Edge-colouring eight-regular planar graphs

Maria Chudnovsky[1]
Columbia University, New York, NY 10027

Katherine Edwards[2], Paul Seymour[3]
Princeton University, Princeton, NJ 08544

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Abstract

It was conjectured by the third author in about 1973 that every d-regular planar graph (possibly with parallel edges) can be d-edge-coloured, provided that for every odd set X of vertices, there are at least d edges between X and its complement. For $d = 3$ this is the four-colour theorem, and the conjecture has been proved for all $d \leq 7$, by various authors. Here we prove it for $d = 8$.

1 Introduction

One form of the four-colour theorem, due to Tait [9], asserts that a 3-regular planar graph can be 3-edge-coloured if and only if it has no cut-edge. But when can d-regular planar graphs be d-edge-coloured?

Let G be a graph. (Graphs in this paper are finite, and may have loops or parallel edges.) If $X \subseteq V(G)$, $\delta_G(X) = \delta(X)$ denotes the set of all edges of G with an end in X and an end in $V(G)\setminus X$. We say that G is *oddly d-edge-connected* if $|\delta(X)| \geq d$ for all odd subsets X of $V(G)$. Since every perfect matching contains an edge of $\delta(X)$ for every odd set $X \subseteq V(G)$, it follows that every d-regular d-edge-colourable graph is oddly d-edge-connected. (Note that for a 3-regular graph, being oddly 3-edge-connected is the same as having no cut-edge, because if $X \subseteq V(G)$, then $|\delta(X)| = 1$ if and only if $|X|$ is odd and $|\delta(X)| < 3$.) The converse is false, even for $d = 3$ (the Petersen graph is a counterexample); but for planar graphs perhaps the converse is true. That is the content of the following conjecture [8], proposed by the third author in about 1973.

1.1 Conjecture. *If G is a d-regular planar graph, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

Some special cases of this conjecture have been proved.

- For $d = 3$ it is the four-colour theorem, and was proved by Appel and Haken [1, 2, 7];
- for $d = 4, 5$ it was proved by Guenin [5];
- for $d = 6$ it was proved by Dvorak, Kawarabayashi and Kral [3];
- for $d = 7$ it was proved by Kawarabayashi and the second author, and appears in the Master's thesis [4] of the latter. The methods of the present paper can also be applied to the $d = 7$ case, resulting in a proof somewhat simpler than the original, and this simplified proof for the $d = 7$ case will be presented in another, four-author paper [6].

Here we prove the next case, namely:

1.2 *Every 8-regular oddly 8-edge-connected planar graph is 8-edge-colourable.*

All these proofs (for $d > 3$), including ours, proceed by induction on d. Thus we need to assume the truth of the result for $d = 7$.

2 An unavoidable list of reducible configurations.

The graph we wish to edge-colour has parallel edges, but it is more convenient to work with the underlying simple graph. If H is d-regular and oddly d-edge-connected, then H has no loops, because for every vertex v, v has degree d, and yet $|\delta_H(v)| \geq d$. (We write $\delta(v)$ for $\delta(\{v\})$.) Thus to recover H from the underlying simple graph G say, we just need to know the number $m(e)$ of parallel edges of H that correspond to each edge e of G. Let us say a *d-target* is a pair (G, m) with the following properties (where for $F \subseteq E(G)$, $m(F)$ denotes $\sum_{e\in F} m(e)$):

- G is a simple graph drawn in the plane;

- $m(e) \geq 0$ is an integer for each edge e;
- $m(\delta(v)) = d$ for every vertex v; and
- $m(\delta(X)) \geq d$ for every odd subset $X \subseteq V(G)$.

In this language, 1.1 says that for every d-target (G, m), there is a list of d perfect matchings of G such that every edge e of G is in exactly $m(e)$ of them. (The elements of a list need not be distinct.) If there is such a list we call it a *d-edge-colouring*, and say that (G, m) is *d-edge-colourable*. For an edge $e \in E(G)$, we call $m(e)$ the *multiplicity* of e. If $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X. We need:

2.1 *Let (G, m) be a d-target, that is not d-edge-colourable, but such that every d-target with fewer vertices is d-edge-colourable. Then*

- $|V(G)| \geq 6$*;*
- *for every $X \subseteq V(G)$ with $|X|$ odd, if $|X|, |V(G) \setminus X| \neq 1$ then $m(\delta(X)) \geq d + 2$; and*
- *G is three-connected, and $m(e) \leq d - 2$ for every edge e.*

Proof. If $m(e) = 0$ for some edge e, we may delete e without affecting the problem; so we may assume that $m(e) > 0$ for every edge e. It is easy to check that G is connected and $|V(G)| \geq 6$ and we omit it. For the second assertion let $X \subseteq V(G)$ with $|X|$ odd and with $|X|, |V(G) \setminus X| \neq 1$. Thus $m(\delta(X)) \geq d$ since (G, m) is a d-target; suppose that $m(\delta(X)) = d$. There is a component of $G|X$ with an odd number of vertices, with vertex set X' say; and so $m(\delta(X')) \geq d$ since (G, m) is a d-target. But $\delta(X') \subseteq \delta(X)$, and $m(e) > 0$ for every edge e; and so $\delta(X') = \delta(X)$. Since G is connected it follows that $X' = X$, and so $G|X$ is connected. Similarly $G|Y$ is connected, where $Y = V(G) \setminus X$. Replace each edge e of G by $m(e)$ parallel edges, forming H; and contract all edges of $H|Y$, forming a d-regular oddly d-edge-connected planar graph H_1 with fewer vertices than H (because $|Y| > 1$). By hypothesis it follows that H_1 is d-edge-colourable. Similarly so is the graph obtained from H by contracting all edges of $H|X$. But these colourings can be combined to give a d-edge-colouring of H, a contradiction. This proves that $m(\delta(X)) > d$. Since $m(\delta(v)) = d$ for every vertex v, it follows that $m(\delta(X))$ has the same parity as $d|X|$, and so $m(\delta(X)) \geq d + 2$. This proves the second assertion.

For the third assertion, suppose that G is not three-connected. Since $|V(G)| > 3$, there is a partition (X, Y, Z) of $V(G)$ where $X, Y \neq \emptyset$ and $|Z| = 2$, such that there are no edges between X and Y. Let $Z = \{z_1, z_2\}$ say. Either both $|X|, |Y|$ are odd, or they are both even. If they are both odd, then since $\delta(X), \delta(Y)$ are disjoint subsets of $\delta(z_1) \cup \delta(z_2)$, and

$$m(\delta(X)), m(\delta(Y)) \geq d = m(\delta(z_1)), m(\delta(z_2)),$$

we have equality throughout, and in particular $m(\delta(X)), m(\delta(Y)) = d$. But then $|X| = |Y| = 1$ from the second assertion, contradicting that $|V(G)| \geq 6$. Now assume $|X|, |Y|$ are both even. Since $\delta(X \cup \{z_1\}), \delta(Y \cup \{z_2\})$ have the same union and intersection as $\delta(z_1), \delta(z_2)$, it follows that $m(\delta(X \cup \{z_1\})) = d$, contrary to the second assertion. Thus G is three-connected. Since $m(e) \geq 1$ for every edge e, and $m(\delta(v)) = d$ for every vertex v, it follows that $m(e) \leq d - 2$ for every edge e. This proves the third assertion, and hence proves 2.1. ∎

A *triangle* is a region of G incident with exactly three edges. If a triangle is incident with vertices u, v, w, for convenience we refer to it as uvw, and in the same way an edge with ends u, v is called uv. Two edges are *disjoint* if they are distinct and no vertex is an end of both of them, and otherwise they *meet*. Let r be a region of G, and let $e \in E(G)$ be incident with r; let r' be the other region incident with e. We say that e is *i-heavy* (for r), where $i \geq 2$, if either $m(e) \geq i$ or r' is a triangle uvw where $e = uv$ and

$$m(uv) + \min(m(uw), m(vw)) \geq i.$$

We say e is a *door* for r if $m(e) = 1$ and there is an edge f incident with r' and disjoint from e with $m(f) = 1$. We say that r is *big* if there are at least four doors for r, and *small* otherwise. A *square* is a region with length four.

Since G is drawn in the plane and is two-connected, every region r is bounded by some cycle which we denote by C_r. In what follows we will be studying cases in which certain configurations of regions are present in G. We will give a list of regions the closure of the union of which is a disc. For convenience, for an edge e in the boundary of this disc, we call the region outside the disc incident with e the "second region" for e; and we write $m^+(e) = m(e)$ if the second region is big, and $m^+(e) = m(e) + 1$ if the second region is small. This notation thus depends not just on (G, m) but on what regions we have specified, so it is imprecise, and when there is a danger of ambiguity we will specify it more clearly.

