

Induced subgraphs of graphs with large chromatic number.
X. Holes of specific residue

Alex Scott

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour¹

Princeton University, Princeton, NJ 08544, USA

January 8, 2017; revised May 20, 2018

¹Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.

Abstract

We prove that, for all integers $\kappa, \ell \geq 0$, every graph with sufficiently large chromatic number contains either a complete subgraph on κ vertices or holes of all lengths modulo ℓ . This unifies and extends results from a number of previous papers.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. We denote the chromatic number of a graph G by $\chi(G)$, and its clique number (the cardinality of its largest clique) by $\omega(G)$. A *hole* in G means an induced subgraph which is a cycle of length at least four.

What can we say about the hole lengths in a graph G with large chromatic number? If G is a complete graph then it has no holes at all, and the question becomes trivial. But if we bound the clique number of G then the question becomes much more interesting, and much deeper. In an influential paper written thirty years ago, Gyárfás [6] made a number of conjectures about induced subgraphs of graphs with large chromatic number and bounded clique number. Three of these conjectures, concerning holes, are particularly well-known:

1.1 For all $\kappa \geq 0$

- there exists c such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or a hole of odd length;
- for all $\ell \geq 0$ there exists c such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or a hole of length at least ℓ ;
- for all $\ell \geq 0$ there exists c such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or a hole whose length is odd and at least ℓ .

All three conjectures are now known to be true: the first was proved by the authors in [8] (see [7] for earlier work); the second jointly with Maria Chudnovsky in [3]; and the third (which is a strengthening of the first two) jointly with Chudnovsky and Sophie Spirkl in [5]. The analogous result for long even holes is also known (it is enough to find two vertices joined by three long paths with no edges between them, and this follows from results of [4] or [10]).

Another intriguing result on holes was shown by Bonamy, Charbit and Thomassé [1], who proved the first case of a conjecture of Kalai and Meshulam by showing the following.

1.2 Every graph with sufficiently large chromatic number contains either a triangle or a hole of length 0 modulo 3.

In this paper we prove the following theorem, which contains all the results mentioned above as special cases.

1.3 For all $\kappa, \ell \geq 0$ there exists c such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or holes of every length modulo ℓ .

Note that this result allows us to demand a *long* hole of length i modulo j by taking $\ell = Nj$ for large N and then choosing a suitable residue. Thus it implies all three Gyárfás conjectures; and it extends the result of Bonamy, Charbit and Thomassé [1] in several ways, allowing us to ask for any size of clique, and a hole of any residue and as long as we want. (Though we cannot demand a hole of any *specific* length: it is well-known that there are triangle-free graphs with arbitrarily large girth.)

In the triangle-free case, an even stronger result is known: we proved in [9] that

1.4 For all $\ell \geq 0$ there exists c such that every triangle-free graph with chromatic number greater than c contains holes of ℓ consecutive lengths.

We conjectured in [9] that the same should be true if we exclude larger cliques:

1.5 Conjecture: For all integers $\kappa, \ell \geq 0$, there exists $c \geq 0$ such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or holes of ℓ consecutive lengths.

This conjecture remains open. However, we make a small step towards it: we will show that under the same hypotheses, there are (long) holes of two consecutive lengths.

1.6 For each $\kappa, \ell \geq 0$ there exists $c \geq 0$ such that every graph with chromatic number greater than c contains either a complete subgraph on κ vertices or holes of two consecutive lengths, both of length more than ℓ .

We have convinced ourselves that with a great deal of work, which we omit, we could get three consecutive “long” holes, but so far that is the best we can do.

As in several other papers of this series, the proof of 1.3 examines whether there is an induced subgraph of large chromatic number such that every ball of small radius in it has bounded chromatic number. Let us make this more precise. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* or *G -distance* between two vertices u, v of G is the length of a shortest path between u, v , or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N_G^\rho(v)$ or $N^\rho(v)$ denotes the set of all vertices u with distance exactly ρ from v , and $N_G^\rho[v]$ or $N^\rho[v]$ denotes the set of all u with distance at most ρ from v . If G is a nonnull graph and $\rho \geq 1$, we define $\chi^\rho(G)$ to be the maximum of $\chi(N^\rho[v])$ taken over all vertices v of G . (For the null graph G we define $\chi^\rho(G) = 0$.) Let \mathbb{N} denote the set of nonnegative integers, and let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function. For $\rho \geq 1$, let us say a graph G is (ρ, ϕ) -controlled if $\chi(H) \leq \phi(\chi^\rho(H))$ for every induced subgraph H of G . Roughly, this says that in every induced subgraph H of G with large chromatic number, there is a vertex v such that $H[N_H^\rho[v]]$ has large chromatic number. Let \mathcal{C} be a class of graphs. We say \mathcal{C} is an *ideal* if every induced subgraph of each member of \mathcal{C} also belongs to \mathcal{C} . If $\rho \geq 2$ is an integer, an ideal \mathcal{C} is ρ -controlled if there is a nondecreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that every graph in \mathcal{C} is (ρ, ϕ) -controlled. For $\ell \geq 4$, an ℓ -hole means a hole of length exactly ℓ . We will prove the following two complementary results, which together imply 1.3:

1.7 Let $\rho \geq 2$ be an integer, and let \mathcal{C} be a ρ -controlled ideal of graphs. Let $\ell \geq 24$ if $\rho = 2$, and $\ell \geq 8\rho^2 + 6\rho$ if $\rho > 2$. Then for all $\kappa \geq 0$, there exists $c \geq 0$ such that every graph $G \in \mathcal{C}$ with $\omega(G) \leq \kappa$ and $\chi(G) > c$ has an ℓ -hole.

1.8 For all integers $\ell \geq 2$ and $\tau \geq 0$ there is an integer $c \geq 0$ with the following property. Let G be a graph such that $\chi^8(G) \leq \tau$, and every induced subgraph J of G with $\omega(J) < \omega(G)$ has chromatic number at most τ . If $\chi(G) > c$ then there are ℓ holes in G with lengths of all possible values modulo ℓ .

Proof of 1.3, assuming 1.7 and 1.8. Let $\kappa, \ell \geq 0$, and let \mathcal{C} be the ideal of graphs with clique number at most κ and with no hole of some length modulo ℓ . By 1.8, for each $\tau \geq 0$ there exists c_τ such that every $G \in \mathcal{C}$ with $\chi^8(G) \leq \tau$ has $\chi(G) \leq c_\tau$, and so \mathcal{C} is 8-controlled. By 1.7 the theorem follows. This proves 1.3. ■

The proof of 1.7 breaks into two parts, the 2-controlled case and the ρ -controlled case when $\rho > 2$ (because if we can be sure that all 2-balls have small chromatic number then it is easier to piece together paths to make holes of any desired length.) We prove the 2-controlled case in the next section, and the other in section 3, deducing 1.7 at the end of section 3. We prove 1.8 in section 4, and the theorem about two consecutive long holes in section 5.

2 2-control

First we handle the 2-controlled case. The proof here is very much like part of the proof of theorem 4.8 of [5]; the main difference is a strengthening of theorem 4.5 of that paper. First we need some definitions. If G is a graph and $B, C \subseteq V(G)$, we say that B *covers* C if $B \cap C = \emptyset$ and every vertex in C has a neighbour in B . We say $A, B \subseteq V(G)$ are *anticomplete* if $A \cap B = \emptyset$ and no vertex in A has a neighbour in B . Let G be a graph, let $x \in V(G)$, let N be some set of neighbours of x , and let $C \subseteq V(G)$ be disjoint from $N \cup \{x\}$, such that x is anticomplete to C and N covers C . In this situation we call (x, N) a *cover* of C in G . For $C, X \subseteq V(G)$, a *multicover* of C in G is a family $(N_x : x \in X)$ such that

- X is stable;
- for each $x \in X$, the pair (x, N_x) is a cover of C ;
- for all distinct $x, x' \in X$, the vertex x' is anticomplete to N_x (and in particular all the sets $\{x\} \cup N_x$ are pairwise disjoint).

Its *length* is $|X|$, and the multicover $(N_x : x \in X)$ is *stable* if each of the sets N_x ($x \in X$) is stable.

