Proof of a conjecture of Bowlin and Brin on four-colouring triangulations

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Abstract

We prove a conjecture of Bowlin and Brin that for all $n \geq 5$, the $n$-vertex biwheel is the planar triangulation with $n$ vertices admitting the largest number of four-colourings.
1 Introduction

All graphs in this paper are finite, and have no loops or parallel edges (except immediately after 1.1). A triangulation is a graph drawn in the 2-sphere $S^2$ such that the boundary of every region is a 3-vertex cycle. A biwheel is a triangulation consisting of a cycle $C$ and two more vertices, each adjacent to every vertex of $C$, and for $n \geq 5$, we denote the $n$-vertex biwheel by $B_n$. For $k > 0$ an integer, a $k$-colouring of a graph $G$ is a map $\phi$ from the vertex set $V(G)$ of $G$ to $\{1, \ldots, k\}$, such that $\phi(u) \neq \phi(v)$ for every edge $uv$. Let $P_k(G)$ denote the number of $k$-colours of a graph $G$. Garry Bowlin and Matt Brin [1, 2] conjectured the following, which is the main result of this note:

1.1 If $G$ is a triangulation with $n \geq 5$ vertices, then $P_4(G) \leq P_4(B_n)$.

The hypothesis that $G$ has no parallel edges is important, and without it the extremal “triangulation” is different. Let us say a pseudo-triangulation is a drawing in $S^2$, possibly with parallel edges but without loops, such that the boundary of every region is a cycle of length three. We claim that every $n$-vertex pseudo-triangulation has at most $3 \cdot 2^n$ 4-colourings. To see this, let $G$ be an $n$-vertex pseudo-triangulation, and order its vertex set $v_1, \ldots, v_n$ such that $v_1, v_2$ are adjacent and for $3 \leq i \leq n$, there is a triangle containing $v_i$ and two of $v_1, \ldots, v_{i-1}$. For $1 \leq i \leq n$, let $G_i = G|\{v_1, \ldots, v_i\}$. (We use $G|X$ to denote the subdrawing of $G$ induced on $X$, when $X \subseteq V(G)$.) Thus $P_4(G_2) = 12$, and for $3 \leq i \leq n$ every 4-colouring of $G_{i-1}$ extends to at most two 4-colourings of $G_i$; and so by induction it follows that for $2 \leq i \leq n$, $P_4(G_i) \leq 3 \cdot 2^i$, and in particular, $P_4(G) \leq 3 \cdot 2^n$. But there is a pseudo-triangulation with $n$ vertices and $3 \cdot 2^n$ 4-colourings, obtained as follows: take a drawing with two vertices $x, y$ and $n - 2$ parallel edges, and for each consecutive pair of parallel edges add a new vertex between them adjacent to $x, y$.

Our proof of 1.1 is based on the same idea of bounding the number of 4-colourings by ordering the vertex set such that each makes a triangle with two of its predecessors, but we need to treat a few vertices as special, and just order the others.

Bowlin and Brin also raised the question of deciding which $n$-vertex triangulation has the second most 4-colourings, and conjectured that the number of 4-colourings of the second-best triangulation is asymptotically half of the number for the biwheel. We do not prove this, but prove that the number of 4-colourings of any non-biwheel on $n$ vertices is asymptotically at most $27/32$ of the number for the biwheel. More precisely, we prove the following, which immediately implies 1.1.

1.2 Let $G$ be a triangulation with $n \geq 5$ vertices.

- If $G$ is a biwheel, then $P_4(G) = 2^n - 8$ if $n$ is odd, and $2^n + 32$ if $n$ is even.
- If $G$ is not a biwheel, then $P_4(G) \leq \frac{27}{32} 2^n \leq 2^n - 8$.

