Excluding a fat tree

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Abstract

Robertson and Seymour proved that for every finite tree H, there exists k such that every finite graph G with no H minor has path-width at most k; and conversely, for every integer k, there is a finite tree H such that every finite graph G with an H minor has path-width more than k. If we (twice) replace "path-width" by "line-width", the same is true for infinite graphs G.

We prove a "coarse graph theory" analogue, as follows. For every finite tree H and every c, there exist k, L, C such that every graph that does not contain H as a c-fat minor admits an (L, C)-quasi-isonetry to a graph with line-width at most k; and conversely, for all k, L, C there exist c and a finite tree H such that every graph that contains H as a c-fat minor admits no (L, C)-quasi-isometry to a graph with line-width at most k.

1 Introduction

Graphs in this paper may be infinite, and have no loops or parallel edges. If G is a graph and $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced on X. If X is a vertex of G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\operatorname{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\operatorname{dist}_G(X, Y) = \infty$.)

Let G, H be graphs, and let $\phi : V(G) \to V(H)$ be a map. Let $L, C \geq 0$; we say that ϕ is an (L, C)-*quasi-isometry* if:

- for all u, v in $V(G)$, if $\operatorname{dist}_G(u, v)$ is finite then $\operatorname{dist}_H(\phi(u), \phi(v)) \leq L \operatorname{dist}_G(u, v) + C$;
- for all u, v in $V(G)$, if $\operatorname{dist}_H(\phi(u), \phi(v))$ is finite then $\operatorname{dist}_G(u, v) \leq L \operatorname{dist}_H(\phi(u), \phi(v)) + C$; and
- for every $y \in V(H)$ there exists $v \in V(G)$ such that $\operatorname{dist}_H(\phi(v), y) \leq C$.

If G is a graph, we write $U(G)$ for $V(G) \cup E(G)$. Let G, H be graphs, and let $c \geq 0$ be an integer. For each $x \in U(H)$, let $\eta(x)$ be a non-null connected subgraph of G, all pairwise vertex-disjoint, such that

- for each $uv \in E(H)$, there is an edge of G between $\eta(u)$ and $\eta(uv)$;
- $\operatorname{dist}_G(\eta(u), \eta(v)) > c$ for all distinct $x, y \in U(H)$, except when one of x, y is in $V(H)$, the other is in $E(H)$, and the edge is incident in H with the vertex.

In these circumstances, we say that G contains H as a *c-fat minor*, and η *exhibits* H as a c-fat minor of G. Sometimes, we are given a subset $X \subseteq V(G)$, and the function η satisfies that $\eta(x) \subseteq G[X]$ for all $x \in U(H)$. In this case, we say that X *contains a c-fat H-minor of G*. Note that this is not the same thing as saying that $G[X]$ contains H as a c-fat minor, because being c-fat depends on a distance function, and we are using the distance function of G rather than that of $G[X]$.

It is easy to see that:

1.1 *Let $L, C, c \geq 0$, and let G contain H as a c-fat minor. Let there be an (L, C)-quasi-isometry from G to some graph G'. If $c \geq L(L + C) + C$ then G' contains H as a minor.*

A. Georgakopoulos and P. Papasoglu [6] conjectured that a converse also holds:

1.2 False conjecture: *For every graph H and all $c \geq 0$, there exists $L, C \geq 0$, such that if a graph G does not contain H as a c-fat minor, then G admits an (L, C)-quasi-isometry to a graph with no H minor.*

This is known to be true when:

- $H = K_3$, by Manning's theorem [8] (and see [6]);
- $H = K_{1,m}$, by Georgakopoulos and Papasoglu [6];
- $H = K_{2,3}$, by Chepoi, Dragan, Newman, Rabinovich, and Vaxes [1];
- $H = K_4^-$ (that is, K_4 with one edge deleted) by Fujiwara and Papasoglu [5].

The conjecture has recently been shown to be false in general, by J. Davies, R. Hickingbotham, F. Illingworth and R. McCarty [3]; but their counterexample H contains a clique of size 15, and there remains hope that 1.2 is true for some interesting graphs H. (Trees? Planar graphs?)

Our result is not exactly of this type, but has the same flavour. A *path-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a path (possibly infinite) and each B_t is a subset of $V(G)$ (called a *bag*), such that:

- $\bigcup_{t \in V(T)} B_t = V(G)$;
- for every edge $e = uv$ of G, there exists $t \in V(T)$ with $u, v \in B_t$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies between t_1, t_3 in T, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a path-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t| - 1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *path-width* of G is the minimum width of a path-decomposition of G.

Robertson and Seymour [11] proved:

1.3 *For every finite tree H, there exists k such that every finite graph G with no H minor has path-width at most k; and conversely, for every integer k, there is a finite tree H such that every finite graph with an H minor has path-width more than k.*

It is important here that G is finite: for instance, when G consists of the disjoint union of countably many infinite paths, then G does not contain the claw $K_{1,3}$ as a minor, and yet G does not have any finite path-width. This can be fixed. Say a *line-decomposition* of G is a family $(B_t : t \in T)$ of subsets of $V(G)$, where T is a linearly-ordered set, satisfying the same three bullets above (with $V(T)$ replaced by T everywhere). The *width* of a line-decomposition $(T, (B_t : t \in T))$ is the maximum of the numbers $|B_t| - 1$ for $t \in T$, or ∞ if there is no finite maximum; and the *line-width* of G is the minimum width of a line-decomposition of G. We proved in [2] that G has line-width at most k if and only if every finite subgraph has path-width at most k. Consequently, if we replace "path-width" by "line-width" in 1.3, then the statement of 1.3 is true even for infinite graphs G.

Our aim in this paper is to give a coarse graph theory analogue of this "line-width" version of 1.3. Our main result says:

1.4 *For every finite tree H and every c, there exist k, L, C such that every graph that does not contain H as a c-fat minor admits an (L, C)-quasi-isometry to a graph with line-width at most k; and conversely, for all k, L, C there exists c and a finite tree H such that every graph that contains H as a c-fat minor admits no (L, C)-quasi-isometry to a graph with line-width at most k.*

The second half is easy: choose $c > L(L + C) + C$, and let H be a finite tree with line-width more than k. If G contains H as a c-fat minor, and there is an (L, C)-quasi-isometry from G to G', then by 1.1, G' contains H as a minor and hence has line-width more than k. The first half is much more difficult and will occupy the whole paper.

As a first step, let us eliminate the (L, C)-quasi-isometry. If $X \subseteq V(G)$, we say X has *quasi-size* at most (k, r) if there is a set $Y \subseteq V(G)$ with $|Y| \le k$, such that $\text{dist}_G(x, Y) \le r$ for each $x \in X$. If $X \subseteq V(G)$, let us say a line-decomposition $(B_t : (t \in T))$ of $G[X]$ is (k, r)-*quasi-bounded* in G if B_t has quasi-size at most (k, r) for each $t \in T$. Let us say X has *quasi-line-width* at most (k, r) if $G[X]$ admits a (k, r)-quasi-bounded line-decomposition.

It is important that the distance function used in the definition of quasi-size and (k,r)-quasi-bounded is that defined by G, not that defined by $G[X]$. This will be the case throughout the paper. Even when speaking of subgraphs of G, we will never use their distance functions: the distance function in use will always be that of G. We sometimes write X for $G[X]$ when $X \subseteq V(G)$. This should cause no confusion since there is always only one graph G under consideration.

It was proved by R. Hickingbotham [7] that:

1.5 *For all k,r, there exist $L,C \geq 1$ such that if G has quasi-line-width at most (k,r), then G admits an (L,C)-quasi-isometry to a graph with line-width at most $2k-1$.*

(In [10], we proved a version of this with path-width instead of line-width, and $2k-1$ replaced by k, but Hickingbotham tells us that his different method of proof yields the result 1.5, and he will add it to his paper.) Thus, to complete the proof of 1.4, it suffices to prove the following:

1.6 *For every finite tree H and every $c \geq 0$, there exist k,r such that every graph that does not contain H as a c-fat minor has quasi-line-width at most (k,r).*

H_0 denotes the tree with one vertex. For $\ell \geq 1$ an integer, let H_ℓ be the finite tree such that every vertex has degree one or three, and for some vertex r (called the *root*) every path from r to a vertex of degree one has length exactly ℓ. Every tree H is a minor of H_ℓ for some choice of ℓ, and then, if G does not contain H as a c-fat minor then it does not contain H_ℓ as a c-fat minor. Consequently it suffices to prove 1.6 when $H = H_\ell$, that is:

1.7 *For every $\ell \geq 1$ and every $c \geq 0$, there exist k,r such that every graph that does not contain H_ℓ as a c-fat minor has quasi-line-width at most (k,r).*

Let $\ell \geq 1$, let r be the root of H_ℓ, let B_1, B_2, B_3 be the three components of $H_\ell \setminus r$, and for $i = 1,2,3$, let e_i be the edge of H_ℓ between r and $V(B_i)$. We say that η exhibits H_ℓ as a *c-superfat* minor of G if η exhibits H_ℓ as a c-fat minor of a graph G, and $\operatorname{dist}_G(x,y) > 3c$ for all i,j with $1 \leq i < j \leq 3$, and all $x \in V(\eta(B_i)) \cup V(\eta(e_i))$, and all $y \in V(\eta(B_j)) \cup V(\eta(e_j))$. When $\ell = 0$, we say that η exhibits H_0 as a c-superfat minor of a graph G if it exhibits H_0 as a c-fat minor of G. If η exhibits H as a c-superfat minor of G, $\eta(H)$ denotes $\bigcup_{x \in U(H)} \eta(x)$.

In order to prove 1.7, it suffices to show:

1.8 *For every $\ell \geq 1$ and every $c \geq 0$, there exist k,r such that every graph that does not contain H_ℓ as a c-superfat minor has quasi-line-width at most (k,r).*

Using superfat instead of fat is helpful for inductive purposes.