Let $(N_x : x \in X)$ be a multicover of C , let $X' \subseteq X$, and for each $x \in X'$ let $N'_x \subseteq N_x$; and let $C' \subseteq C$ be covered by each of the sets N'_x ($x \in X'$). Then $(N'_x : x \in X')$ is a multicover of C' , and we say it is *contained* in $(N_x : x \in X)$.

Again, let $(N_x : x \in X)$ be a multicover of C . Let P be an induced path of G with the following properties:

- P has length three or five;
- the ends of P are in X ;
- no vertex of X not an end of P belongs to or has a neighbour in $V(P)$; and
- every vertex of P belongs to $X \cup \bigcup_{x \in X} N_x \cup C$.

Let us call such a path P an *oddity* for the multicover. If $(N_x : x \in X)$ is a multicover of C , with an oddity P , and $(N'_x : x \in X')$ is a multicover of $C' \subseteq C$ contained in $(N_x : x \in X)$, and $V(P)$ is anticomplete to $X' \cup \bigcup_{x \in X'} N'_x \cup C'$, we say that $(N'_x : x \in X')$ is a multicover of C' *compatible* with P . Let H be the subgraph induced on $\bigcup_{x \in X} N_x$; we call the clique number of H the *cover clique number* of $(N_x : x \in X)$.

First we need to show the following:

2.1 *Let $\tau, \kappa, m', c' \geq 0$ be integers, and let $0 \leq \kappa' \leq \kappa$ be an integer. Then there exist integers $m, c \geq 0$ with the following property. Let G be a graph such that*

- $\omega(G) \leq \kappa$;
- $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$; and
- G admits a stable multicover $(N_x : x \in X)$ with length m , of a set C with $\chi(C) > c$, with cover clique number at most κ' .

Then there is an oddity P for the multicover, and a multicover $(N'_x : x \in X')$ of $C' \subseteq C$ contained in $(N_x : x \in X)$ and compatible with P , such that $|X'| = m'$ and $\chi(C') > c'$.

We proceed by induction on κ' , with τ, κ, m', c' fixed. Thus, inductively, there exist $m_0, c_0 \geq 0$ such that the theorem holds if m, c are replaced by m_0, c_0 respectively, and κ' is replaced by any κ_0 with $0 \leq \kappa_0 < \kappa'$. (Note that possibly $\kappa' = 0$, when this statement is vacuous; in that case take $m_0 = c_0 = 0$.)

Let $m = 4 + 4m_0 + 2m'$. Define $c_m = 4\tau + 2^m(c_0 + c')$, and for $i = m - 1, \dots, 1$ let $c_i = 2c_{i+1} + \tau$. Let $c = 2c_1 + \tau$; we will show that m, c satisfy the theorem.

Let $G, (N_x : x \in X)$ and C be as in the theorem, where $|X| = m$, $\chi(C) > c$ and the cover clique number of $(N_x : x \in X)$ is at most κ' . We may assume (because otherwise the theorem follows from the inductive hypothesis) that:

- (1) *There is no multicover $(N'_x : x \in X')$ of $C' \subseteq C$ contained in $(N_x : x \in X)$ with cover clique number less than κ' , and with $|X'| = m_0$ and $\chi(C') > c_0$.*

Let $X = \{x_1, \dots, x_m\}$, and let us write N_i for N_{x_i} for $1 \leq i \leq m$.

- (2) *For $1 \leq i \leq m$, there exist disjoint $C_i, D_i \subseteq C$ with $\chi(C_i), \chi(D_i) > c_i$, and $A_h \subseteq N_h$ for $1 \leq h \leq i$, such that each A_h covers one of C_i, D_i and is anticomplete to the other.*

If $A \subseteq N_1 \cup \dots \cup N_m$, let $f(A)$ denote the set of vertices in C with a neighbour in A . Since $\chi(C) > c$, there exists $A_1 \subseteq N_1$ minimal such that $f(A_1)$ has chromatic number more than c_1 . Let $C_1 = f(A_1)$ and $D_1 = C \setminus C_1$. From the minimality of A_1 , it follows that $\chi(C_1) \leq c_1 + \tau$. Consequently $\chi(D_1) > \chi(C) - (c_1 + \tau) \geq c_1$. Thus (2) holds for $i = 1$. Now we assume that $i > 1$ and C_{i-1}, D_{i-1} and the sets A_1, \dots, A_{i-1} satisfy (2) for $i - 1$. Choose $A_i \subseteq N_i$ minimal such that one of $\chi(f(A_i) \cap C_{i-1}), \chi(f(A_i) \cap D_{i-1})$ is more than c_i ; say the first (without loss of generality). Let $C_i = f(A_i) \cap C_{i-1}$. Now $\chi(f(A_i) \cap D_{i-1}) \leq c_i + \tau$, from the minimality of A_i , so $\chi(D_i) > c_{i-1} - c_i - \tau \geq c_i$, where $D_i = D_{i-1} \setminus f(A_i)$. Thus A_i covers C_i and is anticomplete to D_i . This proves (2).

From (2) with $i = m$, each A_i covers one of C_m, D_m and is anticomplete to the other. By exchanging C_m, D_m if necessary, we may assume that for at least $m/2$ values of i , A_i covers C_m and is anticomplete to D_m . We may assume (by reordering x_1, \dots, x_m) that A_i covers C_m and is anticomplete to D_m for $1 \leq i \leq m/2$. Let $B_i = N_i \setminus A_i$ for $1 \leq i \leq m/2$.

- (3) *There is an oddity P for $(N_x : x \in X)$ with ends x_1, x_2 and with interior in $B_1 \cup B_2 \cup D_m$.*

Since $\chi(D_m) > c_m \geq \tau$, there is a clique $Z \subseteq D_m$ with $|Z| = \kappa$. Now N_1 covers D_m , but A_1

is anticomplete to D_m , so B_1 covers D_m . Similarly B_2 covers D_m . Choose a vertex $y_1 \in B_1 \cup B_2$ with as many neighbours in Z as possible; and we may assume that $y_1 \in B_1$. Not every vertex of Z is incident with y_1 since $\omega(G) \leq \kappa$; let $z_2 \in Z$ be nonadjacent to y_1 . Choose $y_2 \in B_2$ adjacent to z_2 . From the choice of y_1 , there exists $z_1 \in Z$ adjacent to y_1 and not to y_2 . If y_1, y_2 are nonadjacent, then $x_1-y_1-z_1-z_2-y_2-x_2$ is an oddity, and if y_1, y_2 are adjacent then $x_1-y_1-y_2-x_2$ is an oddity. This proves (3).

Now there are at most four vertices of P that have neighbours in C_m , and so there exists $F \subseteq C_m$ with $\chi(F) > c_m - 4\tau = 2^m(c_0 + c')$ that is anticomplete to $V(P)$. There are two vertices of P in $N_1 \cup N_2$, and those are the only vertices of P that might have neighbours in A_i for $3 \leq i \leq m/2$. Let these vertices be p, q , and for $3 \leq i \leq m/2$ let P_i be the set of vertices in A_i adjacent to p , and Q_i the set adjacent to q .

For each $v \in F$, let $I(v)$ be the set of i with $3 \leq i \leq m/2$ such that v has a neighbour in P_i . For each subset $I \subseteq \{3, \dots, m/2\}$ with $|I| = m_0$, the chromatic number of the set of $v \in F$ with $I \subseteq I(v)$ is at most c_0 , by (1). Since there are at most 2^{m-1} such subsets I , the set of vertices $v \in F$ with $|I(v)| \geq m_0$ has chromatic number at most $2^{m-1}c_0$; and similarly the set of vertices adjacent to neighbours of q in at least m_0 sets A_i has chromatic number at most $2^{m-1}c_0$. Consequently there exists $F' \subseteq F$ with

$$\chi(F') \geq \chi(F) - 2^m c_0 > 2^m c'$$

such that for each $v \in F'$, there are at most $2m_0$ values of $i \in \{3, \dots, m/2\}$ such that v is adjacent to a neighbour of p or q in A_i . There are only at most 2^m possibilities for the set of these values, so there exists $C' \subseteq F'$ with $\chi(C') \geq \chi(F')2^{-m} > c'$ such that all vertices in C' have the same set of values, and in particular there exists $I \subseteq \{3, \dots, m/2\}$ with $|I| = m/2 - 2 - 2m_0 = m'$ such that no vertex in C' has a neighbour adjacent to p or q in any $A_i (i \in I)$. For each $i \in I$, let N'_{x_i} be the set of vertices in A_i nonadjacent to both p, q . Then $(N'_x : x \in \{x_i : i \in I\})$ is a multicover of C' , contained in $(N_x : x \in X)$, and compatible with P . This proves 2.1. \blacksquare

By three successive applications of 2.1 (one for each oddity), we deduce:

2.2 *For all integers $\tau, \kappa \geq 0$, there exist integers $m, c \geq 0$ with the following property. Let G be a graph such that*

- $\omega(G) \leq \kappa$;
- $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$; and
- G admits a stable multicover $(N_x : x \in X)$ of a set C , where $|X| = m$ and $\chi(C) > c$.

