Extending the Gyárfás-Sumner conjecture

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Abstract

Say a set $\mathcal{H}$ of graphs is *heroic* if there exists k such that every graph containing no member of $\mathcal{H}$ as an induced subgraph has cochromatic number at most k. (The *cochromatic number* of G is the minimum number of stable sets and cliques with union $V(G)$.) Assuming an old conjecture of Gyárfás and Sumner, we give a complete characterization of the finite heroic sets.

This is a consequence of the following. Say a graph is *k-split* if its vertex set is the union of two sets A, B, where A has clique number at most k and B has stability number at most k. For every graph H_1 that is a disjoint union of cliques, and every complete multipartite graph H_2, there exists k such that every graph containing neither of H_1, H_2 as an induced subgraph is k-split.

This in turn is a consequence of a bound on the maximum number of vertices in any graph that is minimal not k-split.

1 Introduction

There are infinitely many tournaments H, called "heroes", such that every tournament not containing H as an induced subgraph can be partitioned into a bounded number of transitive tournaments; and in [1] we (with others) were able to explicitly construct all heroes. That was a most enjoyable piece of research, so we looked for an analogue in the world of undirected graphs and induced subgraphs, to do it again. If G, H are graphs, let us say G is *H-free* if no induced subgraph of G is isomorphic to H. (Graphs in this paper are finite and simple.)

Disappointingly there is no non-trivial direct analogue. If we ask "for which graphs H does there exist k such that every H-free graph has chromatic number at most k?", then the answer is, the only such graphs H are cliques with at most two vertices. (To see this, let G_1 be a clique with $k+1$ vertices, and let G_2 be a triangle-free graph with chromatic number at least $k+1$; both G_1 and G_2 must contain H as an induced subgraph.)

Perhaps chromatic number is not the right analogue of partitioning tournaments into a small number of transitive sets; perhaps we should try cochromatic number (the *cochromatic number* of G is the minimum number of stable sets and cliques with union $V(G)$). But if we ask "for which graphs H does there exist k such that every H-free graph has cochromatic number at most k?", the answer is, just the graphs H with at most two vertices. We give a proof, because it allows us to introduce some graphs that we will need later.

- If $n_1, \ldots, n_k \geq 1$ are integers, let $J_{n_1,\ldots,n_k}$ be the graph with k components, all cliques, with $n_1, \ldots, n_k$ vertices respectively; we call such a graph a *clique partition graph*. When $n_1, \ldots, n_k$ are all equal to some number j say, we denote it by J^k_j.
- If $n_1, \ldots, n_k \geq 1$ are integers, let $K_{n_1,\ldots,n_k}$ be the complement of $J_{n_1,\ldots,n_k}$; this is a *complete multipartite graph*. Again, K^k_j denotes the complement of J^k_j.

Now, suppose that every H-free graph has cochromatic number at most k. Then J^{k+1}_{k+1} contains H, because its cochromatic number is $k+1$; and also K^{k+1}_{k+1} contains H, for the same reason. Thus, no induced subgraph of H is a two-edge path or its complement, and so H is a clique or has no edges; and by taking complements if necessary we may assume that H is a clique. Let G be a triangle-free graph with chromatic number at least $2k+2$; then the cochromatic number of G is at least $k+1$, so G contains H as an induced subgraph, and hence $|V(H)| \leq 2$. That completes the proof.

That is a sad end to a promising question; how can we bring it back to life? Here is one way, a suggestion of Bruce Reed; exclude a set of graphs rather than just one graph. If $\mathcal{H}$ is a set of graphs, we say a graph G is *$\mathcal{H}$-free* if G is H-free for each $H \in \mathcal{H}$. Let us say a set of graphs $\mathcal{H}$ is *heroic* if there exists k such that every $\mathcal{H}$-free graph has cochromatic number at most k. Characterizing heroic sets is not trivial, because for instance we do not know whether the following is true (it is essentially an open conjecture independently proposed by Gyárfás [4] and Sumner [6]).

1.1 Conjecture: *For every clique K and every tree T, the set $\{K, T\}$ is heroic.*

Gyárfás and Sumner actually conjectured that for every clique K and tree T, all $\{K, T\}$-free graphs have bounded chromatic number, but it is easy to see that what we stated is equivalent. We shall show that, assuming 1.1, we can characterize exactly all the finite heroic sets. It turns out to be necessary and sufficient that $\mathcal{H}$ contains a clique partition graph, a complete multipartite graph, a forest, and the complement of a forest. (Note that 1.1 is consistent with this; K is simultaneously a clique partition graph, a complete multipartite graph, and the complement of a forest.)