Let us say an 8-target (G, m) is *prime* if

- $m(e) > 0$ for every edge e;
- $|V(G)| \geq 6$;
- $m(\delta(X)) \geq 10$ for every $X \subseteq V(G)$ with $|X|$ odd and $|X|, |V(G) \setminus X| \neq 1$;
- G is three-connected, and $m(e) \leq 6$ for every edge e;

and in addition (G, m) contains none of of the following:

Conf(1): A triangle uvw where u, v both have degree three.

Conf(2): A triangle uvw, where u has degree three and its third neighbour x satisfies

$$m(ux) < m(uw) + m(vw).$$

Conf(3): Two triangles uvw, uwx with $m(uv) + m(uw) + m(vw) + m(ux) \geq 8$.

Conf(4): A square $uvwx$ where $m(uv) + m(vw) + m(ux) \geq 8$ and

$$(m(uv), m(vw), m(wx), m(ux)) \neq (4, 2, 1, 2).$$

Conf(5): Two triangles uvw, uwx where $m^+(uv) + m(uw) + m^+(wx) \geq 7$.

Conf(6): A square $uvwx$ where $m^+(uv) + m^+(wx) \geq 7$.

Conf(7): A triangle uvw with $m^+(uv) + m^+(uw) \geq 7$.

Conf(8): A triangle uvw, where $m(uv) = 3$, $m(uw) = 2$, $m(vw) = 2$, and the second region for one of uv, uw, vw has no door disjoint from uw.

Conf(9): A triangle uvw with $m(uv), m(uw), m(vw) = 2$, such that u has degree at least four, and the second regions for uv, uw both have at most one door, and no door that is disjoint from uvw.

Conf(10): A square $uvwx$ and a triangle wxy, where $m(uv) = m(wx) = m(xy) = 2$, and $m(vw) = 4$.

Conf(11): A square $uvwx$ and a triangle wxy, where $m(uv) \geq 3$, $m(wy) \geq 3$, $m(wx) = 1$, $m(ux) \leq 3$, and $m^+(xy) \geq 3$.

Conf(12): A square $uvwx$ and a triangle wxy, where $m^+(uv) \geq 2$, $m(vw) \geq 2$, $m(wx) = m(wy) = 2$, $m(ux) \leq 3$, and $m^+(xy) \geq 3$.

Conf(13): A region with length five, with edges $e_1, \dots, e_5$ in order, where $m(e_1) \geq \max(m(e_2), m(e_5))$, $m(e_1) + m(e_2) + m(e_3) \geq 8$ and $m^+(e_1) + m^+(e_4) \geq 7$.

Conf(14): A region r and an edge e of C_r, such that $m^+(e) \geq 6$ and at most six edges of C_r disjoint from e are doors for r.

Conf(15): A region r with length at least four, and an edge e of C_r, such that $m^+(e) \geq 4$ and every edge of C_r disjoint from e is 3-heavy.

Conf(16): A region r and an edge uv of C_r, and a triangle uvw, such that $m(uv) + m^+(uw) \geq 4$, and every edge of C_r not incident with u is 3-heavy; moreover, if tu denotes the second edge of C_r incident with u, then either $\max(m(vw), m(tu)) \leq m(uw)$, or r is a triangle and $m(vw) = m(uw) + 1$ and $m(tu) \leq m(tv)$.

Conf(17): A region r with length at least five, and an edge e of C_r, such that $m^+(e) \geq 5$, every edge f of C_r disjoint from e satisfies $m^+(f) \geq 2$, and at most one of them is not 3-heavy.

Conf(18): A region r with length at least four and an edge uv of C_r, and a triangle uvw, such that $m^+(uw) + m(uv) \geq 5$, and $m(vw) \leq m(uw)$, and the second edge of C_r incident with u has multiplicity at most $m(uw)$, and either

- $m(uv) = 3$ and uv is 5-heavy, and every edge f of C_r disjoint from uv satisfies $m^+(f) \geq 2$, and at most one of them is not 3-heavy, or
- $m^+(f) \geq 2$ for every edge f of C_r not incident with u, and at most one such edge is not 3-heavy.

Conf(19): A region r with length at least five and an edge e of C_r, such that $m^+(e) \geq 5$, every edge of C_r disjoint from e is 2-heavy, and at most two of them are not 3-heavy.

We will prove these restrictions are too much, that in fact no 8-target is prime (theorem 3.1). To deduce 1.2, we will show that if there is a counterexample, then some counterexample is prime; but for this purpose, just choosing a counterexample with the minimum number of vertices is not enough, and we need a more delicate minimization. If (G, m) is a d-target, its *score sequence* is the $(d+1)$-tuple $(n_0, n_1, \dots, n_d)$ where n_i is the number of edges e of G with $m(e) = i$. If (G, m) and

(G', m') are d-targets, with score sequences $(n_0, \ldots, n_d)$ and $(n'_0, \ldots, n'_d)$ respectively, we say that (G', m') is *smaller* than (G, m) if either

- $|V(G')| < |V(G)|$, or
- $|V(G')| = |V(G)|$ and there exists i with $1 \leq i \leq d$ such that $n'_i > n_i$, and $n'_j = n_j$ for all j with $i < j \leq d$, or
- $|V(G')| = |V(G)|$, and $n'_j = n_j$ for all j with $0 < j \leq d$, and $n'_0 < n_0$.

(The anomalous treatment of n_0 is just a device to allow d-targets to have edges with $m(e) = 0$, while minimum d-counterexamples have none.) If some d-target is not d-edge-colourable, then we can choose a d-target (G, m) with the following properties:

- (G, m) is not d-edge-colourable
- every smaller d-target is d-edge-colourable.

Let us call such a pair (G, m) a *minimum d-counterexample.* To prove 1.2, we prove two things:

- No 8-target is prime (theorem 3.1), and
- Every minimum 8-counterexample is prime (theorem 4.1).

It will follow that there is no minimum 8-counterexample, and so the theorem is true.

3 Discharging and unavoidability

In this section we prove the following, with a discharging argument.

3.1 *No 8-target is prime.*

The proof is broken into several steps, through this section. Let (G, m) be a 8-target, where G is three-connected. For every region r, we define

$$\alpha(r) = 8 - 4|E(C_r)| + \sum_{e \in E(C_r)} m(e).$$

We observe first:

3.2 *The sum of $\alpha(r)$ over all regions r is positive.*

Proof. Since (G, m) is a 8-target, $m(\delta(v)) = 8$ for each vertex v, and, summing over all v, we deduce that $2m(E(G)) = 8|V(G)|$. By Euler's formula, the number R of regions of G satisfies $|V(G)| - |E(G)| + R = 2$, and so $2m(E(G)) - 8|E(G)| + 8R = 16$. But $2m(E(G))$ is the sum over all regions r, of $\sum_{e \in E(C_r)} m(e)$, and $8R - 8|E(G)|$ is the sum over all regions r of $8 - 4|E(C_r)|$. It follows that the sum of $\alpha(r)$ over all regions r equals 16. This proves 3.2. ∎

Think of $\alpha(r)$ as an initial assignment of charge to each region r. Now we move some small amount of charge between neighbouring regions. Normally we pass one unit of charge from every small region to every big region with which it shares an edge; except that in some exceptional circumstances, sending one unit is too much, and we only send 1/2 or 0. More precisely, for every edge e of G, define $\beta_e(s)$ for each region s as follows. Let r, r' be the two regions incident with e.

- If $s \neq r, r'$ then $\beta_e(s) = 0$.
- If r, r' are both big or both small then $\beta_e(r), \beta_e(r') = 0$.

Henceforth we assume that r is big and r' is small; let f, f' be the edges of $C_r \setminus e$ that share an end with e.

1: If e is a door for r (and hence $m(e) = 1$) then $\beta_e(r) = \beta_e(r') = 0$.

2: If $m(e) = 2$ and $m^+(f) = m^+(f') = 6$ then $\beta_e(r) = \beta_e(r') = 0$.

3: If $m(e) = 2$ and $m^+(f) = 6$ and $m^+(f') = 5$ or vice versa then $\beta_e(r) = -\beta_e(r') = 1/2$.

4: If $m(e) = 3$ and $m^+(f) = m^+(f') = 5$ then $\beta_e(r) = \beta_e(r') = 0$.

5: If $m(e) = 3$ and exactly one of $m^+(f), m^+(f') = 5$, then $\beta_e(r) = -\beta_e(r') = 1/2$.

6: Otherwise $\beta_e(r) = -\beta_e(r') = 1$.

