

# Excluding a fan minor

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## **Abstract**

A “fan” consists of a path together with a vertex adjacent to each vertex of the path. We prove a structure theorem for excluding large fans as minors; roughly, a graph has no large fan minor if and only if each of its blocks admits a tree-decomposition where the tree has bounded height, and each “torso” either has bounded size or is a cycle.

# 1 Introduction

A *tree-decomposition*  $\mathcal{T}$  of a graph  $G$  is a pair  $(T, (W_t : t \in V(T)))$ , where  $T$  is a tree, and  $W_t \subseteq V(G)$  for each  $t \in V(T)$ , such that

- $G = \bigcup_{t \in V(T)} G[W_t]$ , and
- if  $t, t', t'' \in V(T)$  and  $t'$  lies on the path between  $t, t''$ , then  $W_t \cap W_{t''} \subseteq W_{t'}$ .

(See Figure 1.)

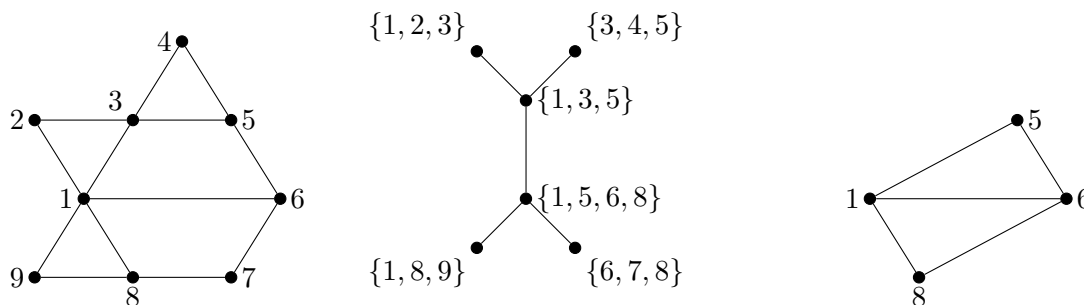


Figure 1: A graph and a tree-decomposition of it. The right-hand graph is one of the torsos.

The *width* of  $\mathcal{T}$  is the maximum of  $|W_t| - 1$  over all  $t \in V(T)$ , and the *tree-width* of  $G$  is the minimum width of its tree-decompositions. There is a well-known theorem of Robertson and Seymour [5]:

**1.1** For every  $H$ , the following are equivalent:

- $H$  is planar;
- there exists  $\ell$  such that every graph not containing  $H$  as a minor has tree-width at most  $\ell$ .

(A graph is a *minor* of another if the first can be obtained from a subgraph of the second by edge-contraction.) See [3] for a simpler proof, and [1] for the best dependence between  $H$  and  $\ell$  known.

This raises the natural question, what if we exclude certain types of planar graphs, are there correspondingly more restricted types of tree-decompositions? Here is one that is well known and easy. (The *height* of a tree-decomposition  $(T, (W_t : t \in V(T)))$  is the length of the longest path in  $T$ .)

**1.2** For every graph  $H$ , the following are equivalent:

- $H$  is a subgraph of a path;
- there exist  $\ell, m$  such that every graph not containing  $H$  as a minor (or equivalently, as a subgraph) admits a tree-decomposition  $(T, (W_t : t \in V(T)))$  of width at most  $\ell$  and height at most  $m$ .

And a less trivial one of Robertson and Seymour [4]:

**1.3** For every graph  $H$ , the following are equivalent:

- $H$  is a forest;
- there exist  $\ell$  such that every graph not containing  $H$  as a minor admits a tree-decomposition  $(T, (W_t : t \in V(T)))$  of width at most  $\ell$  where  $T$  is a path.

(See Diestel [2] for a short proof.)

Before we go on, some more definitions. All graphs in this paper are finite, and have no loops or parallel edges. If  $\mathcal{T} = (T, (W_t : t \in V(T)))$  is a tree-decomposition of  $G$  and  $t \in V(T)$ , the *torso* of  $\mathcal{T}$  at  $t$  is the graph obtained from  $G[W_t]$  by adding edges to make  $W_s \cap W_t$  a clique for all  $s \in V(T)$  adjacent to  $t$ . If  $e = st$  is an edge of  $T$ , the *adhesion* at  $e$  is the set  $W_s \cap W_t$ .

One can easily find theorems like the three above that characterize some simple properties (such as when  $H$  is a subgraph of a matching, or a subgraph of a star, or a subgraph of a cycle, for example). But there are no more known to us that have the depth of 1.3 and 1.1, and this is an appealing area for research, in the hope that the results might be as pretty as 1.3 and 1.1. For instance, here are two nice questions:

- what kind of tree-decomposition corresponds to excluding a series-parallel graph  $H$ ?
- what kind of tree-decomposition corresponds to excluding an apex-tree graph? (An *apex-tree graph* is a graph with a vertex whose deletion makes a tree.)

On this topic, let us mention a result of Rik Sengupta [6], who partially answered the second question above.

**1.4** For every graph  $H$ , the following are equivalent:

- $H$  is a subgraph of an apex-tree graph;
- there exists  $m$  such that every series-parallel graph with no  $H$  minor admits a tree-decomposition of height at most  $m$ , such that each torso is outerplanar, and all adhesions have size at most two.

In this paper, we provide a non-trivial addition to this family of results. A *fan* is a graph obtained from a path by adding another vertex adjacent to every vertex of the path (see Figure 2), and its *length* is the length of the path.

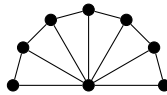


Figure 2: A fan of length six.

We prove:

**1.5** For every graph  $H$ , the following are equivalent:

- $H$  is a subgraph of a fan;
- there exist  $\ell, m$  such that for every graph  $G$  with no  $H$  minor, every maximal 2-connected subgraph of  $G$  admits a tree-decomposition  $(T, (W_t : t \in V(T)))$  of height at most  $m$ , such that for each  $t \in V(T)$ , either  $|W_t| \leq \ell$  or the torso at  $t$  is a cycle.