Then there are three oddities P_1, P_2, P_3 for the multicover, where $V(P_1), V(P_2), V(P_3)$ are pairwise anticomplete.

(The same is true with “three” replaced by any other positive integer, but we only need three.) Next we need:

2.3 *Let $\ell \geq 24$ be an integer. Take the complete bipartite graph $K_{\ell, \ell}$, with bipartition A, B . Add three more edges joining three disjoint pairs of vertices in A . Now subdivide every edge between A and B once, and subdivide each of the three additional edges either two or four times. The graph we produce has a hole of length ℓ .*

We leave the proof to the reader (use the fact that if $x, y, z \in \{3, 5\}$ then ℓ is expressible as a sum of some or none of x, y, z and at least three 4's).

A multicover $(N_x : x \in X)$ of C is said to be *stably k -crested* if there are vertices a_1, \dots, a_k and vertices a_{ix} ($1 \leq i \leq k, x \in X$) of G , all distinct, with the following properties:

- a_1, \dots, a_k and the vertices a_{ix} ($1 \leq i \leq k, x \in X$) do not belong to $X \cup C \cup \bigcup_{x \in X} N_x$;
- for $1 \leq i \leq k$ and each $x \in X$, a_{ix} is adjacent to x , and there are no other edges between the sets $\{a_1, \dots, a_k\} \cup \{a_{ix} : 1 \leq i \leq k, x \in X\}$ and $X \cup C \cup \bigcup_{x \in X} N_x$;
- for $1 \leq i \leq k$ and each $x \in X$, a_{ix} is adjacent to a_i , and there are no other edges between $\{a_1, \dots, a_k\}$ and $\{a_{ix} : 1 \leq i \leq k, x \in X\}$;
- a_1, \dots, a_k are pairwise nonadjacent;
- for all $i, j \in \{1, \dots, k\}$ and all distinct $x, y \in X$, a_{ix} is nonadjacent to a_{jy} .

(Thus the “crest” part is obtained from $K_{k,|X|}$ by subdividing every edge once.) We deduce:

2.4 *Let $\ell \geq 24$, and let $\tau, \kappa \geq 0$. Then there exist $m, c \geq 0$ with the following property. Let G be a graph such that*

- $\omega(G) \leq \kappa$;
- $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$; and
- G admits a stably ℓ -crested stable multicover $(N_x : x \in X)$ of a set C , where $|X| = m$ and $\chi(C) > c$.

Then G has a hole of length ℓ .

Proof. Let m, c satisfy 2.2, choosing $m \geq \ell$ (note that if m, c satisfy 2.2 then so do m', c for $m' \geq m$.) By 2.2, there are three oddities, pairwise anticomplete; and the result follows from 2.3. This proves 2.4. ■

Theorem 4.4 of [5] says:

2.5 *For all $m, c, k, \kappa, \tau \geq 0$ there exist $m', c' \geq 0$ with the following property. Let G be a graph with $\omega(G) \leq \kappa$, such that $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$. Let $(N'_x : x \in X')$ be a multicover in G of some set C' , such that $|X'| \geq m'$ and $\chi(C') > c'$. Then there exist $X \subseteq X'$ with $|X| \geq m$, and $C \subseteq C'$ with $\chi(C) > c$, and a stable multicover $(N_x : x \in X)$ of C contained in $(N'_x : x \in X')$ that is stably k -crested.*

Combining 2.4 and 2.5, we deduce the following (a strengthening of theorem 4.5 of [5]):

2.6 *Let $\ell \geq 24$, and let $\tau, \kappa \geq 0$. Then there exist $m, c \geq 0$ with the following property. Let G be a graph such that*

- $\omega(G) \leq \kappa$;

- $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$; and
- G admits a multicover with length m , of a set C with $\chi(C) > c$.

Then G has a hole of length ℓ .

We need the following, a consequence of theorem 9.7 of [4]. That involves “trees of lamps”, but we do not need to define those here; all we need is that a cycle of length ℓ is a tree of lamps. (Note that what we call a “multicover” here is called a “strongly-independent 2-multicover” in that paper, and indexed in a slightly different way.)

2.7 Let $m, \kappa, c', \ell \geq 0$, and let \mathcal{C} be a 2-controlled ideal, such that for every $G \in \mathcal{C}$:

- $\omega(G) \leq \kappa$;
- G does not admit a multicover of length m of a set with chromatic number more than c' ; and
- G has no hole of length ℓ .

Then there exists c such that all graphs in \mathcal{C} have chromatic number at most c .

Now we prove the main result of this section, that is, 1.7 with $\rho = 2$.

2.8 Let $\ell \geq 24$ and let \mathcal{C} be a 2-controlled ideal of graphs. For all $\kappa \geq 0$ there exists c such that every graph in \mathcal{C} with clique number at most κ and chromatic number more than c has a hole of length ℓ .

Proof. We proceed by induction on κ . The result holds for $\kappa \leq 1$, so we assume that $\kappa \geq 2$ and every graph in \mathcal{C} with clique number less than κ has chromatic number at most τ . Let \mathcal{C}' be the ideal of graphs $G \in \mathcal{C}$ such that $\omega(G) \leq \kappa$ and G has no hole of length ℓ . Choose m, c' to satisfy 2.6 (with c replaced by c'). Choose c to satisfy 2.7 (with \mathcal{C} replaced by \mathcal{C}'). If $G \in \mathcal{C}'$, then by 2.6, G does not admit a multicover with length m of a set with chromatic number more than c' . From 2.7, $\chi(G) \leq c$. This proves 2.8. ■

3 The ρ -controlled case for $\rho \geq 3$.

Let G be a graph. We say a *grading* of G is a sequence (W_1, \dots, W_n) of subsets of $V(G)$, pairwise disjoint and with union $V(G)$. If $w \geq 0$ is such that $\chi(G[W_i]) \leq \tau$ for $1 \leq i \leq n$ we say the grading is τ -colourable. We say that $u \in V(G)$ is *earlier* than $v \in V(G)$ (with respect to some grading (W_1, \dots, W_n)) if $u \in W_i$ and $v \in W_j$ where $i < j$.

Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C . Let $B = \{b_1, \dots, b_m\}$. For $1 \leq i < j \leq m$ we say that b_i is *earlier* than b_j (with respect to the enumeration (b_1, \dots, b_m)). For $v \in C$, let $i \in \{1, \dots, m\}$ be minimum such that b_i, v are adjacent; we call b_i the *earliest parent* of v . An edge uv of $G[C]$ is said to be *square* (with respect to the enumeration (b_1, \dots, b_m)) if the earliest parent of u is nonadjacent to v , and the earliest parent of v is nonadjacent to u . Let $B = \{b_1, \dots, b_m\}$, and let (W_1, \dots, W_n) be a grading of $G[C]$. We say the enumeration (b_1, \dots, b_m)

of B and the grading (W_1, \dots, W_n) are *compatible* if for all $u, v \in C$ with u earlier than v , the earliest parent of u is earlier than the earliest parent of v .

A graph H is a ρ -ball if either $V(H) = \emptyset$ or there is a vertex $z \in V(H)$ such that every vertex of H has H -distance at most ρ from z ; and we call z a *centre* of the ρ -ball. If G is a graph, a subset $X \subseteq V(G)$ is said to be a ρ -ball if $G[X]$ is a ρ -ball. (Note that there may be vertices of G not in X that have G -distance at most ρ from z ; and also, for a pair of vertices in X , their G -distance and their $G[X]$ -distance may be different.)