(Think of β_e as passing some amount of charge between the two regions incident with e.) For each region r, define $\beta(r)$ to be the sum of $\beta_e(r)$ over all edges e. We see that the sum of $\beta(r)$ over all regions r is zero.

The effect of β is passing charge from small regions to big regions with which they share an edge. We need another "discharging" function, that passes charge from triangles to small regions with which they share an edge. If r is a triangle, incident with edges e, f, g, we define its *multiplicity* $m(r) = m(e) + m(f) + m(g)$. A region r is *tough* if r is a triangle, its multiplicity is at least five, and if $r = uvw$ where $m(uv) = 1$ and $m(uw) = m(vw) = 2$, then $m^+(uw) + m^+(vw) \geq 5$. For every edge e of G, define $\gamma_e(s)$ for each region s as follows. Let r, r' be the two regions incident with e.

- If $s \neq r, r'$ then $\gamma_e(s) = 0$.
- If one of r, r' is big, or neither is tough, or they both are tough, then $\gamma_e(r) = \gamma_e(r') = 0$.

Henceforth we assume that r' is tough, and r is small and not tough. Let e, e_1, e_2 be the edges incident with r', and let r_1, r_2 be the regions different from r' incident with e_1, e_2 respectively.

1: If $m(e) = 1$ and $m(e_1), m(e_2) \geq 2$, and $m^+(e_1) + m^+(e_2) \geq 6$ then $\gamma_e(r) = -\gamma_e(r') = 1$.

2: If $m(e) = 1$ and $m^+(e_1) \geq 4$ and $m(e_2) = 1$ and r_2 is small, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

3: If $m(e) = 1$ and $m(e_1) = 3$ and $m(e_2) = 1$ and r_2 is small, and the edge f of $C_r \setminus e$ that shares an end with e, e_1 satisfies $m(f) = 4$, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

4: If $m(e) = 2$ and $m(e_1), m(e_2) \geq 2$ and $m^+(e_1) + m^+(e_2) \geq 5$, and either

– r has more than one door, or

– some door for r is disjoint from e, or

– some edge f of C_r consecutive with e has multiplicity four, and r_1, r_2 are both small,

then $\gamma_e(r) = -\gamma_e(r') = 1$.

5: If $m(e) = 2$ and $m(e_1), m(e_2) = 2$ and some end of e has degree three, incident with e_1 say, and r_1 is small and r_2 is big, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

6: If $m(e) = 3$ and $m(e_1), m(e_2) = 2$ then $\gamma_e(r) = -\gamma_e(r') = 1$.

7: Otherwise $\gamma_e(r) = \gamma_e(r') = 0$.

We observe that, immediately from the rules, we have

3.3 *Let e be incident with regions r, r'. Then $\beta_e(r)$ is non-zero only if exactly one of r, r' is big; and $\gamma_e(r)$ is non-zero only if exactly one of r, r' is tough and neither is big. Thus in all cases, at most one of $\beta_e(r), \gamma_e(r)$ is non-zero. Moreover $|\beta_e(r) + \gamma_e(r)| \leq 1$.*

For each region r, define $\gamma(r)$ to be the sum of $\gamma_e(r)$ over all edges e. Again, the sum of $\gamma(r)$ over all regions r is zero. It follows that the sum over all regions r of $\alpha(r) + \beta(r) + \gamma(r)$ is positive, by 3.2, and so there is a region r for which $\alpha(r) + \beta(r) + \gamma(r) > 0$. By examining the possibilities for such a region r we will deduce that (G, m) is not prime. There now begins a long case analysis, and to save writing we just say "by Conf(7)" instead of "since (G, m) does not contain Conf(7)", and so on.

3.4 *If r is a big region and $\alpha(r) + \beta(r) + \gamma(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime. Let $C = C_r$. Since r is big it follows that $\gamma(r) = 0$, and so $\alpha(r) + \beta(r) > 0$; that is,

$$\sum_{e \in E(C)} (4 - m(e) - \beta_e(r)) < 8.$$

For $e \in E(C)$, define $\phi(e) = m(e) + \beta_e(r)$, and let us say e is *major* if $\phi(e) > 4$. If e is major, then since $\beta_e(r) \leq 1$, it follows that $m(e) \geq 4$; and so $\beta_e(r)$ is an integer, from the β-rules, and therefore $\phi(e) \geq 5$. Moreover, no two major edges are consecutive, since G has minimum degree at least three.

Let D be the set of doors for C. Let

- $\xi = 1$ if there are consecutive edges e, f in C such that $\phi(e) > 5$ and f is a door for r
- $\xi = 2$ if there is no such pair e, f.

(1) *Let e, f, g be the edges of a path of C, in order, where e, g are major. Then*

$$(4 - \phi(e)) + 2(4 - \phi(f)) + (4 - \phi(g)) \geq 2\xi|\{f\} \cap D|.$$

Let r_1, r_2, r_3 be the regions different from r incident with e, f, g respectively. Now $m(e) \leq 6$ since (G, m) is prime, and if $m(e) = 6$ then r_1 is big, by Conf(14), and so $\beta_e(r) = 0$; and so in any case, $\phi(e) \leq 6$. Similarly $\phi(g) \leq 6$. Also, $\phi(e), \phi(g) \geq 5$ since e, g are major. Thus $\phi(e) + \phi(g) \in \{10, 11, 12\}$.

Suppose that $\phi(e) + \phi(g) = 12$. We must show that $\phi(f) \leq 2 - \xi|\{f\} \cap D|$. Now $m(e) \geq 5$, and so $m(f) \leq 2$, since G is three-connected. If $m(f) = 2$ then $f \notin D$, and $\beta_f(r) = 0$ from the β-rules; and so $\phi(f) \leq 2 - \xi|\{f\} \cap D|$. If $m(f) = 1$, then $\beta_f(r) \leq 1$, so we may assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1 \leq 2 - \xi|\{f\} \cap D|$.

Next suppose that $\phi(e) + \phi(g) = 11$. We must show that $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$. Again one of $\phi(e), \phi(g) \geq 6$, say $\phi(e) = 6$; and so $m^+(e) \geq 6$. In particular $m(e) \geq 5$, and so $m(f) \leq 2$. Since $\phi(g) \geq 5$ we have $m^+(g) \geq 5$, and so if $m(f) = 2$, then $\beta_f(r) \leq 1/2$ from the β-rules; and since $f \notin D$ we have $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$. If $m(f) = 1$, then $\phi(f) \leq 2$, and so we may assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1$, and again $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$.

Finally, suppose that $\phi(e) + \phi(g) = 10$. We must show that $\phi(f) \leq 3 - \xi|\{f\} \cap D|$. Suppose that $m(f) \geq 3$. Since $m^+(e), m^+(g) \geq 5$ (because e, g are major), it follows that $m(f) = 3$, and $m(e) = m(g) = 4$ because G is three-connected; but then $\beta_f(r) = 0$ from the β-rules, and since $f \notin D$ we have $\phi(f) \leq 3 - \xi|\{f\} \cap D|$. Next suppose that $m(f) = 2$. Then $\phi(f) \leq 3 = 3 - \xi|\{f\} \cap D|$ as required. Lastly if $m(f) = 1$, then $\phi(f) \leq 2$, so we may assume that $f \in D$; but then $\xi \leq 2$ and $\phi(f) = 1 \leq 3 - \xi|\{f\} \cap D|$. This proves (1).

(2) *Let e, f be consecutive edges of C, where e is major. Then*

$$(4 - \phi(e)) + 2(4 - \phi(f)) \geq 2\xi|\{f\} \cap D|.$$

We have $\phi(e) \in \{5, 6\}$. Suppose that $\phi(e) = 6$. We must show that $\phi(f) \leq 3 - \xi|\{f\} \cap D|$; but $m(f) \leq 2$ since $m(e) \geq 5$, and so $\phi(f) \leq 3$. We may therefore assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1 \leq 3 - \xi|\{f\} \cap D|$. Next, suppose that $\phi(e) = 5$; then we must show that $\phi(f) \leq 7/2 - \xi|\{f\} \cap D|$. Since $m(e) \geq 4$, it follows that $m(f) \leq 3$. If $m(f) = 3$ then $m^+(e) = 5$ and so $\beta_f(r) \leq 1/2$, from the β-rules; but then $\phi(f) \leq 7/2 - \xi|\{f\} \cap D|$. If $m(f) \leq 2$, then $\phi(f) \leq 3$, so we may assume that $f \in D$; but $\xi \leq 2$, and so $\phi(f) = 1 \leq 7/2 - \xi|\{f\} \cap D|$. This proves (2).

