Excluding induced subgraphs

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Abstract

1 Introduction

Given two graphs, G and H, we say that H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and two vertices of H are adjacent if and only if they are adjacent in G. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is called $\mathcal{F}$-free if no member of $\mathcal{F}$ is isomorphic to an induced subgraph of G. A *clique* in a graph is a set of vertices all pairwise adjacent, and a *stable set* is a set of vertices all pairwise non-adjacent. The complement of a graph G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$.

It turns out that many interesting families of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. Perfect graphs is, possibly, one of the most well-known examples. For a graph G, let us denote by $\chi(G)$ the chromatic number of G, and by $\omega(G)$ the size of the largest clique in G. A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. In 1961 Claude Berge conjectured that being perfect is equivalent to the property of being $\mathcal{F}$-free for a certain infinite family $\mathcal{F}$ [2], and in 2002, in joint work with Neil Robertson and Robin Thomas, we were able to prove this conjecture [9]. More precisely, Berge conjectured that a graph is perfect if an only if no induced subgraph of it is an cycle of odd length at least five, or the complement of one. Today such graphs are called *Berge* graphs. The main part of our proof of the conjecture was a more general theorem, that describes the structure of all Berge graphs. More precisely, we proved that every Berge graph either belongs to one of a few well understood families of basic graphs, or admits a certain decomposition (this was conjectured earlier by Conforti, Cornuéjols and Vušković). Having obtained this explicit structural result for all Berge graphs, we were able to verify that all of them are perfect (the other directions of Berge's conjecture is easy, because odd cycles and their complements are not perfect, and every induced subgraph of a perfect graph is).

Theorems following the same general paradigm are known for $\mathcal{F}$-free graphs for other families $\mathcal{F}$. Some of them are easy—for example it is almost immediate to see that if $\mathcal{F}$ consists of a single graph which is an induced two-edge path, then every $\mathcal{F}$-free graph is either complete or disconnected. Others are difficult—take $\mathcal{F}$ to be the set of all even-length cycles, or the set of all cycles of odd length at least five (these are theorems of Conforti, Cornuéjols, Kapoor, and Vušković [21] and of Conforti, Cornuéjols, and Vušković [23], respectively).

One might then ask whether a structural theorem of that kind should exist for every family $\mathcal{F}$. This question is, of course, not well defined, because we do not know yet what graphs should be considered basic, and what kinds of decompositions should be allowed. However, it is of great interest, at least in our opinion, to understand to what extent forbidding an induced subgraph in a graph impacts the global structure of the graph. In the last few years, we have been studying $\mathcal{F}$-free graphs for different families $\mathcal{F}$, in an attempt to get some insight into this question. In this paper we will describe some of the theorems we came up with, and try and emphasize the similarities among them.

Let us now mention a conjecture of Erdős and Hajnal [27], that, in a sense, is concerned with the same question, namely whether forbidding a certain induced subgraph has a global effect on a graph:

Conjecture 1.1 *For every graph H, there exists $\delta(H) > 0$, such that if G is an $\{H\}$-free graph, then G contains either a clique or a stable set of size $|V(G)|^{\delta(H)}$.*

In Section 4 we will describe a structural result, that allowed to solve a special case of 1.1, where H is a "bull" (we will give a precise definition later). The bull was one of the smallest subgraph for which the conjecture had not been known, and thus provided an interesting test case.

Finally, let us mention another problem concerning $\mathcal{F}$-free graphs, and that is the question of their recognition. We will focus on cases, where $\mathcal{F}$ consists of subdivision of a given graph, possibly with parity conditions. It turns out that for some such families $\mathcal{F}$, there exist polynomial time algorithms to test whether a given graph is $\mathcal{F}$-free, while for others the recognition problem has been shown to be NP-complete. At the moment we do not understand what causes this difference, but in the last section of this paper we will survey some related results.

This paper is organized as follows. In Section 2 we describe the decomposition theorem for Berge graphs. Section 3 contains the results about claw-free graphs, there we also try to explain the difference between a "composition" theorem and a "decomposition" theorem, and mention some results concerning coloring. Section 4 deals with bull-free graphs and the solution of the Erdős-Hajnal conjecture for them. In Section 5 we introduce the notion of a "trigraph", which is an object, slightly more general than a graph, which was quite useful to us on a number of occasions. Section 6 is about even-hole-free graphs, there we describe a solution to a conjecture of Reed, and a coloring property of even-hole-free graphs that it implies. Finally, in Section 7 we survey some results on testing for the presence of certain induced subgraphs in a given graph.

2 Perfect Graphs

We start with some definitions. A *hole* in a graph is an induced cycle with at lest four vertices, and an *antihole* in a graph is a hole in its complement. The *length* of a hole is the number of edges in it (and the length of an antihole is the length of its complement.) A *path* in G is an *induced* connected subgraph of G which is either a one-vertex graph, or such that exactly two of its vertices have degree one, and all the others have degree two (this definition is non-standard, but very convenient). An *antipath* is an induced subgraph whose complement is a path. The *length* of a path is the number of edges in it (and the length of an antipath is the number of edges in its complement). If P is a path, P^* denotes the set of internal vertices of P, called the *interior* of P; and similarly for antipaths. A path or a hole is called *even* if it has even length, and *odd* otherwise.

A graph is called *Berge* if every hole and antihole in it is even. The goal of this section is to describe a structural result about Berge graphs, that is used in [9] in order to prove Berge's Strong Perfect Graph Conjecture [2]:

Conjecture 2.1 *A graph is perfect if and only if it is Berge.*

We fist define the basic graphs. We say that G is a *double split graph* if $V(G)$ can be portioned into four sets $\{a_1, \dots, a_m\}$, $\{b_1, \dots, b_m\}$, $\{c_1, \dots, c_n\}$, $\{d_1, \dots, d_n\}$ for some $m, n \geq 2$, such that:

- a_i is adjacent to b_i for $1 \leq i \leq m$, and c_j is nonadjacent to d_j for $1 \leq j \leq n$
- there are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$, and all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, d_{j'}\}$ for $1 \leq j < j' \leq n$
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and these two edges have no common end.

(The name is because such a graph can be obtained from what is called a "split graph" by doubling each vertex). The *line graph* $L(G)$ of a graph G has vertex set the set $E(G)$ of edges of G, and $e, f \in E(G)$ are adjacent in $L(G)$ if they share an end in G. Let us say a graph G is *basic* if either G or $\overline{G}$ is bipartite or is the line graph of a bipartite graph, or is a double split graph. (Note that if G is a double split graph then so is $\overline{G}$.)