## 2 The three-connected case

The proof of 1.5 breaks into two parts: first we handle 3-connected graphs, and then the general case. This section is about the first part.

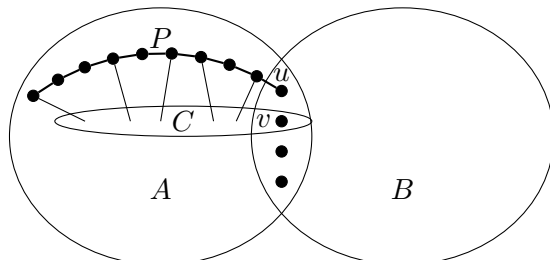


Figure 3: The definition of  $n_{(A,B)}(u,v)$ .

Let  $(A, B)$  be a separation of a graph  $G$ ; that is,  $A, B \subseteq V(G)$ , and  $G[A] \cup G[B] = G$ . For each ordered pair  $(u, v)$  of distinct vertices  $u, v \in A \cap B$ , we define  $n_{(A,B)}(u, v)$  (see Figure 3) to be the largest number  $n$  with the property that there is a path  $P$  of  $G[A]$  with one end  $u$  and with no other vertices in  $A \cap B$ , and a subset  $C \subseteq A \setminus V(P)$ , with  $G[C]$  connected and  $C \cap B = \{v\}$ , such that there are  $n$  distinct vertices of  $P$  with a neighbour in  $C$ . In other words,  $n$  is the maximum length of a fan minor of  $G[(A \setminus B) \cup \{u, v\}]$  using  $u, v$  in the specified way. Let  $n(A, B)$  be the sum of  $n_{(A,B)}(u, v)$  over all distinct  $u, v \in A \cap B$ , and the *score* of  $(A, B)$  is  $n(A, B) - n(B, A)$ . We need the following lemma:

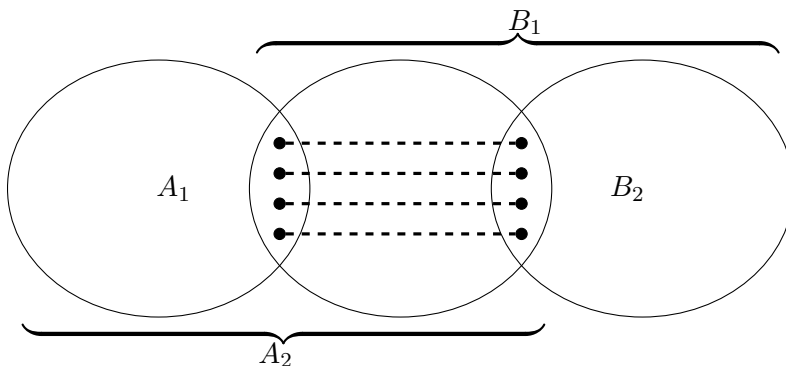


Figure 4: For 2.1. Some of the dashed paths might have length zero.

**2.1** Let  $(A_1, B_1)$  and  $(A_2, B_2)$  be separations of  $G$ , such that

- $A_1 \subseteq A_2$ , and  $B_2 \subseteq B_1$ ;
- $|A_1 \cap B_1| = |A_2 \cap B_2| = k$  say;
- there are  $k$  vertex-disjoint paths of  $G$  between  $A_1 \cap B_1$  and  $A_2 \cap B_2$ , say  $P_1, \dots, P_k$ ;

- the scores of  $(A_1, B_1)$  and  $(A_2, B_2)$  are equal.

(See Figure 4.) For  $1 \leq i \leq k$ , if  $P_i$  has positive length, the component of  $G[B_1 \cap A_2]$  containing  $P_i$  is vertex-disjoint from all  $P_j$  with  $j \neq i$ . If  $P_i$  has length at least two, and not both  $B_1, A_2 = V(G)$ , then  $G$  is not 3-connected.

**Proof.** Let the ends of  $P_i$  be  $s_i \in A_1 \cap B_1$  and  $t_i \in A_2 \cap B_2$ , for  $1 \leq i \leq k$ . Since  $|A_1 \cap B_1| = k$  and  $s_i \in A_1 \cap B_1$ , it follows that no other vertex of  $P_i$  belongs to  $A_1 \cap B_1$ , and similarly  $t_i$  is the only vertex of  $P_i$  in  $A_2 \cap B_2$ . Consequently,  $P_i$  is a subgraph of  $G[B_1 \cap A_2]$  for  $1 \leq i \leq k$ . For all distinct  $i, j \in \{1, \dots, k\}$ ,  $n_{(A_1, B_1)}(s_i, s_j) \leq n_{(A_2, B_2)}(t_i, t_j)$ , since if  $P, C$  are as in the definition of  $n_{(A_1, B_1)}(s_i, s_j)$ , then  $P \cup P_i, C \cup V(P_j)$  play the same role for  $n_{(A_2, B_2)}(t_i, t_j)$ . Similarly,  $n_{(B_1, A_1)}(s_i, s_j) \geq n_{(B_2, A_2)}(t_i, t_j)$  for all distinct  $i, j$ . Since the scores of  $(A_1, B_1)$  and  $(A_2, B_2)$  are equal, it follows that  $n_{(A_1, B_1)}(s_i, s_j) = n_{(A_2, B_2)}(t_i, t_j)$  and  $n_{(B_1, A_1)}(s_i, s_j) = n_{(B_2, A_2)}(t_i, t_j)$  for all distinct  $i, j$ .

