# Proof of a conjecture of Bowlin and Brin on four-colouring triangulations

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## Abstract

We prove a conjecture of Bowlin and Brin that for all $n \geq 5$, the $n$-vertex biwheel is the planar triangulation with $n$ vertices admitting the largest number of four-colourings.

# 1 Introduction

All graphs in this paper are finite, and have no loops or parallel edges (except immediately after 1.1). A *triangulation* is a graph drawn in the 2-sphere $S^2$ such that the boundary of every region is a 3-vertex cycle. A *biwheel* is a triangulation consisting of a cycle $C$ and two more vertices, each adjacent to every vertex of $C$, and for $n \geq 5$, we denote the $n$-vertex biwheel by $B_n$. For $k > 0$ an integer, a *$k$-colouring* of a graph $G$ is a map $\phi$ from the vertex set $V(G)$ of $G$ to $\{1, \dots, k\}$, such that $\phi(u) \neq \phi(v)$ for every edge $uv$. Let $P_k(G)$ denote the number of $k$-colourings of a graph $G$. Garry Bowlin and Matt Brin [1, 2] conjectured the following, which is the main result of this note:

**1.1** *If $G$ is a triangulation with $n \geq 5$ vertices, then $P_4(G) \leq P_4(B_n)$.*

The hypothesis that $G$ has no parallel edges is important, and without it the extremal "triangulation" is different. Let us say a *pseudo-triangulation* is a drawing in $S_2$, possibly with parallel edges but without loops, such that the boundary of every region is a cycle of length three. We claim that every $n$-vertex pseudo-triangulation has at most $3 \cdot 2^n$ 4-colourings. To see this, let $G$ be an $n$-vertex pseudo-triangulation, and order its vertex set $v_1, \dots, v_n$ such that $v_1, v_2$ are adjacent and for $3 \leq i \leq n$, there is a triangle containing $v_i$ and two of $v_1, \dots, v_{i-1}$. For $1 \leq i \leq n$, let $G_i = G|\{v_1, \dots, v_i\}$. (We use $G|X$ to denote the subdrawing of $G$ induced on $X$, when $X \subseteq V(G)$.) Thus $P_4(G_2) = 12$, and for $3 \leq i \leq n$ every 4-colouring of $G_{i-1}$ extends to at most two 4-colourings of $G_i$; and so by induction it follows that for $2 \leq i \leq n$, $P_4(G_i) \leq 3 \cdot 2^i$, and in particular, $P_4(G) \leq 3 \cdot 2^n$. But there is a pseudo-triangulation with $n$ vertices and $3 \cdot 2^n$ 4-colourings, obtained as follows: take a drawing with two vertices $x, y$ and $n - 2$ parallel edges, and for each consecutive pair of parallel edges add a new vertex between them adjacent to $x, y$.

Our proof of 1.1 is based on the same idea of bounding the number of 4-colourings by ordering the vertex set such that each makes a triangle with two of its predecessors, but we need to treat a few vertices as special, and just order the others.

Bowlin and Brin also raised the question of deciding which $n$-vertex triangulation has the second most 4-colourings, and conjectured that the number of 4-colourings of the second-best triangulation is asymptotically half of the number for the biwheel. We do not prove this, but prove that the number of 4-colourings of any non-biwheel on $n$ vertices is asymptotically at most 27/32 of the number for the biwheel. More precisely, we prove the following, which immediately implies 1.1.

**1.2** *Let $G$ be a triangulation with $n \geq 5$ vertices.*

- *If $G$ is a biwheel, then $P_4(G) = 2^n - 8$ if $n$ is odd, and $2^n + 32$ if $n$ is even.*
- *If $G$ is not a biwheel, then $P_4(G) \leq \frac{27}{32} 2^n \leq 2^n - 8$.*