Now we turn to the various kinds of decomposition. If $X \subseteq V(G)$ we denote the subgraph of G induced on X by $G|X$. First, a special case of the "2-join" due to Cornuéjols and Cunningham [25] — a *proper 2-join* in G is a partition (X_1, X_2) of $V(G)$ such that there exist disjoint nonempty $A_i, B_i \subseteq X_i$ ($i = 1, 2$) satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2, and every vertex of B_1 is adjacent to every vertex of B_2,
- there are no other edges between X_1 and X_2,
- for $i = 1, 2$, every component of $G|X_i$ meets both A_i and B_i, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is a path joining the members of A_i and B_i, then it has odd length ≥ 3.

If $X \subseteq V(G)$ and $v \in V(G)$, we say v is X*-complete* if v is adjacent to every vertex in X (and consequently $v \notin X$), and v is X*-anticomplete* if v has no neighbors in X. If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex in X is Y-complete; and being *anticomplete* to Y is defined similarly. Our second decomposition is a slight variation of the "homogeneous pair" of Chvátal and Sbihi [20]. Let A, B be two disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. A *proper homogeneous pair* in G is a homogeneous pair (A, B) such that, if A_1, A_2 respectively denote the sets of all A-complete vertices and all A-anticomplete vertices in $V(G)$, and B_1, B_2 are defined similarly, then:

- every vertex in A has a neighbor in B and a nonneighbor in B, and vice versa
- the four sets $A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_1, A_2 \cap B_2$ are all nonempty.

Let A, B be disjoint subsets of $V(G)$. We say the pair (A, B) is *balanced* if there is no odd path between nonadjacent vertices in B with interior in A, and there is no odd antipath between adjacent vertices in A with interior in B. A set $X \subseteq V(G)$ is *connected* if $G|X$ is connected (so $\emptyset$ is connected); and *anticonnected* if $\overline{G}|X$ is connected. The third kind of decomposition we use is due to Chvátal [19] — a *skew partition* in G is a partition (A, B) of $V(G)$ such that A is not connected and B is not anticonnected. In order for our result to be useful in the proof of the strong perfect graph conjecture, we had to restrict ourselves to only using balanced skew-partition.

The main result of [9] is the following:

Theorem 2.2 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join, or G admits a proper homogeneous pair, or G admits a balanced skew partition.*

Now, since all basic graphs are perfect (for bipartite graphs it is trivial; for line graphs of bipartite graphs it is a theorem of König [28]; for their complements it follows from a theorem of Lovász [29], although originally these were separate theorems of König; and for double split graphs we leave it to the reader.); and none of the decompositions can occur in a minimum size counterexample to 2.1 (for 2-joins this is a result due to to Cornuéjols and Cunningham [25], for proper homogeneous pairs due to Chvátal and Sbihi [20], and for balanced skew partition due to the authors together with Robertson and Thomas [9]), it follows that no graph is a minimum size counterexample to 2.1, and therefore 2.1 is true.

However, one can ask for more from a theorem of the kind of 2.2. While 2.2 provides enough insight into Berge graphs in order to prove 2.1, it does not give a "recipe" that allows to build all Berge (or, equivalently, perfect) graphs, starting from some "easy" basic pieces. (Unlike, say, the easy theorem we mentioned in the introduction, that says that every graph with no path of length two can be built by taking disjoint unions of complete graphs.) The problem lies, unfortunately, in the most elegant of all the decompositions we used, the balanced skew-partition. We have tried, but failed, to "reverse" it, that is turn it into a way to combine two smaller perfect graphs together, to obtain a bigger perfect graph. This is also the reason why 2.2 does not immediately imply the existence of a polynomial time recognition algorithm for Berge graphs (we will come back to this in Section 7).

Another natural question to ask is whether all the basic classes and decompositions used in 2.2 are necessary. The answer to this question turns out to be "no", because the use of the proper homogeneous pair decomposition can be avoided and 2.2 can be strengthened as follows (this is the main result of [3]):

Theorem 2.3 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join, or G admits a balanced skew partition.*

In Section 5 we will explain the main idea of the proof of 2.3, which was to consider more general objects, called "Berge trigraphs".

3 Claw-free Graphs

A *claw* is the complete bipartite graphs $K_{1.3}$ (a vertex with three pairwise non-adjacent neighbors). A graph is called *claw-free* if it is $\{K_{1,3}\}$-free. One well known class of claw-free graphs is the class of line graphs; some properties of line graphs have been generalized to all claw-free graphs. (For example, Edmond's matching algorithm, that allows to find a maximum weight stable set in a line graphs [26], was generalized by Minty to solve the maximum weight stable set problem in claw-free graphs [30].)

However, the question "what does a general claw-free graph look like" remained open, and we are now in the process of writing a series of papers answering it [11], [12],[13],[14], [15]. Unlike in the case of perfect graphs, here we were able to prove a a theorem that says: every claw-free graph can be built starting from graphs that belong to certain explicitly constructed basic classes, and gluing them together by prescribed operations; and all graphs built in this way are claw-free. We do not have a formal way to tell what graphs we should allow to count as basic (can the class of all claw-free graphs be basic?), or what operations are acceptable (is the operation "add a vertex to a graph that has already been constructed provided it does not introduce a claw" allowed?), but we do think that we managed to put our finger on an interesting structural property of claw-free graphs. Informally, all of our basic graphs are "explicit constructions", meaning graphs defined by a list of adjacencies, rather than properties (e.g. being claw-free). For the operations, our criterion was to "eliminate guessing". That means, roughly, that instead of constructing just all claw-free graphs, we constructed pairs (G, X), where G is a claw-free graph, and X is a "handle" (usually a subset of the vertex set of G, or, in some cases, a partition of the vertex set), that will be used when we combine G with another claw-free graph in the construction process. The question of formalizing these ideas is of great interest to us.

The first step in proving the theorem described in the previous paragraphs is obtaining a result similar to 2.2 for the class of claw-free graphs. First we need a number of definitions.

Let G be a graph. If $X \subseteq V(G)$, the graph obtained from G by deleting X is denoted by $G \setminus X$. A clique of size three is a *triangle*, and a stable set of size three is a *triad*. Distinct vertices u, v of G are *twins* (in G) if they are adjacent and have exactly the same neighbors in $V(G) \setminus \{u, v\}$.

