Hadwiger's conjecture for line graphs

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February 1999; revised September 4, 2003

[1]This research was supported by ONR grant N00014-97-1-0512 and NSF grant DMS 9701598

Abstract

We prove that Hadwiger's conjecture holds for line graphs. Equivalently, we show that for every loopless graph G (possibly with parallel edges) and every integer $k \geq 0$, either G is k-edge-colourable, or there are $k+1$ connected subgraphs $A_1, ..., A_{k+1}$ of G, each with at least one edge, such that $E(A_i \cap A_j) = \emptyset$ and $V(A_i \cap A_j) \neq \emptyset$ for $1 \leq i < j \leq k$.

1 Introduction

Hadwiger's conjecture asserts that for every loopless graph G and every integer $k \geq 0$, either G is k-vertex-colourable, or G has K_{k+1} as a minor, that is, there are $k+1$ non-null connected subgraphs $A_1, ..., A_{k+1}$ of G, such that $V(A_i \cap A_j) = \emptyset$ and there is an edge between $V(A_i)$ and $V(A_j)$, for $1 \leq i < j \leq k+1$. This is still open, but in this paper we prove the conjecture for line graphs. (For line graphs of simple graphs the result follows easily from Vizing's theorem and was already known, but here we permit parallel edges.)

Thus, our main result is:

1.1 *For every loopless graph G, and every integer $k \geq 0$ such that G is not k-edge-colourable, there are connected subgraphs $A_1, ..., A_{k+1}$ of G, each with at least one edge, such that $E(A_i \cap A_j) = \emptyset$ and $V(A_i \cap A_j) \neq \emptyset$ for $1 \leq i < j \leq k+1$.*

The referee informs us that Monrad, Stiebitz, Toft and Vizing discussed and obtained a solution to the same problem in September 2002, independent of our work (but knowing that a solution had been obtained). Their solution is similar to ours and they do not intend to publish it.

2 A version of Hadwiger's theorem

We need a version of Vizing's adjacency lemma. Let e_1 be an edge of a loopless graph G (which may have parallel edges), with ends $v_0, v_1 \in V(G)$, let $k \geq 1$ be an integer, and let ϕ be a k-edge-colouring of $G \backslash e_1$. For a vertex v, let

$$\overline{\phi}(v) = \{1, ..., k\} \setminus \{\phi(e) : e \in E(G \backslash e_1) \text{ incident with } v\}.$$

A *Vizing fan* for v_0, e_1, ϕ is a sequence $e_2, ..., e_n \in E(G)$ such that

- for $2 \leq i \leq n$, e_i is incident with v_0; let v_i be its other end
- $v_1, v_2, ..., v_n$ are all distinct
- for all $j \geq 2$ there exists $i < j$ with $i \geq 1$ such that $\phi(e_j) \in \overline{\phi}(v_i)$.

Vizing [1, 2] proved:

2.1 *Let $G, e_1, v_0, v_1, k, \phi$ be as above, where v_0 has degree $\leq k$, and let $e_2, .., e_n$ be a Vizing fan for v_0, e_1, ϕ, where e_i has ends v_0, v_i $(1 \leq i \leq n)$. If G is not k-edge-colourable then the sets*

$$\overline{\phi}(v_0), \overline{\phi}(v_1), ..., \overline{\phi}(v_n)$$

are mutually disjoint.

This has the following corollary. (The number of edges incident with a vertex v is denoted by $\deg(v)$, and if u, v are distinct vertices, $\mu(u, v)$ denotes the number of edges with ends $\{u, v\}$.)

2.2 *Let v_0 be a vertex of a loopless graph G, and let $k \geq 0$ be an integer such that G is not k-edge-colourable, $G\backslash v_0$ is k-edge-colourable, and every vertex of G has degree $\leq k$. There are neighbours $v_1, ..., v_n$ of v_0, all distinct, so that*

$$\sum_{1 \leq i \leq n} (deg(v_i) + \mu(v_0, v_i) - k) \geq 2.$$

Proof. By deleting edges incident with v_0, we may assume that there is an edge e_1 incident with v_0 such that $G\backslash e_1$ is k-edge-colourable and G is not k-edge-colourable. Let e_1 have ends $\{v_0, v_1\}$, let ϕ be a k-edge-colouring of $G\backslash e_1$, and choose a Vizing fan $e_2, ..., e_n$ for v_0, e_1, ϕ, with n maximum. From the maximality of n the set

$$\{\phi(e) : e \in E(G) \text{ incident with } v_0 \text{ but not with any of } v_1, .., v_n\}$$

is disjoint from all the sets $\overline{\phi}(v_1), \overline{\phi}(v_2), ..., \overline{\phi}(v_n)$ (and also trivially from $\overline{\phi}(v_0)$); and by 2.1 the sets $\overline{\phi}(v_0), \overline{\phi}(v_1), ..., \overline{\phi}(v_n)$ are mutually disjoint. Consequently,

$$\left(\deg(v_0) - \sum_{1 \leq i \leq n} \mu(v_0, v_i)\right) + \sum_{0 \leq i \leq n} \big(k - \deg(v_i)\big) + 2 \leq k,$$

that is

$$\sum_{1 \leq i \leq n} \big(\deg(v_i) + \mu(v_0, v_i) - k\big) \geq 2.$$

Finally, we claim that $n \geq 2$. For there exists $c \in \overline{\phi}(v_1)$, because $\deg(v_1) \leq k$ and the edge e_0 is not coloured. Since we cannot properly extend ϕ by giving e_0 the colour c, it follows that $c \notin \overline{\phi}(v_0)$; and hence $n \geq 2$ from maximality. ∎

This in turn has the following corollary.

