Distant digraph domination

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Abstract

A k-*kernel* in a digraph G is a stable set X of vertices such that every vertex of G can be joined from X by a directed path of length at most k. We prove three results about k-kernels.

First, it was conjectured by Erdős and Székely in 1976 that every digraph G with no source has a 2-kernel $|K|$ with $|K| \leq |G|/2$. We prove this conjecture when G is a "split digraph" (that is, its vertex set can be partitioned into a tournament and a stable set), improving a result of Langlois et al., who proved that every split digraph G with no source has a 2-kernel of size at most $2|G|/3$.

Second, the Erdős-Székely conjecture implies that in every digraph G there is a 2-kernel K such that the union of K and its out-neighbours has size at least $|G|/2$. we prove that this is true if $V(G)$ can be partitioned into a tournament and an acyclic set.

Third, in a recent paper, Spiro asked whether, for all $k \geq 3$, every strongly-connected digraph G has a k-kernel of size at most about $|G|/(k+1)$. This remains open, but we prove that there is one of size at most about $|G|/(k-1)$.

1 Introduction

A *digraph* is a finite directed graph with no loops or parallel edges (it may have directed cycles of length two). If G is a digraph, $X \subseteq V(G)$ is *stable* if there is no edge with both ends in X. In a digraph G, if $X, Y \subseteq V(G)$, we say X *k-covers* Y if for each $y \in Y$, there exists $x \in X$ and a directed path of length at most k from x to y. (If X is a singleton $\{x\}$ we write x for $\{x\}$ here, and the same for Y.) A *k-kernel* in a digraph G is a stable set X of vertices that k-covers $V(G)$. [1]

There are many interesting open questions about k-kernels; for instance, not every digraph has a 1-kernel, but every digraph has a 2-kernel [2], and the following was conjectured by P. L. Erdős and L. A. Székely [4] in 1976 (and remains open):

1.1 The small quasi-kernel conjecture: *Every digraph G with no source has a 2-kernel of size at most $|G|/2$.*

(A *source* is a vertex with in-degree zero.) There is a survey on this conjecture in [3], and the best bound on this seems to be a result of Spiro [7], that every digraph G with no source has a 2-kernel of size at most $|G| - \frac{1}{4}(|G| \log |G|)^{1/2}$, which is of course very far from the conjecture.

We will show below that it is enough to prove 1.1 for *oriented graphs*, that is, digraphs with no directed cycle of length two. If G is a counterexample to 1.1, then since it has a 2-kernel, it has a stable set S with $|S| > |G|/2$; and a natural special case is when $G \setminus S$ is a tournament. Let us say G is a *split digraph* if G is an oriented graph and its vertex set admits a partition into a stable set and a tournament. Ai, Gerke, Gutin, Yeo and Zhou [1] proved that 1.1 holds for split digraphs in which all edges between the tournament and the stable set are directed towards the stable set. Langlois, Meunier, Rizzi, Vialette and Zhou [5] proved that every split digraph G with no sources admits a 2-kernel of size at most $2|G|/3$. In section 2, we strengthen these results:

1.2 *Every split digraph G with no sources admits a 2-kernel K with $|K| \leq |G|/2$.*

Our second result concerns a problem of Spiro [7], who observed that 1.1 implies:

1.3 Conjecture: *In every digraph G, there is a 2-kernel K such that at least half the vertices of G belong to K or have an in-neighbour in K.*

We discuss this in section 3, and prove that it holds for split digraphs, and indeed for digraphs with a vertex set that can be partitioned into a tournament and an acyclic subgraph.

Our third result concerns a different problem of Spiro [7], who asked whether:

1.4 Conjecture: *For all integers $k \geq 3$, every strongly-connected digraph G has a k-kernel of size at most $|G|/(k+1) + O_k(1)$.*

It seems that the best known bound in this case is due to Spiro, in the same paper, who proved that under the hypotheses of 1.4, there is a k-kernel of size at most about $|G|/\log k$. Our third result is that there is one of size at most $|G|/(k-1) + O_k(1)$. This is a consequence of 1.5 below.

Let T be a subdigraph with underlying graph a tree, such that for some vertex r of T, every edge of T is directed away from r in the natural sense. We call T an *arborescence*, and r is its *root*. Every strongly-connected digraph has a subdigraph that is a spanning arborescence (*spanning* means that the arborescence contains all vertices of the digraph). In section 4 we will prove:

[1] In some papers a k-kernel is defined with edges reversed: every vertex of G is joined *to* X by a short directed path.

1.5 *For all integers $k \geq 2$, every digraph G with $|G| > 1$ and with a spanning arborescence has a k-kernel of size at most $1 + (|G| - 2)/(k - 1)$.*

This follows easily from a result about acyclic digraphs (*acyclic* means there is no directed cycle):

1.6 *For every integer $k \geq 1$, if G is an acyclic digraph with $|G| \geq 2$ and with only one source, then G has a k-kernel of size at most $1 + (|G| - 2)/k$.*

This result is tight, as can be seen from the digraph shown in figure 1.

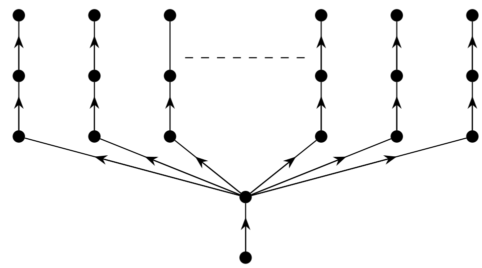

Figure 1: All 3-kernels have size $\geq 1 + (|G| - 2)/3$. For $k > 3$ make the vertical paths longer.