3.1 *Let ϕ be a nondecreasing function and $\rho \geq 3$, and let G be a (ρ, ϕ) -controlled graph. Let $\tau \geq 0$ such that $\chi^{\rho-1}(G) \leq \tau$ and $\chi(J) \leq \tau$ for every induced subgraph J of G with $\omega(J) < \omega(G)$. Let $c \geq 0$ and let (W_1, \dots, W_n) be a τ -colourable grading of G . Let H be a subgraph of G (not necessarily induced) with $\chi(H) > \tau + 1 + \phi(c + \tau)$, and such that $W_i \cap V(H)$ is stable in H for each $i \in \{1, \dots, n\}$. Then there is an edge uv of H , and a ρ -ball X of G , such that*

- u, v are both earlier than every vertex in X ;
- v has a G -neighbour in X , and u does not; and
- $\chi(G[X]) > c$.

Proof. Let us say that $v \in V(G)$ is *internally active* if there is a ρ -ball $X \ni v$ with $\chi(X) > c + \tau$ such that no vertex of X is earlier than v . (Note that $X \cap W_i$ may have more than one element, so there may be vertices in X that are neither earlier nor later than v .) Let R_1 be the set of internally active vertices. We claim first:

$$(1) \chi(G \setminus R_1) \leq \phi(c + \tau).$$

For suppose not. Then since G is (ρ, ϕ) -controlled, there is a ρ -ball $X \subseteq V(G) \setminus R_1$ with $\chi(X) > c + \tau$, which therefore contains an internally active vertex, a contradiction. This proves (1).

Let us say $v \in V(G)$ is *externally active* if there is a ρ -ball X of G with $\chi(X) > c + \tau$ such that every vertex of X is later than v , and v has an H -neighbour in X . Let R_2 be the set of externally active vertices. We claim:

$$(2) R_1 \setminus R_2 \text{ is stable in } H.$$

For suppose that uv is an edge of H with both ends in $R_1 \setminus R_2$. Since each $W_i \cap V(H)$ is stable in H , we may assume that u is earlier than v . Since v is internally active, there is a ρ -ball X containing v with $\chi(X) > c + \tau$ such that no vertex of X is earlier than v ; but then u is externally active, a contradiction. This proves (2).

(3) *There is a subset $Y \subseteq V(H)$ such that $H[Y]$ is connected and has chromatic number more than τ , and a ρ -ball X of G with $\chi(G[X]) > c + \tau$, such that every vertex of Y is earlier than every vertex of X , and some vertex of Y has a H -neighbour in X .*

Since H has chromatic number more than $\tau + 1 + \phi(c + \tau)$, it follows from (1) and (2) that

$\chi(H[R_2]) > \tau$. Let Y be the vertex set of a component of $H[R_2]$ with maximum chromatic number. Choose $v \in Y$ such that no vertex of Y is later than v . Since v is externally active, this proves (3).

Let X, Y be as in (3). If some vertex of Y has no G -neighbour in X , then since $H[Y]$ is connected, there is an edge uv of $H[Y]$ such that v has a G -neighbour in X and u does not, and the theorem holds. We assume then that every vertex of Y has a G -neighbour in X . For each $y \in Y$, let $N(y)$ denote its set of G -neighbours in X . Let z be a centre of X , and for $0 \leq i \leq \rho$ let L_i be the set of vertices in X with $G[X]$ -distance i to z . Thus $L_0 \cup \dots \cup L_\rho = X$. Let Y_0 be the set of all $y \in Y$ with $N(y) \subseteq L_{\rho-1} \cup L_\rho$.

(4) $Y_0 \neq \emptyset$.

Since $\chi(H[Y]) > \tau$, it follows that $\chi(G[Y]) > \tau$, and so some vertex $y \in Y$ has G -distance at least ρ from z . Consequently $N(y) \subseteq L_{\rho-1} \cup L_\rho$. This proves (4).

Choose $y \in Y_0$, if possible with the additional property that $N(y) \cap L_{\rho-1} = \emptyset$. Let U be the set of vertices in L_ρ with a neighbour in $N(y) \cap L_{\rho-1}$.

(5) *There is a vertex y' of Y with $N(y') \not\subseteq N(y) \cup U$.*

For there is a vertex $y' \in Y$ with G -distance at least ρ from y , since $\chi(G[Y]) > \tau$. Since $\rho > 2$, $N(y) \cap N(y') = \emptyset$. If $N(y') \subseteq U$, then $y' \in Y_0$ and $N(y') \cap L_{\rho-1} = \emptyset$; but then $N(y) \cap L_{\rho-1} = \emptyset$ from the choice of y , and so $U = \emptyset$, a contradiction. Thus $N(y') \not\subseteq U$. This proves (5).

Now $X \setminus (N(y) \cup U)$ is a ρ -ball X' say, and some vertex (namely y') of Y has a G -neighbour in it, and another (namely y) has no G -neighbour in it. Since $H[Y]$ is connected, there is an edge uv of $H[Y]$ such that v has a G -neighbour in X' and u does not. But $\chi(X) > c + \tau$, and every vertex in $N(y) \cup U$ has G -distance at most two from y and so $\chi(N(y) \cup U) \leq \tau$, and consequently $\chi(X') \geq \chi(X) - \tau > c$. This proves 3.1. ■

We also need the following, proved in [5]:

3.2 *Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C . Let every induced subgraph J of G with $\omega(J) < \omega(G)$ have chromatic number at most τ . Let the enumeration (b_1, \dots, b_m) of B and the grading (W_1, \dots, W_n) of $G[C]$ be compatible. Let H be the subgraph of G with vertex set C and edge set the set of all square edges. Let (W_1, \dots, W_n) be τ -colourable; then $\chi(G[C]) \leq \tau^2 \chi(H)$.*

We deduce:

3.3 *Let ϕ be a nondecreasing function and $\rho \geq 3$, and let G be a (ρ, ϕ) -controlled graph. Let $\tau \geq 0$ such that $\chi^{\rho-1}(G) \leq \tau$ and $\chi(J) \leq \tau$ for every induced subgraph J of G with $\omega(J) < \omega(G)$. Let $B, C \subseteq V(G)$, where B covers C . Let the enumeration (b_1, \dots, b_m) of B and the grading (W_1, \dots, W_n) of $G[C]$ be compatible. Let (W_1, \dots, W_n) be τ -colourable, and let $\chi(G[C]) > \tau^2(\tau + 1 + \phi(c + \tau))$. Then there is a square edge uv , and a ρ -ball X of G , such that*

- u, v are both earlier than every vertex in X ;

- v has a neighbour in X , and u does not; and
- $\chi(X) > c$.

Proof. Let H be as in 3.2. By 3.2, $\chi(G[C]) \leq \tau^2\chi(H)$. Since $\chi(G[C]) > \tau^2(\tau + 1 + \phi(c + \tau))$ and $\chi^1(G) \leq \tau$, it follows that $\chi(H) > \tau + 1 + \phi(c + \tau)$. By 3.1 applied to $G[C]$ and H , we deduce that there is an edge uv of H , and a ρ -ball X of G , satisfying the theorem. This proves 3.3. \blacksquare

A ρ -comet (\mathcal{P}, X) in a graph G consists of a set \mathcal{P} of induced paths, each with the same pair of ends x, y say, and a ρ -ball X , such that y has a neighbour in X and no other vertex of any member of \mathcal{P} has a neighbour in X . We call x the *tip* of the ρ -comet, and $\chi(X)$ its *chromatic number*, and the set of lengths of members of \mathcal{P} its *spectrum*.

3.4 Let ϕ be a nondecreasing function, and let $\rho \geq 3$ and $\tau \geq 0$. For all integers $c \geq 1$ there exists $c' \geq 0$ with the following property. Let G be a (ρ, ϕ) -controlled graph such that $\chi^{\rho-1}(G) \leq \tau$ and $\chi(J) \leq \tau$ for every induced subgraph J of G with $\omega(J) < \omega(G)$. Let $x \in V(G)$, and let $V(G) \setminus \{x\}$ be a ρ -ball, such that x has a neighbour in $G \setminus x$. Let $\chi(V(G) \setminus \{x\}) > c'$. Then there is a ρ -comet $(\{P, Q\}, C)$ in G with tip x and chromatic number more than c , where $|E(Q)| = |E(P)| + 1$, and $|E(P)| \leq 2\rho + 1$.