Next, let us explain the decompositions. The first is just that there are two vertices in G that are twins, or briefly, "G admits twins". For the second, let (A, B) be a homogeneous pair, such that A, B are both cliques, and A is neither complete nor anticomplete to B. In these circumstances we call (A, B) a *W-join*. (Note that there is no requirement that $A \cup B \neq V(G)$. If the complement of G is bipartite, then G admits a W-join except in degenerate cases.) The pair (A, B) is *nondominating* if some vertex of $G \setminus (A \cup B)$ has no neighbor in $A \cup B$; and it is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and there are no edges between V_1 and V_2. We call the pair (V_1, V_2) a *0-join* in G. Thus G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 partition $V(G)$, and for $i = 1, 2$ there is a subset

$A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a clique, and $A_i, V_i \setminus A_i$ are both nonempty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- for $i = 1, 2$, A_i, B_i are cliques, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty
- A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2, and
- V_0 is a clique; and for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

We call the triple (V_1, V_0, V_2) a *generalized* 2*-join*, and if $V_0 = \emptyset$ we call the pair (V_1, V_2) a 2*-join*. (This is closely related to, but not the same as, the 2-join from the previous section.)

We use one more decomposition, the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

- For $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i
- V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2.
- V_1, V_2 are both nonempty.

In these circumstances we say that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2, B_1 \cup C_2$ and $C_1 \cup A_2$ are three cliques with union $V(G)$, and consequently no graph G with a stable set of size four is expressible as a hex-join.

Next, we list some basic classes of graphs.

- **Line graphs.** We say $G \in \mathcal{S}_0$ if G is isomorphic to a line graph.
- **The icosahedron.** This is the unique planar graph with twelve vertices all of degree five. For $0 \le k \le 3$, $icosa(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. We say $G \in \mathcal{S}_1$ if G is isomorphic to $icosa(0)$, $icosa(-1)$ or $icosa(-2)$.
- **The graphs $\mathcal{S}_2$.** Let G be the graph with vertex set $\{v_1, \dots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6 is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5, and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. We say $H \in \mathcal{S}_2$ if H is isomorphic to $G \setminus X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$.

- **Circular interval graphs.** Let Σ be a circle and let $F_1, \dots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0,1]$, and no three with union Σ. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only $v_1, v_2 \in F_i$ for some i. Such a graph is called a *circular interval graph.* If $\bigcup_{i=1}^{k} \neq \Sigma$, we say that G is a *liner interval graph.* We write $G \in \mathcal{S}_3$ if G is a circular interval graph. .

- **An extension of $L(K_6)$.** Let H be the graph with seven vertices $h_0, \dots, h_6$, in which $h_1, \dots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let G be the graph obtained from the line graph $L(H)$ of H by adding one new vertex, adjacent precisely to the members of $V(L(H)) = E(H)$ that are not incident with h_1 in H. Then G is claw-free. Let $\mathcal{S}_4$ be the class of all graphs isomorphic to induced subgraphs of G.

- **The graphs $\mathcal{S}_5$.** Let $n \geq 0$. Let $A = \{a_1, \dots, a_n\}, B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be three cliques, pairwise disjoint. For $1 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j$, and let c_i be adjacent to a_j, b_j if and only if $i \neq j$. Let d_1, d_2, d_3, d_4, d_5 be five more vertices, where d_1 is $A \cup B \cup C$-complete; d_2 is complete to $A \cup B \cup \{d_1\}$; d_3 is complete to $A \cup \{d_2\}$; d_4 is complete to $B \cup \{d_2, d_3\}$; d_5 is adjacent to d_3, d_4; and there are no more edges. Let the graph just constructed be G. We say $H \in \mathcal{S}_5$ if (for some n) H is isomorphic to $G \setminus X$ for some $X \subseteq A \cup B \cup C$.

- **2-simplicial graphs of antihat type.** Let $n \geq 0$. Let $A = \{a_0, a_1, \dots, a_n\}, B = \{b_0, b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be three cliques, pairwise disjoint. For $0 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. Let the graph just constructed be G. We say $H \in \mathcal{S}_6$ if (for some n) H is isomorphic to $G \setminus X$ for some $X \subseteq A \cup B \cup C$, and then H is said to be 2-*simplicial of antihat type.*

- **Antiprismatic graphs.** Let us say a graph is *antiprismatic* if for every three pairwise nonadjacent vertices u, v, w, every vertex different from u, v, w is adjacent to exactly two of them. Antiprismatic graphs are claw-free, and we gave a structural description of them in the first two papers of the series [11],[12]. We will not include it here for reasons of space.

We can now state the theorem:

Theorem 3.1 *Let G be claw-free. Then either*

- $G \in \mathcal{S}_0 \cup \dots \cup \mathcal{S}_6$, *or*

- *G admits either twins, a nondominating W-join, a coherent W-join, a* 0*-join, a* 1*-join, a generalized* 2*-join, or a hex-join, or*

- *G is antiprismatic.*

Similarly to 2.2, we call 3.1 a "decomposition theorem. But, unlike 2.2, 3.1 can be converted into what we call a "composition theorem", meaning a theorem that allows us to build all claw-free graphs. This is done by "reversions" the decompositions, to obtain "compositions". For example, every claw-free graph that admits twins can be obtained from a smaller claw-free graph by adding a new adjacent copy of an existing vertex. Moreover, given a claw-free graph, one can do this operation, and the resulting graph will be claw-free, no matter what vertex has been replicated (so there is no need to guess the "right" vertex to replicate). Reversing other operations is more difficult, and the general result we obtain for claw-free graphs is quite complicated, and we will not include it here.

Instead, let us consider a subclass of claw-free graphs, the class of "quasi-line" graphs. These are graphs in which the vertex set of the neighborhood of every vertex is the union of two cliques. Let W_i be the graph consisting of an antihole H of length i, and a $V(H)$-complete vertex v (therefore $v \notin V(H)$); and let $\mathcal{F}$ be the family of graphs consisting of the claw, together with all W_i with odd $i \geq 5$. Then G is a quasi-line graph if and only if G is $\mathcal{F}$-free.