Before we pass to our main topic, let us prove the result mentioned earlier, that it is enough to work with oriented graphs instead of general digraphs. We need:

1.7 *Let G be a digraph. Then either*

- *there is a stable set K of G with $|K| \leq |G|/2$ such that K 2-covers every vertex of G that is not a source; or*
- *there is an induced subdigraph G' of G that is a counterexample to the conjecture 1.1.*

Proof. We proceed by induction on $|G|$. Let A be the set of all sources of G. If $A = \emptyset$, then one of the outcomes is true, so we assume that v is a source of G. Let N be the set of all out-neighbours of v. If $N = \emptyset$, then v has no in-neighbours or out-neighbours and the result follows by deleting v and applying the inductive hypothesis. So we assume that $|N| \geq 1$. The vertices in N are not sources, and so $A \cap N = \emptyset$. Let G' be the digraph obtained from G by deleting $N \cup \{v\}$, and let A' be the set of all sources of G'. Thus $A \setminus \{v\} \subseteq A'$. Moreover, each vertex in $A' \setminus A$ has an in-neighbour in G but not in G', and so is 1-covered in G by N and hence 2-covered by v. If there exists a stable set $K' \subseteq V(G')$ with $|K| \leq |G'|/2$ that 2-covers $V(G') \setminus A'$ in G', then $K \cup \{v\}$ satisfies the first outcome of the theorem. If there is no such K, then the second outcome holds for G' and hence for G. This proves 1.7. ∎

We deduce:

1.8 *Suppose that G is a digraph with no source that does not satisfy 1.1. Then it has a subdigraph with no source and with no directed cycle of length two that also does not satisfy 1.1.*

Proof. We proceed by induction on $|E(G)|$. We may assume that there is a directed cycle with vertex set $\{u, v\}$. Let G_1 be obtained by deleting the edge uv. If G_1 has no source, then it does not satisfy 1.1 (because every 2-kernel of G_1 is a 2-kernel of G), and the result follows from the inductive hypothesis applied to G_1. So we assume that G_1 has a source, and hence v is a source of G_1; and therefore v has no in-neighbour in G except u. Similarly, u has no in-neighbour in G except v. Moreover, v is the only source of G_1. Let us apply 1.7 to G_1.

Suppose the first outcome holds, and hence there is a stable set K of G_1 with $|K| \le |G|/2$ such that K 2-covers (in G_1) every vertex of G_1 that is not a source, that is, every vertex except possible v. In particular, K 2-covers u in G_1. Since v is the only in-neighbour of u in G_1, it follows that either $u \in K$, or K 1-covers v in G_1; and in either case K 2-covers v (and hence 2-covers $V(G)$) in G, a contradiction.

Thus the second outcome of 1.7 holds, and there is an induced subdigraph G' of G_1 that is a counterexample to 1.1. But then the result follows from the inductive hypothesis applied to G'. This proves 1.8. ∎

2 Split digraphs

If G is a digraph, we use $G[X]$ to denote the subdigraph induced on $X \subseteq V(G)$. We say "u is adjacent to v" to mean that u is an in-neighbour of v, and "adjacent from" to mean it is an out-neighbour. A *neighbour* of v means a vertex that is either an in-neighbour or an out-neighbour of v. We sometimes use "G-in-neighbour" to mean "in-neighbour in the digraph G", and so on (this is helpful because we sometimes work with different digraphs that have the same vertex set.) For a vertex v of a digraph G, $N^+_G(v)$ denotes the set of all out-neighbours of v, and $N^-_G(v)$ is its set of in-neighbours. A *split* in an oriented graph G is a pair (S,T), where $S \cup T = V(G)$, $S \cap T = \emptyset$, S is a stable set, and $G[T]$ is a tournament. (We will often write T for $G[T]$.)

In this section we prove 1.2, but it is convenient to prove a slightly stronger statement, that the same conclusion holds just assuming that no vertex in S is a source. Now there is a difficulty, because this is false for the 1-vertex digraph with $S = \emptyset$, but this is the only exception. We will prove:

2.1 *Let (S,T) be a split of an oriented graph G, such that $S \neq \emptyset$ and no vertex in S is a source. Then there is a 2-kernel K with $|K| \le |G|/2$.*

For the proof, we begin with some lemmas. A 2-kernel K is *strong* if for every vertex $v \in T$, either there is a vertex in K that 1-covers v, or a vertex in $K \cap T$ that 2-covers v. (We do not know whether 1.2 remains true if we ask for a strong 2-kernel of size at most $|G|/2$.) If $v \in T$, we say $s \in S$ is a *problem* for v if v is adjacent from s, and v does not 2-cover s, and no non-neighbour of v in S 2-covers s. If v has a problem, then v is contained in no 2-kernel.

2.2 *Let G, T, S be as above, and let $v \in T$. If v is contained in no strong 2-kernel, then there exists $w \in V(G) \setminus \{v\}$, adjacent to v, such that $N^-_G(w) \subseteq N^-_G(v)$; and either $w \in S$ and w is a problem for v, or $w \in T$.*

Proof. Since the set consisting of v and all non-neighbours of v in S is not a strong 2-kernel, there exists $w \in V(G) \setminus \{v\}$ such that v does not 2-cover w, and either $w \in T$ and no non-neighbour of v in S 1-covers w, or $w \in S$ and no non-neighbour of v in S 2-covers w. In the first case, since v does not 2-cover w, $N^-_G(w) \cap T \subseteq N^-_G(v)$. If $s \in N^-_G(w) \cap S$, then since no non-neighbour of v in S 1-covers w, it follows that $s \in N^+_G(v) \cup N^-_G(v)$; and since v does not 2-cover w, $s \notin N^+_G(v)$, and so $s \in N_G^-(v)$. This proves that $N_G^-(w) \subseteq N_G^-(v)$ as required. In the second case, w is a problem for v. Moreover, every in-neighbour of w is an in-neighbour of v: because if $u \in T$ is adjacent to w, then u is not adjacent from v since v does not 2-cover w, and so u is adjacent to v. Hence, again, $N_G^-(w) \subseteq N_G^-(v)$. This proves 2.2. ∎