# 2 The main proof

First, we need

**2.1** *If $G$ is a cycle with $n$ vertices then $P_3(G) = 2^n + 2(-1)^n$.*

**Proof.** The result is well-known and elementary, but we give a proof for completeness. For $n \ge 1$, let $\kappa_n = 2^n + 2(-1)^n$. For $n \ge 2$, let $\alpha_n$ be the number of 3-colourings of an $n$-vertex path such that its ends have the same colour, and let $\beta_n$ be the number of 3-colourings such that its ends have different colours. We prove by induction on $n$ that $\alpha_n = \kappa_{n-1}$, and $\beta_n = \kappa_n$. The result is true when $n = 2$, so we assume $n \ge 3$. Now $\alpha_n = \beta_{n-1}$, so the first assertion holds. For the second, let $G$ be a path with vertices $v_1, \dots, v_n$ in order. Each 3-colouring of $G \setminus \{v_n\}$ with $v_1, v_{n-1}$ of different colours extends to a unique 3-colouring of $G$ in which $v_1, v_n$ have different colours, and each 3-colouring of $G \setminus \{v_n\}$ with $v_1, v_{n-1}$ of the same colour extends to two 3-colourings of $G$ in which $v_1, v_n$ have different colours. Consequently

$$\beta_n = \alpha_{n-1} + 2\beta_{n-1} = \kappa_{n-1} + 2\kappa_{n-2} = \kappa_n$$

as required. This proves that $\beta_n = \kappa_n$ for all $n \ge 2$. Now if $G$ is a cycle with $n$ vertices, it follows (by deleting one edge of $G$) that $P_3(G) = \beta_n = \kappa_n$. This proves 2.1. ∎

If $G$ is a triangulation, a *triangle* of $G$ means a region of $G$, and we denote a triangle incident with vertices $a, b, c$ by $abc$. A triangle *touches* another if they are distinct and share an edge. It is convenient to first prove the result when $G$ is 4-connected.

**2.2** *Let $G$ be a 4-connected triangulation, not a biwheel, with $n$ vertices, and with minimum degree $k$ say. (Thus $k \in \{4, 5\}$.) Then $P_4(G) \le 27 \cdot 2^{n-5}$ if $k = 4$, and $P_4(G) \le 45 \cdot 2^{n-6}$ if $k = 5$.*

**Proof.** A *diamond* in $G$ is a set of four vertices of $G$, all pairwise adjacent except for one pair, called the *apices*. A diamond $a, b, c, d$ with apices $a, b$ is *pure* if there is no vertex of $G$ adjacent to $a, b$ and non-adjacent to $c, d$. Let $v \in V(G)$ have degree $k$, and let $N$ be its set of neighbours and $M = V(G) \setminus (N \cup \{v\})$.

(1) *There is a triangle of $G$ with vertex set included in $M$.*

For suppose not. If some vertex in $M$ is adjacent to every vertex of $N$, then $G$ is a biwheel, a contradiction; and at most two vertices of $M$ have $k-1$ neighbours in $N$, by planarity. Moreover, $G|M$ is connected, since $G$ is 4-connected. Since every vertex in $G$ has degree at least four, it follows that at most two vertices in $M$ have degree one in $G|M$. Suppose that $G|M$ is a forest. Then it is a path, with vertices $v_1, \dots, v_n$ in order say; and $v_1, v_n$ both have $k-1$ neighbours in $N$, so $k = 4$ and $G$ is a biwheel, a contradiction. Thus there is a cycle in $G|M$, and hence (1) follows.

(2) *Either $k = 4$ and $n = 8$ and $P_4(G) = 72$, or there is a diamond $D$ of $G$ such that some vertex of $G$ with degree $k$ has no neighbour in $D$.*

For let $xyz$ be a triangle with $x, y, z \in M$; and let $x', y', z'$ be vertices of $G$ different from $x, y, z$ such that there are triangles $x'yz, xy'z, xyz'$. If one of $x', y', z'$ is in $M$ then (2) holds, so we assume that $x', y', z'$ are all in $N$. Since $G|N$ is a cycle of length $k$, we may assume that $x', y'$ are adjacent, and so $z$ has degree four and hence $|N| = k = 4$; and so we may also assume that $y', z'$ are adjacent. It follows that $x, z$ have degree four in $G$. Let $w'$ be the neighbour of $v$ different from $x', y', z'$. Let $px'w'$ touch $vx'w'$. Thus $\{p, v, x', w'\}$ is a diamond, and $x$ is non-adjacent to $v, x', w'$, so we may assume that $v$ is adjacent to $p$, that is, $p = y$. But then $n = 8$ and $P_4(G) = 72$, and the result holds. This proves (2).