2 Some preliminary lemmas

We begin with the reason we use "superfat" instead of "fat":

2.1 *Let $c \geq 2$, let $t \geq 0$, and for $i = 1,2,3$, let η_i exhibit H_t as a c-superfat minor of a graph G, such that $\operatorname{dist}_G(\eta_i(H_t), \eta_j(H_t)) > 5c$ for all distinct $i,j \in \{1,2,3\}$. Let $W \subseteq V(G)$ such that $G[W]$ is connected, and $\operatorname{dist}_G(\eta_i(H), W) = c$ for $h = 1,2,3$; and for $h = 1,2,3$, let P_h be a geodesic from W to $\eta_h(H)$. Then there is a mapping η that exhibits H_{t+1} as a c-superfat minor of G, such that*

$$\eta(H_{t+1}) \subseteq G[W] \cup \eta_1(H_t) \cup \eta_2(H_t) \cup \eta_3(H_t) \cup P_1 \cup P_2 \cup P_3.$$

Proof. Let r be the root of H_t, let B_1, B_2, B_3 be the three components of $H_t \setminus r$, and for $h = 1, 2, 3$, let e_h be the edge of H_t between r and $V(B_h)$. For $h, i \in \{1, 2, 3\}$, let $\eta_h(B_i^+)$ denote $\bigcup_{x \in U(B_i)} \eta_h(x) \cup \eta_h(e_i)$. For $h = 1, 2, 3$, let $w_h \in W$ be the end of P_h in W. We know that for $h = 1, 2, 3$, and all distinct $i, j \in \{1, 2, 3\}$,

$$\mathrm{dist}_G(\eta_h(B_i^+), \eta_h(B_j^+)) \geq 3c + 1.$$

Consequently, there is at most one value of $i \in \{1, 2, 3\}$ such that $\mathrm{dist}_G(P_h, \eta_h(B_i^+)) \leq c$, since P_h has length c. By relabelling, we may assume that $\mathrm{dist}_G(P_h, \eta_h(B_i^+)) > c$ for $i = 1, 2$.

Since $\mathrm{dist}_G(\eta_i(H_t), \eta_j(H_t) > 5c$ and P_i, P_j have length c, it follows that $\mathrm{dist}_G(P_i, \eta_j(H_t)) > 4c$, and $\mathrm{dist}_G(P_i, P_j) > 3c$, for all distinct $i, j \in \{1, 2, 3\}$. Moreover, $\mathrm{dist}_G(P_h, \eta_h(B_i^+)) > c$ for $i = 1, 2$ and $h = 1, 2, 3$.

Let r' be the root of H_{t+1}, and let $B'_1, B'_2, B'_3, e'_1, e'_2, e'_3$ be defined as usual. Thus, for $h = 1, 2, 3$, B'_h is isomorphic to $H_t \setminus V(B_3)$; let ϕ_h be such an isomorphism. Define $\eta(r') = W$, and for $h = 1, 2, 3$, let $\eta(x) = \eta_h(\phi_h(x))$ for each $x \in U(B'_h)$. For $h = 1, 2, 3$, we define $\eta(e'_h)$ as follows. If there is an end y_h of P_h in $\eta_h(r)$, let $\eta(e'_h) = V(P_h) \setminus \{w_h, y_h\}$ (since $c \geq 2$, this set is nonempty). If not, then there is an end of P_h in $\eta_h(B_3^+)$; let $\eta(e'_h) = (V(P_h) \setminus \{w_h\}) \cup \eta_h(B_3^+)$. Then η exhibits H_{t+1} as a c-superfat minor of G. This proves 2.1. ∎

If $X \subseteq V(G)$, $\mathrm{bd}(X)$ denotes the set of vertices in X that have a neighbour in $V(G) \setminus X$, and is called the *boundary* of X. A key idea of the proof is that we work with subsets $X \subseteq V(G)$ such that, simultaneously, $G[X]$ has bounded quasi-line-width and $\mathrm{bd}(X)$ has bounded quasi-size, and it turns out that we can make the two bounds the same with little loss. Let us say $X \subseteq V(G)$ has *quasi-bound* at most (a, b) if $G[X]$ has quasi-line-width at most (a, b) and $\mathrm{bd}(X)$ has quasi-size at most (a, b).

We will need the following lemma about composing line-decompositions.

2.2 *Let $\mathcal{A}$ be a set of disjoint nonempty subsets of $V(G)$, with union W say, such that each $A \in \mathcal{A}$ has quasi-bound at most (a, b). Suppose also that there is a line-decomposition of $G[W]$ in which each bag is the union of at most k members of $\mathcal{A}$. Then $G[W]$ has quasi-line-width at most $((k + 1)a, b)$.*

Proof. Let Z be the union of the sets $\mathrm{bd}(A)$ $(A \in \mathcal{A})$. Then by hypothesis, there is a line-decomposition $(B_t : t \in T)$ of $G[Z]$ such that each bag is the union of at most k sets of the form $\mathrm{bd}(A)$, and so has quasi-size at most (ka, b). For each $A \in \mathcal{A}$, since A is nonempty, there exists $r(A) \in T$ such that $B_{r(A)} \cap A \neq \emptyset$, and hence $A \subseteq B_{r(A)}$ (since $B_{r(A)}$ is a union of boundaries of members of $\mathcal{A}$ and the members of $\mathcal{A}$ are pairwise disjoint). By duplicating points of T, we may assume that the elements $r(A)$ $(A \in \mathcal{A})$ are all distinct; and also by duplicating points of T, we may assume that each $r(A)$ has a successor $s(A) \in T$, that is, an element $s(A) \in T$ different from $r(A)$, such that $r(A) < s(A)$, and there is no $t \in T$ with $r(A) < t < s(A)$. For each $A \in \mathcal{A}$, let $(C_t^A : t \in T^A)$ be an (a, b)-quasi-bounded line-decomposition of $G[A]$. By adding two new elements to T^A, we may assume that T^A has a maximum and minimum element; and so, by inserting T^A into T, we may assume that T^A equals $\{t \in T : r(A) \leq t \leq s(A)\}$. For each $t \in T$, if $t \in T^A$ for some (necessarily unique) $A \in \mathcal{A}$, let $D_t = B_t \cup C_t^A$. If there is no such A, let $D_t = B_t$.

We claim that $(D_t : t \in T)$ is a line-decomposition of $G[W]$. If $v \in W$, choose $A \in \mathcal{A}$ with $v \in A$; then $v \in C_t^A$ for some $t \in T^A$, and hence $v \in D_t$. Next, suppose that $uv \in E(G[W])$. If there exists $A \in \mathcal{A}$ with $u, v \in A$, then $u, v \in C_t^A$ for some $t \in T^A$, and hence $u, v \in D_t$. If there is no such A, choose $A, A' \in \mathcal{A}$ with $u \in A$ and $v \in A'$. Then $u \in \text{bd}(A)$ and $v \in \text{bd}(A')$, and so uv is an edge of $G[Z]$. Choose $t \in T$ such that $u, v \in B_t$; then $u, v \in D_t$.

Finally, suppose that $r < s < t$ are elements of T, and $v \in D_r \cap D_t$. We need to show that $v \in D_s$. Choose $A \in \mathcal{A}$ with $v \in A$. If $v \notin Z$, then each of r, s, t belong to T^A, since no other bags contain v; and then, since $v \in C_r^A \cap C_t^A$, it follows that $v \in C_s^A \subseteq D_s$ as required. Now suppose that $v \in Z$. Then, for $p \in T$, v belongs to D_p if and only if $v \in B_p$; and so $v \in B_r \cap B_t \subseteq B_s \subset D_s$. This proves that $(D_t : t \in T)$ is a line-decomposition of $G[W]$. It is easy to check that each of its bags is $((k+1)a, b))$-bounded. This proves 2.2. ▮

3 Buildings and tie-breakers

In the proof of 1.8, we are given a graph G that does not contain H_ℓ as a c-superfat minor. As the proof proceeds, we will group the vertices of G is different ways, but the graph G itself will remain unchanged. It is helpful also to fix a *tie-breaker* in G, that is, a well-order Λ of the set of all edges of G (this is possible from the well-ordering theorem). If P, Q are distinct finite paths of G, we say P is Λ-*shorter* than Q if either

- $|E(P)| < |E(Q)|$; or
- $|E(P)| = |E(Q)|$, and the first element (under Λ) of $(E(P) \setminus E(Q)) \cup (E(Q) \setminus E(P))$ belongs to P.

This defines a total order on the set of all finite paths of G. A Λ-*geodesic* means a finite path P such that no other path joining its ends is Λ-shorter than P. Every Λ-geodesic of G is a geodesic of G, but the converse is false. (The point of the tie-breaker is that there is only one Λ-geodesic between any two vertices, while this is not true for geodesics.) It is easy to check that if P is a Λ-geodesic then so are all subpaths of P. We will keep Λ fixed, and usually suppress the dependence of other objects on the choice of Λ.

A *building* in a graph G is a subset $X \subseteq V(G)$ such that $G[X]$ is connected. If $\mathcal{T}$ is a set of pairwise vertex-disjoint buildings, in a graph G, we define $V(\mathcal{T}) = \bigcup_{X \in \mathcal{T}} X$. Again, let $\mathcal{T}$ be a set of pairwise vertex-disjoint buildings, in a graph G, now with a tiebreaker Λ. We say that $X \in \mathcal{T}$ is Λ-*closest* to v if there is a path of G between v, X that is Λ-shorter than any other path between v and Y for $Y \in \mathcal{T}$. For each $X \in \mathcal{T}$, let $\Delta_\mathcal{T}(X)$ (or just $\Delta(X)$, when $\mathcal{T}$ is clear) be the set of all $v \in V(G)$ such that X is Λ-closest to v. We call $\Delta(X)$ the *Voronoi cell* of X, and the collection of subsets $\Delta(X)$ $(X \in \mathcal{T})$ is called the *Voronoi partition* defined by $\mathcal{T}$. (It is indeed a partition, provided that each component of G includes a member of $\mathcal{T}$; and this is true for us since we will always assume that G is connected and $\mathcal{T}$ is non-null.) The main purpose of the tie-breaker is to make the Voronoi partition defined by $\mathcal{T}$ well-defined. We see that:

- $X \subseteq \Delta(X)$ for each $X \in \mathcal{T}$;
- the sets $\Delta(X)$ $(X \in \mathcal{T})$ are pairwise disjoint and have union $V(G)$;
- for each $X \in \mathcal{T}$ and each $v \in \Delta(X)$, $\text{dist}_G(v, X) \leq \text{dist}_G(v, Y)$ for each $Y \in \mathcal{T}$, and there is a path of $G[\Delta(X)]$ between v and X of length $\text{dist}_G(v, X)$.