2.3 *Let G, T, S be as above, and suppose that G, S, T form a smallest counterexample to 2.1. Suppose also that $v \in T$ is contained in no strong 2-kernel, and let w be as in 2.2. If $w \in T$, then there is no problem for w.*

Proof. Suppose that $w \in T$, and $s \in S$ is a problem for w. Let $A = N_G^+(v)$. Since $N_G^-(w) \subseteq N_G^-(v)$, no vertex in A is adjacent to w, and in particular $s \notin A$. Make a digraph G' from G by deleting v and making w complete to A. So G' has no sources.

(1) *$N_{G'}^-(w) \subseteq N_G^-(v)$.*

Let $u \in N_{G'}^-(w)$. So $u \notin A$, and so $u \in N_G^-(w) \subseteq N_G^-(v)$. This proves (1).

Let K be a 2-kernel of G'. We will show that K is also a 2-kernel of G. Certainly it is stable in G.

(2) *$w \notin K$.*

Suppose that $w \in K$. Then $s \notin K$, so there is a directed path P of G', of length one or two, from some $x \in K$ to s. Since s is a problem for w in G, some edge of P is not an edge of G, which is impossible since $s \notin A$. This proves (2).

So $w \notin K$. Since K 2-covers w in G', (1) implies that K 2-covers v in G, and 1-covers v in G if it 1-covers w in G'. Let $a \in A$. We must show that K 2-covers a in G. If $a \in K$ this is true, so we assume there is a directed path P of G' of length one or two, from some $x \in K$ to a. If P is a path of G then K 2-covers a in G, so we may assume that the last edge of P is an edge of G' not in G. But $w \notin K$ and $x \in K$, so $w \neq x$, and therefore P has length two with middle vertex w. By (1), x-v-a is a path of G, so K 2-covers a in G.

This proves that every 2-kernel of G' is a 2-kernel of G. Since G, S, T form a smallest counterexample to 2.1, and G' has fewer vertices than G, and $(S, T \setminus \{v\})$ is a split for G', with $S \neq \emptyset$, and no vertex in S is a source in G', it follows that G' has a 2-kernel of size at most $|G'|/2$; but this is also a 2-kernel for G, which is impossible. This proves that there is no problem for w, and so proves 2.3. ∎

Now we prove the main theorem, which we restate:

2.4 *Let (S, T) be a split of an oriented graph G, such that $S \neq \emptyset$ and no vertex in S is a source. Then there is a 2-kernel K with $|K| \leq |G|/2$.*

Proof. We may assume that G, S, T form a smallest counterexample. Let B be the set of all vertices in T with problems. For each $b \in B$, select a problem z_b for b, and let Z be the set $\{z_b : b \in B\}$. Let Q be the set of all $q \in S \setminus Z$ with $N_G^-(q) \subseteq B$. For each $q \in Q$, it has an in-neighbour in B, since it is not a source; select one such in-neighbour b_q. Similarly, for each $s \in S \setminus (Q \cup Z)$, choose some $t_s \in T \setminus B$ adjacent to s.

For each $z \in Z$, let $\Phi(z)$ be the set of $q \in Q$ such that $z = z_{b_q}$. For each $t \in T \setminus B$, let $\Phi(t)$ be the union of $\{t\}$ and the set of $s \in S \setminus (Q \cup Z)$ such that $t = t_s$. Thus, the sets $\Phi(v)$ $(v \in Z \cap (T \setminus B))$ are pairwise disjoint and have union $V(G) \setminus (B \cup Z)$. Some of the sets $\Phi(z)$ $(z \in Z)$ may be empty.

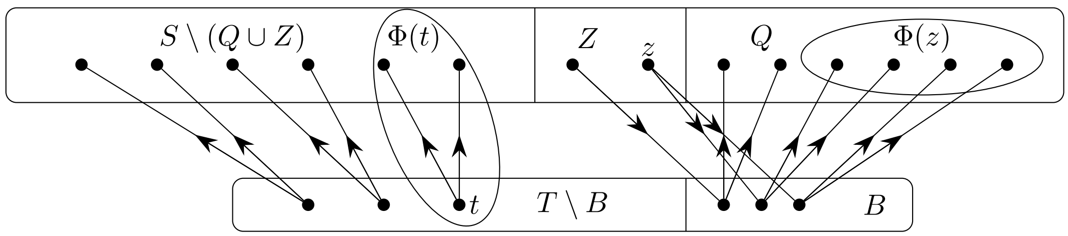

Figure 2: Definitions of $\Phi(z)$ and $\Phi(t)$.

Let H be the oriented graph obtained from $G[(T \setminus B) \cup Z]$ by adding all possible edges from $T \setminus B$ to Z; that is, if $t \in T \setminus B$ and $z \in Z$ are nonadjacent in G then we add an edge tz.