To prove the assertion of the lemma, we may assume that  $i = 1$  from the symmetry. Suppose that  $P_1$  has positive length, and let  $D$  be the component of  $G[B_1 \cap A_2]$  that contains  $P_1$ . Suppose that  $D$  intersects (and hence includes) some  $P_j$  with  $j \geq 2$ , and choose a minimal path  $Q$  of  $D$  between  $V(P_1)$  and  $V(P_2 \cup \dots \cup P_k)$ , with ends  $x \in V(P_1)$  and  $y \in V(P_2)$  say. Thus, no vertices of  $Q$  belongs to  $(A_1 \cap B_1) \cup (A_2 \cap B_2)$  except possibly  $x, y$ . Suppose that  $x \neq s_1$ , and choose  $P, C$  as in the definition of  $n_{(A_1, B_1)}(s_1, s_2)$ . Let  $C' = C \cup V(P_2) \cup V(Q \setminus x)$ . Then  $G[C']$  is connected, and  $C' \cap B_2 = \{t_2\}$ . There are  $n_{(A_1, B_1)}(s_1, s_2)$  vertices in  $P$  with a neighbour in  $C$ , and therefore there are at least  $n_{(A_1, B_1)}(s_1, s_2) + 1$  vertices in  $P \cup P_1$  with a neighbour in  $C'$ , namely all those in  $P$  together with  $x$ . So  $n_{(A_2, B_2)}(s_1, s_2) > n_{(A_1, B_1)}(s_1, s_2)$ , a contradiction. This proves that  $x = s_1$ . But then  $x \neq s_2$ , and by the same argument  $n_{(B_1, A_1)}(s_1, s_2) > n_{(B_2, A_2)}(t_1, t_2)$ , again a contradiction. This proves that  $D$  is vertex-disjoint from  $P_2 \cup \dots \cup P_k$ . Now suppose that  $P_1$  has length at least two. If  $G \setminus \{s_1, t_1\}$  is not connected, then  $G$  is not 3-connected as required, so we may assume that  $G \setminus \{s_1, t_1\}$  is connected, and hence  $D = G$ . But then  $k = 1$ , and  $B_1 = A_2 = V(G)$ . This proves 2.1. ■

If  $\mathcal{T} = (T, (W_t : t \in V(T)))$  is a tree-decomposition of  $G$ , we say  $\mathcal{T}$  is *linked* if:

- for all distinct edges  $e_1, e_2$  of  $T$ , if the adhesions at  $e_1$  and at  $e_2$  have the same size  $k$ , and every edge of the path of  $T$  between  $e_1, e_2$  has an adhesion of size at least  $k$ , then there are  $k$  vertex-disjoint paths in  $G$  between the adhesions at  $e_1$  and at  $e_2$ .

It is sometimes helpful to designate some vertex of a tree  $T$  as its *root*. If  $Q$  is a subpath of a path of  $T$  starting at the root, we say that  $Q$  is an *ancestral* path. Let  $r \in V(T)$  be designated as the root; we say that a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  is *r-settled* if it is linked, and in addition:

- if  $s, t \in V(T)$  are adjacent then  $W_s \not\subseteq W_t$ ; and
- no two edges of any ancestral path of  $T$  have the same adhesion. (To clarify, we are just asking that the two adhesions are not the same set; they might have the same size.)

Here is a way to convert a linked tree-decomposition  $(T, (W_t : t \in V(T)))$  to one that is *r-settled*, where  $r \in V(T)$ . The *height-sum* of the tree-decomposition is the sum of the lengths of all paths of  $T$  that have the root as one end. Here are two operations:

- If there are adjacent  $s, t \in V(T)$  with  $W_s \subseteq W_t$ , contract the edge  $st$  to a vertex  $u$  and define  $W_u = W_t$ ; this gives another linked tree-decomposition, and with a smaller height-sum.
- If there are distinct edges  $e_1, e_2$  of  $T$  with the same adhesion, and  $e_1$  is between  $e_2$  and the root  $r$ , let  $e_i = s_i t_i$  for  $i = 1, 2$ , where there is a path of  $T$  passing through  $r, s_1, t_1, s_2, t_2$  in order (possibly  $r = s_1$  or  $s_2 = t_1$ ). Let  $T'$  be the tree obtained from  $T$  by deleting  $e_2$  and adding a new edge  $s_1 t_2$ . Then one can check that  $(T', (W_t : t \in V(T)))$  is a linked tree-decomposition with smaller height-sum.

Let us call this process (of repeatedly applying these two operations until they can no longer be applied) *settling*. The settling process terminates, since the height-sum is a non-negative integer, and at each step the height-sum decreases.

Turning to 1.5; for 3-connected graphs, we have the following:

**2.2** *Let  $F$  be a fan. Then there exists  $\ell \geq 0$  such that every 3-connected graph with a path of length  $\ell$  contains  $F$  as a minor.*

**Proof.** Since  $F$  is planar, there exists  $w$  such that every graph not containing  $H$  as a minor has tree-width at most  $w$ . Let  $K = 2w^2|V(F)|$ , and  $n = (2w + 3)^w K^w$ , and choose  $\ell$  such that no graph containing a path of length  $\ell$  admits a tree-decomposition of width at most  $w$  and height at most  $2n$ . (This is possible by 1.2.) Let  $G$  be a graph not containing  $F$  as a minor. Some component of  $G$  has a path of length  $\ell$ , so it suffices to show that this component contains  $F$  as a minor; that is, we may assume that  $G$  is connected. By a theorem of Thomas [7],  $G$  admits a linked tree-decomposition of width at most  $w$ . (Thomas in fact proved a stronger property called being “lean” that we do not need and that would not work here.) Choose a root  $r$  in the tree, and apply the settling process until it terminates, obtaining a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$ . From the definition of the settling process, this tree-decomposition also has width at most  $w$ .

Let  $K = 2w^2|V(F)|$ , and  $n = (2w + 3)^w K^w$ .