Circular interval graphs are quasi-line graphs, but there is another way to construct quasi-line graphs, that we explain next. A vertex $v \in V(G)$ is *simplicial* if the set of neighbors of v is a clique. A *strip* (G, a, b) consists of a claw-free graph G together with two designated simplicial vertices a, b called the *ends* of the strip. For instance, if G is a linear interval graph, with vertices $v_1, \ldots, v_n$ in order and with $n > 1$, then v_1, v_n are simplicial, and so (G, v_1, v_n) is a strip, called a *linear interval strip*.

Suppose that (G, a, b) and (G', a', b') are two strips. We compose them as follows. Let A, B be the set of vertices of $G \setminus \{a, b\}$ adjacent in G to a, b respectively, and define A', B' similarly. Take the disjoint union of $G \setminus \{a, b\}$ and $G' \setminus \{a', b'\}$; and let H be the graph obtained from this by adding all possible edges between A and A' and between B and B'. Then H is claw-free.

This method of composing two strips is symmetrical between (G, a, b) and (G', a', b'), but we do not use it in a symmetrical way. We use it as follows. Start with a graph G_0 with an even number of vertices and which is the disjoint union of complete graphs, and pair the vertices of G_0. Let the pairs be $(a_1, b_1), \ldots, (a_n, b_n)$, say. For $i = 1, \ldots, n$, let (G'_i, a'_i, b'_i) be a strip. For $i = 1, \ldots, n$, let G_i be the graph obtained by composing (G_{i-1}, a_i, b_i) and (G'_i, a'_i, b'_i); then the resulting graph G_n is called a *composition* of the strips (G'_i, a'_i, b'_i) $(1 \leq i \leq n)$. For instance, if for each of the strips (G'_i, a'_i, b'_i) , G'_i is a 3-vertex path with ends a'_i, b'_i, then the effect of composing with (G'_i, a'_i, b'_i) is the identification of a_i, b_i; and so the graphs that are compositions of such 3-vertex path strips are precisely line graphs.

It is easy to check that every graph that is the composition of linear interval strips is a quasi-line graph, so this gives us a second construction for quasi-line graphs (and this includes line graphs, since the 3-vertex strip mentioned above is a linear interval strip).

We can prove the following decomposition theorem for quasi-line graphs [16]:

Theorem 3.2 *For every quasi-line graph G, either G is a circular interval graph, or G is a composition of linear interval strips, or G admits a 0-join, or a W-join.*

It is clear how to "reverse" the 0-join decomposition: all one needs to do is take a disjoint union. The W-join decomposition is trickier, but, it turns out, that one can avoid it at the expense of expanding the list of basic graphs. (In order to do that, we use the same idea as in eliminating proper homogeneous pairs from 2.2, and we will explain it later).

Let us now describe the expanded list of basic graphs. We say that a graph G is a *fuzzy circular interval graph* if:

- there is a map ϕ from $V(G)$ to a circle C (not necessarily injective), and
- there is a set of intervals from C, none including another, and such that no point of C is an end of more than one of the intervals, so that
- for u, v in G, if u, v are adjacent then $\{u, v\}$ is a subset of one of the intervals, and if u, v are nonadjacent then u, v are both ends of any interval including both of them (and in particular, if $\phi(u) = \phi(v)$ then u, v are adjacent).

(If also we required ϕ to be injective, this would be equivalent to the definition of a circular interval graph.) If x, y are ends of an interval and one of the sets $\phi^{-1}(x), \phi^{-1}(y)$ has at least two members, then the pair $(\phi^{-1}(x), \phi^{-1}(y))$ is a homogeneous pair of cliques; and these turn out to be the only kinds of homogeneous pairs of cliques that we need. (Fuzzy linear interval strips are defined analogously, with the additional condition that if a, b are the ends of the strip then $\phi(a), \phi(b)$ are different from $\phi(v)$ for all other vertices v of G.)

We prove [16]:

Theorem 3.3 *For every quasi-line graph G, either G is a fuzzy circular interval graph, or G is a composition of fuzzy linear interval strips, or G admits a 0-join.*

This immediately implies the following composition theorem:

Theorem 3.4 *Every quasi-line graph G can be obtained by taking disjoint unions of fuzzy circular interval graphs and graphs that are compositions of fuzzy linear interval strips. Moreover, every graph obtained this way is a quasi-line graph.*

Finally, let us mention, that, similarly to the case of Berge graphs, the property of being claw-free implies that the chromatic number of a graph (and therefore all its induced subgraphs) is bounded by a function of the size of its largest clique. It is easy to see that for a claw-free graph G, $\chi(G) \leq \omega(G)^2$, and this is tight since every graph with no triad is claw-free. However, if we restrict our attention to graphs that are, in some sense, "far away" from being triad-free, a much better bound is true [17]:

Theorem 3.5 *Let G be a connected claw-free graph that contains a triad. Then $\chi(G) \leq 2\omega(G)$.*

The proof of 3.5 uses our structure theorem for claw-free graphs, but if we replace $\chi(G) \leq 2\omega(G)$ by $\chi(G) \leq 4\omega(G)$, there is an easy elementary proof. However, the factor of 2 is tight. 3.5 can be strengthened further if we assume that G is a quasi-line graph [8]:

Theorem 3.6 *Let G be a quasi-line graph. Then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

The proof of 3.6 relies on 3.4, and the factor of $\frac{3}{2}$ is tight.

Curiously, we also a get a theorem similar to 3.5 for graphs whose complements are claw-free [17], and here the proof does not use any of the heavy machinery described earlier in this section.

Theorem 3.7 *Let G be the complement of a connected claw-free graph that contains a triad. Then $\chi(G) \leq 2\omega(G)$.*

4 Bull-free Graphs

The *bull* is the graph B with vertex set

$$\{x_1, x_2, x_3, y, z\}$$

and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

A graph is called *bull-free* if it is $\{B\}$-free. Obvious examples of bull free graphs are graphs with no triangle and graphs with no triad; but there are others. Let us call a graph G an *ordered split graph* if there exists an integer n such that the vertex set of G is the union of a clique $\{k_1, \ldots, k_n\}$ and a stable set $\{s_1, \ldots, s_n\}$, and s_i is adjacent to k_j if and only if $i + j \leq n + 1$. It is easy to see that every ordered split graph is bull-free. A large ordered split graph contains a large clique and a large stable set, and therefore the three classes (triangle-free, triad-free and ordered split graphs) are significantly different.