For each $v \in V(H)$, let $N^0_H(v)$ be the set of vertices that are neither out- nor in-neighbours of v (including v itself). Thus $N^0_H(v) = Z$ if $v \in Z$, and $N^0_H(v) = \{v\}$ if $v \in T \setminus B$. Define $\phi^+(v) = \sum_{u \in N^+_H(v)} |\Phi(u)|$ and define $\phi^-(v), \phi^0(v)$ similarly. We call $\phi^-(v) + \phi^0(v)/2$ the *score* of v. If $V(H) = \emptyset$, then $T \setminus B = \emptyset$ and $B = \emptyset$ (since $Z = \emptyset$); so $T = \emptyset$, which implies that $S = \emptyset$ (since there are no sources in S), a contradiction. So $V(H) \neq \emptyset$. We have

$$\sum_{u \in V(H)} |\Phi(u)| \phi^+(u) = \sum_{uw \in E(H)} |\Phi(u)||\Phi(w)| = \sum_{w \in V(H)} |\Phi(w)| \phi^-(w),$$

and therefore

$$\sum_{u \in V(H)} |\Phi(u)|(\phi^-(u) - \phi^+(u)) = 0.$$

We claim that there exists $v \in V(H)$ such that $\phi^+(v) \geq \phi^-(v)$. If $|\Phi(u)|(\phi^-(u) - \phi^+(u)) \neq 0$ for some $u \in V(H)$, then $|\Phi(u)|(\phi^-(u) - \phi^+(u)) > 0$ for some $u \in V(H)$ and the claim is true. If not, then either $|\Phi(u)| = 0$ for each $u \in V(H)$, or $\phi^-(u) - \phi^+(u) = 0$ for some $u \in V(H)$, and in either case the claim is true. This proves that there exists $v \in V(H)$ such that $\phi^+(v) \geq \phi^-(v)$.

Since

$$\phi^+(v) + \phi^-(v) + \phi^0(v) = |G| - |Z| - |B| \leq |G| - 2|Z|,$$

it follows that $\phi^-(v) + \phi^0(v)/2 \leq |G|/2 - |Z|$. Choose $v \in V(H)$ with score as small as possible (and consequently with score at most $|G|/2 - |Z|$).

A vertex in T is *pure-up* if it has no in-neighbour in S. The case when v has score exactly $|G|/2 - |Z|$ is troublesome, so let us first handle that.

(1) *We may assume that either v has score strictly less than $|G|/2 - |Z|$, or $v \in Z$ and $\Phi(v) \neq \emptyset$, or $|\Phi(v)| \geq 2$.*

We assume that v has score exactly $|G|/2-|Z|$. It follows that $|B|=|Z|$, and every vertex $u \in V(H)$ has score at least $|G|/2-|Z|$, and so satisfies $\phi^+(u) \le \phi^-(u)$. But

$$\sum_{u \in V(H)} |\Phi(u)|(\phi^-(u)-\phi^+(u)) = 0.$$

It follows that for every $u \in V(H)$, $|\Phi(u)|(\phi^-(u)-\phi^+(u)) = 0$, so either $\Phi(u)=\emptyset$ (and hence $u \in Z$) or $\phi^+(u)=\phi^-(u)$ (and hence u has the same score as v). In particular, if $\Phi(u) \neq \emptyset$ for some $u \in Z$, then we may replace v by u and the claim holds. Similarly, if some $u \in T \setminus B$ satisfies $|\Phi(u)| \ge 2$, we can replace v by u. So we may assume that $\Phi(u)=\emptyset$ for all $u \in Z$ (and hence $Q=\emptyset$), and $\Phi(u)=\{u\}$ for each $u \in T \setminus B$ (and hence $S \setminus (Q \cup Z)=\emptyset$). Consequently, $S=Z$. Since $|Z| \le |G|/2$ (because $|Z|=|B|$), we may assume that there exists $p_0 \in T$ not 2-covered by Z. Thus p_0 is pure-up, and so $P \neq \emptyset$, where P is the set of pure-up vertices. Choose $p \in P$ that 2-covers P. (Any vertex of maximum out-degree in $T[P]$ has this property.) Let Z' be the set of vertices in Z that are not adjacent from p; so $Z' \cup \{p\}$ is stable. We claim it is a 2-kernel. Certainly $Z' \cup \{p\}$ 2-covers Z; each vertex in T 1-covered by $Z \setminus Z'$ is 2-covered by p; every other vertex of T 1-covered by Z is 1-covered by Z'; and each vertex of T not 1-covered by Z is in P, and hence is 2-covered by p. So $Z' \cup \{p\}$ is a 2-kernel, and therefore we may assume its size is more than $|G|/2$. Since $|Z|=|B|$, it follows that $|T \setminus B|=1$ and hence $T \setminus B = P = \{p\}$, since $P \cap B = \emptyset$; and so $p_0=p$. Since Z 1-covers B and does not 2-cover $p_0=p$, it follows that p is adjacent to every vertex in B. But then $\{p\}$ is a 2-kernel (because every vertex in $S=Z$ has an in-neighbour, since it is not a source). This proves (1).

(2) *If $v \in Z$ then the theorem holds.*

Let J be the set of vertices in $S \setminus Z$ that are 2-covered by v. (Possibly $J \cap Q \neq \emptyset$.) Let $A = S \setminus (J \cup Q \cup Z)$, and $F = (T \setminus B) \setminus N_G^+(v)$. Since $N_H^-(v)=F$, and therefore the union of the sets $\Phi(u)$ ($u \in N_H^-(v)$) includes $F \cup A$, it follows that $\phi^-(v) \ge |F|+|A|$. Moreover,

$$\phi^0(v) = \sum_{z \in Z} |\Phi(z)| = |Q|.$$

Consequently, the score of v is at least $|F|+|A|+|Q|/2$, and so the latter is at most $|G|/2-|Z|$.