2.3 *Let G be a loopless graph, and let $k \geq 0$ be an integer such that G is not k-edge-colourable and every vertex has degree $\leq k$. Then there exist distinct vertices u, v, w such that*

$$\min\big(\deg(u), \deg(v)\big) + \mu(v, w) \geq k + 1.$$

Proof. Choose $v_0 \in V(G)$ of maximum degree; we may assume that $G\backslash v_0$ is k-edge-colourable, for otherwise we may delete v_0 and repeat. Let $v_1, ..., v_n$ be as in 2.2, with $n \geq 2$. Then (writing v_{n+1} for v_1)

$$\sum_{1 \leq i \leq n} (\deg(v_i) + \mu(v_0, v_{i+1}) - k) \geq 2$$

and so there exists i with $1 \leq i \leq n$ such that

$$\deg(v_i) + \mu(v_0, v_{i+1}) \geq k + 1.$$

Let $u = v_1, v = v_0, w = v_{i+1}$; then u, v, w are distinct (since $n \geq 2$), and

$$\min\left(\deg(u), \deg(v)\right) + \mu(v, w) = \deg(v_i) + \mu(v_0, v_{i+1}) \geq k + 1$$

as required. ∎

3 The main proof

Proof of 1.1: We proceed by induction on $|V(G)|$. We clain first that we may assume that

(1) *For every two distinct vertices* v_1, v_2*, if* $d = \min(\deg(v_1), \deg(v_2))$ *then there are* d *paths of* G *between* v_1 *and* v_2*, pairwise edge-disjoint.*

For by Menger's theorem there is a partition (X_1, X_2) of $V(G)$ with $v_1 \in X_1$ and $v_2 \in X_2$, such that there are $|\delta(X_1, X_2)|$ pairwise edge-disjoint paths of G between v_1 and v_2, where $\delta(X_1, X_2)$ denotes the set of edges of G with one end in X_1 and the other in X_2. Suppose that $|X_1|, |X_2| \geq 2$. For $i = 1, 2$ let G_i be the graph obtained from G by deleting all edges with both ends in X_i and then identifying all the vertices of X_i in a new vertex. Since G is not k-edge-colourable, it follows that at least one of G_1, G_2 is not k-edge-colourable, say G_1. Since $|X_1| > 1$, it follows that $|V(G_1)| < |V(G)|$, and so from the inductive hypothesis there are pairwise edge-disjoint connected subgraphs $A'_1, ..., A'_{k+1}$ of G_1, each with at least one edge, such that $V(A'_i \cap A'_j) \neq \emptyset$ $(1 \leq i < j \leq k+1)$. From the choice of (X_1, X_2), there are paths $P(e)$ $(e \in \delta(X_1, X_2))$ of G_2, pairwise edge-disjoint, such that $e \in E(P(e))$ $(e \in \delta(X_1, X_2))$ and v_2 belongs to every $P(e)$. For $1 \leq i \leq k+1$, let A_i be the subgraph of G formed by all the edges in A'_i, and the edges in $P(e)$ for each $e \in E(A'_i)$, and all vertices incident with these edges. Then $A_1, ..., A_{k+1}$ satisfy the theorem. So we may assume that $\min(|X_1|, |X_2|) = 1$; but then (1) holds. This proves (1).

If some vertex v has degree $\geq k+1$, let $A_1, ..., A_{k+1}$ be pairwise edge-disjoint connected subgraphs, each with $v \in V(A_i)$ and $E(A_i) \neq \emptyset$; then the theorem is satisfied. We may therefore assume that every vertex has degree $\leq k$. By ref2.3, there are distinct vertices u, v, w such that

$$\min(\deg(u), \deg(v)) + \mu(v, w) \geq k + 1.$$

Let $d = \min(\deg(u), \deg(v))$; then by (1) there are d edge-disjoint paths between u and $\{v, w\}$, and we may choose them so that no edge between v and w belongs to any of them. Then these d paths, together with the $\mu(v, w)$ edges between v and w, form $k + 1$ edge-disjoint connected subgraphs that pairwise intersect, as required. ∎

Remark: In fact this proof shows that if G is not k-edge-colourable and yet every vertex has degree $\leq k$, then there are three distinct vertices u, v, w and $k + 1$ edge-disjoint paths each between two of u, v, w.

References

[1] V.G. Vizing, "On an estimate of the chromatic class of a p-graph", *Diskret. Analiz.* 3 (1964), 25-30.

[2] V.G. Vizing. "Critical graphs with a given chromatic class", *Diskret. Analiz.* 5 (1965), 9-17.