We say that distinct sets $X, Y \subseteq V(G)$ *touch* if either $X \cap Y \neq \emptyset$ or there is an edge between X and Y. If $\mathcal{T}$ is a set of pairwise vertex-disjoint buildings, and $X, Y \in \mathcal{T}$ are distinct, we say that X *adjoins* Y if $\Delta_{\mathcal{T}}(X)$ touches $\Delta_{\mathcal{T}}(Y)$. A set $C \subseteq \mathcal{T}$ is *adjoin-connected* if for every partition of C into two nonempty sets A, B, some member of A adjoins some member of B.

4 Realms

Let $c \geq 2$ and $\ell \geq 1$ be integers, fixed throughout the paper. (We will be concerned with graphs that do not contain H_ℓ as a c-superfat minor.) We also want to fix throughout the paper a large number d_0, depending on c, ℓ. Actually we will take $d_0 = 5c \cdot 3^{2\ell(\ell+1)}$, but its exact value will not matter until the final section.

A *century* is an integer k with $0 \leq k \leq \ell$. For each century k, a *space requirement* is a function $\delta : \{0, \ldots, 2\ell\} \to \mathbb{N}$ satisfying:

- $\delta(i) \geq 2\delta(i+1) + 2$ for each i with $2k - 2 \leq i < 2\ell$;
- $\delta(i) \leq d_0$ for all $i \in \{0, \ldots, 2\ell\}$; and
- $\delta(2\ell) \geq 5c$.

A *budget* is a pair (α, β) of positive integers. For each century k, let us select a space requirement δ and a budget (α, β): both will be fixed until the century changes, and the century will be fixed until the final section. The notation α, β, δ does not show the dependence of these quantities on k: this is for convenience, since they will all be fixed for a long time.

Let Λ be a tie-breaker in a connected graph G. A *kth-century society* in G is a set $\mathcal{T}$ of pairwise vertex-disjoint buildings in G, where each member of $\mathcal{T}$ is assigned to be a "house" or "fort" of $\mathcal{T}$, satisfying the following (where $\text{rk}(X) = k - 1$ if X is a house and $\text{rk}(X) = k$ if X is a fort):

- For all $v \in V(G)$, there exists $X \in \mathcal{T}$ such that $\text{dist}_G(v, X) \leq d_0$;
- Every two members X, Y of $\mathcal{T}$ have distance more than $\delta(\text{rk}(X) + \text{rk}(Y))$;
- Every fort of $\mathcal{T}$ contains a c-superfat H_k-minor of G;
- For each fort X of $\mathcal{T}$, $\text{bd}(X)$ has quasi-bound at most (α, β);
- If C is a maximal adjoin-connected set of houses of $\mathcal{T}$, then $\bigcup_{X \in V(C)} \Delta_{\mathcal{T}}(X)$ has quasi-bound at most $(\alpha, \beta - d_0)$.

We plan to combine terms in a society to make new societies, and when we do this, the sets $\Delta(X)$ may change, even for sets X that were not involved in the combination, so the final condition in the definition of "society" is hard to maintain. Our first goal is to simplify it, and we also want to introduce "castles", which are combinations of three or four forts. Let us say a *kth century realm* is a set $\mathcal{T}$ of pairwise vertex-disjoint buildings in G, where each member of $\mathcal{T}$ is assigned to be a house, fort or castle of $\mathcal{T}$, satisfying the following (where $\text{rk}(X) = k - 1, k$ or $k + 1$ depending whether X is a house, fort or castle):

- For all $v \in V(G)$, there exists $X \in \mathcal{T}$ such that $\text{dist}_G(v, X) \leq d_0$;

- If $X, Y \in \mathcal{T}$ then $\mathrm{dist}_G(X, Y) > \delta(\mathrm{rk}(X) + \mathrm{rk}(Y))$;
- Every fort or castle X of $\mathcal{T}$ contains a c-superfat $H_{\mathrm{rk}(X)}$-minor of G;
- Every fort of $\mathcal{T}$ has quasi-bound at most (α, β);
- Every castle of $\mathcal{T}$ has quasi-bound at most $(8\alpha, \beta + (\ell - k)d_0)$;
- For every community C of $\mathcal{T}$, $V(C)$ has quasi-bound at most (α, β). (A *community* of $\mathcal{T}$ is an adjoin-connected set of houses of $\mathcal{T}$, not necessarily maximal.)

If X belongs to a realm $\mathcal{T}$, then it is a house, fort or castle of T, and we call this its *class* under $\mathcal{T}$.

We claim:

4.1 *Let $\mathcal{T}$ be a kth-century society in G. Then there is a kth-century realm in G.*

Proof. Let us say a kth-century society $\mathcal{T}_2$ is an *extension* of another, $\mathcal{T}_1$, if each house of $\mathcal{T}_1$ is a subset of a house of $\mathcal{T}_2$, and each fort of $\mathcal{T}_1$ is a fort of $\mathcal{T}_2$. By Zorn's lemma, since there is a kth-century society, there is a kth-century society $\mathcal{T}$ such that no kth-century society $\mathcal{T}' \neq \mathcal{T}$ is an extension of $\mathcal{T}$.

(1) $\mathrm{dist}_G(v, \mathrm{bd}(\Delta(X))) \leq d_0$ *for every house X of $\mathcal{T}$ and every $v \in \mathrm{bd}(X)$.*

Let $v \in \mathrm{bd}(X)$, and choose a neighbour u of v with $u \notin X$. Since $\mathrm{dist}_G(X, Y) > \delta(\mathrm{rk}(X)+\mathrm{rk}(Y)) \geq 1$ for all $Y \in \mathcal{T}$ with $Y \neq X$, it follows that $u \notin V(\mathcal{T})$. Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by removing X and adding a new house $X \cup \{u\}$. From the choice of $\mathcal{T}$, $\mathcal{T}'$ is not a society, and so either there is a house or fort $Y \in \mathcal{T}$ different from X with $\mathrm{dist}_G(u, Y) \leq \delta(k - 1 + \mathrm{rk}(Y)) \leq \delta(2k - 2)$, or $\Delta_{\mathcal{T}'}(X \cup \{u\}) \neq \Delta_{\mathcal{T}}(X)$. In the first case, there is a path between v, Y of length at most $\delta(2k - 2) + 1 \leq d_0$, and so a path between $v, \mathrm{bd}(\Delta_{\mathcal{T}}(X))$ of length at most d_0. In the second case, there is a vertex $w \in \Delta_{\mathcal{T}'}(X \cup \{u\}) \setminus \Delta_{\mathcal{T}}(X)$. Choose $Y \in \mathcal{T}$ such that $\mathrm{dist}_G(u, Y) \leq d_0$. Since $\mathrm{dist}_G(u, w) \leq \mathrm{dist}_G(u, Y)$ (because $w \in \Delta_{\mathcal{T}'}(X \cup \{u\}) \setminus \Delta_{\mathcal{T}}(X)$), it follows that $\mathrm{dist}_G(u, w) \leq d_0$. There is an internal vertex x of the shortest path between u, w in $\mathrm{bd}(\Delta_{\mathcal{T}}(X))$, and so $\mathrm{dist}_G(u, x) < d_0$, and therefore $\mathrm{dist}_G(v, x) \leq d_0$. This proves (1).

We claim that $\mathcal{T}$ is a kth-century realm. Every community of $\mathcal{T}$ is a subset of a maximal adjoin-connected subset of the set of houses of $\mathcal{T}$, so it suffices to check that if C is a maximal adjoin-connected subset of the set of houses of $\mathcal{T}$, then $\mathrm{bd}(V(C))$ has quasi-size at most (α, β). We know that $\mathrm{bd}(\bigcup_{X \in V(C)} \Delta_{\mathcal{T}}(X))$ has quasi-size at most $(\alpha, \beta - d_0)$; but by (1), each vertex in $\mathrm{bd}(V(C))$ has distance at most d_0 from some vertex in $\mathrm{bd}(\bigcup_{X \in V(C)} \Delta_{\mathcal{T}}(X))$, and so $\mathrm{bd}(V(C))$ indeed has quasi-size at most (α, β). This proves 4.1. ∎

We observe that:

4.2 *In every connected graph G with a tie-breaker Λ, there is a 0th-century society in G.*

Proof. By Zorn's lemma, there exists $S \subseteq V(G)$ maximal such that $\mathrm{dist}_G(u, v) > d_0$ for all distinct $u, v \in S$. Let $\mathcal{T} = \{\{v\} : v \in S\}$ where each member of $\mathcal{T}$ is a fort of $\mathcal{T}$. We claim that $\mathcal{T}$ is a 0th-century society. Let $v \in S$. Then $G[\{v\}]$ has quasi-line-width at most (α, β), since $\alpha \geq 1$. Moreover $\mathrm{bd}(\{v\}) \subseteq \{v\}$, and so has quasi-size at most (α, β), since $\alpha \geq 1$. Also, trivially, X contains a c-superfat H_0-minor of G.