(1) *Every path in  $T$  starting from the root has length less than  $n$ .*

To see this, suppose it is false; and let  $e_1, e_2, \dots, e_n$  be the edges in order of some path  $Q$  of  $T$  starting from the root. Each of  $e_1, \dots, e_n$  has an adhesion of some size between  $1, w$  (since  $G$  is connected). For  $1 \leq h \leq w$ , let  $I_h$  be the set of  $i \in \{1, \dots, n\}$  such that  $e_i$  has an adhesion of size  $h$ . Since  $|I_1 \cup \dots \cup I_w| = n = (2w + 3)^w K^w$ , there exists  $h \in \{1, \dots, w\}$  minimum such that  $|I_1 \cup \dots \cup I_h| \geq (2w + 3)^h K^h$ . Thus, there are fewer than  $(2w + 3)^{h-1} K^{h-1}$  edges in  $I_1 \cup \dots \cup I_{h-1}$  between the first and last edges in  $I_h$ , and hence there are  $(2w + 3)K$  successive members of  $I_h$  with no members of  $I_1 \cup \dots \cup I_{h-1}$  between them. More exactly, there is a subpath  $Q'$  of  $Q$  that contains  $(2w + 3)K$  members of  $I_h$  and no members of  $I_1 \cup \dots \cup I_{h-1}$ . For each  $i \in I_h$ , let  $T_i^1, T_i^2$  be the components of  $T \setminus e_i$ , where  $r \in V(T_i^1)$ ; and let  $A_i, B_i$  be respectively  $\bigcup_{t \in V(T_i^1)} W_t$  and  $\bigcup_{t \in V(T_i^2)} W_t$ . Then  $A_i \cap B_i$  is the adhesion at  $e_i$  and so has size  $h$ . Since there are fewer than  $K$  possible values for the score of  $(A_i, B_i)$  (since  $G$  does not contain  $F$  as a minor), there are at least  $2w + 3$  edges  $e \in I_h \cap E(Q')$  corresponding to separations with the same score. Let  $f_1, \dots, f_{2w+3}$  be such edges, numbered in their order in  $Q'$ . Since  $\mathcal{T}$  is linked, there are  $h$  vertex-disjoint paths  $P_1, \dots, P_h$  between the adhesions of  $f_1$  and  $f_{2w+2}$ . Moreover, since the adhesions of  $f_1, \dots, f_{2w+2}$  are all different, there is at least one edge of  $P_1 \cup \dots \cup P_h$  in the subgraph between the adhesions of  $f_j$  and  $f_{j+1}$ , for

$1 \leq j \leq 2w + 1$ ; and so the sum of the lengths of  $P_1, \dots, P_h$  is at least  $2w + 1$ , and since  $h \leq w$ , we may therefore assume that some  $P_i$  has length at least two. From 2.1 we have a contradiction, since  $G$  is 3-connected and there is a vertex in the adhesion of  $f_{2w+3}$  that is not in the adhesion of  $f_{2w+2}$ . This proves (1).

So  $G$  admits a tree-decomposition of width at most  $w$  and height at most  $2n$ . From the choice of  $\ell$  it follows that  $G$  contains no path of length  $\ell$ . This proves 2.2. ■

### 3 Reducing the 2-connected case to the 3-connected case

Now we prove the following, which complements 2.2:

**3.1** *Let  $F$  be a fan. Then every two-connected graph  $G$  with no  $F$  minor admits a tree-decomposition with height less than  $32|V(F)|$ , such that every adhesion has size two, and every torso is either 3-connected or a cycle.*

**Proof.** Let  $G$  be a 2-connected graph with no  $F$  minor. A *triangle* means a cycle of length three. We claim that:

(1) *There is a tree-decomposition  $(T, (W_t : t \in V(T)))$  of  $G$  such that:*

- *every adhesion has size two;*
- *if  $s, t \in V(T)$  are adjacent, then  $W_s \not\subseteq W_t$ ; and*
- *every torso is either 3-connected or a triangle.*

Choose a tree-decomposition  $(T, (W_t : t \in V(T)))$  such that all adhesions have size at most two, and the set of all pairs  $\{u, v\}$  such that  $\{u, v\} \subseteq W_t$  for some  $t \in V(T)$  is minimal. (We call this set the *set of nonseparated pairs*.) We may choose  $(T, (W_t : t \in V(T)))$  such that in addition, if  $s, t \in V(T)$  are adjacent, then  $W_s \not\subseteq W_t$ , because applying the first settling operation does not change the set of nonseparated pairs. Since  $G$  is 2-connected, it follows that all adhesions have size exactly two (because if  $s, t \in V(T)$  are adjacent, then  $W_s \setminus W_t, W_t \setminus W_s$  are both nonempty and no component of  $G \setminus (W_s \cap W_t)$  intersects both of them).

We claim that every torso is either 3-connected or a triangle. Let  $t \in V(T)$ , and suppose that the torso of  $\mathcal{T}$  at  $t$  is not 3-connected and not a triangle. Consequently, there exists  $X \subseteq W_t$  with  $|X| \leq 2$ , and a partition  $(Y_1, Y_2)$  of  $W_t \setminus X$  such that  $Y_1, Y_2 \neq \emptyset$  and there are no edges of the torso between  $Y_1, Y_2$ . Hence we may partition the set of edges of  $T$  incident with  $t$  into two sets  $E_1, E_2$ , such that for  $i = 1, 2$ , if  $st \in E_i$  then  $W_s \cap W_t \subseteq Y_i \cup X$ . Let  $T'$  be obtained from  $T$  by: deleting  $t$ ; adding two new vertices  $t_1, t_2$ , adjacent to each other; and for  $i = 1, 2$  and each  $st \in E_i$ , adding an edge between  $s, t_i$ . Then  $T'$  is a tree; define  $W_{t_i} = X \cup Y_i$  for  $i = 1, 2$ . Then  $(T', (W_t : t \in V(T)))$  is a tree-decomposition of  $G$ , all its adhesions have size at most two, and its set of nonseparated pairs is strictly reduced, a contradiction. This proves that every torso is either 3-connected or a complete graph, and so proves (1).