Choose $X \subseteq S$ minimal such that $A \cup Z \cup X$ 1-covers every vertex of T that is not pure-up. Thus $|X| \le |F|$, since Z 1-covers $B \cup (T \cap N_G^+(v))$. Let $K = A \cup Z \cup X$. We claim that K is a 2-kernel. It certainly 2-covers S, since Z 2-covers Q, and $A \cup \{v\}$ 2-covers $S \setminus (Q \cup Z)$. It 1-covers all vertices in T that are not pure-up, from the choice of X. Suppose it does not 2-cover some $p \in T \setminus B$. Then p is pure-up, so $p \notin B$; and p is complete to all vertices in T that are not pure-up, since K 1-covers all such vertices and does not 2-cover p. Moreover, each vertex in Z is adjacent from p in H. Thus, every H-in-neighbour of p is also pure-up, and so is adjacent to v in H. Consequently

$$|\Phi(p)| + \sum_{u \in N_H^-(p)} |\Phi(u)| \le \sum_{u \in N_H^-(v)} |\Phi(u)|;$$

and since $|\Phi(p)| \ge 1$ it follows that p has smaller score than v, a contradiction.

So K is a 2-kernel. But

$$|K| \le |X|+|A|+|Z| \le |F|+|A|+|Z| \le |G|/2-|Q|/2.$$

It follows that $|K| \le |G|/2$. This proves (2).

Henceforth we assume that $v \in T \setminus B$ and, by (1), either v has score strictly less than $|G|/2 - |Z|$, or $|\Phi(v)| \ge 2$.

(3) *v extends to a strong 2-kernel.*

Suppose not. By 2.2, there exists $t \in T$, adjacent to v, such that every G-in-neighbour of t is a G-in-neighbour of v, and $t \in T \setminus B$ by 2.3. A vertex of H is a G-in-neighbour of v if and only if it is an H-in-neighbour of v, and the same is true for in-neighbours of t; so every H-in-neighbour of t is an H-in-neighbour of v. Hence $\phi^-(v) \ge \phi^-(t) + |\Phi(t)|$. Since $\phi^0(v) = |\Phi(v)|$ and $\phi^0(t) = |\Phi(t)|$, it follows that

$$\phi^-(v) + \phi^0(v)/2 \ge \phi^-(t) + |\Phi(t)| + |\Phi(v)|/2 > \phi^-(t) + \phi^0(t)/2,$$

and so the score of t is strictly less than that of v, contradicting the choice of v. This proves (3).

Let $Q' = \bigcup_{z \in Z \setminus N^-(v)} \Phi(z)$, and $Q'' = \bigcup_{z \in Z \cap N^-(v)} \Phi(z)$; so $Q'' = Q \setminus Q'$. Let J be the set of vertices in $S \setminus Q$ that are 2-covered by v in $G \setminus B$. So, J, Z are both subsets of $S \setminus Q$, but they might intersect each other. S is also partitioned into three subsets, $S \cap N_G^+(v)$, $S \cap N_G^-(v)$ and $S \setminus N_G(v)$, where we define $N_G(v) = N_G^+(v) \cup N_G^-(v)$. (See figure 3.) We intend to find a 2-kernel containing v of size at most $|G|/2$, but we must be careful only to add vertices in $S \setminus N_G(v)$, to keep the set stable.

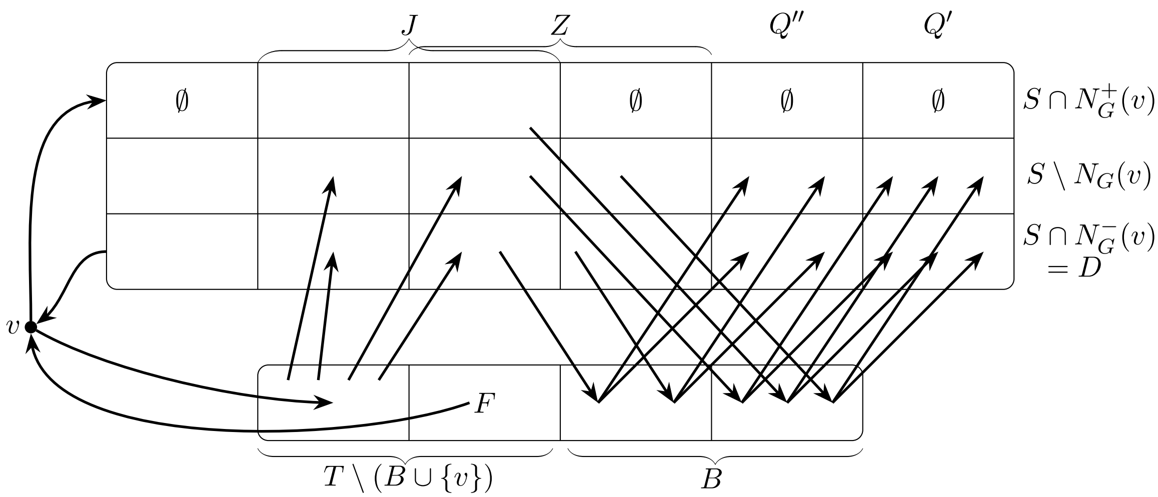

Figure 3: v is adjacent to everything in the top row of boxes, and from everything in the third. Its adjacency to B is not specified in the figure. It has no out-neighbours in Q since $v \notin B$, and so all its out-neighbours in S belong to J.