For $i = 0, 1, 2$, let E_i be the set of edges $f \in E(C)$ such that f is not major, and f meets exactly i major edges in C. Let D be the set of doors for C. By (1), for each $f \in E_2$ we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) + \frac{1}{2}(4 - \phi(g)) \geq \xi|\{f\} \cap D|$$

where e, g are the major edges meeting f. By (2), for each $f \in E_1$ we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) \geq \xi|\{f\} \cap D|$$

where e is the major edge consecutive with f. Finally, for each $f \in E_0$ we have

$$4 - \phi(f) \geq \xi|\{f\} \cap D|$$

since $\phi(f) \leq 4$, and $\phi(f) = 1$ if $f \in D$. Summing these inequalities over all $f \in E_0 \cup E_1 \cup E_2$, we deduce that $\sum_{e \in E(C)}(4 - \phi(e)) \geq \xi|D|$. Consequently

$$8 > \sum_{e \in E(C)} (4 - m(e) - \beta_e(r)) \geq \xi|D|.$$

But $|D| \geq 4$ since r is big, and so $\xi = 1$ and $|D| \leq 7$, a contradiction by Conf(14). This proves 3.4. ∎

3.5 *If r is a triangle that is not tough, and $\alpha(r)+\beta(r)+\gamma(r)>0$, then (G,m) is not prime.*

Proof. Suppose (G,m) is prime, and let $r=uvw$. Suppose first that r has multiplicity five; and hence, since it is not tough, we may assume that $m(uv)=1$ and $m(uw)=m(vw)=2$, and the second regions for uw, vw are both big. Thus from the β-rules, $\beta_{uw}(r), \beta_{vw}(r)=-1$, and since $\gamma_{uw}(r), \gamma_{vw}(r)=0$ from the γ-rules and $\beta_{uv}(r)+\gamma_{uv}(r)\leq 1$ from 3.3, we deduce by adding that $\beta(r)+\gamma(r)\leq -1$. But

$$\alpha(r)=-4+m(uv)+m(vw)+m(uw)=1,$$

contradicting that $\alpha(r)+\beta(r)+\gamma(r)>0$. Thus r has multiplicity at most four.

Since $\alpha(r)=-4+m(uv)+m(vw)+m(uw)\leq 0$, and $\beta(r)\leq 0$, it follows that $\gamma(r)>0$.

(1) *$m(e)=1$ for every edge e incident with r such that $\gamma_e(r)>0$.*

For suppose that $m(e)>1$ and $\gamma_e(r)>0$, where $e=uv$. Since r has multiplicity at most four it follows that $m(e)=2$. Since $\gamma_e(r)>0$, there is a vertex $x\neq w$ such that uvx is a triangle, and $m(ux), m(vx)\geq 2$, and one of $m^+(ux), m^+(vx)$ is at least three, say $m^+(ux)\geq 3$; and r has two doors. By Conf(5), $m^+(vw)=1$, and so $\beta_{vw}(r)=-1$ and $\beta_{uw}(r)\leq 0$, and hence $\beta(r)\leq -1$; yet $\gamma(r)\leq 1$, contradicting that $\alpha(r)+\beta(r)+\gamma(r)>0$. This proves (1).

(2) *There is no edge e incident with r and with a big region such that $m(e)=1$.*

Let r be incident with edges e,f,g, and suppose that $m(e)=1$ and e is incident with a big region. Thus $\beta(r)\leq -1$, and so $\gamma(r)>1$; and consequently $\gamma_f(r), \gamma_g(r)>0$, and therefore $m(f)=m(g)=1$ from (1). But then $\alpha(r)=-1$, and yet $\gamma(r)\leq 2$, contradicting that $\alpha(r)+\beta(r)+\gamma(r)>0$. This proves (2).

Choose e with $\gamma_e(r)>0$, say $e=uv$. Thus $m(uv)=1$, and there is a tough triangle $r'=uvx$ say. By Conf(3), r' has multiplicity at most six.

(3) *We may assume that $m^+(ux)\leq 3$ and $m^+(vx)\leq 3$.*

For suppose that $m^+(ux)\geq 4$. By (2), $m^+(vw)\geq 2$, contrary to Conf(5). This proves (3).

Now $\gamma_{uv}(r)>0$, and from (1), (3), it follows that $\gamma_{uv}(r)$ is determined by the first γ-rule. In particular, $m^+(ux)=3$, and $m^+(vx)=3$. Suppose that vw is 3-heavy. By Conf(16) it follows that $m(vx)>m(ux)$, and so $m(vx)=3$ and $m(ux)=2$; but then by Conf(3), $m(uw)=m(vw)=1$, contrary to Conf(16). Thus vw and similarly uw are not 3-heavy, and so by the same argument $\gamma_{uw}(r)=0$ and $\gamma_{vw}(r)=0$; and so $\gamma(r)=1$. Consequently $\alpha(r)>-1$, and so we may assume that $m(uw)=2$. Let r_1 be the second region for uw. Now $m(ux)+m(uv)+m(uw)\leq 6$, and so there is an edge f incident with r_1 and u different from uw, ux. Moreover, $m(f)\leq 3$, since $m(ux)+m(uv)+m(uw)\geq 5$; and so if r_1 is big then $\beta_{uw}(r)=-1$, a contradiction. Thus r_1 is small, contrary to Conf(5). This proves 3.5. ∎

3.6 *If r is a tough triangle with $\alpha(r)+\beta(r)+\gamma(r)>0$, then (G,m) is not prime.*

Proof. Suppose (G,m) is prime, and let $r = uvw$. Now $\alpha(r) = m(uv) + m(vw) + m(uw) - 4$, so

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4.$$

Let r_1, r_2, r_3 be the regions different from r incident with uv, vw, uw respectively. It follows that $\beta_e(r), \gamma_e(r) \leq 0$ for every edge e of r.

(1) *If r_1 is big then $\beta_{uv}(r) = -1$.*

For let us examine the β-rules. Certainly uv is not a door for r_1, since r is a triangle; so the first rule does not apply. Let f, f' be the edges incident with r_1 different from uv that are incident with u, v respectively. If the second β-rule applies then $m(uv) = 2$ and $m(f), m(f') \geq 5$, which implies that $m(uw), m(vw) = 1$, contradicting that uvw has multiplicity at least five. If the third rule applies, then $m(uv) = 2$ and $m^+(f) = 6$ and $m^+(f') = 5$ say; but then $m(uw) = 1$ and $m(vw) = 2$, contrary to Conf(1). The fourth rule does not apply, by Conf(1). Thus we assume that the fifth rule applies. Let $m(uv) = 3$, $m^+(f) = 5$, and $m^+(f') < 5$. Hence $m(f) = 4$, and so u has degree three, and $m(vw) = 1$ by Conf(2), and r_3 is small, and $\beta_{uv}(r) = -1/2$. Since

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4$$

it follows that

$$\beta_{uw}(r) + \beta_{vw}(r) + \gamma_{uw}(r) + \gamma_{vw}(r) \geq 0,$$

and since all the terms on the left are non-positive it follows that they are all zero. Now r_2 is not big since $\beta_{vw}(r) = 0$, and r_3 is not a triangle by Conf(2), so the third γ-rule applies to uw, a contradiction since $\gamma_{uw}(r) = 0$. This proves (1).

Let $X = \{u, v, w\}$. Since (G,m) is prime, it follows that $|V(G) \setminus X| \geq 3$, and $m(\delta(X)) \geq 10$. But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so $10 \leq 8+8+8-2m(uv)-2m(uw)-2m(vw)$, that is, r has multiplicity at most seven. Suppose first that r has multiplicity seven. By Conf(3), none of r_1, r_2, r_3 is a triangle. Now $\beta(r) + \gamma(r) > -3$. Consequently we may assume that $\beta_{uv}(r) + \gamma_{uv}(r) > -1$, and hence r_1 is small by (1). By Conf(7), $m(uv) + m(uw) < 6$ and hence $m(vw) \geq 2$; and similarly $m(uw) \geq 2$. Now $\gamma_{uv}(r) > -1$, and so the first, fourth and sixth γ-rules do not apply to uv. Since the first γ-rule does not apply, $m(uv) > 1$. Since the sixth γ-rule does not apply, one of $m(uw), m(vw) > 2$, say $m(uw) \geq 3$, and so $m(uv) = 2$, $m(uw) = 3$ and $m(vw) = 2$. Since the fourth γ-rule does not apply, r_1 has no door disjoint from uv, contrary to Conf(8).