The tree-decomposition given by (1) is linked, since all its adhesions have size two and  $G$  is 2-connected. Choose a root  $r$  for  $T$ , and let us apply the second settling operation to the tree-decomposition given by (1) until the process stops. This operation preserves all the properties of (1), so we obtain a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  as in (1) that is  $r$ -settled.

For each edge  $e$  of  $T$ , let  $T_1, T_2$  be the two components of  $T \setminus e$  with the root in  $V(T_1)$ , and define  $A_e = \bigcup_{s \in V(T_1)} W_s$  and  $B_e = \bigcup_{s \in V(T_2)} W_s$ . Thus,  $A_e \cap B_e$  is the adhesion at  $e$ , and so has size two, for each  $e \in E(T)$ . We define the *score* of  $e$  to be the score of  $(A_e, B_e)$ . For each  $t \in V(T)$  different from the root  $r$ , let  $e_t$  be the edge of  $T$  incident with  $t$  in the path between  $t$  and  $r$ . If  $e_t$  has ends  $s, t$ , we call  $s$  the *parent* of  $t$ . A path of  $T$  is *traditional* if it is ancestral and all its edges have the same score.

(2) *Let  $Q$  be a traditional path of  $T$  with length at least two. Let its edges be  $e_1, \dots, e_n$  in order, where  $e_1$  is closest to the root. Then there is no path in  $G[B_{e_1} \cap A_{e_n}]$  between the two members of  $A_{e_1} \cap B_{e_1}$  or between the two members of  $A_{e_n} \cap B_{e_n}$ .*

Since  $G$  is 2-connected, there are two vertex-disjoint paths  $P_1, P_2$  of  $G[B_{e_1} \cap A_{e_n}]$  between  $A_{e_1} \cap B_{e_1}$  and  $A_{e_n} \cap B_{e_n}$ . For  $i = 1, 2$ , Since the two adhesions are different, one of  $P_1, P_2$  has nonzero length, and so by 2.1,  $P_1, P_2$  belong to different components of  $H$ . This proves (2).

We say a vertex  $t$  of  $T$  is *sheltered* if  $t$  is the middle vertex of a traditional path of length two, and an edge  $e \in E(T)$  is *sheltered* if  $e$  is the middle edge of a traditional path of length three. If an edge is sheltered then both its ends are also sheltered.

(3) *If  $t \in V(T)$  is sheltered, then the torso of  $\mathcal{T}$  at  $t$  is a triangle.*

Let  $s-t-u$  be a traditional path of length two. Let  $B_s \cap A_t = \{x_1, y_1\}$  and  $B_t \cap A_u = \{x_2, y_2\}$ . Then  $x_1y_1, x_2y_2$  are edges of the torso of  $\mathcal{T}$  at  $t$ ; and deleting these two edges from the torso makes it disconnected, by (2). Hence this torso is not 3-connected, and so it is a triangle. This proves (3).

If  $S$  is a subtree of  $T$ , let  $H(S)$  be the graph with vertex set  $\bigcup_{s \in V(S)} W_s$ , in which distinct  $u, v \in V(H_s)$  are adjacent if either  $uv \in E(G)$  or  $\{u, v\} = W_s \cap W_t$  for some edge  $st$  of  $T$  with  $s \in V(S)$  and  $t \notin V(S)$ . A subtree  $S$  of  $T$  is *sheltered* if all its vertices and edges are sheltered.

(4) *If  $S$  is a sheltered subtree of  $T$ , let  $s$  be the vertex of  $S$  closest to the root  $r$ ; then  $s \neq r$ , and all edges of  $S$  have the same score as  $e_s$ .*

Let  $s$  be the vertex of  $S$  closest to the root  $r$ . Since  $s$  is sheltered, it follows that  $s \neq r$ . Let  $e \in E(S)$ . Then there is an ancestral path with first edge  $e_s$  and last edge  $e$ , and all its edges except the first are sheltered. Consequently, every two consecutive edges of this path have the same score, and so  $e$  has the same score as  $e_s$ . It follows that all edges of  $S$  have the same score. This proves (4).

(5) *If  $S$  is a sheltered subtree of  $T$ , then  $H(S)$  is a cycle.*

We proceed by induction on  $|V(S)|$ . By (4), all edges in  $S$  have the same score. Let  $s$  be the vertex of  $S$  closest to the root  $r$ ; then by (4),  $s \neq r$ . Let  $p$  be its parent. By (3),  $|W_s| = 3$ ; let

$W_s = \{v_1, v_2, w\}$ , where  $\{v_1, v_2\}$  is the adhesion of  $\mathcal{T}$  at  $e_s$ . For each edge  $e \neq e_s$  of  $T$  incident with  $s$ , the adhesion of  $\mathcal{T}$  at  $e$  is one of the sets  $\{v_1, w\}, \{v_2, w\}$  (since no such edge has adhesion  $\{v_1, v_2\}$  because all adhesions of edges in an ancestral path are different). Let  $E_i$  be the set of edges  $e$  incident with  $s$  with adhesion  $\{v_i, w\}$ , for  $i = 1, 2$ .

For each edge  $e \in E_1 \cup E_2$ , since  $G$  is 2-connected and  $B_e \neq A_e \cap B_e$ , there is a path of  $G[B_e]$  between the two members of  $A_e \cap B_e$ , say  $P_e$ .

Suppose that  $E_i$  contains an edge  $st$  of  $S$ , for some  $i \in \{1, 2\}$ . By (2) applied to the traditional path  $p-s-t$ , there is no path in  $G[B_{e_s} \cap A_{st}]$  between  $v, w$ , and so  $v, w$  are nonadjacent, and there is no path  $P_f$  for  $f \in E_i \setminus \{st\}$ . Hence  $v_i, w$  are nonadjacent and  $E_i = \{st\}$ .