2 The main proof

First, we need

2.1 If $G$ is a cycle with $n$ vertices then $P_3(G) = 2^n + 2(-1)^n$. 


Proof. The result is well-known and elementary, but we give a proof for completeness. For \( n \geq 1 \), let \( \kappa_n = 2^n + 2(-1)^n \). For \( n \geq 2 \), let \( \alpha_n \) be the number of 3-colourings of an \( n \)-vertex path such that its ends have the same colour, and let \( \beta_n \) be the number of 3-colourings such that its ends have different colours. We prove by induction on \( n \) that \( \alpha_n = \kappa_{n-1} \) and \( \beta_n = \kappa_n \). The result is true when \( n = 2 \), so we assume \( n \geq 3 \). Now \( \alpha_n = \beta_{n-1} \), so the first assertion holds. For the second, let \( G \) be a path with vertices \( v_1, \ldots, v_n \) in order. Each 3-colouring of \( G \setminus \{v_n\} \) with \( v_1, v_{n-1} \) of different colours extends to a unique 3-colouring of \( G \) in which \( v_1, v_n \) have different colours, and each 3-colouring of \( G \setminus \{v_n\} \) with \( v_1, v_{n-1} \) of the same colour extends to two 3-colourings of \( G \) in which \( v_1, v_n \) have different colours. Consequently

\[
\beta_n = \alpha_{n-1} + 2\beta_{n-1} = \kappa_{n-1} + 2\kappa_{n-2} = \kappa_n
\]

as required. This proves that \( \beta_n = \kappa_n \) for all \( n \geq 2 \). Now if \( G \) is a cycle with \( n \) vertices, it follows (by deleting one edge of \( G \)) that \( P_3(G) = \beta_n = \kappa_n \). This proves 2.1.

If \( G \) is a triangulation, a triangle of \( G \) means a region of \( G \), and we denote a triangle incident with vertices \( a, b, c \) by \( abc \). A triangle touches another if they are distinct and share an edge. It is convenient to first prove the result when \( G \) is 4-connected.

2.2 Let \( G \) be a 4-connected triangulation, not a biwheel, with \( n \) vertices, and with minimum degree \( k \) say. (Thus \( k \in \{4,5\} \).) Then \( P_4(G) \leq 27 \cdot 2^{n-5} \) if \( k = 4 \), and \( P_4(G) \leq 45 \cdot 2^{n-6} \) if \( k = 5 \).

Proof. A diamond in \( G \) is a set of four vertices of \( G \), all pairwise adjacent except for one pair, called the apices. A diamond \( a, b, c, d \) with apices \( a, b \) is pure if there is no vertex of \( G \) adjacent to \( a, b \) and non-adjacent to \( c, d \). Let \( v \in V(G) \) have degree \( k \), and let \( N \) be its set of neighbours and \( M = V(G) \setminus (N \cup \{v\}) \).

(1) There is a triangle of \( G \) with vertex set included in \( M \).

For suppose not. If some vertex in \( M \) is adjacent to every vertex of \( N \), then \( G \) is a biwheel, a contradiction; and at most two vertices of \( M \) have \( k-1 \) neighbours in \( N \), by planarity. Moreover, \( G|M \) is connected, since \( G \) is 4-connected. Since every vertex in \( G \) has degree at least four, it follows that at most two vertices in \( M \) have degree one in \( G|M \). Suppose that \( G|M \) is a forest. Then it is a path, with vertices \( v_1, \ldots, v_n \) in order; and \( v_1, v_n \) both have \( k-1 \) neighbours in \( N \), so \( k = 4 \) and \( G \) is a biwheel, a contradiction. Thus there is a cycle in \( G|M \), and hence (1) follows.

(2) Either \( k = 4 \) and \( n = 8 \) and \( P_4(G) = 72 \), or there is a diamond \( D \) of \( G \) such that some vertex of \( G \) with degree \( k \) has no neighbour in \( D \).