Moreover, $\mathrm{dist}_G(X, Y) > d_0$ for all distinct $X, Y \in \mathcal{T}$. For each $v \in V(G)$, either $v \in S$ or there exists $u \in S$ with $\mathrm{dist}_G(u, v) \le d_0$, from the maximality of S, and so in either case there exists $u \in S$ with $\mathrm{dist}_G(u, v) \le d_0$. This proves 4.2. ∎

Our strategy to prove 1.8 is, for each century k, we will give a carefully-chosen space requirement $\delta(k)$ and budget (α_k, β_k). We are given a graph G that does not contain H_ℓ as a c-superfat minor. We may assume that G is connected, and so admits a 0th-century society by 4.2. We will prove that for $0 \le k < \ell$, if G admits a k-th century society and hence a k-th century realm, then it also admits $(k+1)$st-century society; and hence G admits a ℓ-century society $\mathcal{T}$. No buildings in $\mathcal{T}$ are forts or castles, since G contains no c-superfat H_ℓ-minor of G, so they are all houses, and the result follows.

5 Optimizing within a century

Before we move into the $(k+1)$st century, we need to choose a "good" kth-century realm, by which we mean, roughly, one with the maximum number of castles; but since G and the realm might be infinite, we need to formulate this more carefully. Let us say a kth-century realm $\mathcal{T}_2$ is an *extension* of a kth-century realm $\mathcal{T}_1$ if for each member $X_1 \in \mathcal{T}_1$ there exists $X_2 \in \mathcal{T}_2$ such that:

- $X_1 \subseteq X_2$; and
- either X_2 is a building of higher class than X_1 (that is, either X_1 is a house of $\mathcal{T}_1$ and X_2 is a fort or castle of $\mathcal{T}_2$, or X_1 is a fort of $\mathcal{T}_1$ and X_2 is a castle of $\mathcal{T}_2$), or $X_1 = X_2$ and X_1 has the same class under $\mathcal{T}_1$ and under $\mathcal{T}_2$.

Let us say a kth-century realm $\mathcal{T}$ is *optimal* if no other kth-century realm is an extension of $\mathcal{T}$. It follows from Zorn's lemma that if there is a kth-century realm, then one of its extensions is optimal. In this section we will prove some properties of optimal realms.

Let $X \in \mathcal{T}$, and let $W \subseteq V(G)$ be disjoint from X. We say a path P is a *c-leg* on X in W if one end p of P is in X, and $V(P) \subseteq W \cup \{p\}$, and P is a geodesic of length exactly c, and $\mathrm{dist}_G(v, X) > c$ for each $v \in W \setminus V(P)$.

The use of optimality is the following:

5.1 *Let $\mathcal{T}$ be an optimal kth-century realm, in a graph G that does not contain H_ℓ as a c-fat minor. Then there do not exist distinct forts X_1, X_2, X_3 of $\mathcal{T}$, and a subset $W \subseteq V(G)$, with the following properties, where $Z = W \cup X_1 \cup X_2 \cup X_3$:*

- *$G[W]$ is connected and disjoint from $X_1 \cup X_2 \cup X_3$;*
- *W has a c-leg on X_i for $i = 1, 2, 3$;*
- *Z has quasi-bound at most $(8\alpha, \beta + (\ell - k)d_0)$;*
- *for each $Y \in \mathcal{T}$, either $\mathrm{dist}_G(W, Y) > \delta(k + 1 + \mathrm{rk}(Y))$, or $Y \subseteq Z$ and $\mathrm{rk}(Y) \le k$.*

Proof. Let $\mathcal{T}'$ consist of the set of members of $\mathcal{T}$ that are not included in Z, together with Z, where Z is a castle of $\mathcal{T}'$, and each other member of $\mathcal{T}'$ has the same class in $\mathcal{T}'$ that it has in $\mathcal{T}$. We claim, first, that:

(1) *$\mathcal{T}'$ is a kth-century realm.*

We must check that the members of $\mathcal{T}'$ are pairwise vertex-disjoint buildings (which is true, since any member of $\mathcal{T}'$ that intersects Z is included in Z), and:

- For all $v \in V(G)$, there exists $X \in \mathcal{T}'$ such that $\text{dist}_G(v, X) \le d_0$;
- If $X, Y \in \mathcal{T}'$ then $\text{dist}_G(X, Y) > \delta(\text{rk}(X) + \text{rk}(Y))$;
- Every fort or castle X of $\mathcal{T}'$ contains a c-superfat $H_{\text{rk}(X)}$-minor of G.

The first is clear. For the second, we may assume that $X = Z$, so $\text{rk}(X) = k + 1$. Since $Y \in \mathcal{T}'$ and therefore $Y \not\subseteq Z$, it follows that $\text{dist}_G(W, Y) > \delta(k + 1 + \text{rk}(Y))$; and

$$\text{dist}_G(X_i, Y) > \delta(k + \text{rk}(Y)) \ge \delta(k + 1 + \text{rk}(Y))$$

for $1 \le i \le 3$. Consequently the second bullet holds. For the third, for $i = 1, 2, 3$ let P_i be a c-leg of W on X_i, and let P_i have ends p_i, w_i, where $p_i \in X_i$. It follows (from the final condition in the definition of a c-leg) that each vertex of $P_i \setminus w_i$ has no neighbour in W except for its one or two neighbours in P_i. In particular, P_1, P_2, P_3 are vertex-disjoint except that w_1, w_2, w_3 might be equal. Let W' be obtained from W by deleting the internal vertices of P_1, P_2, P_3. (We recall that $p_1, p_2, p_3 \notin W$.) It follows that $G[W']$ is connected, and $\text{dist}_G(W', X_i) = c$ for $1 \le i \le 3$. From 2.1 applied to W', there is a function η that exhibits H_{k+1} as a c-superfat minor of G, with

$$\eta(H_{k+1}) \subseteq W \cup \eta_1(H_k) \cup \eta_2(H_k) \cup \eta_3(H_k).$$

This proves the third bullet and so proves (1).

Since by hypothesis, Z has quasi-bound at most $(8\alpha, \beta + (\ell - k)d_0)$, it follows that $\mathcal{T}'$ is a kth-century realm. But this contradicts the optimality of $\mathcal{T}$, and so proves 5.1. ∎

6 Passages

Let $\mathcal{T}$ be a kth-century realm, in the usual notation. A *passage* is an induced path P of G with the following properties:

- there exist distinct $X_1, X_2 \in \mathcal{T}$, each either a house or fort of $\mathcal{T}$, such that one end of P is in X_1 and the other in X_2 (and consequently P has length more than $\delta(2k) \ge 5c$);
- no internal vertex of P belongs to $X_1 \cup X_2$;
- $\text{dist}_G(P, Y) > \delta(k + 1 + \text{rk}(Y))$ for each $Y \in \mathcal{T}$ with $Y \ne X_1, X_2$;
- for $i \in \{1, 2\}$, let P_i be the subpath of P with one end in X_i of length c; then P_i is a geodesic, and for each $v \in V(P)$, $\text{dist}_G(v, X_i) \le c$ if and only if $v \in V(P_i)$.

We say that P *joins* X_1, X_2, and P is *incident* with X_1, X_2. We will only need passages of bounded length.

6.1 *Let $\mathcal{T}$ be an optimal kth-century realm in a graph G. There do not exist forts $X_1, \ldots, X_{m_1}$ and communities $Y_1, \ldots, Y_{m_2}$ of $\mathcal{T}$, and passages $P_1, \ldots, P_{m_3}$, with the following properties:*

- *$m_1, m_3 \geq 3$, and $m_1 + m_2 \leq 7$;*
- *each of $P_1, \ldots, P_{m_3}$ has length at most $(\ell - k)d_0 + 1$, and joins two members of*
$$\{X_1, \ldots, X_{m_1}\} \cup Y_1 \cup \cdots \cup Y_{m_2};$$
- *for $1 \leq i \leq 3$, P_i is incident with X_i, and none of $P_1, \ldots, P_{m_3}$ is incident with X_i except P_i; and*
- *$G[W]$ is connected, where*
$$W = (X_4 \cup \cdots \cup X_{m_1} \cup V(Y_1) \cup \cdots \cup V(Y_{m_2}) \cup V(P_1) \cup \cdots \cup V(P_{m_3})) \setminus (X_1 \cup X_2 \cup X_3).$$

Proof. Suppose that such $X_1, \ldots, X_{m_1}, Y_1, \ldots, Y_{m_2}, P_1, \ldots, P_{m_3}$ exist. Let $Z = W \cup X_1 \cup X_2 \cup X_3$. We claim that:

- $G[W]$ is connected and disjoint from $X_1 \cup X_2 \cup X_3$;
- W has a c-leg on X_i for $i = 1, 2, 3$;
- Z has quasi-line-width at most $(8\alpha, \beta + (\ell - k)d_0))$, and $\mathrm{bd}(Z)$ has quasi-size at most $(8\alpha, \beta + (\ell - k)d_0))$;
- for each $X' \in \mathcal{T}$, either $\mathrm{dist}_G(W, X') > \delta(k + 1 + \mathrm{rk}(X'))$, or $X' \subseteq Z$.

The first is clear, and the second and fourth hold from the definition of a passage. To see the third, let
$$A = X_1 \cup \cdots \cup X_{m_1} \cup V(Y_1) \cup \cdots \cup V(Y_{m_2}).$$
Since each of $X_1, \ldots, X_{m_1}, V(Y_1), \ldots, V(Y_{m_2})$ has quasi-line-width at most (α, β), so does A (because there are no edges between the sets of $\mathcal{T}$). Since the boundary of each X_i and of each $V(Y_i)$ has quasi-size at most (α, β), and $m_1 + m_2 \leq 7$, it follows that $\mathrm{bd}(A)$ has quasi-size at most $(7\alpha, \beta)$, and hence $\mathrm{bd}(A) \cup V(P_1) \cup \cdots \cup V(P_{m_3})$ has quasi-size at most $(7\alpha, \beta + (\ell - k)d_0)$, since each P_i has length at most $(\ell - k)d_0 + 1$ and has both ends in $\mathrm{bd}(A)$.