For *infinite* heroic sets, the problem of characterizing them is more difficult; for instance, a different conjecture of Gyárfás [5], also open, asserts that for every clique K and every integer t, there exists k such that every graph not containing K and with no induced cycle of length at least t has chromatic number at most k. In our language, this is equivalent to the assertion that the set consisting of K and all cycles of length at least t is heroic. If so, this is an example of a heroic set that does not include a minimal heroic set. Indeed, we do not know any minimal heroic sets that are infinite.

2 Graphs of bounded splitness

If G is a graph, $\omega(G)$ and $\alpha(G)$ denotes its *clique number* (the cardinality of the largest clique of G) and its *stability number* (the cardinality of the largest stable set). For $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X; $\omega(X)$ denotes $\omega(G|X)$, and $\alpha(X)$ denotes $\alpha(G|X)$. For $k \geq 0$ an integer, let us say a graph G is *k-split* if $V(G)$ can be partitioned into two sets A, B, such that $\omega(A) \leq k$ and $\alpha(B) \leq k$; and *non-k-split* otherwise. Thus a graph is 1-split if and only if it is a split graph in the usual sense.

It was proved by Földes and Hammer [3] that a graph is a split graph if and only if no induced subgraph is a cycle of length four or five, or J_2^2; and in particular all minimal non-1-split graphs have at most five vertices. We need an upper bound for the size of minimal non-k-split graphs in general, proved below.

First we need the following version of Ramsey's theorem.

2.1 *If G is a graph with no clique of cardinality r and no stable set of cardinality s then*

$$|V(G)| < \binom{r+s-2}{r-1}.$$

2.2 *Let $k \geq 1$ be an integer, and let G be minimal non-k-split. Then*

$$|V(G)| \leq (k+2)^{2^{2k+1}}.$$

Proof. Let $d = \binom{2k+1}{k}$. Since $(k+2)^{2d} \leq (k+2)^{2^{2k+1}}$, we suppose for a contradiction that $|V(G)| \geq (k+2)^{2d}$. Let $v \in V(G)$; then by hypothesis, $G \setminus \{v\}$ is k-split. By adding v to the "A"-side of the corresponding partition, we deduce that there exists $A_v \subseteq V(G)$ with $v \in A_v$, such that

- $\omega(A_v) \leq k+1$, and every clique of cardinality $k+1$ in A_v contains v
- $\alpha(V(G) \setminus A_v) \leq k$.

Fix some $w \in V$, and let $A_w = A$.

(1) *For all $v \in V(G)$, $|A_v \setminus A| \leq d-1$, and $|A \setminus A_v| \leq d-1$.*

For let $X = A_v \setminus A$. Since $X \subseteq A_v$ it follows that $\omega(X) \le k+1$; and $\alpha(X) \le k$ since $X \subseteq V(G) \setminus A_w$. By 2.1, $|X| < \binom{2k+1}{k} = d$. Similarly $|A \setminus A_v| \le d-1$. This proves (1).

Choose $P \subseteq A$ and $Q \subseteq V(G) \setminus A$, with $P \cup Q$ maximal such that there exists $X \subseteq V(G)$ with the following properties:

- for each $v \in X$, $Q \subseteq A_v$ and $P \cap A_v = \emptyset$, and
- $|X| \ge (k+2)^{2d-|P|-|Q|}$.

(This is possible, since taking $X = V(G)$ and $P, Q = \emptyset$ satisfies the two conditions above.) Let $A' = (A \cup Q) \setminus P$ and $B' = V(G) \setminus A'$.

(2) $\alpha(B') \le k$.

For suppose that there is a stable set S of cardinality $k+1$ included in B'. For each $v \in X$, $S \not\subseteq V(G) \setminus A_v$ since $\alpha(V(G) \setminus A_v) \le k$; and so there exists $s_v \in S \cap A_v$. It follows that $s_v \notin P$ (since $P \cap A_v = \emptyset$), and $s_v \notin A'$ (since $v \in S \subseteq B'$ and B' is disjoint from A'). Thus $s_v \notin A' \cup P = A \cup Q$. Since there are only $k+1$ possible values for s_v (since $|S| = k+1$), there exists $X' \subseteq X$ with $|X'| \ge |X|/(k+1)$, such that s_v is the same for all $v \in X'$, say $s_v = s$ for all $v \in X'$. The pair $P, Q \cup \{s\}$ contradicts the choice of P, Q. This proves (2).