For let \( xyz \) be a triangle with \( x, y, z \in M \); and let \( x', y', z' \) be vertices of \( G \) different from \( x, y, z \) such that there are triangles \( x'yz, xy'z, xy'z' \). If one of \( x', y', z' \) is in \( M \) then (2) holds, so we assume that \( x', y', z' \) are all in \( N \). Since \( G|N \) is a cycle of length \( k \), we may assume that \( x', y' \) are adjacent, and so \( z \) has degree four and hence \( |N| = k = 4 \); and so we may also assume that \( y', z' \) are adjacent. It follows that \( x, z \) have degree four in \( G \). Let \( w' \) be the neighbour of \( v \) different from \( x', y', z' \). Let \( px'w' \) touch \( vx'w' \). Thus \( \{p, v, x', w'\} \) is a diamond, and \( x \) is non-adjacent to \( v, x', w' \), so we may
assume that \( v \) is adjacent to \( p \), that is, \( p = y \). But then \( n = 8 \) and \( P_4(G) = 72 \), and the result holds. This proves (2).

In view of (2), we may assume that there is a diamond \( \{a, b, c, d\} \) in \( M \), with apices \( a, b \).

(3) There is a pure diamond included in \( M \).

For we may assume that \( \{a, b, c, d\} \) is not pure, and so there is a vertex \( p \) adjacent to \( a, b \) and not to \( c, d \). From the symmetry between \( c, d \), we may assume that the cycle with vertex set \( \{a, c, b, p\} \) divides \( S^2 \) into two open discs \( D_1, D_2 \), one containing \( d \) and the other containing \( v \), say \( d \in D_1 \). Let \( bdq \) touch \( bdc \). Then \( q \neq p \) since \( q \) is adjacent to \( d \), and so \( q \in D_1 \), and in particular \( q \in M \). Suppose that the diamond \( \{c, q, b, d\} \) is not pure; then there is a vertex \( r \) adjacent to \( c, q \) and not to \( b, d \), which is impossible by planarity. This proves (3).

In view of (3) we may assume that \( \{a, b, c, d\} \) is pure.

(4) We can order \( V(G) \setminus \{a, b, c, d\} \) and \( \{v_1, \ldots, v_{n-4}\} \) in such a way that \( v_1 = v \), \( N = \{v_2, \ldots, v_{k+1}\} \), and for \( k + 2 \leq i \leq n - 4 \) there is a triangle containing \( v_i \) and two of \( v_1, \ldots, v_{i-1} \).

For let \( G' \) be the drawing obtained from \( G \) by deleting \( a, b, c, d \), and let \( D \) be the region of \( G' \) containing \( a, b, c, d \). Then \( D \) is an open disc, and so there is a closed walk tracing its boundary. Since \( G \) is 4-connected and the diamond \( \{a, b, c, d\} \) is pure, it follows that no vertex appears twice in this closed walk, and so \( D \) is bounded by a cycle \( C \) say. Choose a sequence \( v_1, \ldots, v_j \) of distinct members of \( V(G) \setminus \{a, b, c, d\} \), where \( v_1 = v \), \( N = \{v_2, \ldots, v_{k+1}\} \), and for \( k + 2 \leq i \leq j \) there is a triangle containing \( v_i \) and two of \( v_1, \ldots, v_{i-1} \), with \( j \) maximum. Let \( X = \{v_1, \ldots, v_j\} \) and \( Y = V(G) \setminus (\{a, b, c, d\} \cup X) \). Let \( R \) be the set of all triangles with vertex set included in \( X \). Let \( S \) be the closure of the union of the members of \( R \); thus \( S \) is some closed subset of \( S^2 \), with boundary the closure of some set of edges of \( G \). Let \( e \in E(G) \) be an edge of \( G \) in the boundary of \( S \), where \( e = xy \) say, and let \( xyz \in R \) touch some region \( xyz' \notin R \). Thus \( z' \notin X \) from the definition of \( R \), and so from the choice of \( j \) it follows that \( z' \in \{a, b, c, d\} \), and consequently \( e \in E(C) \). Consequently every edge in the boundary of \( S \) belongs to \( E(C) \), and since every vertex of \( G \) is incident with an even number of such edges, it follows that \( C \) is the boundary of \( S \). Consequently \( S \) is a closed disc, and hence contains all vertices of \( G \) not in \( \{a, b, c, d\} \). It follows that \( j = n - 4 \). This proves (4).