In view of (2), we may assume that there is a diamond $\{a, b, c, d\}$ in $M$, with apices $a, b$.

(3) *There is a pure diamond included in $M$.*

For we may assume that $\{a, b, c, d\}$ is not pure, and so there is a vertex $p$ adjacent to $a, b$ and not to $c, d$. From the symmetry between $c, d$, we may assume that the cycle with vertex set $\{a, c, b, p\}$ divides $S^2$ into two open discs $D_1, D_2$, one containing $d$ and the other containing $v$, say $d \in D_1$. Let $bdq$ touch $bdc$. Then $q \neq p$ since $q$ is adjacent to $d$, and so $q \in D_1$, and in particular $q \in M$. Suppose that the diamond $\{c, q, b, d\}$ is not pure; then there is a vertex $r$ adjacent to $c, q$ and not to $b, d$, which is impossible by planarity. This proves (3).

In view of (3) we may assume that $\{a, b, c, d\}$ is pure.

(4) *We can order $V(G) \setminus \{a, b, c, d\}$ and $\{v_1, \ldots, v_{n-4}\}$ in such a way that $v_1 = v$, $N = \{v_2, \ldots, v_{k+1}\}$, and for $k + 2 \leq i \leq n - 4$ there is a triangle containing $v_i$ and two of $v_1, \ldots, v_{i-1}$.*

For let $G'$ be the drawing obtained from $G$ by deleting $a, b, c, d$, and let $D$ be the region of $G'$ containing $a, b, c, d$. Then $D$ is an open disc, and so there is a closed walk tracing its boundary. Since $G$ is 4-connected and the diamond $\{a, b, c, d\}$ is pure, it follows that no vertex appears twice in this closed walk, and so $D$ is bounded by a cycle $C$ say. Choose a sequence $v_1, \ldots, v_j$ of distinct members of $V(G) \setminus \{a, b, c, d\}$, where $v_1 = v$, $N = \{v_2, \ldots, v_{k+1}\}$, and for $k + 2 \leq i \leq j$ there is a triangle containing $v_i$ and two of $v_1, \ldots, v_{i-1}$, with $j$ maximum. Let $X = \{v_1, \ldots, v_j\}$ and $Y = V(G) \setminus (\{a, b, c, d\} \cup X)$. Let $\mathcal{R}$ be the set of all triangles with vertex set included in $X$. Let $S$ be the closure of the union of the members of $\mathcal{R}$; thus $S$ is some closed subset of $S^2$, with boundary the closure of some set of edges of $G$. Let $e \in E(G)$ be an edge of $G$ in the boundary of $S$, where $e = xy$ say, and let $xyz \in \mathcal{R}$ touche some region $xyz' \notin \mathcal{R}$. Thus $z' \notin X$ from the definition of $\mathcal{R}$, and so from the choice of $j$ it follows that $z' \in \{a, b, c, d\}$, and consequently $e \in E(C)$. Consequently every edge in the boundary of $S$ belongs to $E(C)$, and since every vertex of $G$ is incident with an even number of such edges, it follows that $C$ is the boundary of $S$. Consequently $S$ is a closed disc, and hence contains all vertices of $G$ not in $\{a, b, c, d\}$. It follows that $j = n - 4$. This proves (4).