It turns out, however, that, similarly to claw-graphs, there is a composition theorem for bull-free graphs; all bull-free graphs can be built starting from graphs that belong to a few basic classes, gluing them together by certain operations [4]. The basic classes we need are triangle-free graphs, triad-free graphs, a certain generalization of the ordered split graphs, and a couple of others, that we will not describe here. Let $\mathcal{B}$ denote the set of all bull-free graphs that belong to one of the basic classes. Next we describe some operations, that are used to combine two smaller bull-free graphs together, to obtain a new, larger, bull-free graph.

Operation $\mathcal{O}_1$ is the operation of complementation. The input of $\mathcal{O}_1$ is a graph G_1, and the output is the complement of G_1.

Operation $\mathcal{O}_2$ is the operation of taking disjoint union of two graphs. The input of $\mathcal{O}_2$ is a pair of graphs G_1, G_2, and the output is a new graph G_3, with $V(G_3) = V(G_1) \cup V(G_2)$ and $E(G_3) = E(G_1) \cup E(G_2)$.

Operation $\mathcal{O}_3$ is defined as follows. The input of $\mathcal{O}_3$ is a pair of graphs G_1, G_2, and ordered subsets A_1, B_1 of $V(G_1)$ and A_2, B_2 of $V(G_2)$, with the following properties:

- A_1, B_1, A_2, B_2 are stable sets, with $|A_1| = |A_2|$ and $|B_1| = |B_2|$.
- A_1 is complete to B_1, and A_2 to B_2.
- For $i = 1, 2$ let G'_i be the graph obtained from G_i by adding two new vertices a_i, b_i such that $\{a_i\}$ is complete to A_i and $\{b_i\}$ to B_i, and there are no other edges incident with a_i, b_i. Then both G'_1 and G'_2 is bull free.

Under these circumstances, the result of applying $\mathcal{O}_3$ to $G_1, G_2, A_1, B_1, A_2, B_2$ is the graph G_3, obtained from the disjoint union of G_1 and G_2 by identifying the corresponding vertices of A_1 and A_2, and the corresponding vertices of B_1 and B_2.

Operation $\mathcal{O}_4$ is the operation of substitution. The input of $\mathcal{O}_4$ is a pair of graphs G_1, G_2 and a vertex $v \in V(G_1)$. The output is a new graph G_3, with $V(G_3) = V(G_1) \cup V(G_2) \setminus \{v\}$ and $E(G_3) = E(G_1 \setminus \{v\}) \cup E(G_2) \cup \{xy : x \in V(G_1) \setminus \{v\},\ y \in V(G_2),$ and $xv \in E(G_1)\}$. Please note that unlike all the previous operations, $\mathcal{O}_4$ is not symmetric between G_1 and G_2.

The main result of [4] is the following:

Theorem 4.1 *Let G be a bull-free graph. Then either $G \in \mathcal{B}$, or G can be obtained starting from graphs in $\mathcal{B}$, by repeated applications of operations $\mathcal{O}_1, \ldots, \mathcal{O}_4$. Conversely, every graph obtained in this way is bull-free.*

As in the case of claw-free graphs, we start by proving a "decomposition" theorem for bull-free graphs, that is a theorem that says that every bull-free graph is either basic, or admits a decomposition. Reversing the decompositions yields the operations $\mathcal{O}_1, \ldots, \mathcal{O}_4$. Another similarity with claw-free graphs (and quasi-line graphs) is that one can state a decomposition theorem for bull-free graphs that uses very few basic classes, but needs a decomposition similar to a W-join. The conditions under which introducing a homogeneous pair in a bull-free graph produces another bull-free graph are quite complicated, and do not seem to be far from saying "add a vertex if it does not create a bull". But again, by considering the more general structure of "bull-free trigraphs", we were able to eliminate the use of homogeneous pairs, at the expense of expanding the list of basic classes.

In [10] Safra and the first author use 4.1 to settle the Erdös-Hajnal conjecture for the case when H is the bull, by proving the following:

Theorem 4.2 *Let G be a bull-free graph. Then G contains a stable set or a clique of size $|V(G)|^{\frac{1}{4}}$.*

In order to prove 4.2, it is shown inductively, using 4.1, that every bull-free graph G can be covered by at most $|V(G)|^{\frac{1}{2}}$ induced subgraphs of G, each of which is perfect. It follows that there exists an induced subgraph H of G, containing at least $|V(G)|^{\frac{1}{2}}$ vertices, and such that H is perfect. Consequently, H contains a stable set or a clique of size $|V(H)|^{\frac{1}{2}} \geq |V(G)|^{\frac{1}{4}}$, and 4.2 follows.

5 Trigraphs

The goal of this section is to introduce the notion of a "trigraph". A *trigraph* T is a 4-tuple $(V(T), E(T), S(T), N(T))$ where V is the *vertex set* of T and every unordered pair of vertices belongs to one of the three disjoint sets: the *strong edges* $E(T)$, the *strong non-edges* $N(T)$ and the *switchable pairs* $S(T)$, and such that every vertex of T belongs to at most one switchable pair. Let us say that two vertices u, v of T are *strongly adjacent* if $\{u, v\}$ is a strong edge, *strongly non-adjacent* if $\{u, v\}$ is a strong non-edge, and *semi-adjacent* if $\{u, v\}$ is a switchable pair. In this notation a graph can be viewed as a trigraph with an empty set of switchable pairs. A *realization* of a trigraph $T = (V(T), E(T), S(T), N(T))$ is a graph $G = (V(G), E(G))$ such that $V(G) = V(T)$, and $E(T) \subseteq E(G) \subseteq S(T)$.

Thus trigraphs are objects, generalizing graphs, and on a number of occasions, considering them instead of graphs, when dealing with classes of graphs defined by forbidding certain induced subgraphs, allowed us to prove stronger theorems for the class of *graphs* we were interested in.

We use trigraphs while dealing with Berge graphs, claw-free graphs, and bull-free graphs. In all three cases the situation is as follows: we are able to prove a theorem that said "every Berge (claw-free, bull-free) graph either belongs to one of a few basic classes, or admits one of a few decompositions", where one of the decompositions was a "homogeneous pair decomposition" (these are really a few different decomposition, depending on the exact class of graphs in question, but all of them have in common that the graph admits a homogeneous pair). In all cases, it is possible to define an operation that is the "reverse" of the homogeneous pair decomposition, let us call it an *expansion*. Given a list $\mathcal{L}$ of basic graphs, we call an *expanded basic* graph every graph that can be obtained from a graph in $\mathcal{L}$ by performing expansions. Now we would like to strengthen the theorem, and prove that every Berge (claw-free, bull-free) graph is either an expanded basic graph, or admits one of a few decompositions (none of which is a homogeneous pair decomposition). The last step is to describe explicitly all expanded basic graphs, thus eliminating the use of homogeneous pairs.