For \( 1 \leq i \leq n - 4 \), let \( G_i = G \setminus \{v_1, \ldots, v_i\} \). For \( k + 2 \leq i \leq n - 4 \), \( P_4(G_i) \leq 2P_4(G_{i-1}) \), and so \( P_4(G_{n-4}) \leq 2^{n-k-5}P_4(G_{k+1}) \). But every 4-colouring of \( G_{n-4} \) can be extended to at most six 4-colourings of \( G \) (this is easy to check, and we leave it to the reader), and so \( P_4(G) \leq 6 \cdot 2^{n-k-5}P_4(G_{k+1}) \). By 2.1, if \( k = 4 \) then \( P_4(G_{k+1}) = 72 \), and if \( k = 5 \) then \( P_4(G_{k+1}) = 120 \). This proves 2.2.

2.3 Let \( n \geq 6 \) be such that every triangulation with \( n' \) vertices admits at most \( 2^{n'} + 32 \) 4-colourings, for \( 5 \leq n' \leq n - 1 \). Let \( G \) be a triangulation with \( n \) vertices.

- If \( G \) has a vertex of degree three, then \( P_4(G) \leq 2^{n-1} + 32 \), and \( P_4(G) = 24 \) if \( n = 6 \).
If $G$ has no vertex of degree three and $G$ is not 4-connected, then $n \geq 9$ and $P_4(G) \leq 2^{n-1} + 128$.

Suppose first that some vertex $v$ has degree three. Now $G' = G \setminus v$ is a triangulation, so from the hypothesis $P_4(G') \leq 2^{n-1} + 32$, and $P_4(G') = 24$ if $n = 6$. But every 4-colouring of $G'$ extends to a unique 4-colouring of $G$, and the result follows.

Now we assume that $G$ has no vertex of degree three, but is not 4-connected. Consequently there is a cycle of length three in $G$ that does not bound a region; and so there are two triangulations $G_1, G_2$ in $S$ with union $G$, intersecting just in this cycle, and each with at least four vertices. Let $G_i$ have $n_i + 3$ vertices for $i = 1, 2$. Since $G$ has no vertex of degree three it follows that $n_1, n_2 \geq 3$, and so $n \geq 9$. Moreover, $P_4(G) = P_4(G_1)P_4(G_2)/24$, and from the hypothesis, $P_4(G_i) \leq 2^{n_i+3} + 32$ for $i = 1, 2$. It follows that

$$P_4(G) \leq (2^{n_1+3} + 32)(2^{n_2+3} + 32)/24 \leq (2^{n_1} + 32)(2^{n_2} + 32)/24 = 2^{n-1} + 128,$$

since $n_1, n_2 \geq 3$ and sum to $n - 3$. This proves 2.3.

Proof of 1.2. The first assertion of 1.2 is easy using 2.1 and we leave it to the reader. Note also that $\frac{27}{32} 2^n \leq 2^n - 8$ if $n \geq 6$, and every triangulation with five vertices is a biwheel. Thus for the second assertion, we proceed by induction on $n$, and we may therefore assume that every triangulation with $n'$ vertices admits at most $2^{n'} + 32$ 4-colourings, for $5 \leq n' \leq n - 1$. Let $G$ be a triangulation with $n$ vertices, not a biwheel, and so $n \geq 6$. If $n \geq 7$ and $G$ has a vertex of degree three, then by 2.3,

$$P_4(G) \leq 2^{n-1} + 32 \leq \frac{27}{32} 2^n$$

as required; while if $n = 6$ and $G$ has a vertex $v$ of degree three, then by 2.3,

$$P_4(G) = 24 \leq \frac{27}{32} 2^n.$$

Thus we may assume that $G$ has no vertex of degree three. If $G$ is not 4-connected, then by 2.3 $n \geq 9$ and

$$P_4(G) \leq 2^{n-1} + 128 \leq \frac{27}{32} 2^n$$

as required. If $G$ is 4-connected and has no vertex of degree four, then by 2.2,

$$P_4(G) \leq 45 \cdot 2^{n-6} \leq \frac{27}{32} 2^n,$$

while if $G$ is 4-connected and has a vertex of degree four then the result follows from 2.2. This proves 1.2.

References