But $\mathrm{bd}(Z) \subseteq \mathrm{bd}(A) \cup V(P_1) \cup \cdots \cup V(P_{m_3})$, and so $\mathrm{bd}(Z)$ has quasi-size at most $(7\alpha, \beta + (\ell - k)d_0)$. Since A has quasi-line-width at most (α, β) and $Z = A \cup V(P_1) \cup \cdots \cup V(P_{m_3})$, it follows that Z has quasi-line-width at most $(8\alpha, \beta + (\ell - k)d_0)$. This proves our claim that the four bullets above hold; but this contradicts 5.1. So there are no such $X_1, \ldots, X_{m_1}, Y_1, \ldots, Y_{m_2}, P_1, \ldots, P_{m_3}$. This proves 6.1. ∎

A community *adjoins* a fort X if one of its members adjoins X. If $A \subseteq \mathcal{T}$, we denote the set of members of A that are houses or forts by A^0. Let $C \subseteq \mathcal{T}^0$. If $X_1, X_2 \in C$ are both forts, we say they *semiadjoin (in C)* if either X_1 adjoins X_2, or there is a community D of $\mathcal{T}$ included in C that adjoins them both. We say $X \in C$ is *C-peripheral* if either:

- X is a fort, and semiadjoins (in C) at most one other fort in C; or

- X is a house, and for every community $D \subseteq C$ containing X, D adjoins at most one fort in C, and any such fort is C-peripheral.

A community included in C is *C-peripheral* if one (and hence all) of its members are C-peripheral.

An *integer interval* is a set I of integers such that if $a, b \in I$ then I contains all integers between a, b. We say (a, b) is an *end-set* of an integer interval I if $a, b \in I$, and $a-1, b+1 \notin I$ (and hence I is finite).

6.2 *Let G be a graph that does not contain a c-superfat H_ℓ-minor, and let $\mathcal{T}$ be an optimal kth-century realm in G. Each community adjoins at most two forts, and each fort semiadjoins (in $\mathcal{T}$) at most two other forts. Consequently, if C is an adjoin-connected subset of $\mathcal{T}^0$ that contains a fort, then the forts in C can be numbered as X_i ($i \in I$), where I is an integer interval, with the following properties:*

- *for each $i \in I$, if $i+1 \in I$ then X_i semiadjoins (in C) X_{i+1}, and no other pair of forts in C semiadjoin (in $\mathcal{T}$) each other except possibly X_a, X_b, where (a, b) is an end-set of I and $b \geq a+2$;*
- *for each community S with $S \subseteq C$, S adjoins at most two forts in C.*

Moreover, if C, D are disjoint adjoin-connected subsets of $\mathcal{T}^0$, and P is a passage of length at most $(\ell - k)d_0 + 1$ joining some $X \in C$ and some $Y \in \mathcal{D}$, then either X is C-peripheral or D is a community.

Proof. Let $S \subseteq C$ be a community, and suppose that it adjoins three forts $X_1, X_2, X_3 \in C$. Thus for $i = 1, 2, 3$, there is a passage P_i between X_i and some $Y_i \in S$, with length at most $2d_0 + 1$. Choose more passages $P_4, \ldots, P_{m_3}$ joining pairs of members of S, of length at most $2d_0 + 1$, so that the union of their vertex sets with $V(S)$ induces a connected subgraph of G (this is possible since S is a community). This contradicts 6.1, taking $m_1 = 3$ and $m_2 = 1$.

Now let X_4 be a fort, and suppose that there are three other forts X_1, X_2, X_3 that C-semiadjoin X_4. Thus for $1 \leq i \leq 3$, either X_4 adjoins X_i, or there is a community $S_i \subseteq C$ that adjoins both X_4, X_i. For $i = 1, 2, 3$, if X_4 adjoins X_i, let P_i be a passage of length at most $2d_0 + 1$ joining them and let $S_i = \emptyset$; and otherwise let S_i be a community in C that adjoins both X_4, X_i, and let P_i be a passage joining some member of S_i and X_i. Then $\{X_4\} \cup S_1 \cup S_2 \cup S_3$ is adjoin-connected, so we can choose passages $P_4, \ldots, P_{m_3}$, each joining two members of $\{X_4\} \cup S_1 \cup S_2 \cup S_3$ and each of length at most $2d_0 + 1$, such that the union of their vertex sets and $X_4 \cup V(S_1) \cup V(S_2) \cup V(S_3)$ induces a connected subgraph of G. But this contradicts 6.1, taking $m_1 = 4$ and $m_2 = 3$.

Let H be the graph with vertex set the set of forts in C, in which two forts are adjacent if they semiadjoin. Since C is adjoin-connected, it follows that H is connected, and H has maximum degree at most two from the argument above. Consequently H is a path (possibly one-way or two-way infinite) or a cycle (necessarily finite), and so the forst in C can be numbered as in the theorem.

Finally, suppose that C, D are disjoint adjoin-connected subsets of $\mathcal{T}^0$, and P is a passage of length at most $(\ell - k)d_0 + 1$ joining some $X \in C$ and some $Y \in \mathcal{D}$. Suppose that X is not C-peripheral, and D is not a community. Thus either

- X is a fort, and X is semiadjoins (in C) two other forts of C; or

- X is a house, and there is a community containing X and included in C that adjoins two forts of C; or

- X is a house, and there is a community containing X and included in C that adjoins a fort of C that is not C-peripheral.

Moreover, the end of P in $V(D)$ either belongs to a fort of D, or it belongs to a member of a community included in D that adjoins a fort of D. This gives a contradiction to 6.1 as before (we omit the details, since there are several cases, all similar to what we already did twice). This proves 6.2. ∎

6.3 *Let G be a graph that does not contain a c-superfat H_ℓ-minor, and let $\mathcal{T}$ be an optimal kth-century realm in G, and let $S = \bigcup_{X\in\mathcal{T}^0} \Delta_{\mathcal{T}}(X)$. Then S has quasi-line-width at most $(5\alpha, \beta + d_0)$.*

Proof. Let J be the graph with vertex set the set of forts of $\mathcal{T}$ together with the set of all maximal communities of $\mathcal{T}$, where we say $X, Y \in V(J)$ are adjacent if X adjoins Y (and consequently at least one of X, Y is a fort). From 6.2, each maximal community in J has degree at most two in J, and its neighbours in J are forts. By 6.2, J has line-width at most three. For each $X \in V(J)$, if X is a fort of $\mathcal{T}$ let $A(X) = \Delta_{\mathcal{T}}(X)$, and if X is a maximal community of $\mathcal{T}$, let $A(X)$ be the union of the sets $\Delta_{\mathcal{T}}(Y)$ $(Y \in X)$. Thus the sets $A(X)$ $(X \in V(J))$ are pairwise disjoint and have union S. For each $X \in V(J)$, $A(X)$ has quasi-line-width at most $(\alpha, \beta + d_0)$, and its boundary has quasi-size at most $(\alpha, \beta + d_0)$. By 2.2, S has quasi-line-width at most $(5\alpha, \beta + d_0)$. This proves 6.3. ∎

7 Governments

Let $\mathcal{T}$ be an optimal kth-century realm in a graph G, with a tie-breaker Λ.

For $k + 1 \le t \le \ell - 1$, a *province* A of *type* t is a subset of $\mathcal{T}$ such that:

- there are at most $3^{t-k} - 1$ forts that are A^0-peripheral;

- the set of houses that are A^0-peripheral is the union of $3^{t-k} - 2$ communities; and

- there are exactly 3^{t-k-1} castles in A.

For instance, each castle of $\mathcal{T}$ forms a singleton province with type $k + 1$. If $\mathcal{T}$ is an optimal kth-century realm, we say a *government* for $\mathcal{T}$ is a set $\mathcal{G}$ of pairwise vertex-disjoint provinces, together with the choice of a set $F(A) \subseteq V(G)$ for each $A \in \mathcal{G}$, called the *framework* of A, satisfying:

- If $A \in \mathcal{G}$, then $G[F(A)]$ is connected and includes the union of the members of A;

- If $A \in \mathcal{G}$ has type t, then every vertex in $F(A)$ is joined to some member of A by a path of $G[F(A)]$ of length at most $d_0(t - k - 1)$;

- If $A_1, A_2 \in \mathcal{G}$ are distinct, of types i, j respectively, then $\mathrm{dist}_G(F(A_1), F(A_2)) > \delta(i + j)$;

- Each castle of $\mathcal{T}$ is contained in some province in $\mathcal{G}$;

- If $A \in \mathcal{G}$ has type t and $X \in \mathcal{T}$ is not included in any province in $\mathcal{G}$, then $\mathrm{dist}_G(F(A), X) > \delta(t + \mathrm{rk}(X))$;
- If $A \in \mathcal{G}$ has type t, then $F(A)$ contains a c-superfat H_t-minor of G.

If A is a province of $\mathcal{G}$, we denote its type by type(A). The members of $\mathcal{T}$ that are not included in provinces of $\mathcal{G}$ are called *rebels.* Thus, there may be rebel houses and rebel forts, but no castles are rebels.

We will fix some optimal kth-century realm $\mathcal{T}$, and will consider different governments for $\mathcal{T}$. There is a government for $\mathcal{T}$ each member of which is a singleton province (taking $F(X) = X$ for all $X \in \mathcal{T}$ of rank $> k$); we call this the *primordial government* of $\mathcal{T}$.