Proof. Let $c' = 2\tau^2(\tau + 1 + \phi(c + \tau))$, and let G, x be as in the theorem. Since $V(G) \setminus \{x\}$ is a ρ -ball, every vertex of G has G -distance at most $2\rho + 1$ from x ; for $0 \leq k \leq 2\rho + 1$ let L_k be the set of vertices in $C \cup \{x\}$ with $G[C \cup \{x\}]$ -distance exactly k from x . Since $\chi(V(G) \setminus \{x\}) > c'$, there exists k such that $\chi(L_k) > c'/2$. Since $\chi^2(G) \leq \tau$ it follows that $k \geq 3$. Let (b_1, \dots, b_n) be an enumeration of L_{k-1} , and for $1 \leq i \leq n$ let W_i be the set of vertices in L_k that are adjacent to b_i but not to b_1, \dots, b_{i-1} . Then (W_1, \dots, W_n) is a τ -colourable grading of $G[L_k]$, compatible with (b_1, \dots, b_n) .

Since $\chi(L_k) > \tau^2(\tau + 1 + \phi(c + \tau))$, by 3.3 there is a square edge uv of $G[L_k]$, and a ρ -ball C of $G[L_k]$, such that

- u, v are both earlier than every vertex in C ;
- v has a neighbour in C , and u does not; and
- $\chi(C) > c$.

Let u', v' be the earliest parents of u, v respectively. Let P consist of the union of the path $v-v'$ and a path of length $k - 1$ between v', x with interior in L_1, \dots, L_{k-2} ; and let Q consist of the union of the path $v-u-u'$ and a path of length $k - 1$ between u', x with interior in L_1, \dots, L_{k-2} . Then $|E(Q)| = |E(P)| + 1$ and $|E(P)| \leq 2\rho$. Moreover, no vertex in X has a neighbour in $P \cup Q$ different from v , since all vertices in X are later than u, v . This proves 3.4. \blacksquare

By repeated application of 3.4 we deduce:

3.5 Let ϕ be a nondecreasing function, and let $\rho \geq 3$, $\ell \geq \rho(8\rho + 6)$ and $\tau \geq 0$. For all integers $c \geq 1$ there exists $c' \geq 0$ with the following property. Let G be a (ρ, ϕ) -controlled graph such that $\chi^{\rho-1}(G) \leq \tau$ and $\chi(J) \leq \tau$ for every induced subgraph J of G with $\omega(J) < \omega(G)$. Let $x \in V(G)$, and let $V(G) \setminus \{x\}$ be a ρ -ball, such that x has a neighbour in $G \setminus x$. Then there is a ρ -comet (\mathcal{P}, X) in G with tip x and chromatic number more than c , such that its spectrum includes $\{\ell + i : 0 \leq i \leq 2\rho + 3\}$.

Proof. Let $c_{\ell+1} = c$, and for $i = \ell, \dots, 1$ let 3.4 be satisfied setting $c = c_{i+1}$ and $c' = c_i$. Let $c' = c_1$.

(1) For all $k \geq 1$ there exists p_k with $1 \leq p_k \leq 2\rho$ and a ρ -comet in G with tip x , chromatic number more than c_k , and spectrum including $\{p_1 + \dots + p_k + i : 1 \leq i \leq k\}$.

By hypothesis there is a ρ -comet in G with chromatic number more than c_1 , tip x and spectrum $\{1\}$, so the statement holds when $k = 1$, setting $p_1 = 0$; and it follows for $k \geq 2$ by repeated application of 3.4. This proves (1).

Now p_1, \dots, p_ℓ exist and sum to at least ℓ , so there exists $k \leq \ell$ maximum such that

$$p_1 + \dots + p_k \leq \ell.$$

Since $p_1 + \dots + p_{4\rho+3} < 2(4\rho+3)\rho \leq \ell$, it follows that $k \geq 4\rho+3$. From the maximality of k , and since $p_{k+1} \leq 2\rho$, it follows that $p_1 + \dots + p_k > \ell - 2\rho$. Consequently the spectrum of the corresponding ρ -comet contains $\{\ell + i : 0 \leq i \leq 2\rho + 3\}$. This proves 3.5. \blacksquare

3.6 Let ϕ be a nondecreasing function, and let $\rho \geq 3$, $\ell \geq 8\rho^2 + 6\rho$, and $\tau \geq 0$. Then there exists c with the following property. Let G be a (ρ, ϕ) -controlled graph with $\chi(G) > c$ such that $\chi^{\rho-1}(G) \leq \tau$ and $\chi(J) \leq \tau$ for every induced subgraph J of G with $\omega(J) < \omega(G)$. Then there is an ℓ -hole in G .

Proof. Define $c_4 = \ell(2\rho + 4)\tau$. Choose c_3 such that 3.5 is satisfied replacing c, c', ℓ by $c_4, c_3, \ell - 6\rho$ respectively. Let $c_2 = \rho\tau + \phi(c_3)$, $c_1 = \tau\phi(c_2)$, $c = \phi(c_1)$, and let G be as in the theorem with $\chi(G) > c$. Since $\chi(G) > c$, there exists $z \in V(G)$ such that, denoting the set of vertices of G with G -distance i from z by L_i , we have $\chi(L_\rho) > c_1$. Since L_1 is τ -colourable, there is a stable subset A of L_1 such that the set B of vertices in L_ρ that are descendants of vertices in A has chromatic number more than $c_1/\tau = \phi(c_2)$. Consequently there is a ρ -ball $C \subseteq B$ with $\chi(C) > c_2$. Choose $D \subseteq A$ minimal such that every vertex in C has an ancestor in D . Let $v_1 \in D$; then there exists $v_{\rho-1} \in L_{\rho-1}$ with a neighbour in C such that v_1 is its only ancestor in D . Let $v_1 - v_2 - \dots - v_{\rho-1}$ be an induced path, where $v_i \in L_i$ for $1 \leq i \leq \rho - 1$. The set of vertices in C with distance less than ρ from one of $v_1, v_2, \dots, v_{\rho-1}$ has chromatic number at most $\rho\tau$, and so the set E of vertices in C with distance at least ρ from each of $v_1, v_2, \dots, v_{\rho-1}$ has chromatic number more than $c_2 - \rho\tau = \phi(c_3)$. Consequently there is a ρ -ball $F \subseteq E$, with chromatic number more than c_3 .

Since C is a ρ -ball and $v_{\rho-1}$ has a neighbour in C , there is an induced path P of $G[C \cup \{v_\rho\}]$ from $v_{\rho-1}$ to some vertex $x \in C$ with a neighbour in F , of length at most 2ρ , such that no vertex of P different from x has a neighbour in F . By 3.5 applied to x, F , since $\chi(F) > c_3$, there is a vertex $v \in F$, $2\rho + 4$ induced paths $P_0, \dots, P_{2\rho+3}$ of $G[F \cup \{v_{\rho-1}\}]$ between x, v , and a ρ -ball $X \subseteq F$, such that:

- $|E(P_i)| = \ell - 6\rho + i$ for $0 \leq i \leq 2\rho + 3$;
- $V(P_i) \cap X = \emptyset$ for $0 \leq i \leq 2\rho + 3$;
- v has a neighbour in X and no other vertex of P_i has a neighbour in X , for $0 \leq i \leq 2\rho + 2$; and
- $\chi(X) > c_4$.

Now every vertex of X has distance at least ρ from each of $v_1, \dots, v_{\rho-1}$, but there may be vertices in X with distance less than ρ to a vertex in P or in one of $P_0, \dots, P_{2\rho+3}$. The union of these paths has at most $\ell(2\rho+4)$ vertices (in fact much fewer), and since $\chi(X) > \ell(2\rho+4)\tau$, there exists a vertex in X with G -distance at least ρ from all vertices of these paths. Let $y \in L_{\rho-1}$ be adjacent to this vertex, and let Q be an induced path between y, x with interior in X , of length at most $2\rho+2$. Let R be a path of length $\rho-1$ between y, z with interior in $D \cup L_2 \cup L_3 \cup \dots \cup L_{\rho-2}$.

