbours in M_{i-1}. Since s_{i+1} has no neighbour in J_i it follows that $u, v \neq s_{i+1}$, so they have the same neighbours in J_{i-1} by (4), and if one is adjacent to s_i then they both are, by (1).

This proves 5.1. ∎

Now we complete the proof of 1.3.

Proof. For purposes of induction it is better to prove the slightly stronger statement

$$\chi(G) \le \frac{2^{2^{\omega(G)+2}}}{48(\omega(G)+2)}.$$

We proceed by induction on $\omega(G)$. If $\omega(G) = 1$ then $\chi(G) = 1$ and the result holds; so we may assume that $\omega(G) > 1$, and $\chi(H) \le n$ for every induced subgraph H of G with $\omega(H) < \omega(G)$, where

$$n = \frac{2^{2^{(\omega+1)}}}{48(\omega+1)}$$

(writing ω for $\omega(G)$ henceforth). We may assume that G is connected; choose a vertex s_0, and let L_i be the set of all vertices v such that the shortest path from s_0 to v has i edges, for all $i \ge 0$ such that some such vertex v exists. Fix $k \ge 1$. By 5.1 with $\ell = 1$ there is a levelling $(M_0, \dots, M_k)$ in G satisfying the conclusion of 5.1, and such that $\chi(M_k) \ge \chi(L_k)/6$. But by 4.3, $\chi(M_k) \le 4n^2\omega$. Consequently $\chi(L_k) \le 24n^2\omega$. Since this holds for all k, it follows that

$$\chi(G) \le 48n^2\omega = 48\omega \left\{ \frac{2^{2^{(\omega+1)}}}{48(\omega+1)} \right\}^2 \le \frac{2^{2^{\omega+2}}}{48(\omega+2)}.$$

This proves 1.3. ∎

One of the curious features of the proof of the strong perfect graph theorem [5] was that while it proved that every graph G either has an odd hole or has an odd hole in its complement or admits an $\omega(G)$-colouring, there was apparently no way to convert the proof to a polynomial-time algorithm that actually finds one of these three things (although a polynomial-time algorithm was found shortly afterwards, using a different method [3]). That is not the case with the result of this paper: it can easily be converted to a polynomial-time algorithm, which, with input a graph G, outputs either

- an odd hole in G, or
- a clique C in G and a $2^{2^{|C|+2}}$-colouring of G.

We leave the details to the reader. In fact, with some work we were able to bring its running time down to $O(|V(G)|^3 \log(|V(G)|))$.

References

[1] M. Bonamy, P. Charbit and S. Thomassé, "Graphs with large chromatic number induce $3k$-cycles", submitted for publication, arXiv:1408.2172, manuscript August 2014.

[2] A. Brandstadt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey*, Philadelphia, SIAM, 1999.

[3] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, "Recognizing Berge graphs", *Combinatorica*, 25 (2005), 143–186.

[4] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "K_4-free graphs with no odd holes", *J. Combinatorial Theory, Ser. B*, 100 (2010), 313–331.

[5] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "The strong perfect graph theorem", *Annals of Math.* 164 (2006), 51–229.

[6] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás' conjectures", submitted for publication, arXiv:1411.6465, manuscript October 2014.

[7] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. III. Long holes", submitted for publication, arXiv:1506.02232, manuscript May 2015.

[8] Blanche Descartes, "A three colour problem", *Eureka* (April 1947). (Solution March 1948.)

[9] Blanche Descartes, "Solution to Advanced Problem no. 4526", *Amer. Math. Monthly* 61 (1954), 352.

[10] P. Erdős, "Graph theory and probability", *Canad. J. Math* 11 (1959), 34–38.

[11] A. Gyárfás, "On Ramsey covering-numbers", *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), 10.

[12] A. Gyárfás, "Problems from the world surrounding perfect graphs", *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* 19 (1987), 413–441.

[13] A. Gyárfás, E. Szemerédi and Zs. Tuza, "Induced subtrees in graphs of large chromatic number", *Discrete Math.* 30 (1980), 235-244.

[14] H.A. Kierstead and S.G. Penrice, "Radius two trees specify χ-bounded classes", *J. Graph Theory* 18 (1994), 119-129.

[15] H.A. Kierstead and Y. Zhu, " Radius three trees in graphs with large chromatic number", *SIAM J. Discrete Math.* 17 (2004), 571–581.

[16] A.D. Scott, "Induced trees in graphs of large chromatic number", *J. Graph Theory* 24 (1997), 297–311.

[17] A.D. Scott, "Induced cycles and chromatic number", *J. Combinatorial Theory, Ser. B*, 76 (1999), 150–154.

[18] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes", in preparation (manuscript January 2015).

[19] D.P. Sumner, "Subtrees of a graph and chromatic number", in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.