Let $\mathcal{G}$ be a government for $\mathcal{T}$. A *stronghold* of $\mathcal{G}$ is a set $X \subseteq V(G)$ such that either $X = F(A)$ for some $A \in \mathcal{G}$, or $X \in \mathcal{T}$ is a rebel of $\mathcal{G}$; and $V(\mathcal{G})$ denotes the union of the strongholds of $\mathcal{G}$. If X is a stronghold, a vertex of G is *X-local* if its Λ-geodesic to X is Λ-shorter than its Λ-geodesic to any other stronghold of $\mathcal{G}$. For each stronghold X, let $L_{\mathcal{G}}(X)$ be the set of X-local vertices. If X is a stronghold, a path P of G with ends p, q is *X-local* if P is a geodesic from p to $V(\mathcal{G})$, and $q \in X$. Thus q is the only vertex of P in $V(\mathcal{G})$, and all the vertices of P are X-local, and P has length at most d_0. (Thus, whether a vertex or path is X-local depends on the government.) If X, Y are strongholds, a *channel* between X, Y is a path R of G, with an edge $uv \in E(R)$, such that the two components of $R \setminus \{uv\}$ are respectively X-local and Y-local. Note that passages are defined just with reference to $\mathcal{T}$, and so are government-independent, while channels are defined with reference to the current government, and will change if the government changes. We say a stronghold *talks to* another stronghold if there is a channel joining them. For each rebel X, $L_{\mathcal{G}}(X) \subseteq \Delta_{\mathcal{T}}(X)$, and so if two rebels talk to each other then they adjoin each other (but the converse may not be true).

A set C of rebels is *in communication* if for every partition D, E of C with $D, E \neq \emptyset$, some member of D talks to some member of E. A *network* of $\mathcal{G}$ is a maximal set of rebel houses that is in communication. Thus, every network is a community; but whether a set is a community is determined by $\mathcal{T}$ and is independent of the current government, while whether a set is a network depends on the government. We say a set C of rebels is *organized* if the set of houses in C is a union of networks.

7.1 *In the same notation, let C be an adjoin-connected set of rebels that contains at least one fort, and let $A \in \mathcal{G}$. Let R be the set of $u \in V(G)$ such that there exist $X \in C$ with $u \in L_{\mathcal{G}}(X)$, and there exists $v \in L_{\mathcal{G}}(F(A))$ adjacent to u. Then R has quasi-size at most*

$$\left(i\left(14 \cdot 3^{\ell-k-2} - 3\right)\alpha, \beta + (\ell - k)d_0\right).$$

Proof. Let A have type $t < \ell$, Let $u \in R$, and choose $X \in C$ with $u \in L_{\mathcal{G}}(X)$, and $v \in L_{\mathcal{G}}(F(A))$ adjacent to u. Since u is X-local, there is a geodesic P between u, X that is X-local. Since v is $F(A)$-local,

$$\mathrm{dist}_G(v, X) \geq \mathrm{dist}(X, F(A))/2 \geq \frac{1}{2}\delta(t + \mathrm{rk}(X)) \geq c + 1,$$

and so P has length at least c. There is a path in $L_{\mathcal{G}}(F(A))$ between $v, F(A)$ of length at most d_0, and every vertex of $F(A)$ is joined to $V(A)$ by a path of $G[F(A)]$ of length at most $d_0(t - k - 1)$. So there is a path of $G[L_{\mathcal{G}}(F(A))]$ between v and $V(A)$ of length at most $d_0(t - k) \leq d_0(\ell - k - 1)$. Let

Q be the shortest path in $G[L_{\mathcal{G}}(F(A))]$ between $v, V(A)$, and let Q have ends v, q where $q \in Y \in A$. Thus Q has length at most $(\ell - k - 1)d_0$. Since u is X-local, it follows that

$$\mathrm{dist}_G(v, Y) \geq \frac{1}{2}\mathrm{dist}_G(X, Y) - 1 > \frac{1}{2}\delta(\mathrm{rk}(X) + t) - 1 \geq c.$$

Let Q' be the subpath of Q of length c with one end q, and let its other end be q'. It follows that Q' is a geodesic of G (because every vertex with distance at most c from Y has distance more than c from every other member of $\mathcal{T}$), and every vertex of Q not in $V(Q')$ has distance more than c from Y. So the union of P, Q and the edge uv is a passage of length at most $(\ell - k)d_0 + 1$, containing the edge uv. We have proved then that for each such choice of uv, there is a passage of length at most $(\ell - k)d_0 + 1$, containing the edge uv, joining some $X \in C$ and some $Y \in A$. Since C is not a community by hypothesis, it follows from 6.2 that Y is special, where we say a member of the province A is *special* if either it is a castle, or it is A^0-peripheral. The special houses in A can be partitioned into at most $3^{t-k} - 2$ communities, and there are at most $3^{t-k} - 1$ special forts in A, and 3^{t-k-1} special castles. For each of these communities, the boundary of its union has quasi-size at most (α, β); and each of the forts has boundary with quasi-size at most (α, β) and each of the castles has boundary of quasi-size at most $(8\alpha, \beta + (\ell - k)d_0)$. Consequently, the boundary of the union of the special members of A has quasi-size at most

$$\left(\left(3^{t-k} - 2\right)\alpha + \left(3^{t-k} - 1\right)\alpha + 3^{t-k-1}(8\alpha), \beta + (\ell - k)d_0\right) = \left(\left(14 \cdot 3^{t-k-1} - 3\right)\alpha, \beta + (\ell - k)d_0\right).$$

Since $t < \ell$, this proves 7.1. ∎

8 Revolution

Again, let G be a graph that does not contain H_ℓ as a c-superfat minor. Let $\mathcal{T}$ be an optimal kth-century realm in a graph G, with a tie-breaker Λ, and let $\mathcal{G}$ be a government for $\mathcal{T}$. We say that a set C of rebels is a *cabal* if:

- C is in communication, and organized;
- some (at most two) forts in C are designated as "leaders"; and some (at most three) networks included in C are designated as "leading networks";
- every element of C that is C-peripheral is either a leader or belongs to a leading network;
- if $X \in C$ talks to some rebel not in C, then either X is a leader, or X belongs to a leading network;
- for some $j \in \{k+1, \dots, \ell - 1\}$ there are three provinces $A_1, A_2, A_3 \in \mathcal{G}$ of type j, such that for $1 \leq i \leq 3$, some member of C talks to $F(A_i)$; and
- either $|C| = 1$, or C is a network, or for each $j \in \{k+1, \dots, \ell - 1\}$ there are at most four provinces $A \in \mathcal{G}$ of type j such that some member of C talks to $F(A)$.

8.1 *With $\mathcal{T}, \mathcal{G}$ as above, suppose that C is a cabal. Let $W = \bigcup_{X \in C} L_{\mathcal{G}}(X)$. Then* $\mathrm{bd}(W)$ *has quasi-size at most*

$$(56(\ell-k-1)3^{\ell-k-2}\alpha, \beta+(\ell-k)d_0).$$

Proof. Since C is a cabal, it follows that $k \le \ell-2$. If $|C|=1$, say $C=\{X\}$, then $\mathrm{bd}(X)$ has quasi-size at most (α,β), and so $\mathrm{bd}(W)$ has quasi-size at most $(\alpha, \beta+d_0)$, since every vertex in $L_{\mathcal{G}}(X)$ has distance at most d_0 from some vertex in $\mathrm{bd}(X)$. Similarly, if C is a network, and hence a community, then $\mathrm{bd}(V(C))$ has quasi-size at most (α,β), and hence again $\mathrm{bd}(W)$ has quasi-size at most $(\alpha,\beta+d_0)$. Thus we may assume that neither of these is true, and so for each $j \in \{k+1,\dots,\ell-1\}$ there are at most four provinces $A \in \mathcal{G}$ of type j such that some member of C talks to $F(A)$. Let $\mathcal{Q}$ be the set of all such provinces in $\mathcal{G}$; so $|\mathcal{Q}| \le 4(\ell-k-1)$. If some $X \in C$ talks to some rebel Y not in C, then either X is a leader, or X belongs to a leading network; and the boundary of each leader has quasi-size at most (α,β), and so does the boundary of the union of the members of each leading network. Consequently, $\mathrm{bd}(W)$ is the union of at most five sets of quasi-size at most $(\alpha,\beta+d_0)$, and at most $4(\ell-k-1)$ further sets each with quasi-size at most

$$\left(\left(14 \cdot 3^{\ell-k-2}-3\right)\alpha, \beta+(\ell-k)d_0\right),$$

by 7.1. It follows that $\mathrm{bd}(W)$ has quasi-size at most

$$\left((5+56(\ell-k-1)3^{\ell-k-2}-12(\ell-k-1))\alpha, \beta+(\ell-k)d_0\right).$$

This proves 8.1. ∎

8.2 *With $\mathcal{T}, \mathcal{G}$ as before, suppose that C is a cabal, and let j, A_1, A_2, A_3 be as in the fifth bullet in the definition of a cabal. Let $A = \{A_1, A_2, A_3\} \cup C$, and define $F(A)$ to be the set of all vertices that are X-local for some $X \in C \cup \{F(A_1), F(A_2), F(A_3)\}$. Let $\mathcal{G}' = (\mathcal{G} \setminus \{A_1,A_2,A_3\}) \cup \{A\}$. Then $\mathcal{G}'$ is a government, and $F(A)$ has quasi-bound at most*

$$\left(7(\ell-k)3^{\ell-k}\alpha, \beta+2(\ell-k-1)d_0\right).$$

Proof. First we show that:

(1) *A is a province of type $j+1$.*

To show this, we need to check that

- there are at most $3^{j+1-k}-1$ forts that are A^0-peripheral;
- there is a set of at most $3^{j+1-k}-2$ communities, each included in C, such that every A^0-peripheral house belongs to one of them;
- exactly 3^{j-k} castles belong to A.

For the first, there are at most $3(3^{j-k}-1)$ forts that are A^0-peripheral and belong to $A_1 \cup A_2 \cup A_3$, and only two in C, so in total at most $3^{j+1-k}-1$ as required. For the second, for $i = 1, 2, 3$ there is a set of at most $3^{j-k}-2$ communities, each included in A_i, such that every house in A_i that is A^0-peripheral belongs to one of them; and at most three more suffice for C, so in total we need at most $3(3^{j-k}-2)+3 \leq 3^{j+1-k}-2$. The third bullet is clear. This proves (1).