The union of the four paths $z-v_1-v_2-\dots-v_{\rho-1}, P, Q, R$ has length at most 6ρ , and at least $4\rho-3$, since P, Q have lengths at least ρ and at least $\rho-1$ respectively. Let their union have length j where $4\rho-3 \leq j \leq 6\rho$. Let $i = 6\rho-j$; then $0 \leq i \leq 2\rho+3$, and so P_i is defined and has length $\ell-6\rho+i+j$. Consequently the union of the five paths $z-v_1-v_2-\dots-v_{\rho-1}, P, P_i, Q$ and R is a cycle H_i of length ℓ , and we claim it is induced. Certainly it is a cycle; suppose that it is not induced, and so there is an edge ab say that joins two nonconsecutive vertices of H_i . It follows that a, b do not both belong to any of its five constituent paths. Certainly $a, b \neq z$. Suppose that $a = v_i$ for some i . Since every vertex in E has distance at least ρ from v_i , it follows that $b \notin V(P_i)$ and $b \notin V(Q)$. Also $b \notin V(P)$ since every vertex of $V(P) \setminus \{v_{\rho-1}\}$ belongs to L_k and P is an induced path containing $v_{\rho-1}$. Thus $b \in R$. Since the distance between y, a is at least $\rho-1$, and R has length $\rho-1$, it follows that $b \in L_1$ and so $i \in \{1, 2\}$; but $i \neq 1$ since A is stable, and $i \neq 2$ since $v_{\rho-1}$, and hence v_2 , has a unique ancestor in D . This proves that $a, b \notin \{v_1, \dots, v_{\rho-1}\}$.

Next suppose that $a \in V(R)$, and so either $b \in V(P) \setminus \{v_{\rho-1}\}$, or $b \in V(P_i \cup Q \setminus \{y\})$. In either case $b \in L_k$, and so $a = y$; hence $b \notin V(Q)$ since Q is induced and $y \in V(Q)$, and $a \notin V(P \cup P_i)$ since y has a neighbour in X with distance at least ρ from each vertex of $V(P \cup P_i)$, and $\rho \geq 3$. This proves that $a, b \notin V(R)$. Next suppose that $a \in V(P) \setminus \{x\}$. Then $b \in F$; but no vertex of P except x has a neighbour in F , a contradiction. Finally, suppose that $a \in V(P_i)$ and $b \in V(Q)$. No vertex of P_i has a neighbour in X except v , and $v \in V(Q)$, a contradiction. This proves that H_i is a hole, and so G has a hole of length ℓ . This proves 3.6. \blacksquare

Let us deduce 1.7, which we restate:

3.7 *Let $\rho \geq 2$ be an integer, and let \mathcal{C} be a ρ -controlled ideal of graphs. Let $\ell \geq 24$ if $\rho = 2$, and $\ell \geq 8\rho^2 + 6\rho$ if $\rho > 2$. Then for all $\kappa \geq 0$, there exists $c \geq 0$ such that every graph $G \in \mathcal{C}$ with $\omega(G) \leq \kappa$ and $\chi(G) > c$ has an ℓ -hole.*

Proof. By induction on κ we may assume that there exists τ_1 such that every graph in \mathcal{C} with clique number less than κ and no ℓ -hole has chromatic number at most τ_1 . Let \mathcal{C}_2 be the ideal of $G \in \mathcal{C}$ with clique number at most κ and no ℓ -hole. We suppose that there are graphs in \mathcal{C}_2 with arbitrarily large chromatic number, and so \mathcal{C}_2 is not 2-controlled, by 2.8. Consequently there exists τ_2 such that if \mathcal{C}_3 denotes the class of graphs $G \in \mathcal{C}_2$ with $\chi^2(G) \leq \tau_2$, there are graphs in \mathcal{C}_3 with arbitrarily large chromatic number. Hence by 3.6 with $\rho = 3$, \mathcal{C}_3 is not 3-controlled, and so on; and we deduce that there is an ideal \mathcal{C}_ρ of graphs in \mathcal{C} that is not ρ -controlled, a contradiction. This proves 3.7. \blacksquare

4 Controlling 8-balls

In this section we prove 1.8. We use the following relative of 3.1, proved in [5]:

4.1 Let $\tau, c \geq 0$ and let (W_1, \dots, W_n) be a τ -colourable grading of a graph G . Let H be a subgraph of G (not necessarily induced) with $\chi(H) > \tau + 2(c + \chi^1(G))$. Then there is an edge uv of H , and a subset X of $V(G)$, such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X ;
- v has a neighbour in X , and u does not; and
- $\chi(X) > c$.

We deduce a version of 3.3 that has no assumption of ρ -control:

4.2 Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C . Let every induced subgraph J of G with $\omega(J) < \omega(G)$ have chromatic number at most τ . Let the enumeration (b_1, \dots, b_m) of B and the grading (W_1, \dots, W_n) of $G[C]$ be compatible. Let (W_1, \dots, W_n) be τ -colourable, and let $\chi(G[C]) > \tau^2(2c + 3\tau)$. Then there is a square edge uv , and a subset X of $V(G)$, such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X ;
- v has a neighbour in X , and u does not; and
- $\chi(X) > c$.

Proof. Let H be as in 3.2; then by 3.2, $\chi(G[C]) \leq \tau^2\chi(H)$. Since $\chi(G[C]) > \tau^2(2c + 3\tau)$ and $\chi^1(G) \leq \tau$, it follows that $\chi(H) > \tau + 2(c + \chi^1(G))$. By 4.1 applied to $G[C]$ and H , we deduce that there is an edge uv of H , and a subset X of $V(G)$, satisfying the theorem. This proves 4.2. ■

A *shower* in G is a sequence $(L_0, L_1, \dots, L_k, s)$ where L_0, L_1, \dots, L_k are pairwise disjoint subsets of $V(G)$ and $s \in L_k$, such that

- $|L_0| = 1$;
- L_{i-1} covers L_i for $1 \leq i < k$;
- for $0 \leq i < j \leq k$, if $j > i + 1$ then no vertex in L_j has a neighbour in L_i ; and
- $G[L_k]$ is connected.

(Note that we do not require that L_{k-1} covers L_k .) We call the vertex in L_0 the *head* of the shower, and s its *drain*, and $L_0 \cup \dots \cup L_k$ is its *vertex set*. (Thus the drain can be any vertex of L_k .) The set of vertices in L_k with a neighbour in L_{k-1} is called the *floor* of the shower.

Let \mathcal{S} be a shower with head z_0 , drain s and vertex set V . An induced path of $G[V]$ between z_0, s is called a *jet* of \mathcal{S} . The set of all lengths of jets of \mathcal{S} is called the *jetset* of \mathcal{S} . For integers $\ell \geq 2$ and $1 \leq k \leq \ell$, we say a jetset is (k, ℓ) -complete if there are k jets J_0, \dots, J_{k-1} such that $|E(J_j)| = |E(J_0)| + j$ modulo ℓ for $0 \leq j \leq k - 1$.

We will prove:

4.3 Let $\tau \geq 0$ and $\ell \geq 2$. For all integers t with $1 \leq t \leq \ell$ there exists $c_t \geq 0$ with the following property. Let G be a graph such that $\chi^3(G) \leq \tau$, and such that every induced subgraph J of G with $\omega(J) < \omega(G)$ has chromatic number at most τ . Let $\mathcal{S} = (L_0, L_1, \dots, L_k, s)$ be a shower in G , with floor of chromatic number more than c_t . Then the jetset of \mathcal{S} is (t, ℓ) -complete.

Proof. If a shower has a nonempty floor then it has at least one jet, so we make take $c_1 = 0$. Hence inductively we assume that $t \geq 2$, and c_{t-1} exists. Let $c_t = 2(\ell + 1)\tau^2(2c_{t-1} + 7\tau)$, and let G be a graph, and let $\mathcal{S} = (L_0, L_1, \dots, L_k, s)$ be a shower in G with floor of chromatic number more than c_t . Let F be its floor. For $i \geq 0$, let M_i be the set of vertices with $G[L_k]$ -distance exactly i from s . Then there exists $d \geq 0$ such that $\chi(F \cap M_d) \geq \chi(F)/2$; and $d \geq 3$ since $\chi^2(G) \leq \tau < \chi(F)/2$. For $0 \leq i \leq d$, each vertex $v \in M_i$ is joined to s by an induced path of length i with interior in $M_1 \cup \dots \cup M_{i-1}$; let us call such a path a *bloodline* of v .