Now there are three cases, depending how many of  $E_1, E_2$  contain an edge of  $S$ . Suppose first that neither of  $E_1, E_2$  contain an edge of  $S$ . Thus  $V(S) = \{s\}$ , and  $H(S)$  equals the torso of  $\mathcal{T}$  at  $t$ . Consequently,  $H(S)$  is a triangle by (3), and therefore a cycle and the claim holds. So we may assume that at least one of  $E_1, E_2$  contains an edge of  $S$ .

Suppose that  $E_1, E_2$  both contain edges of  $S$ . Then  $|E_1| = |E_2| = 1$ , and so  $s$  has degree three in  $T$ ; and moreover,  $w$  is nonadjacent to both  $v_1, v_2$ . Let  $S_1, S_2$  be the components of  $S \setminus \{s\}$ , where the edges between  $s$  and  $S_i$  belongs to  $E_i$  for  $i = 1, 2$ . Thus  $S_1, S_2$  are both sheltered, and so from the inductive hypothesis,  $H(S_1), H(S_2)$  are both cycles. But  $H(S)$  is obtained from  $H(S_1) \cup H(S_2)$  by deleting the edges  $v_i x$  for  $i = 1, 2$  (since  $|E_i| = 1$  and  $v_i, w$  are nonadjacent) and adding the edge  $v_1 v_2$ ; and so  $H(S)$  is a cycle.

Finally, we assume that  $E_1$  contains an edge of  $S$  and  $E_2$  does not. Let  $S_1 = S \setminus s$ . Then  $|E_1| = 1$ , and  $w$  is nonadjacent to both  $v_1$ . Since  $G$  is 2-connected, there is a path of  $G \setminus v_1$  between  $w, v_2$ , and so either  $v_2, w$  are adjacent in  $G$ , or  $E_2 \neq \emptyset$ ; and in either case,  $H(S)$  is obtained from  $H(S_1)$  by deleting  $v_1 w$  and adding  $v_2$  and the edges  $v_1 v_2, v_2 w$ . Since  $H(S_1)$  is a cycle from the inductive hypothesis, it follows again that  $H(S)$  is a cycle. This proves (5).

(6) *Let  $Q$  be an ancestral path of  $T$ . There are fewer than  $16|V(F)|$  edges of  $Q$  that are not sheltered.*

For every edge  $e$  of  $T$ , its score is between  $-4(|V(F)| - 2)$  and  $4(|V(F)| - 2)$ , since  $G$  does not contain  $F$  as a minor. Let  $X$  be the set of internal vertices  $x$  of  $Q$  such that the two edges of  $Q$  incident with  $x$  have different scores. Then  $|X| \leq 8(|V(F)| - 2) + 1$ , since the scores of the edges in  $Q$  monotonically (non-strictly) increase as we move along  $Q$  from the root. For every edge  $e$  of  $Q$  that is not sheltered, either it is the first or last edge of  $Q$ , or one of its ends is in  $X$ ; and so at most  $2|X| + 2 < 16|V(F)|$  edges of  $Q$  are not sheltered. This proves (6).

Let  $T'$  be the tree obtained from  $T$  by contracting all sheltered edges of  $T$ . By (6),  $T'$  has height less than  $32|V(F)|$ . For each  $x \in V(T')$ , let  $S_x$  be the subtree of  $T$  whose vertices were identified under contraction to form  $x$ . It follows that all edges of  $S_x$  are sheltered. For each  $x \in V(T')$ , define  $U_S = \bigcup_{t \in V(S_x)} W_t$ ; then  $\mathcal{T}' = (T', (U_S : S \in V(T')))$  is a tree-decomposition of  $G$  of height at most  $32|V(F)|$ , and all its adhesions have size two. For each  $x \in V(T')$ , if  $|S_x| = 1$  then the torso of  $\mathcal{T}'$  at  $x$  equals the torso of  $\mathcal{T}$  at the vertex of  $S_x$ , and so is either 3-connected or a triangle, since  $\mathcal{T}$  satisfies (1). If  $|S_x| \geq 2$  then all edges of  $S_x$  are sheltered, and so  $S_x$  is sheltered, and consequently the torso of  $\mathcal{T}'$  at  $x$  is a cycle by (5). This proves 3.1. ■

## 4 Combining the parts

By combining 2.2 and 3.1, we deduce:

**4.1** *For every fan  $F$ , there exists  $\ell, m$  with the following property. Every 2-connected graph  $G$  not containing  $F$  as a minor admits a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  of height at most  $m$  such that for every  $t \in V(T)$ , either  $|W_t| \leq \ell$  or the torso of  $\mathcal{T}$  at  $t$  is a cycle.*

**Proof.** By 2.2, there exists  $p$  such that every 3-connected graph with a path of length  $p$  contains  $F$  as a minor. By 1.2, there exist  $\ell, m_1$  such that every graph with no path of length  $p$  admits a tree-decomposition of width at most  $\ell$  and height at most  $m_1$ . Let  $m = 32|V(F)|m_1$ ; we will show that  $\ell, m$  satisfy the theorem.