Next, suppose that r has multiplicity six. Thus $\beta(r) + \gamma(r) > -2$, and so by (1), at most one of r_1, r_2, r_3 is big. Suppose that $m(uv) = 4$; then $m(vw), m(uw) = 1$. Since at most one of r_1, r_2, r_3 is big, it follows from Conf(7) that r_1 is big, and hence r_2, r_3 are small. By Conf(3), r_2, r_3 are not tough. By the second γ-rule, $\gamma_{vw}(r) = \gamma_{uw}(r) = -1/2$, and since $\beta_{uv}(r) = -1$ by (1), this contradicts $\beta(r) + \gamma(r) > -2$. Thus $m(uv) \leq 3$. Suppose next that $m(uv) = 3$; then from the symmetry we may assume that $m(uw) = 2$ and $m(vw) = 1$. Since one of r_1, r_3 is small, and r_2 is not tough by Conf(3), the first γ-rule implies that $\beta_{vw}(r) + \gamma_{vw}(r) \leq -1$. Since $\beta(r) + \gamma(r) > -2$, it follows from (1) that neither of r_1, r_3 is big, contrary to Conf(7). Thus $m(uv) \leq 2$, and similarly $m(uw), m(vw) \leq 2$, and so

$m(uv), m(uw), m(vw) = 2$. Since $\beta(r)+\gamma(r) > -2$, it follows that $\beta_e(r)+\gamma_e(r) \leq -1$ for at most one edge e incident with r; and so we may assume that $\beta_{uv}(r)+\gamma_{uv}(r) > -1$ and $\beta_{uw}(r)+\gamma_{uw}(r) > -1$. By (1), r_1, r_3 are both small. By Conf(3), r_1, r_3 are not tough, and since the fourth γ-rule does not apply, it follows that r_1 has at most one door, and no door disjoint from uv, and r_3 has at most one door, and no door disjoint from uw, and u has degree at least four, contrary to Conf(9).

Finally, suppose that r has multiplicity five. Now $\beta(r)+\gamma(r) > -1$, and hence $\beta_e(r)+\gamma_e(r) > -1$ for every edge e incident with r; and so by (1) r_1, r_2, r_3 are all small. Suppose that $m(uv) = 3$, and hence $m(uw), m(vw) = 1$. If neither of r_2, r_3 is tough, then by the second γ-rule, $\gamma_{uw}(r) = \gamma_{vw}(r) = -1/2$, a contradiction. Thus we may assume that r_3 is a tough triangle uwx. By Conf(5), $m(wx) = 1$, and so $m(ux) \geq 3$ since r_3 is tough, contrary to Conf(3). Thus we may assume that $m(uv) \leq 2$; and so from the symmetry we may assume that $m(uv) = m(uw) = 2$ and $m(vw) = 1$. The first γ-rule does not apply to vw, and so r_2 is a tough triangle vwx. By Conf(3), $m(vx), m(wx) \leq 2$, and so $m(vx), m(wx) = 2$. Since r_2 is tough, one of vx, wx is incident with a small region different from uvx, contrary to Conf(5). This proves 3.6. ∎

3.7 *If r is a small region with length at least four and with $\alpha(r) + \beta(r) + \gamma(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime. Let $C = C_r$. Note that for each $e \in E(C)$, $-1 \leq \beta_e(r) \leq 0$ and $0 \leq \gamma_e(r) \leq 1$ Since $\alpha(r) = 8 - 4|E(C)| + \sum_{e \in E(C)} m(e)$, it follows that

$$8 - 4|E(C)| + \sum_{e \in E(C)} m(e) + \sum_{e \in E(C)} (\beta_e(r) + \gamma_e(r)) > 0,$$

that is,

$$\sum_{e \in E(C)} (m(e) + \beta_e(r) + \gamma_e(r) - 4) > -8.$$

For each $e \in E(C)$, let

$$\phi(e) = m(e) + \beta_e(r) + \gamma_e(r).$$

It follows that $|\phi(e) - m(e)| \leq 1$ for each e by 3.3. For each integer i, let E_i be the set of edges of C such that $\phi(e) \in \{i, i - \frac{1}{2}\}$.

(1) *For every $e \in E(C)$, $\phi(e)$ is one of $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$, and hence $E(C)$ is the union of E_0, E_1, E_2, E_3, E_4.*

For let $e \in E(C)$. Since $m(e) \geq 1$ and $\beta_e(r) \geq -1$ it follows that $\phi(e) \geq 0$. Next we show that $\phi(e) \leq 4$. Now $m(e) < 6$ by Conf(14). Suppose that $m(e) = 5$. Then the second region incident with e is big, by Conf(14); and hence $\beta_e(r) = -1$ from the β-rules, and $\gamma_e(r) = 0$ and so $\phi(e) \leq 4$. Now suppose that $m(e) = 4$. Then by the γ-rules, $\gamma_e(r) = 0$, and so $\phi(e) \leq 4$. Finally, if $m(e) \leq 3$ then $\phi(e) \leq 4$ since $\gamma_e(r) \leq 1$. Thus $\phi(e) \leq 4$ in all cases. Finally, suppose that $\phi(e) = 7/2$, and hence $m(e) = 3$ or 4. If $m(e) = 3$ then $\gamma_e(r) = 1/2$, contrary to the γ-rules; while if $m(e) = 4$ then $\beta_e(r) = -1/2$, contrary to the β-rules. This proves (1).

(2) *Let $e \in E(C)$; then $e \in E_4$ if and only if either $m^+(e) \geq 5$, or $m(e) = 3$ and e is 5-heavy.*

Moreover, no two edges in E_4 are consecutive in C.

The first assertion is immediate from the β- and γ-rules. For the second, suppose that $e, f \in E_4$ share an end v. Since v has degree at least three, it follows that $m(e) + m(f) \leq 7$ and so we may assume that $m(e) = 3$. Let e have ends u, v; then from the first assertion there is a triangle uvw where $m(uw), m(vw) = 2$. Hence $m(f) = 3$, and so there is similarly a triangle containing f, with third vertex x. Consequently $w = x$; but this contradicts Conf(3) and hence proves (2).

(3) *If $e \in E_4$, and $f \in E(C)$ is disjoint from e, and every edge in $E(C) \setminus \{f\}$ disjoint from e is 3-heavy, and there is no edge of C with multiplicity one disjoint from f, then $f \in E_0$.*

For by Conf(6) if $|E(C)| = 4$ and $m^+(e) \geq 5$, or by Conf(17) or Conf(18) otherwise, it follows that $m^+(f) = 1$. Since there is no edge of C with multiplicity one disjoint from f, it follows that $\beta_f(r) = -1$ from the β-rules, and so $f \in E_0$. This proves (3).

For $0 \leq i \leq 4$, let $n_i = |E_i|$.

(4) *If $e \in E(C)$ satisfies $m(e) = 2$, and $n_4 = 0$, and r has at most one door, and no door disjoint from e, then $\phi(e) \leq 2$.*

For if not, then $\gamma_e(r) > 0$, and so from the γ-rules, there is a triangle uvw with $e = uv$, and some edge f of C consecutive with e satisfies $m^+(f) = 5$; but then $f \in E_4$, contradicting that $n_4 = 0$. This proves (4).

(5) *If u, v, w are consecutive vertices in C, and $uv \in E_4$ and $m(uv) = 3$, then $\phi(vw) \leq 2$.*

For since $uv \in E_4$, by (2) there is a triangle uvx with $m(ux) = m(vx) = 2$. From Conf(2) it follows that $m(vw) \leq 2$; and since w is not adjacent to x by Conf(3), and hence vw is not 4-heavy, the γ-rules imply that $\phi(vw) \leq 2$. This proves (5).

Let C have vertices $v_1, \ldots, v_k$ in order, and let v_{k+1} mean v_1. For $1 \leq i \leq k$ let e_i be the edge $v_i v_{i+1}$, and let r_i be the region incident with e_i different from r.

Since

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8,$$

we have $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$, that is,

$$3n_0 + 2n_1 + n_2 + k - n_4 \leq 7,$$

since $n_0 + n_1 + n_2 + n_3 + n_4 = k$. But by (2), $n_4 \leq k/2$ and so

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7.$$

Since $k \geq 4$ it follows that $3n_0 + 2n_1 + n_2 \leq 5$, and hence $n_0 + n_1 \leq 2$.

Case 1: $n_0 + n_1 = 2$.

Since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, we have $n_4 \geq n_0 + n_2 + k - 3$. Thus $n_4 > 0$. If $k = 4$, let $e \in E_4$; then by (3) the edge f of C disjoint from e belongs to E_0, and so by (2), $n_4 = 1$; but this contradicts $n_0 + n_2 + k - 3 \leq n_4$.

Thus $k \geq 5$. Since

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7,$$

and $2n_0 + 2n_1 = 4$ and $k/2 \geq 5/2$, it follows that $n_0 = n_2 = 0$ and $n_1 = 2$ and $k \leq 6$.