Let $D = N_G^-(v) \cap S$, and $F = (T \setminus B) \cap N_G^-(v)$. Thus

$$N_H^-(v) = F \cup (Z \cap D).$$

The union of the sets $\Phi(t)$ $(t \in F)$ includes $F \cup (S \setminus (Q \cup J \cup Z))$, and $\bigcup_{z \in Z \cap D} \Phi(z) = Q''$. Consequently

$$\phi^-(v) \geq |F| + |S \setminus (Q \cup J \cup Z)| + |Q''|,$$

and so the score of v is at least

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + \phi^0(v)/2.$$

Since $\phi^0(v) \geq 1$, and either $\phi^0(v) \geq 2$ or the score of v is strictly less than $|G|/2 - |Z|$, it follows that

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 + |Z| \leq |G|/2.$$

Since v extends to a strong 2-kernel, for each $u \in T \setminus B$ that is not 2-covered by v, there is an in-neighbour of u in $S \setminus N_G(v)$; choose $X \subseteq S \setminus N_G(v)$ minimal 1-covering each vertex in F that is not 2-covered by v. Thus $|X| \leq |F|$. For each $u \in D$, since v extends to a 2-kernel, there exists $t \in S \setminus N_G(v)$ that 2-covers u; let $Y \subseteq S \setminus N_G(v)$ be minimal 2-covering $D \setminus (J \cup Q')$. Thus $|Y| \leq |D \setminus (J \cup Q')|$.

Let

$$K = \{v\} \cup (Z \setminus (D \cup N_G^+(v))) \cup (S \setminus (Q \cup J \cup Z \cup D)) \cup X \cup Y \cup (Q'' \setminus D).$$

We claim that K is a 2-kernel. Certainly it is stable.

(4) *K 2-covers S.*

Let $s \in S$, and assume first that $s \notin Q$. If $s \in J$ then v 2-covers s; if $s \in D \setminus J$ then Y 2-covers s; if $s \in Z \setminus (J \cup D)$ then $s \in K$; and if $s \notin Z \cup J \cup D$ then $s \in K$. So in this case K 2-covers s. Next assume that $s \in Q$. So $s \notin J \cup N_G^+(v)$. If $s \in Q'' \setminus D$ then $s \in K$, and if $s \in Q'' \cap D$ then Y 2-covers s, so we assume that $s \in Q'$, and so $z_{b_s} \in Z \setminus D$. If $z_{b_s} \notin N_G^+(v)$ then $z_{b_s} \in K$ and so K 2-covers s, so we assume that $z_{b_s} \in N_G^+(v)$. Then b_s is adjacent from v (because b_s does not 2-cover z_{b_s} since z_{b_s} is a problem for b_s) and so K 2-covers s. This proves (4).

(5) *K 2-covers T, and hence K is a 2-kernel.*

Let $t \in T$. We may assume that $t \in N_G^-(v)$. If $t \in T \setminus B$ then $t \in F$ and X 1-covers t, so we assume that $t \in B$. If $z_t \notin N_G(v)$ then $z_t \in K$ and 1-covers t, so we assume that $z_t \in N_G(v)$. Since v is adjacent from t and z_t is a problem for t, it follows that $z_t \notin N_G^+(v)$, so $z_t \in N_G^-(v)$. Choose $y \in Y$ such that y 2-covers z_t, and choose $u \in T$ such that y-u-z_t is a directed path. Since z_t is a problem for t, it follows that t is adjacent from u, and so y 2-covers t. This proves (5).

Now let us bound the size of K. We have

$$|K| \leq 1 + |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |X| + |Y| + |Q'' \setminus D|.$$

We know that

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 \leq |G|/2 - |Z|,$$

and $|X| \leq |F|$, and $|Y| \leq |D \setminus (J \cup Q')|$. Adding, we deduce that:

$$\begin{aligned}
&|K| + |F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 + |X| + |Y| \\
&\leq 1 + |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |X| + |Y| + |Q'' \setminus D| \\
&+ |G|/2 - |Z| + |F| + |D \setminus (J \cup Q')|.
\end{aligned}$$

This simplifies to:

$$\begin{aligned}
|K| + |S \setminus (Q \cup J \cup Z)| + |Q''| \leq |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |Q'' \setminus D| \\
+ |G|/2 - |Z| + |D \setminus (J \cup Q')|.
\end{aligned}$$

Since

$$|S \setminus (Q \cup J \cup Z)| = |S \setminus (Q \cup J \cup Z \cup D)| + |D \setminus (Q \cup J \cup Z)|,$$

we deduce

$$|K| + |D \setminus (Q \cup J \cup Z)| + |Q''| \leq |Q'' \setminus D| + |D \setminus (J \cup Q')| + |Z \setminus D| + |G|/2 - |Z|.$$

Since

$$|D \setminus (J \cup Q')| - |D \setminus (Q \cup J \cup Z)| = |(D \setminus J) \cap (Q'' \cup (Z \setminus Q'))| \leq |D \cap (Q'' \cup Z)|,$$

this further simplifies to:

$$|K| + |Q'' \cap D| \leq |D \cap (Q'' \cup Z)| - |Z \cap D| + |G|/2,$$

and so $|K| \leq |G|/2$. This proves 2.4. ∎

3 Large 2-kernels

In this section, we turn to a second topic, Spiro's question 1.3. While it seems to be asking for something close to the opposite of 1.1, Spiro observed that 1.1 implies 1.3. Here is his argument: to prove 1.3 for a digraph G, choose a large number n. If G has a source v, delete v and all its out-neighbours and apply induction; while if G has no sources, for each vertex v of G, add n new vertices adjacent from v and with no other neighbours. Applying 1.1 with n sufficiently large implies that G satisfies 1.3.