The approach we use is as follows. Let $\mathcal{F}$ be a family of graphs. Let us say that the family $\mathcal{T}$ of trigraphs is $\mathcal{F}$-free, if every graph that is a realization of a trigraph in $\mathcal{T}$ is $\mathcal{F}$-free. Now, instead of considering Berge (claw-free, bull-free) graphs, we turn to Berge (claw-free, bull-free) trigraphs. For every decomposition we expect to use for the class of $\mathcal{F}$-free graphs, we define its trigraph analogue, in such a way that if two vertices of a graph were specified as being adjacent in the graph decomposition, they are specified as being strongly adjacent in the trigraph decomposition, and the same for pairs that were specified to be non-adjacent. For example, the graph decomposition "G is disconnected", becomes the trigraph decomposition "$V(T)$ can be partition into two non-empty subsets V_1 and V_2, such that every vertex of V_1 is strongly non-adjacent to every vertex of V_2". For every basic class $\mathcal{C}$ of graphs, the corresponding basic class of trigraphs consists of all $\mathcal{F}$-free trigraphs T, such that some graph of $\mathcal{C}$ is a realization of T.

Now suppose we were able to prove that every $\mathcal{F}$-free graph is either basic or admits one of the decompositions $D_1, \ldots, D_k$, or admits a homogeneous pair. In all the cases that we have considered, we could then prove that every $\mathcal{F}$-free trigraph is either basic (in the trigraph sense explained above) or admits (the trigraph analogue of) one of the decompositions $D_1, \ldots, D_k$, or admits (the trigraph analogue of) a homogeneous pair. So far this is the same theorem, only in slightly grater generality. It turns out, however, that this more general version allows us to prove the strengthened theorem for graphs that we are interested in.

It is enough to prove that every $\mathcal{F}$-free trigraph is either basic (in the trigraph sense) or admits (the trigraph analogue of) one of the decompositions $D_1, \ldots, D_k$. Here is the outline of the proof. Suppose this is false and let T be a trigraph that is not basic, and does not admits any of the decomposition $D_1, \ldots, D_k$, and subject to that with $|V(T)|$ minimum. By the theorem we know for trigraphs, T admits (the trigraph analogue of) a homogeneous pair (A, B). So every vertex of $V(T) \setminus (A \cup B)$

is either strongly adjacent to every vertex of A, or strongly anti-adjacent to every vertex of A, and the same for B. Let T' be the trigraph obtained from T by replacing the set A by a new vertex a, and the set B by a new vertex b, such that

- a is semi-adjacent to b in T'
- for every vertex $v \in V(T) \setminus (A \cup B)$, v is strongly adjacent to a in T' if v is strongly adjacent to every vertex of A in T, and v is strongly anti-adjacent to a in T' if v is strongly anti-adjacent to every vertex of A in T, and
- for every vertex $v \in V(T) \setminus (A \cup B)$, v is strongly adjacent to b in T' if v is strongly adjacent to every vertex of B in T, and v is strongly anti-adjacent to b in T' if v is strongly anti-adjacent to every vertex of B in T

By the minimality of $|V(T)|$, it follows that T' is either basic, or admits one of the decompositions $D_1, \ldots, D_k$. But then, since the pair $\{a, b\}$ is a switchable pair of T', the adjacency between the vertices a and b was not specified in the definition of the decomposition, and therefore T exhibits the same kind of behavior as T', meaning that T is either basic, or admits the same kind of decomposition as T' (this is true with a few exceptions due to certain non-triviality conditions, but the "bad" cases can be dealt with separately). This, however, is a contradiction to the way T was chosen. This completes the proof.

At first it seems that instead of using trigraphs, one could redefine the decompositions and say the whole proof in terms of graphs only. We would like to remark that despite a certain amount of effort invested in this approach, we were unable to come up with a consistent set of definitions, and so the idea of using trigraphs seems crucial.

6 Even-hole-free Graphs

In this section we discuss the family of even-hole-free graphs; these are $\mathcal{F}$-free graphs where $\mathcal{F}$ is the family of all cycles of even length. (Similarly, *odd-hole-free graphs* are graphs with no induced odd cycles of length at least five). Unfortunately, for even-hole-free graphs we do not have a composition theorem similar to 3.4 or 4.1. The best known result of this kind is a theorem similar to 2.2, due to Conforti, Cornuéjols, Kapoor and Vušković [21], that states that every even-hole-free graph is either basic or admits a decomposition. This theorem was then used in [22] to design a polynomial time recognition algorithm for the class of even-hole-free graphs.

However, the following conjecture of Reed [32] remained open (a *bisimplicial* vertex in a graph is a vertex whose set of neighbors is the union of two cliques):

Conjecture 6.1 *Every non-null even-hole-free graph has a bisimplicial vertex.*

This conjecture is proved by Addario-Berry, Havet, Reed and the authors in [1]. At first we directed out effort to trying to find a composition theorem for even-hole-free graphs, but were unsuccessful. It still seemed, however, that proving a statement stronger than 6.1, that would contain some information about the location of the bisimplicial vertices in the graph, would allow us to apply induction and prove 6.1. This direction was a lot more fruitful, and eventually lead to a proof of 6.1, that we now outline.

Let us start with some definitions. Let G be a graph and let S be a subset of $V(G)$. The *neighborhood* of S, denoted by $N_G(S)$, is S together with the set of all vertices of $V(G) \setminus S$ with a neighbor in S. The *non-neighborhood* of S is the set $V(G) \setminus N_G(S)$. If S consists of a single vertex s, we write $N_G(s)$ instead of $N_G(\{s\})$. A set S of vertices in a graph G is called *dominating (in G)* if $N_G(S) = V(G)$, and *non-dominating* otherwise. An induced subgraph H of G is *dominating* if $V(H)$ is dominating, and *non-dominating* otherwise; we denote by $N_G(H)$ the set $N_G(V(H))$. The stronger statement we ended up proving is the following:

Theorem 6.2 *Let G be an even-hole-free graph. Then both the following statements hold:*

1. *If H is a non-dominating hole in G, then some vertex of $V(G) \setminus N_G(H)$ is bisimplicial in G.*

2. *If K is a non-dominating clique in G of size at most two, then some vertex of $V(G) \setminus N_G(K)$ is bisimplicial in G.*

Clearly the second statement of 6.2 with $K = \emptyset$ implies 6.1. We remark that the second statement of 6.2 is false if we replace "at most two" by "at most three". The graph obtained from K_4 by choosing a vertex and subdividing once the edges incident with it is a counterexample.