Suppose that $k = 6$; then $n_4 = 3$ since $n_4 \geq n_0 + n_2 + k - 3$, so we may assume that $e_1, e_3, e_5 \in E_4$. By Conf(17) and Conf(18), it follows that $m^+(e_4) = 1$, and hence $e_4 \in E_0 \cup E_1$, and similarly $e_6, e_2 \in E_0 \cup E_1$, a contradiction since $n_0 + n_1 = 2$. Thus $k = 5$, and so $n_4 \geq 2$, and by (2) $n_4 = 2$ and we may assume that $e_1, e_3 \in E_4$. By Conf(17) and Conf(18), $m^+(e_4) = 1$, and similarly $m^+(e_5) = 1$. Since $n_1 = 2$, and $n_0, n_2 = 0$, it follows that $m(e_2) > 1$. But then $e_4 \in E_0$ by (3), contradicting that $n_0 = 0$.

Case 2: $k = 4$ and $n_0 + n_1 = 1$ and $n_4 > 0$.

Let $e_4 \in E_4$; by (3), $e_2 \in E_0$ and so $m(e_2) = 1$. By (2) and Conf(2) and Conf(4), it follows that $m(e_1), m(e_3) \leq 2$. Now e_2 is the only edge of C that is not 2-heavy, since $n_0 + n_1 = 1$, and in particular r has at most one door. Since $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$ and $n_0 = 1$, it follows that $n_2 \leq 1$, so we may assume that $e_1 \notin E_2$. Thus $\phi(e_1) > 2$, and hence $m(e_1) = 2$. By (2) and (5), $m^+(e_4) \geq 5$, so by Conf(4), $m(e_4) = 4$. Since $\phi(e_1) > 2$, it follows from the γ-rules that r_1 is a triangle v_1v_2w say, where $m(v_1w), m(v_2w) \geq 2$. Consequently $m(v_1w) = 2$. Since $e_3 \notin E_1$, it follows that $m^+(e_3) \geq 2$; so $m(v_2w) = m^+(v_2w) = 2$ by Conf(18) (taking v_2, v_1, w to be the vertices called u, v, w in Conf(18) respectively). From Conf(10) it follows that $m(e_3) = 1$. From the γ-rules it follows that $\phi(e_1) = 5/2$. Since $\sum_{e \in E(C)} \phi(e) > 8$ and $\phi(e_2) + \phi(e_4) \leq 4$, it follows that $\phi(e_3) \geq 2$. Since $m(e_3) = 1$, the γ-rules imply that e_3 is 3-heavy, contrary to Conf(16) (taking v_2, v_1, w to be the vertices called u, v, w in Conf(16) respectively).

Case 3: $k = 4$ and $n_0 + n_1 = 1$ and $n_4 = 0$.

Let $e_4 \in E_0 \cup E_1$, and so $m(e_4) \leq 2$. Since every edge of C that is not 2-heavy belongs to $E_0 \cup E_1$, it follows that e_1, e_2, e_3 are 2-heavy. Since $n_4 = 0$, it follows that $m^+(e_i) \leq 4$ for $i = 1, 2, 3, 4$.

Suppose that $\phi(e_1) \geq 3$, and hence $\phi(e_1) = 3$ by (1) since $n_4 = 0$. By (4) it follows that $m(e_1) \geq 3$. If $m^+(e_1) = 3$, then from the β-rules, the edge xv_2 of r_1 incident with v_2 and different from e_1 has multiplicity four and hence $m(e_2) = 1$; and since x, v_3 are non-adjacent by Conf(2), this contradicts that e_2 is 2-heavy. Thus $m^+(e_1) \geq 4$. By Conf(6), $m^+(e_3) \leq 2$, and so $\phi(e_3) \leq 2$ by (4). Since $\phi(e_2) \leq 3$, and $\phi(e_4) \leq 1$, and $\sum_{e \in E(C)} \phi(e) > 8$, it follows that $\phi(e_2) \geq 5/2$ (and so e_2 is 3-heavy), and $\phi(e_3) \geq 3/2$, and $\phi(e_4) \geq 1/2$ (and so $m^+(e_4) \geq 2$). By Conf(2), it is not the case that $m(e_3) = 2$ and the edge of r_3 consecutive with e_3 and incident with v_3 has multiplicity four; and so, since $\phi(e_3) \geq 3/2$, the β-rules imply that $m(e_3) = 1$ and r_3 is a triangle v_3v_4y say. Now by Conf(15), not both $m(v_3y), m(v_4y) \geq 2$; and $m(e_2) \leq 3$ by Conf(4), so by Conf(18), $m^+(v_3y), m^+(v_4y) \leq 3$. But then the γ-rules imply that $\phi(e_3) \leq 1$, a contradiction. This proves that $\phi(e_1) \leq 5/2$; and similarly $\phi(e_3) \leq 5/2$.

Since $\sum_{e \in E(C)} \phi(e) > 8$, and $\phi(e_2) \leq 3$ (because $n_4 = 0$) it follows that $\phi(e_1) + \phi(e_3) \geq 9/2$, and $\phi(e_4) \geq 1/2$; and from the symmetry we may assume that $\phi(e_1) = 5/2$ and $\phi(e_3) \geq 2$. The β-

and γ-rules imply that $m(e_1) = 3$ (since $m^+(e_2) \le 4$). Since $\phi(e_2) + \phi(e_3) \ge 5$, and $\phi(e_3) \le 5/2$, it follows that $\phi(e_2) \ge 5/2$ (and hence $m(e_2) \ge 2$).

Suppose that $m(e_3) = 1$. Since $\phi(e_3) \ge 2$, the first γ-rule applies, and so r_3 is a triangle v_3v_4y, and $m(v_3y), m(v_4y) \ge 2$, and $m^+(v_3y) + m^+(v_4y) \ge 6$. By Conf(4), $m(e_2) \le 3$, so by Conf(18), $m^+(v_3y), m^+(v_4y) \le 3$, and hence equality holds for both. By Conf(11), $m(v_3y), m(v_4y) = 2$; but this is contrary to Conf(16).

So $m(e_3) \ge 2$, and by Conf(4), $m(e_2) = m(e_3) = 2$. If $m^+(e_3) = 2$, then from the β-rules it follows that both edges of r_3 consecutive with e_3 have multiplicity five; but this is impossible since $m(e_2) = 2$. So $m^+(e_3) = 3$. Since $\phi(e_2) \ge 5/2$ it follows that r_2 is a triangle v_2v_3x, $m(v_2x), m(v_3x) \ge 2$, and one of $m^+(v_2x), m^+(v_3x) \ge 3$, and e_4 is a door for r. Since $\phi(e_4) > 0$, we deduce that $m^+(e_4) \ge 2$. By Conf(2), $m(v_2x) = 2$. By Conf(12), $m^+(v_3x) = 2$ and $m^+(v_2x) = 2$, a contradiction.

Case 4: $k = 4$ and $n_0 + n_1 = 0$.

Since $n_0, n_1 = 0$, it follows that $\phi(e_i) \ge 3/2$ and hence e_i is 2-heavy, for $1 \le i \le 4$. Consequently $n_4 = 0$, from (3). Since $\sum_{e \in E(C)} \phi(e) > 8$, we may assume because of the symmetries of the square that $\phi(e_1) + \phi(e_3) \ge 9/2$, and $\phi(e_1) \ge \phi(e_3)$, and therefore $\phi(e_1) \ge 5/2$. Thus $m(e_1) \ge 3$ from (4). If some edge f of the boundary of r_1 consecutive with e_1 satisfies $m(f) = 4$, say $f = v_1x$, then $m(e_4) = 1$ and v_1 has degree three; but since e_4 is 2-heavy, it follows that x, v_4 are adjacent, contrary to Conf(2). Thus there is no such f, and so by the β-rules, $m^+(e_1) \ge 4$.

Suppose that $m(e_3) \ge 2$. By Conf(6) it follows that $m^+(e_3) = 2$, and in particular r_3 is big. Since $\phi(e_3) \ge 3/2$, the β-rules imply that some edge f of the boundary of r_3 consecutive with e_3 satisfies $m(f) = 5$, say $f = v_4x$; and since x, v_1 are nonadjacent by Conf(2) it follows that $e_4 \in E_0 \cup E_1$, a contradiction. Thus $m(e_3) = 1$. Since e_3 is 2-heavy it follows that r_3 is a triangle v_3v_4x say.

By Conf(4), $m(e_2), m(e_4) \le 3$. By Conf(15), we may assume that $m(v_3x) = 1$; and by Conf(18), $m^+(v_4x) \le 3$. Since $m(e_4) \le 3$, the γ-rules imply that $\phi(e_3) \le 1$, a contradiction.