If G is a digraph and $X \subseteq V(G)$, let $N_G^+[X]$ denote the set of vertices that either belong to X or are adjacent from a vertex in X. The same construction (adding $nf(v)$ new out-leaves for each vertex) shows that 1.1 implies a slightly stronger statement ($\mathbb{Z}_+$ denotes the set of non-negative integers, and $f(X)$ denotes $\sum_{v \in X} f(v)$):

3.1 Conjecture: *In every digraph G, and for every map $f : V(G) \to \mathbb{Z}_+$ there is a 2-kernel K such that $f(N_G^+[K]) \geq f(V(G))/2$.*

In this section we show that 3.1 is true for split digraphs, and indeed for a somewhat more general class of graphs. If G is an oriented graph, let us say a *break* of G is a partition (S, T) of $V(G)$ such that $G[S]$ is *acyclic* (that is, has no directed cycles), and $G[T]$ is a tournament. We will show:

3.2 *In every oriented graph G that admits a break, and for every map $f : V(G) \to \mathbb{Z}_+$, there is a 2-kernel K such that $f(N^+_G[K]) \ge f(V(G))/2$.*

The greater generality given by the function f will be useful for the inductive proof, allowing us to delete vertices without changing $f(V(G))$. We need a result of von Neumann and Morgenstern [6]:

3.3 *Every acyclic digraph has a unique 1-kernel.*

In order to prove 3.2, we prove a stronger statement (by the *non-neighbourhood* of a vertex v, we mean the digraph induced on the set of vertices different from and nonadjacent with v):

3.4 *Let (S,T) be a break of an oriented graph G, and let $f : V(G) \to \mathbb{Z}_+$ be a map. Then there is a 2-kernel K such that $f(N^+_G[K]) \ge f(V(G))/2$, where either $K \subseteq S$, or K consists of some $v \in T$ together with the unique 1-kernel of its non-neighbourhood.*

Proof. We assume the result holds for all oriented graphs that admit breaks (S',T') with $2|S'| + |T'| < 2|S| + |T|$. For each $X \subseteq S$, let $A(X)$ be the unique 1-kernel of $G[X]$ (which exists by 3.3); and for each $v \in T$, let $M(v)$ be its non-neighbourhood. Let us say a 2-kernel K of G is *special for* (G,S,T) if either $K \subseteq S$, or $K = \{v\} \cup A(M(v))$ for some $v \in T$.

(1) *We may assume that $\{v\} \cup A(M(v))$ is a 2-kernel for each $v \in T$.*

Suppose not. Certainly $\{v\} \cup A(M(v))$ is stable, so there is a vertex $w \ne v$ such that $\{v\} \cup A(M(v))$ does not 2-cover w. We claim that $N^-_G(w) \subseteq N^-_G(v)$. For suppose that $s \in N^-_G(w) \setminus N^-_G(v)$. Since $s \notin \{v\} \cup N^+_G(v)$ (because $\{v\} \cup A(M(v))$ does not 2-cover w), it follows that v, s are nonadjacent, and so $s \in M(v) \subseteq S$. But then s is 1-covered by $A(M(v))$, and so w is 2-covered by $\{v\} \cup A(M(v))$, a contradiction. This proves that $N^-_G(w) \subseteq N^-_G(v)$. Thus every 2-kernel of $G' = G \setminus v$ is also a 2-kernel of G. Define $f'(w) = f(w) + f(v)$, and $f'(x) = f(x)$ for all $x \in V(G) \setminus \{v,w\}$. Applying the inductive hypothesis to G' and f', we deduce there is a 2-kernel K of G' (and hence of G), special for $(G', S, T \setminus \{v\})$ (and hence special for (G,S,T)), such that $f'(N^+_{G'}[K]) \ge f'(V(G))/2 = f(V(G))/2$. But $N^+_{G'}[K] \subseteq N^+_G[K]$, and if $w \in N^+_{G'}[K]$ then $v, w \in N^+_G[K]$, and so $f'(N^+_{G'}[K]) \le f(N^+_G[K])$. Hence $f(N^+_G[K]) \ge f(G)/2$. This proves (1).

A *sink* of G is a vertex that has no out-neighbours.

(2) *Let $s \in S$ be a sink of $G[S]$. We may assume that s is a neighbour of every vertex in T.*

For each $t \in T$, if s,t are nonadjacent, let us add the edge ts, forming an oriented graph G'. Suppose the theorem holds for G', with the same function f, and let K' be a 2-kernel of G', special for (G', S, T), with $f(N^+_{G'}[K']) \ge f(V(G'))/2 = f(V(G))/2$. For each $v \in T$, let $M'(v)$ be the non-neighbourhood of v in G'. There are four cases:

- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ adjacent from s in G;
- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ adjacent to s in G;
- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ nonadjacent with s in G;

- $K' \subseteq S$.

In the first two cases, $M'(v) = M(v)$, and $\{v\} \cup A(M(v))$ is a 2-kernel of G by (1); and $N^+_{G'}[K'] = N^+_G[K']$, and so K' satisfies the theorem. In the third case, $M'(v) = M(v) \setminus \{s\}$. If $A(M'(v))$ 1-covers s, then $A(M'(v)) = A(M(v))$ and so K' satisfies the theorem. If $A(M'(v))$ does not 1-cover s, then $A(M(v)) = A(M'(v)) \cup \{s\}$ (because s is a sink of $G[S]$), and so $K = \{v\} \cup A(M(v))$ satisfies the theorem. Finally, in the fourth case, $K' \subseteq S$. If K' is a 2-kernel of G then it satisfies the theorem, so we assume it is not; and since K' is a 2-kernel of G', it follows that K' does not 2-cover s. But then $K' \cup \{s\}$ satisfies the theorem. This proves (2).