Let $|P| + |Q| = i$ say.

(3) $|X| - (k+1) \ge (k+1)(k+2)^{2d-1-i}$.

For since $X \ne \emptyset$, (1) implies that $|P|, |Q| \le d-1$, and so $i \le 2d-2$. Hence

$$(k+2)^{2d-i} = (k+1)(k+2)^{2d-1-i} + (k+2)^{2d-1-i} \ge (k+1)(k+2)^{2d-1-i} + (k+1).$$

This proves (3).

Since G is not k-split, (2) implies that $\omega(A') > k$; let $T \subseteq A'$ be a clique of cardinality $k+1$. For each $v \in X$, every clique of cardinality $k+1$ in A_v contains v; and so for each $v \in X$, either $v \in T$ or $T \setminus A_v \ne \emptyset$. If $T \setminus A_v \ne \emptyset$, choose $t_v \in T \setminus A_v$. Since $v \notin A_v$, and $Q \subseteq A_v$, it follows that $v \notin Q$; and since $v \in T \subseteq A'$, we deduce that $v \in A' \setminus Q = A \setminus P$. Now there are at most $k+1$ values of $v \in X$ with $v \in T$, and so by (3) there are at least $(k+1)(k+2)^{2d-i-1}$ values of $v \in X$ such that t_v is defined. Since each $t_v \in T$ and $|T| = k+1$, there exists $X' \subseteq X$ with $|X'| \ge (k+2)^{2d-i-1}$ such that t_v is defined for all $v \in X'$ and they are all equal, say $t_v = t$ for each $v \in X'$. The pair $P \cup \{t\}, Q$ contradicts the choice of P, Q. This proves 2.2. ∎

The main result of this section is the following.

2.3 *Let H_1 be a clique partition graph, and let H_2 be a complete multipartite graph. Then there exists p such that every $\{H_1, H_2\}$-free graph is p-split.*

Proof. For $k \geq 1$, let $p(1, b, k) = p(a, 1, k) = k - 1$ for all $a, b \geq 1$, and for $a, b \geq 2$, define $p(a, b, k)$ inductively as follows:

$$p(a,b,k) = (k+1)^{k2^{2k-1}}(p(a-1,b,k) + p(a,b-1,k)) + (k+1)^{2^{2k-1}}.$$

Since every clique partition graph is an induced subgraph of J_s^a for some a, s, it is enough to prove the theorem when $H_1 = J_s^a$ (for all a, s); and similarly it is enough to prove the theorem when $H_2 = K_t^b$ (for all b, t). Moreover, by replacing s, t by their maximum, it is enough to prove the result when $s = t$. Thus it suffices to prove the following:

(1) *For all $a, b, k \geq 1$, every $\{J_k^a, K_k^b\}$-free graph is $p(a, b, k)$-split.*