Case 5: $k \ge 5$ and $n_0 + n_1 = 1$.

Since $3n_0 + 2n_1 + n_2 + k - n_4 \le 7$, we have $n_4 \ge n_0 + n_2 + k - 5$. Let $E_0 \cup E_1 = \{e_k\}$.

Suppose that $n_4 = 0$. Then since $n_4 \ge n_0 + n_2 + k - 5$ it follows that $k = 5$. Since

$$\sum_{e \in E(C)} \phi(e) > 4k - 8 = 12,$$

and $\phi(e_5) \le 1$, and $\phi(e_i) \le 3$ for $i = 1, 2, 3, 4$ (by (1), since $n_4 = 0$) it follows that $\phi(e_i) \ge 5/2$ for $i = 1, 2, 3, 4$, and hence $e_1, \ldots, e_4$ are 3-heavy. If $m(e_1) \le 2$, then since $\phi(e_1) \ge 5/2$ it follows from the γ-rules that $m(e_2) = 4$ and r_2 is small; but then $e_2 \in E_4$, a contradiction. Thus $m(e_1) \ge 3$; so $m(e_1) = m^+(e_1) = 3$ by Conf(15). Since $m(e_2) \ge 2$, it follows that not both edges of r_1 consecutive with e_1 have multiplicity four, and so from the β-rules, $\phi(e_1) \le 5/2$. Similarly $\phi(e_4) \le 5/2$, contradicting that $\sum_{e \in E(C)} \phi(e) > 12$. This proves that $n_4 > 0$.

Suppose that $n_2 = 0$. Thus $e_1, \ldots, e_4$ are 3-heavy. Since $n_4 > 0$, (3) implies that $n_0 = 1$. Since $\phi(e_1) > 2$, the β- and γ-rules imply that either:

- $m(e_1) = 2$ and r_1 is a triangle v_1v_2w say; and $m(v_1w), m(v_2w) \ge 2$, and $m(e_2) = 4$. Consequently $m(v_2w) = 2$, contrary to Conf(16).

- $m(e_1) = 3$ and r_1 is big, and, if u_1-v_1-v_2-u_2 is the three-edge path of C_{r_1} with middle edge e_1, then one of $m(u_1v_1), m(u_2v_2) = 4$ and is incident with a small region. But if $m(u_1v_1) = 4$ then the second region incident with it is r_k, and this is not small since $n_0 = 1$; and if $m(u_2v_2) = 4$ then v_2 has degree three and $m(e_2) = 1$, and since e_2 is 3-heavy it follows that u_2, v_3 are adjacent, and $m(u_2v_3) \geq 2$, contrary to Conf(2).
- $m^+(e_1) \geq 4$; but this is contrary to Conf(15).

This proves that $n_2 \geq 1$.

Since $3n_0 + 2n_1 + n_2 + k/2 \leq 7$, we have $n_0 + n_2 + k/2 \leq 5$, and in particular $n_2 \leq 2$. If $e \in E(C)$ is not 3-heavy, then $\phi(e) \leq 2$ from the γ-rules, and so at most two edges of $E(C)$ not in $E_0 \cup E_1$ are not 3-heavy. By Conf(8) and Conf(19) it follows that $e_1, e_{k-1} \notin E_4$, so every edge in E_4 is disjoint from e_k. Since there are three consecutive edges of C not in E_4, and no two edges in E_4 are consecutive by (2), it follows that $n_4 \leq k/2 - 1$; and since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, it follows that $n_0 + n_2 + k/2 \leq 4$, and so $n_2 = 1$, and $n_0 = 0$, and $k \leq 6$. In particular, from (5) every edge $e \in E_4$ has $m(e) \geq 4$.

Suppose that $k = 6$. Since $n_4 \geq n_0 + n_2 + k - 5$ and $n_4 \leq k/2 - 1$, it follows that $n_4 = 2$; and so $E_4 = \{e_2, e_4\}$, since the members of E_4 are disjoint from e_6 and from each other. Since $e_2 \in E_4$, (3) implies that e_5 is not 3-heavy, and so $e_5 \in E_2$; and similarly $e_1 \in E_2$, a contradiction since $n_2 = 1$.

Thus $k = 5$. Since $n_4 \leq k/2 - 1$ it follows that $n_4 = 1$, so we may assume that $E_4 = \{e_2\}$. By (3), e_4 is not 3-heavy, and so $\phi(e_4) \leq 2$. Consequently $E_2 = \{e_4\}$, and $\phi(e_1) + \phi(e_3) \geq 11/2$. Since $\phi(e_4), \phi(e_5) > 0$, it follows that $m^+(e_4), m^+(e_5) \geq 2$, and since $m^+(e_2) \geq 5$, two applications of Conf(13) imply that $m(e_3) + m(e_4) \leq 3$ and $m(e_1) + m(e_5) \leq 3$. Since $m(e_1), m(e_3) \geq 2$ (because $\phi(e_1), \phi(e_3) > 2$) it follows that $m(e_1), m(e_3) = 2$ and e_1, e_3 are 4-heavy; and $m(e_4), m(e_5) = 1$. Since $\phi(e_4) > 1$, r_4 is a triangle v_4v_5x say. Since e_4 is not 3-heavy, one of $m(v_4x), m(v_5x) = 1$. If $m(v_4x) = 1$ then by Conf(16), $m(xv_5) \leq 2$; but then $\phi(e_4) = 1$ from the γ-rules, a contradiction. So $m(v_5x) = 1$. Since $\phi(e_4) > 1$, the γ-rules imply that $m^+(v_4x) \geq 4$. But this contradicts Conf(18).

Case 6: $k \geq 5$ and $n_0 + n_1 = 0$.

Since $n_0, n_1 = 0$, it follows that $\phi(e_i) \geq 3/2$ and hence e_i is 2-heavy, for $1 \leq i \leq k$. Since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, we have $n_4 \geq n_2 + k - 7$.

Suppose first that $n_4 > 0$. By (2) and Conf(8) and Conf(19), every edge in E_4 is disjoint from at least three edges that are not 3-heavy and that therefore belong to E_2. In particular $n_2 \geq 3$. Let $e \in E_4$; then e is disjoint from all the other edges in E_4, and from at least three edges in E_2, so $k - 3 \geq n_4 - 1 + 3$, that is, $k \geq n_4 + 5$. But $n_4 \geq n_2 + k - 7 \geq k - 4$, a contradiction.

This proves that $n_4 = 0$, and so $E(C) = E_2 \cup E_3$. Since $n_4 \geq n_2 + k - 7$, it follows that $n_2 + k \leq 7$. In particular, $k \in \{5, 6, 7\}$. From (4), every edge $e \in E(C)$ with $m(e) = 2$ belongs to E_2, since $n_4 = 0$ and there are no doors for r. Consequently every $e \in E_3$ satisfies $m(e) \geq 3$. Suppose that $m^+(e) = 3$ for some $e \in E_3$, say $e = e_1$. Thus r_1 is big, and $\beta_e(r) > -1$ since $\phi(e) > 2$. Hence from the β-rules, some edge of C_{r_1} consecutive with e_1 has multiplicity four, say v_1x. Hence $m(e_k) = 1$, and since $n_0, n_1 = 0$, it follows that r_k is a triangle, and therefore x, v_k are adjacent, contrary to Conf(2). This proves that $m^+(e) \geq 4$ for every $e \in E_3$.

By Conf(15), every edge in E_3 is disjoint from some edge in E_2, and in particular $n_2 \geq 2$. Since $n_2 + k \leq 7$, we have $k = 5$ and $n_2 = 2$. Every edge in E_3 is disjoint from one of the edges in E_2, so we may assume that $e_1, e_2 \in E_2$, and $e_3, e_4, e_5 \in E_3$. Since $m^+(e_3), m^+(e_4), m^+(e_5) \geq 4$, Conf(13) implies that $m^+(e_1) \leq 2$; and by Conf(15), e_1 is not 3-heavy. From the γ-rules, $\phi(e_1) \leq 3/2$, and similarly $\phi(e_2) \leq 3/2$. But for $i = 3, 4, 5$, $\phi(e_i) \leq 3$ since $n_4 = 0$; and so $\sum_{e \in E(C)} \phi(e) \leq 12$, contradicting our initial assumption that

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8.$$

This completes the proof of 3.7. ∎

Proof of 3.1. Suppose that (G, m) is a prime 8-target, and let α, β, γ be as before. Since the sum over all regions r of $\alpha(r) + \beta(r) + \gamma(r)$ is positive, there is a region r with $\alpha(r) + \beta(r) + \gamma(r) > 0$. But this is contrary to one of 3.4, 3.5, 3.6, 3.7. This proves 3.1. ∎

4 Reducibility

Now we begin the second half of the paper, devoted to proving the following.