To show that $\mathcal{G}'$ is a government, we need to show in addition that $G[F(A)]$ is connected (which is clear), and that

- every vertex in $F(A)$ is joined to some member of A by a path of $G[F(A)]$ of length at most $d_0(j+1-k)$;
- if $A' \in \mathcal{G}$ with $A' \neq A$, then $\text{dist}_G(F(A), F(A')) > \delta(\text{type}(A') + j + 1)$;
- if $Y \in \mathcal{T}$ is not included in any province in $\mathcal{G}'$, then $\text{dist}_G(F(A), Y) > \delta(\text{rk}(Y) + j + 1)$;
- $j \leq \ell - 2$, and $F(A)$ contains a c-superfat H_{j+1}-minor of G.

For the first bullet, let $v \in F(A)$. Thus, v is X-local for some $X \in C \cup \{F(A_1), F(A_2), F(A_3)\}$. If $X \in C$, then v is joined to X by an X-local path, which therefore has length at most d_0, as required; so we may assume that $X = F(A_1)$. Let P be an $F(A_1)$-local path between v and $F(A_1)$, and let p be the end of P in $F(A_1)$. Thus, P has length at most d_0. Since $p \in F(A_1)$, p is joined to some member of A_1 by a path Q of $G[F(A_1)]$ of length at most $d_0(j-k)$; and its union with P provides the path we need. This proves the first bullet.

For the second bullet, let $A' \in \mathcal{G}$ with $A' \neq A$. Let $v \in F(A)$; we need to show that $\text{dist}_G(v, F(A')) > \delta(\text{type}(A') + j + 1)$. Choose $X \in C \cup \{F(A_1), F(A_2), F(A_3)\}$ such that v is X-local. Thus $\text{dist}_G(v, F(A')) \geq \text{dist}_G(v, X)$. If $X \in C$, then

$$\text{dist}_G(v, F(A')) + \text{dist}_G(v, X) > \delta(\text{type}(A') + \text{rk}(X)),$$

so

$$\text{dist}_G(v, F(A')) > \delta(\text{type}(A') + \text{rk}(X))/2 > \delta(\text{type}(A') + j + 1)$$

as required. If $X = F(A_1)$ say, then

$$\text{dist}_G(v, F(A')) + \text{dist}_G(v, X) > \delta(\text{type}(A') + j),$$

so

$$\text{dist}_G(v, F(A')) > \delta(\text{type}(A') + j)/2 > \delta(\text{type}(A') + j + 1)$$

as required. This proves the second bullet.

For the third, suppose that $Y \in \mathcal{T}$ is not included in any province in $\mathcal{G}'$. Let $v \in F(A)$; we need to show that $\text{dist}_G(v, Y) > \delta(\text{rk}(Y) + j + 1)$. Choose $X \in C \cup \{F(A_1), F(A_2), F(A_3)\}$ such that v is X-local. Thus $\text{dist}_G(v, Y) \geq \text{dist}_G(v, X)$. If $X \in C$, then

$$\text{dist}_G(v, Y) + \text{dist}_G(v, X) > \delta(\text{rk}(Y) + \text{rk}(X)),$$

so

$$\text{dist}_G(v, Y) > \delta(\text{rk}(Y) + \text{rk}(X))/2 > \delta(\text{rk}(Y) + j + 1)$$

as required. If $X = F(A_1)$ say, then $\mathrm{dist}_G(v,Y) + \mathrm{dist}_G(v,X) > \delta(\mathrm{rk}(Y)+j)$, so

$$\mathrm{dist}_G(v,Y) > \delta(\mathrm{rk}(Y)+j)/2 > \delta(\mathrm{rk}(Y)+j+1)$$

as required. This proves the third bullet.

Since $\delta(2j) \geq 5c$ for all $j \in \{k+1,\dots,\ell-1\}$. 2.1 implies that $F(A)$ contains a c-superfat H_{j+1}-minor of G; and hence $j \leq \ell-2$ since G does not contain H_ℓ as a c-superfat minor. This proves the fourth bullet, and so proves that $\mathcal{G}'$ is a government.

For the quasi-bound of $F(A)$, let $W = \bigcup_{X\in C} L_{\mathcal{G}}(X)$, and let B be the set of castles in A. Thus $|B| \leq 3^{\ell-k-2}$. Every vertex in $F(A)$ is joined to some member of A by a path of $G[F(A)]$ of length at most $d_0(\ell-k-2)$, and hence every vertex in $F(A) \setminus (V(A) \cup (W \setminus \mathrm{bd}(W)))$ is joined to $\mathrm{bd}(V(B)) \cup \mathrm{bd}(W)$ by a path of length at most $d_0(\ell-k-2)$. Since $\mathrm{bd}(V(B))$ has quasi-size at most

$$\left(8\cdot 3^{\ell-k-2}\alpha, \beta+(\ell-k)d_0\right),$$

and $\mathrm{bd}(W)$ has quasi-size at most

$$\left(56(\ell-k-1)3^{\ell-k-2}\alpha, \beta+(\ell-k)d_0\right)$$

by 8.1, it follows that $F(A) \setminus (V(A) \cup (W \setminus \mathrm{bd}(W)))$ has quasi-size at most

$$\left(56(\ell-k-1)3^{\ell-k-2}\alpha, \beta+2(\ell-k-1)d_0\right).$$

Since $\mathrm{bd}(F(A)) \subseteq (A) \setminus (V(A) \cup (W \setminus \mathrm{bd}(W)))$, it follows that $\mathrm{bd}(F(A))$ has quasi-size at most the same.

Since W has quasi-line-width at most $(5\alpha, \beta+d_0)$ by 6.3, and each member of B has quasi-line-width at most $(8\alpha, \beta+(\ell-k-1)d_0)$, and there are no edges between any two of the sets in $B\cup\{W\}$, it follows that $W \cup V(B)$ has quasi-line-width at most

$$(8\alpha, \beta+(\ell-k-1)d_0).$$

Since every vertex of $F(A)$ not in $W \cup V(B)$ belongs to $F(A) \setminus (V(A) \cup (W \setminus \mathrm{bd}(W)))$, it follows that $F(A)$ has quasi-line-width at most

$$\left(8\cdot 3^{\ell-k-2}\alpha + 56(\ell-k-1)3^{\ell-k-2}\alpha, \beta+2(\ell-k-1)d_0\right)$$

and hence at most

$$\left(7(\ell-k)3^{\ell-k}\alpha, \beta+2(\ell-k-1)d_0\right).$$

This proves 8.2. ∎

If $\mathcal{G}, \mathcal{G}'$ are related as in 8.2, we say that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by a *revolution.* Let us say a government $\mathcal{G}$ is *small* if for each province $A \in \mathcal{G}$, $F(A)$ has quasi-bound at most

$$\left(7(\ell-k)3^{\ell-k}\alpha, \beta+3(\ell-k-1)d_0\right).$$

The primordial government is small, and we want to modify it to make the best possible small government. To do so, let us say a government $\mathcal{G}_2$ *extends* a government $\mathcal{G}_1$ if (using $F_1(A)$, and $F_2(A)$ as the respectively framework functions):

- every rebel of $\mathcal{G}_1$ either is a rebel of $\mathcal{G}_2$, or is a subset of a province of $\mathcal{G}_2$;
- every province X_1 of $\mathcal{G}_1$ is a subset of a province X_2 of $\mathcal{G}_2$, where $F_1(X_1) \subseteq F_2(X_2)$, and either $X_1 = X_2$ and $F_1(X_1) = F_2(X_2)$, or the type of X_2 in $\mathcal{G}_2$ is greater than the type of X_1 in $\mathcal{G}_1$.

Let us say a small government $\mathcal{G}$ is *stable* if no other small government extends $\mathcal{G}$. Since the primordial government is small, it follows from Zorn's lemma that there is a stable small government. If a small government is stable, then (by the final statement of 8.2) no other government can be obtained from it by a revolution, and therefore there are no cabals.

8.3 *Let $\mathcal{T}$ be an optimal kth-century realm, and let $\mathcal{G}$ be a stable small government for $\mathcal{T}$. Let C be a set of rebels, in communication and maximal with this property; and let W be the union of the sets $L_{\mathcal{G}}(X)$ ($X \in C$). Then W has quasi-line-width at most $(5\alpha, \beta + d_0)$, and* $\mathrm{bd}(W)$ *has quasi-size at most*

$$\left((\ell - k)3^{\ell-k+1}\alpha, \beta + 2(\ell - k + 1)d_0\right).$$

Proof. The first claim follows from 6.3, since W is a subset of the set S of 6.3. For the second claim, if C is a community, then $\mathrm{bd}(V(C))$ has quasi-size at most (α, β), and hence $\mathrm{bd}(W)$ has quasi-size at most $(\alpha, \beta + d_0)$, since each vertex in $\mathrm{bd}(W)$ has distance at most d_0 from $\mathrm{bd}(V(C))$. So we may assume that C is not a community.

We say that $C' \subseteq C$ is *dangerous* if C' is in communication, and there exist $j > k$ and three members $A_1, A_2, A_3 \in \mathcal{G}$, all of type j, such that for $1 \le i \le 3$, some member of C' talks to $F(A_i)$. Suppose that C is not dangerous. Then for $k + 1 \le j \le \ell - 1$, there are at most two provinces $A \in \mathcal{G}$ such that $F(A)$ talks to some member of C. Let $\mathcal{Q}$ be the set of all such provinces in $\mathcal{G}$; so $|\mathcal{Q}| \le 2(\ell - k)$. But since C is a maximal set of rebels in communication, it follows that for each $u \in \mathrm{bd}(W)$ there exists $A \in \mathcal{Q}$ such that u has a neighbour in $L_{\mathcal{G}}(F(A))$; and so by 7.1, $\mathrm{bd}(W)$ has quasi-size at most

$$\left(2(\ell - k)\left(7 \cdot 3^{\ell-k-2} - 3\right)\alpha, \beta + 2(\ell - k + 1)d_0\right)$$

and the theorem holds.

If some network included in C is dangerous, then it is a cabal (designating itself as the only leading network), a contradiction. So no network included in C is dangerous.