Let  $G$  be a 2-connected graph not containing  $F$  as a minor. By 3.1, there is a tree-decomposition  $(T_1, (W_t^1 : t \in V(T_1)))$  of  $G$  with height less than  $32|V(F)|$  such that all its adhesions have size two, and each torso is either 3-connected or a cycle. For each  $t \in V(T_1)$ , let  $H(t)$  be the torso of  $(T_1, (W_t^1 : t \in V(T_1)))$  at  $t$ . Since  $G$  is 2-connected,  $H(t)$  is a minor of  $G$  for each  $t \in V(T_1)$ . If  $H(t)$  is 3-connected for some  $t \in V(T_1)$ , then since  $H(t)$  does not contain  $F$  as a minor, it has no path of length  $p$  from the choice of  $p$ , and so admits a tree-decomposition of width at most  $\ell$  and height at most  $m_1$  from the choice of  $\ell, m_1$ . By substituting such a tree-decomposition for  $H(t)$  in the natural way whenever  $H(t)$  is 3-connected, we obtain a tree-decomposition of  $G$  of height at most  $32|V(F)|m_1 = m$  such that each of its torsos either has at most  $\ell$  vertices or is a cycle. This proves 4.1. ■

Now we deduce our main result.

**Proof of 1.5.** If  $H$  is a subgraph of a fan, then the second bullet of 1.5 holds, because of 4.1 applied to each maximal 2-connected subgraph of  $G$ ; so it remains to show the reverse implication. In other words, we need to show that if  $H$  is not a subgraph of a fan, then for all  $\ell, m$  there is a 2-connected graph  $G$  with no  $H$  minor that admits no tree-decomposition  $(T, (W_t : t \in V(T)))$  of height at most  $m$  such that for each  $t \in V(T)$ , either  $|W_t| \leq \ell$  or the torso at  $t$  is a cycle. To see this, let  $\ell, m \geq 0$ , and we may assume that  $\ell \geq 3$  by increasing  $\ell$  if necessary. Choose  $n$  such that no graph with a path of length  $n$  admits a tree-decomposition of width at most  $\ell$  and height at most  $m$ . Let  $G$  be a fan of length  $n$ . Then  $G$  has no  $H$  minor, since  $H$  is not a subgraph of a fan, and all minors of a fan are subgraphs of a fan. Suppose that  $G$  admits a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  of height at most  $m$  such that for each  $t \in V(T)$ , either  $|W_t| \leq \ell$  or the torso at  $t$  is a cycle. Since  $G$  is a chordal graph, all torsos of  $\mathcal{T}$  are also chordal, and so any torso that is a cycle has length three, and hence has at most  $\ell$  vertices. Thus  $\mathcal{T}$  has width at most  $\ell$ , and so  $G$  has no path of length  $n$ , a contradiction. This proves 1.5 ■

## 5 An application

Let  $H$  be a graph obtained from a cycle and a path by identifying one end of the path with a vertex of the cycle. We call such a graph a *stopsign*. Every stopsign is a subgraph of a fan, so we can use 1.5 to give information about the graphs that do not contain a given stopsign. We claim:

**5.1** *For every graph  $H$ , the following are equivalent:*

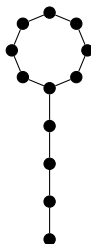


Figure 5: A stopsign.

- $H$  is a subgraph of a stopsign;
- there exist  $\ell, m$  such that every connected graph with no  $H$  minor admits a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  such that either:
  - $\mathcal{T}$  has height at most  $m$ , and one torso is a cycle, and all other torsos have size at most  $\ell + 1$ ; or
  - $\mathcal{T}$  has width at most  $\ell$ , and every path of  $T$  of length  $m$  contains an edge with adhesion of size 1.

**Proof.** We show first that the first bullet implies the second. We assume that  $H$  is a subgraph of a stopsign; and hence we may assume that  $H$  is a stopsign. Let  $F$  be a fan containing  $H$  as a minor. Choose  $\ell_1, m_1$  such that setting  $\ell = \ell_1$  and  $m = m_1$  satisfies 1.5. Choose  $\ell_2, m_2$  such that every graph with no path of length  $|V(F)|^2$  admits a tree-decomposition of height at most  $m_2$  and width at most  $\ell_2$ . Let  $\ell = \max(\ell_1, \ell_2) + 1$  and  $m = \max(m_1, m_2) + 1$ . We claim that  $\ell, m$  satisfy 5.1.

To see this, let  $G$  be a connected graph with no  $H$  minor. Suppose first that  $G$  has no cycle of length at least  $|V(F)|$ . Then each maximal 2-connected component has no path of length  $|V(F)|^2$ , and so admits a tree-decomposition of height at most  $m_2$  and width at most  $\ell_2$ . By combining these tree-decompositions, we deduce that  $G$  admits a tree-decomposition of width at most  $\ell_2$  such that every path of  $T$  of length  $m_2 + 1$  contains an edge with adhesion of size 1, and so the theorem holds.

Thus we may assume that some maximal 2-connected component  $G'$  has a cycle of length at least  $|V(F)|$ . For each  $v \in V(G')$ , let  $B_v$  be the component of  $G \setminus v$  that contains  $G' \setminus v$ , and let  $A_v = G \setminus B_v$  (thus,  $v \in V(A_v)$ ). Hence the subgraphs  $A_v$  ( $v \in V(G')$ ) are pairwise vertex-disjoint, and their union, together with  $G'$ , equals  $G$ . Since  $G$  does not contain  $H$ , it follows that for each  $v \in V(G')$ , there is no path in  $A_v$  of length  $|V(H)|$  with one end  $v$ , and hence there is no path in  $A_v$  of length  $2|V(F)|$ . Consequently,  $A_v$  admits a tree-decomposition of height at most  $m_2$  and width at most  $\ell_2$ .