For $1 \leq i \leq n - 4$, let $G_i = G|\{v_1, \ldots, v_i\}$. For $k + 2 \leq i \leq n - 4$, $P_4(G_i) \leq 2P_4(G_{i-1})$, and so $P_4(G_{n-4}) \leq 2^{n-k-5} P_4(G_{k+1})$. But every 4-colouring of $G_{n-4}$ can be extended to at most six 4-colourings of $G$ (this is easy to check, and we leave it to the reader), and so $P_4(G) \leq 6 \cdot 2^{n-k-5} P_4(G_{k+1})$. By 2.1, if $k = 4$ then $P_4(G_{k+1}) = 72$, and if $k = 5$ then $P_4(G_{k+1}) = 120$. This proves 2.2. ∎

**2.3** *Let $n \geq 6$ be such that every triangulation with $n'$ vertices admits at most $2^{n'} + 32$ 4-colourings, for $5 \leq n' \leq n - 1$. Let $G$ be a triangulation with $n$ vertices.*

- *If $G$ has a vertex of degree three, then $P_4(G) \leq 2^{n-1} + 32$, and $P_4(G) = 24$ if $n = 6$.*

- *If $G$ has no vertex of degree three and $G$ is not 4-connected, then $n \geq 9$ and $P_4(G) \leq 2^{n-1}+128$.*

Suppose first that some vertex $v$ has degree three. Now $G' = G \setminus v$ is a triangulation, so from the hypothesis $P_4(G') \leq 2^{n-1} + 32$, and $P_4(G') = 24$ if $n = 6$. But every 4-colouring of $G'$ extends to a unique 4-colouring of $G$, and the result follows.

Now we assume that $G$ has no vertex of degree three, but is not 4-connected. Consequently there is a cycle of length three in $G$ that does not bound a region; and so there are two triangulations $G_1, G_2$ in $S$ with union $G$, intersecting just in this cycle, and each with at least four vertices. Let $G_i$ have $n_i + 3$ vertices for $i = 1, 2$. Since $G$ has no vertex of degree three it follows that $n_1, n_2 \geq 3$, and so $n \geq 9$. Moreover, $P_4(G) = P_4(G_1)P_4(G_2)/24$, and from the hypothesis, $P_4(G_i) \leq 2^{n_i+3} + 32$ for $i = 1, 2$. It follows that

$$P_4(G) \leq (2^{n_1+3} + 32)(2^{n_2+3} + 32)/24 \leq (2^{n-3} + 32)(2^6 + 32)/24 = 2^{n-1} + 128,$$

since $n_1, n_2 \geq 3$ and sum to $n - 3$. This proves 2.3. ∎

**Proof of 1.2.** The first assertion of 1.2 is easy using 2.1 and we leave it to the reader. Note also that $\frac{27}{32}2^n \leq 2^n - 8$ if $n \geq 6$, and every triangulation with five vertices is a biwheel. Thus for the second assertion, we proceed by induction on $n$, and we may therefore assume that every triangulation with $n'$ vertices admits at most $2^{n'} + 32$ 4-colourings, for $5 \leq n' \leq n - 1$. Let $G$ be a triangulation with $n$ vertices, not a biwheel, and so $n \geq 6$. If $n \geq 7$ and $G$ has a vertex of degree three, then by 2.3,

$$P_4(G) \leq 2^{n-1} + 32 \leq \frac{27}{32}2^n$$

as required; while if $n = 6$ and $G$ has a vertex $v$ of degree three, then by 2.3,

$$P_4(G) = 24 \leq \frac{27}{32}2^n.$$

Thus we may assume that $G$ has no vertex of degree three. If $G$ is not 4-connected, then by 2.3 $n \geq 9$ and

$$P_4(G) \leq 2^{n-1} + 128 \leq \frac{27}{32}2^n$$

as required. If $G$ is 4-connected and has no vertex of degree four, then by 2.2,

$$P_4(G) \leq 45 \cdot 2^{n-6} \leq \frac{27}{32}2^n;$$

while if $G$ is 4-connected and has a vertex of degree four then the result follows from 2.2. This proves 1.2. ∎

## References


[1] Garry Bowlin and Matthew Brin, "Coloring planar maps via colored paths in the associahedra", manuscript, January 2013 (arXiv:1301.3984).

[2] Matthew Brin, "Maps with a large number of four-colorings", posted on MathOverflow August 14 2012 (http://mathoverflow.net/questions/104722/maps-with-a-large-number-of-4-colorings).