Let us now describe the proof of 6.2. The proof uses induction. Let G be a graph such that 6.2 holds for all smaller graphs. First we suppose that G fails to satisfy the first statement, that is there is a non-dominating hole H in G, but there is no bisimplicial vertex in the non-neighborhood of $V(H)$. Now the idea is to examine the neighborhood of $V(H)$ and try to find what we call a "useful cutset" in G, that is, a subset C of $V(G)$ and an edge e with both ends in C such that

- $V(G) \setminus C$ is the disjoint union of two non-empty sets, L and R, anticomplete to each other

- $C \subseteq N(e)$ and the non-neighborhood of e in the graph $G|(C \cup R)$ is a non-empty subset of the non-neighborhood of $V(H)$ in G.

If we find such a cutset C, then it follows, from the minimality of G, that R contains a vertex v which is bisimplicial in $G|(C \cup R)$; and since L is anticomplete to R, it follows that v is a bisimplicial vertex of G, which is a contradiction.

Unfortunately, we do not always succeed in finding a useful cutset; sometimes we have to make do with a set C and a list $u_1, .., u_k, v_1, .., v_k$ of vertices of C (possibly with repetitions) where u_i is non-adjacent to v_i in G for every $1 \leq i \leq k$, such that:

- $V(G) \setminus C$ is the disjoint union of two non-empty sets, L and R, anticomplete to each other

- the graph G' obtained from $G|(R \cup C)$ by adding the edge u_iv_i for every $1 \leq i \leq k$ is even-hole-free

- For some edge e of G', $C \subseteq N_{G'}(e)$, and the non-neighborhood of e in the G' is a non-empty subset of the non-neighborhood of $V(H)$ in G

- if v is a bisimplicial vertex of G' contained in the non-neighborhood of e, then v is bisimplicial in G.

Having found such a set C etc, the same argument as in the case of a "genuine" useful cutset leads to a contradiction.

So G satisfies the first statement of 6.2. Suppose it fails to satisfy the second. This means that there is a non-dominating clique K of size at most two in G with no bisimplicial vertex in its non-neighborhood. An easy argument shows that there is a hole H of G such that K is included in $V(H)$. Since the first assertion of the theorem holds for G, we deduce that H is dominating in G. Now we can examine the structure of G relative to H, and again find variations on the idea of a useful cutset, such as the one described above, that lead to a contradiction. So G satisfies the second statement of 6.2 too. This completes the inductive proof.

A graph G is called *odd-signable* if there exists a function $f : E(G) \to \{0, 1\}$ such that $\sum_{e \in E(H)} f(e)$ is odd for every hole H of G. It is natural to ask whether 6.1 is true if we replace "even-hole-free" by "odd-signable". The answer to this question is "no", and the six vertex graph which is the 1-skeleton of the cube is a counterexample.

Finally, we would like to point out an easy corollary of 6.1, that, similarly to the case of perfect graphs, claw-free graphs and quasi-line graphs, establishes a connection between the property of being $\mathcal{F}$-free, and the fact the the chromatic number of the graph (and therefore if all induced subgraphs) is bounded by a function of the size of the largest clique.

Theorem 6.3 *Let G be an even hole free graph. The $\chi(G) \leq 2\omega(G)/$*

Proof. The proof is by induction on $|V(G)|$. By 6.1 there exists a bisimplicial vertex v in G. The graph G' obtained from G by deleting v is another even hole free graph, $\omega(G') \leq \omega(G)$, and, inductively, G' can be properly colored with at most $2\omega(G)$ colors. Let c be such a coloring of G'. Since v is bisimplicial in G, $|N_G(v)| \leq 2\omega(G)$ and at least one of the 2ω colors does not appear in $N_G(v) \setminus \{v\}$ in c. Now v can be colored with this color, thus extending c to a proper coloring of G with at most $2\omega(G)$ colors. This proves 6.3.

7 Detecting Induced Subgraphs

Given an infinite family $\mathcal{F}$ of graphs, it is natural to ask whether one can test in polynomial time if a given graph G is $\mathcal{F}$-free. In this section, will survey some known results in this direction. For brevity, let us say "testing for $\mathcal{F}$" when we mean "testing for being $\mathcal{F}$-free". In all cases the family $\mathcal{F}$ we consider consists of subdivisions of a given graph, possibly with some parity conditions. It turns out that even in this restricted setting, testing for $\mathcal{F}$ can be done in polynomial time for families $\mathcal{F}$, and can be shown to be NP-complete for others. At the moment we do not know what the reason for this difference in behavior is.

A *pyramid* is a graph consisting of a triangle $\{b_1, b_2, b_3\}$, called the *base*, a vertex $a \notin \{b_1, b_2, b_3\}$, called the *apex*, and three paths $P1, P_2, P_3$, such that for $i, j \in 1, 2, 3$

- the ends of P_i are a and b_i,

- if $i \neq j$ then P_i is disjoint from P_j and the only edge between $V(P_i)$ and $V(P_j)$ are b_ib_j, and
- at most one of P_1, P_2, P_3 has length one.

In this case we say that the pyramid is *formed* by the paths P_1, P_2, P_3.

Let $\mathcal{P}$ be the family of all pyramids. It turns out that testing for $\mathcal{P}$ is relatively easy, and can be done in time $O(|V(G)|^9)$ [5]. The idea is as follows. If G contains a pyramid, then it contains a pyramid P with the number of vertices smallest. We are going to "guess" some of the vertices of P in G, then find shortest paths in G between pairs of vertices that we guessed that were joined by a path in P, and then test whether the subgraph of G formed by the union of these shortest paths is a pyramid. If the answer is "yes", then G contains a pyramid, and we stop. Surprisingly, it turns out, that choosing the shortest paths with a little bit of care, we can guarantee that if the answer is "no", then there is no pyramid in P. We call this general strategy of testing for a family $\mathcal{F}$ a *shortest-paths detector* for $\mathcal{F}$.