Fix k; we shall prove (1) by induction on $a + b$. If $a = 1$ then the result holds, since every J_k^1-free graph G satisfies $\omega(G) < k$, and so is $(k-1)$-split. Similarly the result holds if $b = 1$, so we assume $a, b \geq 2$. Let G be $\{J_k^a, K_k^b\}$-free. Let $n = (k+1)^{2^{2k-1}}$. If G is $(k-1)$-split then the theorem holds, since $p(a, b, k) \geq k - 1$. Thus we assume G is not $(k-1)$-split, and therefore contains some induced subgraph, $G|W$ say, that is minimal non-$(k-1)$-split. By 2.2, $|W| \leq n$. Let $\mathcal{P}$ be the set of all k-element cliques that are subsets of W, and let $\mathcal{Q}$ be the set of all k-element stable sets in W. For each $P \in \mathcal{P}$, let $M(P)$ be the set of all vertices in $V(G) \setminus W$ with no neighbours in P; and for each $Q \in \mathcal{Q}$, let $N(Q)$ be the set of all vertices in $V(G) \setminus W$ that are adjacent to every vertex in Q. Now for every vertex $v \in V(G) \setminus W$, since $G|W$ is not $(k-1)$-split, either the set of neighbours of v in W includes a k-element stable set, or the set of non-neighbours of v in W includes a k-element clique. Thus either $v \in M(P)$ for some $P \in \mathcal{P}$, or $v \in N(Q)$ for some $Q \in \mathcal{Q}$. Consequently the union of the sets $M(P)$ ($P \in \mathcal{P}$), $N(Q)$ ($Q \in \mathcal{Q}$), and W equals $V(G)$. Now for each $P \in \mathcal{P}$, $G|M(P)$ does not contain J_k^{a-1}, since if it did we could add P to it to obtain a copy of J_k^a in G. Since also $G|M(P)$ does not contain K_k^b, from the inductive hypothesis it follows that $G|M(P)$ is $p(a-1, b, k)$-split. Similarly $G|N(Q)$ is $p(a, b-1, k)$-split, for each $Q \in \mathcal{Q}$. Since $|W| \leq n$, and $G|W$ is therefore n-split, and since $|\mathcal{P}|, |\mathcal{Q}| \leq n^k$, it follows that G is $(n^k(p(a-1, b, k) + n^k p(a, b-1, k) + n)$-split, and hence $p(a, b, k)$-split. This proves (1), and hence completes the proof of 2.3. ∎

We point out that 2.3 is best possible in the following sense: if $\mathcal{H}$ is a set of graphs, and there exists p such that every $\mathcal{H}$-free graph has cochromatic number at most p, then $\mathcal{H}$ contains both a clique partition graph and a complete multipartite graph. (To see this, observe that J_{p+1}^{p+1} must contain a member of $\mathcal{H}$, and so must K_{p+1}^{p+1}.)

3 Heroic sets

Let us apply what we just proved to characterizing heroic sets. First, we observe:

3.1 *If $\mathcal{H}$ is a heroic set of graphs, then*

- *some member of $\mathcal{H}$ is a clique partition graph*
- *some member of $\mathcal{H}$ is a complete multipartite graph*
- *for each $g \geq 1$ some member of $\mathcal{H}$ has no cycle of length less than g*

- *for each $g \geq 1$ the complement of some member of $\mathcal{H}$ has no cycle of length less than g.*

In particular, if $\mathcal{H}$ is finite then it contains a forest and the complement of a forest.

Proof. Let k be such that every $\mathcal{H}$-free graph has cochromatic number less than k. Consequently J_k^k contains a member of $\mathcal{H}$; this proves the first statement. The second follows by taking complements. For the third, let G be a graph of girth at least g and chromatic number at least $2g$; such graphs exist, by a theorem of Erdős [2]. The cochromatic number of G is at least k, so G contains a member of $\mathcal{H}$; this proves the third statement. The fourth follows by taking complements. This proves 3.1. ∎

We propose:

3.2 Conjecture: *A finite set of graphs is heroic if and only if it contains a clique partition graph, a complete multipartite graph, a forest, and the complement of a forest.*

3.3 *Conjecture 3.2 is equivalent to the Gyárfás-Sumner conjecture 1.1.*

Proof. It is easy to see that 3.2 implies 1.1; let us prove the converse. The "only if" half of 3.2 is proved in 3.1, and we need to prove the "if" half. Thus, let $\mathcal{H}$ be a set of graphs containing a clique partition graph H_1, a complete multipartite graph H_2, a forest H_3, and a graph H_4 that is the complement of a forest. By 2.3 there exists p such that every $\{H_1, H_2\}$-free graph is p-split.

By 1.1, there exists q such that every $\{K_{p+1}, H_3\}$-free graph has chromatic number at most q; and by applying 1.1 in the complement, there exists r such that, for every graph containing neither a $(p+1)$-vertex stable set nor H_4, $V(G)$ is the union of r cliques. We claim that every $\mathcal{H}$-free graph has cochromatic number at most $q + r$.

For let G be $\mathcal{H}$-free. Since G is $\{H_1, H_2\}$-free, it is p-split; let (A, B) be a partition of $V(G)$ such that $\omega(A) \leq p$ and $\alpha(B) \leq p$. Then $G|A$ is $\{K_{p+1}, H_3\}$-free, and so its chromatic number is at most q. Similarly B is the union of at most r cliques, and so the cochromatic number of G is at most $q + r$. Consequently $\mathcal{H}$ is heroic. This proves 3.3. ∎

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