If $S = \emptyset$, then G is a tournament and the result holds, so we assume that $S \neq \emptyset$, and hence contains a sink of $G[S]$. By (2), then $(S \setminus \{s\}, T \cup \{s\})$ is also a break of G, and from the inductive hypothesis, there is a 2-kernel K of G such that $f(N^+_G[K]) \geq f(V(G))/2$, and K is special for $(G, S \setminus \{s\}, T \cup \{s\})$. But then K is also special for (G, S, T). This proves 3.4. ∎

What happens to 3.1 if we assume that $V(G)$ can be partitioned into two sets S, T where T is a tournament and S is small? By 3.4, the conjecture holds if $|S| \leq 2$, and in hope of finding a counterexample, we worked on the case when $|S| = 3$. But the conjecture is also true in this case (by an *ad hoc* argument that does not seem capable of any generalization, and we omit the details).

There is a natural refinement of the conjectures 1.1 and 1.3, equivalent to 1.1 and implying 1.3, that:

3.5 Conjecture: *In every digraph G, and for every map $f : V(G) \to \mathbb{Z}_+$ there is a 2-kernel K such that $|K| + f(V(G))/2 \leq |G|/2 + f(N^+_G(K))$.*

To deduce this from 1.1, add $f(v)$ out-leaves to each vertex v. It implies 1.1 by taking $f(v) = 0$ for all v, and it implies 1.3 by scaling f to be very large. Perhaps the proof of 2.1 can be modified to show that split graphs satisfy 3.5, but we have not seriously attempted this.

4 k-kernels

Now we turn to the proof of our third result, 1.5. We begin with:

4.1 *For all integers $k \geq 0$, if G is an acyclic digraph with only one source, then there exists $X \subseteq V(G)$ with $|X| \leq 1 + (|G| - 1)/(k + 1)$ that k-covers $V(G)$. Moreover, either $|G| = 1$ or $|X| \leq 1 + (|G| - 2)/(k + 1)$ or X is not stable.*

Proof. Let r be the unique source. If $|G| \leq k$, we may take $X = \{r\}$; then $|X| \leq 1 + (|G| - 2)/(k+1)$ unless $|G| = 1$, so the result holds. We assume then that $|G| > k$, and proceed by induction on $|G|$. For each $v \in V(G)$, let A_v be the set of vertices that are joined by a directed path (of any length) from v; and choose v with $|A_v|$ minimal such that $|A_v| \geq k + 1$. (This is possible since $|A_r| \geq k + 1$.) For each $w \in A_v$, there is a directed path P from v to w, and if P has length more than k then we may replace v by its outneighbour in P, contradicting the minimality of A_v. Thus every vertex in A_v is joined from v by a path of length at most k. If $v = r$ then we may take $X = \{r\}$ and win as before, so we assume that $v \neq r$. Let G' be the digraph obtained by deleting A_v. Every vertex of G' has an in-neighbour in G' except r, so G' has a unique source; and from the inductive hypothesis, there exists $X' \subseteq V(G')$ such that $|X'| \le 1 + (|G'| - 1)/(k + 1)$ and X' k-covers $V(G')$. Moreover, either $|G'| = 1$ or $|X'| \le 1 + (|G'| - 2)/(k + 1)$ or X' is not stable. Let $X = X' \cup \{v\}$. Thus X k-covers $V(G)$. Moreover, since $|A_v| \ge k+1$, it follows that $|X| \le 1+(|G|-1)/(k+1)$, and if either $|X'| \le 1+(|G'|-2)/(k+1)$ or X' is not stable, then correspondingly either $|X| \le 1+(|G|-2)/(k+1)$ or X is not stable. So we assume that $|G'| = 1$, and so $V(G') = \{r\}$. Since G has a unique source, it follows that v is adjacent from r, and so X is not stable. This proves 4.1. ∎

We deduce:

4.2 *For every integer $k \ge 1$, if G is an acyclic digraph with $|G| > 1$ and with only one source, then G has a k-kernel of size at most $1 + (|G| - 2)/k$.*

Proof. By 4.1 applied to G with k replaced by $k-1$, there exists $X \subseteq V(G)$ with $|X| \le 1+(|G|-1)/k$ that $(k-1)$-covers $V(G)$. The digraph $G[X]$ is acyclic and hence has a 1-kernel Y, by 3.3. Hence Y is a k-kernel in G. Moreover, since $|G| \ge 2$, either $|X| \le 1 + (|G| - 2)/k$ (when $|Y| \le |X|$ and the result is true), or X is not stable (when $|Y| \le |X| - 1 \le (|G| - 1)/k$ and again the result is true). This proves 4.2. ∎

As we said before, this result is tight (see figure 1). Now let us deduce 1.5, which we restate:

4.3 *For all integers $k \ge 2$, every digraph G with $|G| > 1$ and with a spanning arborescence has a k-kernel of size at most $1 + (|G| - 2)/(k - 1)$.*

Proof. Since G has a spanning arborescence, its vertex set can be numbered $\{v_1, \dots, v_n\}$ in such a way that for $2 \le j \le n$ there exists $i \in \{1, \dots, j - 1\}$ such that v_iv_j is an edge. Let A be the set of all edges v_iv_j of G with $i < j$, and let $B = E(G) \setminus A$. Let G_A be the subgraph with vertex set $V(G)$ and edge set A, and define G_B similarly. Both G_A, G_B are acyclic, and G_A has a unique source. By 4.2 applied to G_A with k replaced by $k - 1$, G_A has a $(k - 1)$-kernel X of size at most $1 + (|G| - 2)/(k - 1)$. Now X is stable in G_A, and $G_B[X]$ is acyclic, and so has a 1-kernel Y, by 3.3. But then Y is a k-kernel in G, and $|Y| \le |X| \le 1 + (|G| - 2)/(k - 1)$. This proves 4.3. ∎

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