Let us now be more precise. For $u, v \in V(G)$ we denote by $d_G(u, v)$ the length of the shortest path of G between u and v. If P is a pyramid, formed by three paths P_1, P_2, P_3, with apex a and base $\{b_1, b_2, b_3\}$, we say its *frame* is the 10-tuple

$$a, b_1, b_2, b_3, s_1, s_2, s_3, m_1, m_2, m_3,$$

where

- for $i = 1, 2, 3$, s_i is the neighbor of a in P_i
- for $i = 1, 2, 3$, $m_i \in V(P_i)$ satisfies $d_{P_i}(a, m_i) - d_{P_i}(m_i, b_i) \in \{0, 1\}$.

A pyramid P in G is *optimal* if there is no pyramid P' with $|V(P')| < |V(K)|$.

Theorem 7.1 *[5] Let P be an optimal pyramid, with frame $a, b_1, b_2, b_3, s_1, s_2, s_3, m_1, m_2, m_3$. Let S_1, T_1 be the subpaths of P_1 from m_1 to s_1, b_1 respectively. Let F be the set of all vertices nonadjacent to each of s_2, s_3, b_2, b_3.*

1. *Let Q be a path between s_1 and m_1 with interior in F, and with minimum length over all such paths. Then a-s_1-Q-m_1-T_1-b_1 is a path (say P_1'), and P_1', P_2, P_3 form an optimal pyramid.*
2. *Let Q be a path between m_1 and b_1 with interior in F, and with minimum length over all such paths. Then a-s_1-S_1-m_1-Q-b_1 is a path (say P_1'), and P_1', P_2, P_3 form an optimal pyramid.*

Analogous statements hold for P_2, P_3.

7.1 can be used to design an algorithm to test for $\mathcal{P}$:

- guess the frame $a, b_1, b_2, b_3, s_1, s_2, s_3, m_1, m_2, m_3$ of an optimal pyramid P of G,
- find shortest paths between m_1 and b_1, and between m_1 and s_2, not containing any neighbors of $s_2, s_3, b_2, and b_3$; do the same for $m_2.b_2, s_2$ and m_2, b_3, s_3,

- test if the union of the six shortest paths, together with the vertex a forms a pyramid.

Now, by 7.1, the answer if "yes", if and only if G contains a pyramid. The algorithm in [5] is similar, it was modified a little to bring the running time down to $O(|V(G)|)^9$.

The main result of [5] is a polynomial time algorithm for testing if a graph is Berge (and therefore perfect). Since every pyramid contains an odd hole, it follows that every odd-hole-free, and therefore every Berge, graph is $\mathcal{P}$-free.

Even though the algorithm in [5] was found after 2.2 had been proved, it does not use 2.2. The idea in [5] is to use the shortest-path detector for odd holes. Unfortunately, there does not seem to be a theorem similar to 7.1 for odd holes, and so, first, the graph needs to be "prepared" for using a shortest-paths detector. The first step is to test for $\mathcal{P}$, and a few other families $\mathcal{F}$, that are easy to test for, and such that every Berge graphs is $\mathcal{F}$-free. Now we can assume that the graph in question is $\mathcal{F}$-free for all these $\mathcal{F}$. The next step is applying "cleaning", a technique fist proposed in [24]. The idea of cleaning is to find, algorithmically, polynomially many subsets $X_1, \ldots, X_k$ of $V(G)$, such that if G contains an odd hole, then for at least one value of $i \in \{1, \ldots, k\}$ the graph $G_i = G \setminus X_i$ contains an odd hole that can be found using a shortest-paths detector. Finally, applying a shortest-paths detector for odd hole to each of $G_1, \ldots, G_k$, we detect and odd hole if and only if G contains one.

In addition to the algorithm just described, [5] contains another algorithm to test for Bergeness, that instead of a shortest-paths detector for odd holes, uses a decomposition theorem for odd-hole-free graphs from [23], but we will not describe this algorithm here. We remark, that both algorithms in [5] test for Bergeness, and not for the family of odd holes. The question of testing if a graph contains an odd hole is still open. On the other hand, the problem of testing if a graph contains an even hole can be solved in polynomial time. There are two known algorithms. One, due to Conforti, Cornuéjols, Kapoor, and Vušković [22], and the others due to Kawarabayshi and the authors [7]. Both algorithms use cleaning, and then, the former uses a decomposition theorem of [21] for even-hole-free graphs, and the latter a shortest-paths detector.

There are two other kinds of graphs that are somewhat similar to the pyramid, called a "theta" and a "prism". A *theta* in a graph consisting of two nonadjacent vertices s, t and three paths P, Q, R, each between s and t, such that the sets $V(P) \setminus \{s, t\}$, $V(Q) \setminus \{s, t\}$, and $V(R) \setminus \{s, t\}$ are pairwise disjoint, the union of every pair of them is a hole. A *prism* is a graph consisting of two disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three paths P_1, P_2, P_3, with the following properties:

- for $i = 1, 2, 3$, the ends of P_i are a_i and b_i,
- P_1, P_2, P_3 are pairwise disjoint , and
- for $1 \le i < j \le 3$, there are precisely two edges between $V(P_i)$ and $V(P_j)$, namely $a_i a_j$ and $b_i b_j$.

Let $\mathcal{T}$ be the family of all thetas, and $\mathcal{P}r$ the family of all prisms. Then every even-hole-free graph is $\mathcal{T} \cup \mathcal{P}r$-free, and so prisms and thetas play a similar role for even-hole-free graphs to the one that pyramids play for odd-hole-free graphs. It turns out, however, that, unlike $\mathcal{P}$, the problem of testing for $\mathcal{P}r$ is NP-complete (this is a theorem due to Maffray and Trotignon [31]). On the other hand, testing for $\mathcal{T}$ can be done in polynomial time [18]. The problems of testing for $\mathcal{P} \cup \mathcal{P}r$ and testing for $\mathcal{T} \cup \mathcal{P}r$ can also be solved in polynomial time (see [31] and [6], respectively).

All the algorithms mentioned above use variations on the ideas of cleaning and shortest paths detectors (or decomposition theorems), except one, and that is the algorithm for testing for $\mathcal{T}$. There our approach is different. We prove that a graph is $\mathcal{T}$-free if and only if it admits a certain structure, that can be tested for in polynomial time. This result is particularly pleasing from our point of view, because this is the first time that a composition theorem and an algorithm appear together in the study of graphs with forbidden induced subgraphs.

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