Now, since  $G'$  is 2-connected and does not contain  $F$  as a minor, 1.5 implies that  $G'$  admits a tree-decomposition  $\mathcal{T} = (T, (W_t : t \in V(T)))$  of height at most  $m_1$  such that for every  $t \in V(T)$ , either  $|W_t| \leq \ell_1 + 1$  or the torso of  $\mathcal{T}$  at  $t$  is a cycle. If each  $|W_t| \leq \ell$ , then by combining this tree-decomposition with that of  $A_v$  for each  $v \in V(G')$ , we deduce that  $G$  admits a tree-decomposition of width at most  $\ell$  such that every path of length more than  $\max(m_1, m_2)$  in  $T$  contains an edge with adhesion of size one, and so the theorem holds. Thus, we assume that at least one torso of  $\mathcal{T}$  is a cycle of length at least  $\ell$ . Since  $G'$  is 2-connected, there exists  $X = \{x_1, \dots, x_n\} \subseteq V(G')$  of  $G'$ , with  $n \geq \ell$ , such that (reading subscripts modulo  $n$ ):

- for each component  $D$  of  $G' \setminus X$ , there exist  $i \in \{1, \dots, n\}$  such that  $x_i, x_{i+1}$  have neighbours in  $V(D)$  and no other members of  $X$  have neighbours in  $V(D)$ ; and
- for  $1 \leq i \leq n$  there is a component  $D$  of  $G' \setminus X$  such that  $x_i, x_{i+1}$  have neighbours in  $V(D)$ .

Consequently, the same is true for  $G$ . Hence, for  $1 \leq i \leq n$  there is a subgraph  $C_i$  of  $G$  with the following properties, (again, reading subscripts modulo  $n$  (see Figure 6):

- $x_i, x_{i+1} \in V(C_i)$  and there is a path in  $C_i$  between  $x_i, x_{i+1}$ ;
- for  $1 \leq i < j \leq n$ , if  $j \neq i + 1$  and  $(i, j) \neq (1, n)$  then  $C_i \cap C_j$  is null;
- $C_i \cap C_{i+1} = \{x_{i+1}\}$  for  $1 \leq i \leq n$ ; and
- $C_1 \cup \dots \cup C_n = G$ .

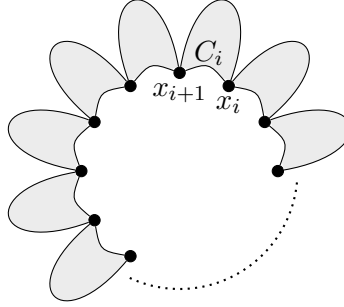


Figure 6: For 5.1.

Choose  $n$  maximum such that there is such a set  $X = \{x_1, \dots, x_n\}$  and  $C_1, \dots, C_n$ . It follows that for  $1 \leq i \leq n$ , either  $x_i, x_{i+1}$  are adjacent, or there are two paths of  $C_i$  between  $x_i, x_{i+1}$  with disjoint interiors. We claim that for  $1 \leq i \leq n$ ,  $C_i$  has no path of length  $|V(F)|^2$ . Suppose that  $i = 1$  say, and there is a path  $P$  in  $C_1$  of length  $|V(F)|^2$ . There is a path in  $C_1$  between  $x_1, x_2$ , say  $Q$ . There is no path  $R$  of  $C$  with  $|V(R)| \geq |V(F)|$  with one end in  $Q$  and no other vertex in  $Q$ , since  $G$  does not contain  $H$  as a minor. (Let us call this fact the “forbidden path condition”.) If  $P \cap Q$  is null, then the union of  $P$  and a path of  $C_1$  between  $P, Q$  gives a path violates the forbidden path condition, a contradiction. Thus  $P \cap Q$  is non-null. If any component of  $P \setminus V(Q)$  has length at least  $|V(F)|$ , then again the forbidden path condition is violated. So there is no such component, and therefore  $|V(P \cap Q)| \geq |V(F)|$ , and in particular  $|V(Q)| \geq |V(F)|$ . Hence every path of  $C_1$  between  $x_1, x_2$  has at least  $|V(F)|$  vertices. So  $x_1, x_2$  are nonadjacent, and there are two paths of  $C_1$  between  $x_1, x_2$  with disjoint interiors, both of length at least  $|V(F)|$ , and again the forbidden path condition is violated. This proves that  $C_i$  has no path of length  $|V(F)|^2$ , for  $1 \leq i \leq n$ . Hence  $C_i$  admits a tree-decomposition of height at most  $m_2$  and width at most  $\ell_2$ , for  $1 \leq i \leq n$ . Consequently,  $G$  admits a tree-decomposition of height at most  $m_2 + 2$  such that one torso is a cycle and all others have at most  $\ell_2 + 1$  vertices, and the theorem holds. This proves the more difficult part of 5.1.

To complete the proof, we need to show the reverse implication. Thus, let  $\ell, m \geq 0$ , and choose  $n$  such that no graph with a path of length  $n$  admits a tree-decomposition of width at most  $\ell$  and height at most  $m$ . Let  $G$  be a stopsign made identifying one end of a path of length  $n + 1$  with a vertex of a cycle of length  $n + 1$ . All minors of  $G$  are subgraphs of stopsigns, so  $H$  is not a minor of  $G$ . Suppose that  $G$  admits a tree-decomposition  $\mathcal{T}$  such that either:

- $\mathcal{T}$  has height at most  $m$ , and one torso is a cycle, and all other torsos have size at most  $\ell + 1$ ;  
or
- $\mathcal{T}$  has width at most  $\ell$ , and every path of  $T$  of length  $m$  contains an edge with adhesion of size 1.

In the second case, all cycles of  $G$  have bounded length, so if  $n$  is sufficiently large then  $G$  admits no such decomposition. In the first case,  $G$  admits a decomposition as shown in Figure 6: there are subgraphs  $C_1, \dots, C_n$  for some  $n \geq 3$ , arranged in a circular order, each sharing one vertex with the next and otherwise disjoint, with union  $G$ , and the longest path in each  $C_i$  is at most  $n$ . In particular, no  $C_i$  has a cycle of length at least  $n + 1$ , so the (unique) cycle of  $G$  contains all the vertices that belong to more than one of the  $C_i$ 's. But there is a path of  $G$  of length  $n$  disjoint from the cycle of  $G$ , so it is a subgraph of some  $C_i$ , which is impossible. This proves 5.1. ■

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