If $u \in F \cap M_d$ and $v \in M_0 \cup \dots \cup M_{d-1}$, we say that v is a *grandparent* of u if there exists $b \in L_{k-1}$ adjacent to both u, v . (Note that some vertices in M_d may have no grandparents; those with no neighbours in L_{k-1} at all, and those whose neighbours in L_{k-1} have no neighbours in $M_0 \cup \dots \cup M_{d-2}$.) Let C be the set of vertices in M_d with a grandparent in $M_0 \cup \dots \cup M_{d-2}$.

(1) If $\chi(C) > \ell\tau^2(2c_{t-1} + 7\tau)$ then the theorem holds.

Let (v_1, \dots, v_n) be an enumeration of $M_0 \cup \dots \cup M_{d-2}$, where the vertex in M_0 is first, followed by the vertices in M_1 in some order, and so on; more precisely, for $1 \leq i < j \leq n$, if $v_i \in M_a$ and $v_j \in M_b$ where $a, b \in \{0, \dots, d-2\}$, then $a \leq b$. Let B_0 be the set of vertices in L_{k-1} that have neighbours in $M_0 \cup \dots \cup M_{d-2}$, and let (b_1, \dots, b_m) be an enumeration of B_0 , enumerating the members of B_0 with the earliest neighbours in (v_1, \dots, v_n) first; more precisely, for $1 \leq i < j \leq m$, if b_j is adjacent to v_q for some $q \in \{1, \dots, n\}$, then there exists $p \in \{1, \dots, q\}$ such that b_i is adjacent to v_p . We say v_i is the *earliest grandparent* of $u \in M_d$ if i is minimum such that v_i is a grandparent of u . For $1 \leq i \leq n$, let W_i be the set of vertices in M_d whose earliest grandparent is v_i . Let $C = W_1 \cup \dots \cup W_n$; so (W_1, \dots, W_n) is a grading of $G[C]$. It is $\chi^2(G)$ -colourable, since $W_i \subseteq N_G^2[v_i]$ for each i , and hence τ -colourable. For $u \in W$, if v is the earliest grandparent of u , then the earliest parent of u is adjacent to v . Consequently the enumeration (b_1, \dots, b_m) and the grading (W_1, \dots, W_n) are compatible; because if $u, v \in W$ with u earlier than v , the earliest grandparent of u is earlier than the earliest grandparent of v , and consequently the earliest parent of u is earlier than the earliest parent of v .

For $0 \leq j < \ell$, let C_j be the set of vertices $u \in C$ such that $i - j$ is a multiple of ℓ , where i is the length of the bloodline of the earliest grandparent of u . Thus $C_0 \cup \dots \cup C_{\ell-1} = C$. Choose j such that $\chi(C_j) > \tau^2(2c_{t-1} + 7\tau)$. Then by 4.2, there is a square edge uv with $u, v \in C_j$, and a subset X of C_j , such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X ;
- v has a G -neighbour in X , and u does not; and
- $\chi(X) > c_{t-1} + 2\tau$.

Let the earliest parents of u, v be u', v' respectively, and let their earliest grandparents be u'', v'' respectively. Let P be the induced path between v, s consisting of the path $v-v'-v''$ and a bloodline of v'' . Thus P has length $j + 2$ modulo ℓ . Let Q be the path between v, s consisting of the edge uv , the path $u-u'-u''$, and a bloodline of u'' . Note that Q is induced, since v is not adjacent to u' (because uv is square). Moreover, Q has length $j + 3$ modulo ℓ .

Let Z be the set of vertices in X with distance at least four from both u', v' , and let L'_{k-1} be the set of vertices in L_{k-1} with a neighbour in Z . Let L'_{k-2} be the set of vertices in L_{k-2} with a neighbour in L'_{k-1} , and for $0 \leq i \leq k-3$ let $L'_i = L_i$. It follows that L_{i-1} covers L_i for $0 \leq i \leq k-2$, and so $(L'_0, \dots, L'_{k-1}, X \cup \{v\}, v)$ is a shower \mathcal{S}' . Let V' be its vertex set.

We claim that $V' \cap V(P \cup Q) = \{v\}$, and every edge between V' and $V(P \cup Q)$ is incident with v . For suppose that $a \in V'$ and $b \in V(P \cup Q)$ are equal or adjacent, and $a, b \neq v$. If $b = u$, then $a \neq b$ since $u \notin V'$; $a \notin X$ since u has no neighbour in X ; and so $a \in L'_i$ for some $i' < k$. Then $i = k-1$ since $b \in L_k$, and so a has a neighbour in Z from the definition of L'_{k-1} ; and so the G -distance between a, u is at least two from the definition of Z , a contradiction. Thus $b \neq u$. Next suppose that $b \in \{u', v'\}$. Since u', v' are the earliest parents of u, v respectively, and u, v are earlier than every vertex in X , it follows that no vertex in X is adjacent to b ; and so $a \neq b$, and $a \notin X$. Hence $a \in L'_i$ for some $i < k$, and since $b \in L_{k-1}$ it follows that $k-2 \leq i \leq k-1$. But then the distance between a and some vertex of Z is at most two, and since the distance between b, Z is at least four, it follows that a, b are nonadjacent, a contradiction. So $b \notin \{u', v'\}$ and so $b \in M_j$ for some $j \leq d-2$, and so b is a vertex of a bloodline of the earliest grandparent of one of u, v (say u'', v'' respectively). Consequently $a \neq b$, and $a \notin X$, and so $a \in L'_{k-1}$. Choose $z \in Z$ adjacent to a ; then u, v are earlier than z , and so u'', v'' are both earlier than the earliest grandparent of z . It follows that no parent of z has a neighbour in a bloodline of either of u'', v'' , and so a, b are nonadjacent, a contradiction. This proves our claim that $V' \cap V(P \cup Q) = \{v\}$, and every edge between V' and $V(P \cup Q)$ is incident with v .

The floor of \mathcal{S}' includes Z , and $\chi(Z) \geq \chi(X) - 2\chi^3(G) > c_{t-1}$. Consequently the jetset of \mathcal{S}' is $(t-1, \ell)$ -complete; let J_0, \dots, J_{t-2} be the corresponding jets of \mathcal{S}' . But for $0 \leq j \leq t-2$, both $J_j \cup P$ and $J_j \cup Q$ are jets of \mathcal{S} , and so the jetset of \mathcal{S} is (t, ℓ) -complete. This proves (1).

(2) If $\chi((F \cap M_d) \setminus C) > \tau^2(2c_{t-1} + 7\tau)$ then the theorem holds.

Let us write $C' = (F \cap M_d) \setminus C$; then C' is the set of vertices in M_d that have neighbours in L_{k-1} , but every such neighbour has no neighbour in $M_0 \cup \dots \cup M_{d-2}$. The neighbours in L_{k-1} might or might not have neighbours in M_{d-1} . Take an arbitrary enumeration (b_1, \dots, b_n) of M_{d-1} , and for $1 \leq i \leq n$ let W_i be the set of vertices in C' adjacent to b_i and nonadjacent to b_1, \dots, b_{i-1} . Thus (W_1, \dots, W_n) is a grading of $G[C']$, compatible with (b_1, \dots, b_n) , and it is $\chi^1(G)$ -colourable and hence τ -colourable. By 4.2, there is a square edge uv of $G[C']$, and a subset X of C' , such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X ;
- v has a neighbour in X , and u does not; and
- $\chi(X) > c_{t-1} + 2\tau$.

Let u', v' be the earliest parents of u, v respectively. Let P be an induced path between v, s consisting of the edge vv' and a bloodline of v' ; and let Q be the path consisting of the edges uv, uu' , and a bloodline of u' . They are both induced. Let Z be the set of all vertices in X with G -distance at least four from both of u, v . Let L'_{k-1} be the set of vertices in L_{k-1} with a neighbour in Z , and for $0 \leq i \leq k-2$ let $L'_i = L_i$. Then $(L'_0, \dots, L'_{k-1}, X \cup \{v\}, v)$ is a shower \mathcal{S}' . Moreover, its vertex set V' satisfies $V' \cap V(P \cup Q) = \{v\}$, and every edge between V' and $V(P \cup Q)$ is incident with v (the proof is as in (1), and we omit it). Its floor includes Z , and $\chi(Z) \geq \chi(X) - 2\chi^3(G) > c_{t-1}$; and so the jetset of \mathcal{S}' is $(t-1, \ell)$ -complete. But for each jet J of \mathcal{S}' , $J \cup P, J \cup Q$ are both jets of \mathcal{S} , and so \mathcal{S} is (t, ℓ) -complete. This proves (2).