Thus we may assume that C is dangerous and not a community. Since C is in communication, it is also adjoin-connected, and so we may number the forts in C as X_i ($i \in I$) where I is an integer interval, as in 6.2. Since C is a maximal set of rebels that is in communication, it follows that C is organized.

For $i_1, i_2 \in I$ with $i_1 \le i_2$, let $C(i_1, i_2)$ be the union of $\{X_{i_1}, \dots, X_{i_2}\}$ and all networks that talk to one of $X_{i_1}, \dots, X_{i_2}$. It follows that $C(i_1, i_2)$ is organized. Since C is dangerous, we may choose i_1, i_2 with $i_2 - i_1$ minimal such that $C(i_1, i_2)$ is dangerous. (Possibly $i_1 = i_2$.) Let D be the union of $\{X_{i_1}, \dots, X_{i_2}\}$ and all networks that talk to one of $X_{i_1+1}, \dots, X_{i_2-1}$. Again, D is organized. Each member of $C(i_1, i_2) \setminus D$ is a house, and belongs to a network that talks to one or both of X_{i_1}, X_{i_2} and to none of $X_{i_1+1}, \dots, X_{i_2-1}$.

Since $C(i_1, i_2)$ is dangerous, there exist $j > k$ and three members $A_1, A_2, A_3 \in \mathcal{G}$, all of type j, such that for $1 \le i \le 3$, some member of $C(i_1, i_2)$ talks to $F(A_i)$. For $i = 1, 2, 3$, $F(A_i)$ talks to either some member of D, or to some network included in $C(i_1, i_2) \setminus D$. So by adding to D at most three of the networks included in $C(i_1, i_2) \setminus D$, we can construct a set D' that is dangerous, in communication, and organized. Let us construct such a set D' by adding to D as few networks as possible; so in particular, if D is dangerous then $D' = D$. We designate X_{i_1}, X_{i_2} as leaders of D', and the (at most three) networks with union $D' \setminus D$ as leading networks of D'. We claim that D' is a cabal. To show this, we must check that

- D' is in communication, and organized;
- some (at most two) forts of D' are designated as leaders; and some (at most three) networks of D' are designated as leading networks;
- every element of D' that is D'-peripheral is either a leader or belongs to a leading network;
- if $X \in D'$ talks to some rebel not in D', then either X is a leader, or X belongs to a leading network;
- for some $j \in \{k+1, \dots, \ell-1\}$ there are three provinces $A_1, A_2, A_3 \in \mathcal{G}$ of type j, such that for $1 \leq i \leq 3$, $F(A_i)$ talks to some member of D'; and
- either $|D'| = 1$, or D' is a network, or for each $j \in \{k+1, \dots, \ell-1\}$ there are at most four provinces $A \in \mathcal{G}$ of type j such that $F(A)$ talks to a member of D'.

The first two bullets are clear. The third holds since no member of D is D-peripheral except possibly X_{i_1}, X_{i_2}.

For the fourth bullet, suppose that $X \in D'$ talks to some rebel $Y \notin D'$. If X is a fort, then $X = X_i$ for some $i \in \{i_1, \dots, i_2\}$; and since $Y \notin D$, it follows that $i \in \{i_1, i_2\}$ and so X is a leader. Now we assume that X is a house; let B be the network of $\mathcal{G}$ that contains X. Since D' is organized. it follows that $B \subseteq D'$. Since $Y \notin D'$, it follows that $Y \notin B$, and so B talks to Y and hence Y is a fort. Choose $i \in \{i_1, \dots, i_2\}$ such that B talks to X_i, with $i \notin \{i_1, i_2\}$ if possible. If $i \neq i_1, i_2$, then X_i semiadjoins X_{i-1}, X_{i+1} and Y, contrary to 6.2. Hence we cannot choose $i \notin \{i_1, i_2\}$; so $B \not\subseteq D$, and therefore B is a leading network. This proves the fourth bullet.

The fifth bullet holds since D' is dangerous. Finally, for the sixth bullet, if $D' \neq D$, then, since we added as few networks to D as possible to make a dangerous set, and no network is dangerous, the sixth bullet holds. So we may assume that $D' = D$, and so D is dangerous. If $i_1 \neq i_2$, then for every province $A \in \mathcal{G}$, if $F(A)$ talks to some member of D then it also talks to a member of one of $C(i_1+1, i_2), C(i_1, i_2-1)$, and since $C(i_1+1, i_2)$ and $C(i_1, i_2-1)$ are not dangerous (by the minimality of $i_2 - i_1$), it follows that the sixth bullet holds. So we may assume that $i_1 = i_2$. Since $D = D'$, it follows that $D = \{X_{i_1}\}$ and so $|D| = 1$, and again the sixth bullet holds. This proves that D' is a cabal, a contradiction since $\mathcal{G}$ is a stable small government. This proves 8.3. ∎

9 A new century

So far, we have kept fixed the century k, and its space requirement δ and its budget (α, β), but now we approach another century, and we need more flexible notation. For $0 \leq k \leq \ell$, let the space requirement of the kth century be the function δ_k, defined by

$$\delta_k(i) = 5c \cdot 3^{2\ell(\ell+1-k)-i}$$

for $0 \le i \le 2\ell$. Note that this is indeed a space requirement, for each $k \in \{0, \dots, \ell\}$, from the choice of $d_0 = 5c \cdot 3^{2\ell(\ell+1)}$. Let the budget for the kth century be (α_k, β_k), where $(\alpha_0, \beta_0) = (1, 1)$, and inductively for $0 \le k < \ell$,

$$\alpha_{k+1} = 7(\ell - k)3^{\ell-k}\alpha_k$$
$$\beta_{k+1} = \beta_k + 2(\ell - k)d_0.$$

We claim:

9.1 *Let G be a graph that contains no c-superfat H_ℓ-minor. Suppose that G admits a kth-century society, where $k < \ell$. Then G admits a $(k+1)$st-century society.*

Proof. Since G admits a kth-century society, it also admits a kth-century realm, by 4.1, and hence an optimal kth-century realm $\mathcal{T}$ say. Since there is a primordial government for $\mathcal{T}$, there is a stable small government $\mathcal{G}$ for $\mathcal{T}$. Let $\mathcal{T}'$ be the set of strongholds of $\mathcal{G}$. We assign each rebel house or rebel fort of $\mathcal{G}$ to be a house of $\mathcal{T}'$, and for each $A \in \mathcal{G}$, we assign $F(A)$ to be a fort of $\mathcal{T}'$. If $X \in \mathcal{T}'$, we write $\mathrm{rk}'(X) = k$ or $k+1$ depending whether X is a house or fort of $\mathcal{T}'$. We claim that $\mathcal{T}'$ is a $(k+1)$st-century society.

We must check that

1. The members of $\mathcal{T}'$ are pairwise vertex-disjoint and induce connected subgraphs of G.
2. For all $v \in V(G)$, there exists $X \in \mathcal{T}'$ such that $\mathrm{dist}_G(v, X) \le d_0$.
3. If $X, Y \in \mathcal{T}'$ are distinct then $\mathrm{dist}_G(X, Y) > \delta_{k+1}(\mathrm{rk}'(X) + \mathrm{rk}'(Y))$.
4. Every fort of $\mathcal{T}'$ contains a c-superfat H_{k+1}-minor of G.
5. For each fort X of $\mathcal{T}'$, $\mathrm{bd}(X)$ has quasi-bound at most $(\alpha_{k+1}, \beta_{k+1})$.
6. If C is a maximal adjoin-connected subset of the set of houses of $\mathcal{T}'$, then $\mathrm{bd}(\bigcup_{X \in V(C)} \Delta_{\mathcal{T}'}(X))$ has quasi-bound at most $(\alpha_{k+1}, \beta_{k+1} - d_0)$.

Statement 1 is true since the strongholds of $\mathcal{G}$ are pairwise disjoint, and statement 2 holds since each member of $\mathcal{T}$ is a subset of some member of $\mathcal{T}'$. For the third statement, if $X, Y \in \mathcal{T}'$, then

$$\mathrm{dist}_G(X, Y) \ge \delta_k(2\ell) \ge \delta_{k+1}(2k) \ge \delta_{k+1}(\mathrm{rk}'(X) + \mathrm{rk}'(Y)),$$

and so this statement holds. The fourth statement holds since if X is a fort of $\mathcal{T}'$, then $X = F(A)$ for some $A \in \mathcal{G}$; and since A has type $t > k$, $F(A)$ contains a c-superfat H_t-minor of G, and so also contains a c-superfat H_{k+1}-minor of G. Statement 5 holds since $\mathcal{G}$ is a small government. Finally, for statement 6, let C is a maximal adjoin-connected subset of the set of houses of $\mathcal{T}'$. If X, Y are rebel houses or forts of $\mathcal{G}$, then X talks to Y if and only if $\Delta_{\mathcal{T}'}(X)$ touches $\Delta_{\mathcal{T}'}(Y)$; so C is a network of $\mathcal{G}$, and hence the statement follows from 8.3. This proves 9.1. ∎

Now we can prove our main result, 1.8, which we restate in a slightly stronger form (recalling that ℓ, c are fixed):

9.2 *Every graph that does not contain H_ℓ as a c-superfat minor has quasi-line-width at most $(\alpha_\ell, \beta_\ell - d_0)$.*

Proof. Let G be a graph that does not contain H_ℓ as a c-superfat minor. We may assume that G is connected. By 4.2, G admits a 0th-century society; so by 9.1, G admits an ℓth-century society $\mathcal{T}$. Since G does not contain H_ℓ as a c-superfat minor, all members of $\mathcal{T}$ are houses; and since G is connected and therefore $\mathcal{T}$ is adjoin-connected, we deduce that $V(G) = \bigcup_{X \in V(C)} \Delta_{\mathcal{T}}(X)$ has quasi-bound at most $(\alpha_\ell, \beta_\ell - d_0)$. This proves 9.2. ∎

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