Since $\chi(F) > c_t$, it follows that $\chi(F \cap M_d) > (\ell + 1)\tau^2(2c_{t-1} + 7\tau)$, and so either $\chi(C) > \ell\tau^2(2c_{t-1} + 7\tau)$ or $\chi((F \cap M_d) \setminus C) > \tau^2(2c_{t-1} + 7\tau)$. Hence the result follows from (1) or (2). This proves 4.3. ■

A *recirculator* for a shower $(L_0, L_1, \dots, L_k, s)$ with head z_0 is an induced path R with ends s, z_0 such that no internal vertex of R belongs to V and no internal vertex of R has any neighbours in $V \setminus \{s, z_0\}$. We need the following, proved in [9]:

4.4 *Let $c \geq 0$ be an integer, and let G be a graph such that $\chi(G) > 44c + 4\chi^8(G)$. Then there is a shower in G , with floor of chromatic number more than c , and with a recirculator.*

We deduce 1.8, which we restate:

4.5 *For all integers $\ell \geq 2$ and $\tau \geq 0$ there is an integer $c \geq 0$ with the following property. Let G be a graph such that $\chi^8(G) \leq \tau$, and every induced subgraph J of G with $\omega(J) < \omega(G)$ has chromatic number at most τ . If $\chi(G) > c$ then there are ℓ holes in G with lengths of all possible values modulo ℓ .*

Proof. Let c_ℓ be as in 4.3 with $k = \ell$, and let $c = 44c_\ell + 4\tau$. Let G be a graph such that $\chi^8(G) \leq \tau$, and every induced subgraph J of G with $\omega(J) < \omega(G)$ has chromatic number at most τ , with $\chi(G) > c$. By 4.4 there is a shower in G , with floor of chromatic number more than c_ℓ , and with a recirculator. By 4.3 this shower is (ℓ, ℓ) -complete. Thus adding the recirculator to each of the corresponding jets gives the ℓ holes we need. This proves 4.5. ■

5 Two consecutive holes

Finally let us prove 1.6, which we restate.

5.1 *For each $\kappa, \ell \geq 0$ there exists $c \geq 0$ such that every graph G with $\omega(G) \leq \kappa$ and $\chi(C) > c$ has holes of two consecutive lengths, both of length more than ℓ .*

Proof. We may assume that $\ell \geq 8$. By induction on κ , there exists τ_1 such that every graph with clique number less than κ and chromatic number more than τ_1 has two holes of consecutive length more than ℓ . Let \mathcal{C}_2 be the ideal of graphs with clique number at most κ and with no two holes of consecutive lengths more than ℓ . By 2.8, \mathcal{C}_2 is not 2-controlled, and so for some τ_1 there are graphs

G in \mathcal{C}_2 with arbitrarily large chromatic number and $\chi^2(G) \leq \tau$. Let \mathcal{C}_3 be the ideal of graphs in G with $\chi^2(G) \leq \tau$. By 3.6 with $\rho = 3$, \mathcal{C}_3 is not 3-controlled, and so on; and we deduce that there is an ideal \mathcal{C}_ℓ of graphs in \mathcal{C}_2 , with unbounded chromatic number, and a number τ such that $\chi^\ell(G) \leq \tau$ for each $G \in \mathcal{C}_\ell$.

Let $c' = 14\tau^3$, and let $c = 44c' + 4\tau$; and choose $G \in \mathcal{C}_\ell$ with $\chi(G) > c$. By 4.4, there is a shower $\mathcal{S} = (L_0, \dots, L_k, s)$ in G with a recirculator and with floor F with chromatic number more than $14\tau^3$. Define d, M_0, \dots, M_d and C as in the proof of 4.3.

(1) *If $\chi(C) > 7\tau^3$ then the theorem holds.*

Define the enumeration (v_1, \dots, v_n) of $M_0 \cup \dots \cup M_{d-2}$, and B_0 and the enumeration (b_1, \dots, b_m) of B_0 , as in the proof of 4.3. Let $b'_{m+1-i} = b_i$; so $(b'_1, \dots, b'_m) = (b_m, \dots, b_1)$ is also an enumeration of the same set. For $i = 1, \dots, m$ let W_i be the set of vertices in M_d that are adjacent to b'_i and to none of b'_1, \dots, b'_{i-1} . Thus $W_1 \cup \dots \cup W_m = C$, and (W_1, \dots, W_m) is a $\chi^1(G)$ -colourable (and hence τ -colourable) grading of $G[C]$, compatible with (b'_1, \dots, b'_m) . By 4.2 (taking $c = 2\tau$), there is a square edge uv of $G[C]$, and a subset X of C , such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X ;
- v has a neighbour in X , and u does not; and
- $\chi(X) > 2\tau$.

Let $u', v' \in B_0$ be the earliest parents of u, v respectively. Since $\chi(X) > 2\tau$, there exists $x \in X$ with G -distance at least four from each of u', v' . Let $x' \in B_0$ be its earliest parent, and let R be an induced path of $G[X \cup \{v, x'\}]$ between v and x' . Now no vertex of the interior of R is adjacent to either of u', v' since u, v are earlier than every member of X and u', v' are their earliest parents. Also, x' is nonadjacent to u, v, u', v' since the G -distance between x and u', v' is at least four. Since uv is square, the path Q' obtained from R by adding the edge vv' is induced, and since u has no neighbour in X , so is the path P' obtained from R by adding the path $v-u-u'$. Now there are paths between the apex of \mathcal{S} (say a) and u' , and between a, v' , both of length $k-1 \geq \ell$. No vertex of L_k has a neighbour different from u', v' in these paths. Also x' has no neighbour in $P' \cup Q'$; because any such neighbour would be in $L_k \cup L_{k-1} \cup L_{k-2}$, and hence would have distance at most one from one of u', v' , which is impossible since the G -distance between x and u', v' is at least four. Thus by taking the union of the first of these with P' and the second with Q' , we obtain two paths P, Q , both induced and both between a, x' , with consecutive lengths. Choose i minimum such that x' is adjacent to v_i , and let R be a bloodline of v_i (defined as in 4.3). Let $u' = b'_f, v' = b'_g, x' = b'_h$. Since u, v are earlier than x , it follows that $f, g < h$. Now u', v' are nonadjacent to v_i since the G -distance between $u'v'$ and x is at least four. Since $u' = b_{m+1-f}$ and $x' = b_{m+1-h}$, and $m+1-f > m+1-h$, it follows that no neighbour of u' belongs to $V(R)$, and similarly for v' . Thus the union of P with the edge $x'v_i$ and R is induced, and so is the union of Q with $x'v_i$ and R . Consequently there are jets of \mathcal{S} with consecutive lengths. By taking their unions with the recirculator, we obtain holes of consecutive lengths. This proves (1).

(2) If $\chi((F \cap M_d) \setminus C) > 7\tau^3$ then the theorem holds.

The proof of this is the same as that for step (2) of the proof of 4.3 and we omit it.

From (1) and (2) the result follows. ■

References

- [1] M. Bonamy, P. Charbit and S. Thomassé, “Graphs with large chromatic number induce $3k$ -cycles”, submitted for publication (manuscript August 2014).
- [2] M. Chudnovsky, A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás’ conjectures”, *J. Combinatorial Theory, Ser. B*, 118 (2016), 109–128.
- [3] M. Chudnovsky, A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. III. Long holes”, *Combinatorica*, to appear, arXiv:1506.02232.
- [4] M. Chudnovsky, A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings”, submitted for publication (manuscript September 2015), arXiv:1609.00314.
- [5] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, “Induced subgraphs of graphs with large chromatic number. VIII. Long odd holes”, submitted for publication (manuscript December 2016), arXiv:1701.07217.
- [6] A. Gyárfás, “Problems from the world surrounding perfect graphs”, *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* 19 (1987), 413–441.
- [7] A. Scott, “Induced cycles and chromatic number”, *J. Combinatorial Theory, Ser. B*, 76 (1999), 150–154
- [8] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. I. Odd holes”, *J. Combinatorial Theory, Ser. B*, 121 (2016), 68–84.
- [9] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes”, submitted for publication (manuscript January 2015), arXiv:1509.06563.
- [10] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. VI. Banana trees”, submitted for publication (manuscript January 2017), arXiv:1701.05597.