4.1 *Every minimum* 8*-counterexample is prime.*

Again, the proof is broken into several steps. Clearly no minimum 8-counterexample (G, m) has an edge e with $m(e) = 0$, because deleting e would give a smaller 8-counterexample; and by 2.1, every minimum 8-counterexample satisfies the conclusions of 2.1. Thus, it remains to check that (G, m) contains none of Conf(1)–Conf(19). Sometimes it is just as easy to prove a result for general d instead of $d = 8$, and so we do so.

4.2 *If* (G, m) *is a minimum* d*-counterexample, then every triangle has multiplicity less than* d.

Proof. Let uvw be a triangle of G, and let $X = \{u, v, w\}$. Since $|V(G)| \geq 6$, 2.1 implies that $m(\delta(X)) \geq d + 2$. But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so $d + 2 \leq d + d + d - 2m(uv) - 2m(uw) - 2m(vw)$, that is, $m(uv) + m(uw) + m(vw) \leq d - 1$. This proves 4.2. ∎

If C is a cycle of length four in G, say with vertices u, v, w, x in order, let m' be defined as follows: $m'(uv) = m(uv) - 1$, $m'(vw) = m(vw) + 1$, $m'(wx) = m(wx) - 1$, $m'(ux) = m(ux) + 1$, and $m'(e) = m(e)$ for all other edges e. If (G, m) is a minimum d-counterexample, then because of the second statement of 2.1, it follows that (G, m') is a d-target. (Note that possibly $m'(uv), m'(wx)$ are zero; this is the reason to permit $m(e) = 0$ in a d-target.) We say that (G, m') is obtained from (G, m) by *switching on the sequence* u-v-w-x-u. If (G, m') is smaller than (G, m), we say that the sequence u-v-w-x-u is *switchable.*

4.3 *No minimum* d*-counterexample contains Conf(1).*

Proof. Suppose that (G, m) is a minimum d-counterexample, with a triangle uvw, where u, v have degree three. Let the neighbours of u, v not in $\{u, v, w\}$ be x, y respectively. Let H be a simple graph obtained from G by adding new edges if necessary to make w, x, y pairwise adjacent, and extend m to $E(H)$ by setting $m(e) = 0$ for every new edge. Thus (H, m) is not d-edge-colourable, and although it may not be a minimum d-counterexample, no d-counterexample has fewer vertices.

Define $f(w) = m(uw) + m(vw)$, $f(x) = m(ux)$, and $f(y) = m(vy)$. Since $m(\delta(\{u, v\}))$ is even, it follows that $f(w) + f(x) + f(y)$ is even. Define

$$\begin{aligned} n(wx) &= \frac{1}{2}(f(x) + f(w) - f(y)) \\ n(wy) &= \frac{1}{2}(f(y) + f(w) - f(x)) \\ n(xy) &= \frac{1}{2}(f(x) + f(y) - f(w)). \end{aligned}$$

It follows that $n(wx), n(wy), n(xy)$ are integers. Since $m(\delta(\{u, v, w\})) \geq d$ and $m(\delta(w)) = d$, it follows that $m(ux)+m(vy) \geq m(uw)+m(vw)$ and hence $n(xy) \geq 0$. Similarly, since $m(\delta(\{u, v, x\})) \geq d$ and $m(\delta(x)) = d$, it follows that $n(wy) \geq 0$, and similarly $n(wx) \geq 0$.

Let $G' = H \setminus \{u, v\}$. For each edge e of G', define $m'(e)$ as follows. If e is incident with a vertex different from x, y, w let $m'(e) = m(e)$. For $e = xy, wx, wy$ let $m'(e) = m(e) + n(e)$. We claim that (G', m') is a d-target. To show this, let $X \subseteq V(G')$ with $|X|$ odd; we must show that $m'(\delta_{G'}(X)) \geq d$. By replacing X by its complement if necessary (which also is odd, since $|V(G)|$ is even), we may assume that X contains at most one of w, x, y. But then from the choice of $f(w), f(x), f(y)$, it follows that $m'(\delta_{G'}(X)) = m(\delta_G(X)) \geq d$ as required. Thus (G', m') is a d-target. Since $|V(G')| < |V(G)|$, there are d perfect matchings $F'_1, \ldots, F'_d$ of G' such that every edge $e \in E(G')$ is in exactly $m'(e)$ of them. Now each of $F'_1, \ldots, F'_d$ contains at most one of the edges wx, wy, xy. Let I_1, I_2, I_3, I_0 be the sets of $i \in \{1, \ldots, d\}$ such that F'_i contains wx, wy, xy or none of the three, respectively. Thus $|I_1| = m'(wx) = m(wx) + n(wx)$. For $n(wx)$ values of $i \in I_1$ let $F_i = (F'_i \setminus \{wx\}) \cup \{ux, vw\}$, and for the remaining $m(wx)$ values let $F_i = F'_i \cup \{uv\}$. Thus F_i is a perfect matching of G for each $i \in I_1$. Define F_i $(i \in I_2)$ similarly. For $n(xy)$ values of $i \in I_3$ let $F_i = (F'_i \setminus \{xy\}) \cup \{ux, vy\}$, and for the others let $F_i = F_i \cup \{uv\}$. For $i \in I_0$ let $F_i = F'_i \cup \{uv\}$. Then $F_1, \ldots, F_d$ are perfect matchings of G, and we claim that every edge e is in exactly $m(e)$ of them. This is clear if e has an end different from u, v, w, x, y; and true from the construction if both ends of e are in $\{w, x, y\}$. From the symmetry we may therefore assume that e is incident with u. If $e = ux$, then e belongs to $n(wx) + n(xy)$ of $F_1, \ldots, F_d$; but

$$n(wx) + n(xy) = \frac{1}{2}(f(x) + f(w) - f(y)) + \frac{1}{2}(f(x) + f(y) - f(w)) = f(x) = m(ux)$$

as required. The other two cases are similar. This is a contradiction, since (G, m) is a minimum d-counterexample, and so there is no such triangle uvw. This proves 4.3. ∎

Incidentally, a similar proof would show that G is four-connected except for cutsets of size three that cut off just one vertex, but we do not need this.

If (G, m) is a d-target, and x, y are distinct vertices both incident with some common region r, we define $(G, m) + xy$ to be the d-target (G', m') obtained as follows:

- If x, y are adjacent in G, let $(G', m') = (G, m)$.

- If x, y are non-adjacent in G, let G' be obtained from G by adding a new edge xy, extending the drawing of G to one of G' and setting $m'(e) = m(e)$ for every $e \in E(G)$ and $m'(xy) = 0$.

4.4 *No minimum d-counterexample contains Conf(2).*

Proof. Let (G, m) be a minimum d-counterexample, with a triangle uvw, and suppose that u has only one other neighbour x, and $m(ux) < m(uw) + m(vw)$. Let $(G', m'') = ((G, m) + vx) + wx$. For each $e \in E(G')$, define $m'(e)$ as follows. If $e \neq ux, uw, vw, vx$ let $m'(e) = m(e)$. Let

$$\begin{aligned} m'(vx) &= m''(vx) + m(vw) \\ m'(vw) &= 0 \\ m'(ux) &= m(ux) - m(vw) \\ m'(uw) &= m(uw) + m(vw). \end{aligned}$$

Since $m(uv) + m(uw) + m(ux) = d$ and $m(uv) + m(uw) + m(vw) \leq d$ since $m(\delta(\{u, v, w\})) \geq d$, it follows that $m(ux) \geq m(vw)$, and so $m'(e) \geq 0$ for every edge e. Moreover, $m'(\delta(z)) = d$ for every vertex z, from the construction. We claim that (G', m') is a d-target. For let $X \subseteq V(G')$ with $|X|$ odd; and we may assume that $u \notin X$. We must show that $m'(\delta(X)) \geq d$. If X contains at most one of v, w, x then $m'(\delta(X)) = m(\delta(X)) \geq d$ as required, so we may assume that X contains at least two of v, w, x. If $v, w, x \in X$ then $m'(\delta(X)) \geq m'(\delta(u)) = d$ as required. If $X \cap \{v, w, x\} = \{v, w\}$ then $m'(\delta(X)) = m(\delta(X)) + 2m(vw) \geq d$, and if $X \cap \{v, w, x\} = \{w, x\}$ then $m'(\delta(X)) = m(\delta(X)) \geq d$, so we may assume that $X \cap \{v, w, x\} = \{v, x\}$, and hence $m'(\delta(X)) = m(\delta(X)) - 2m(vw)$. We must therefore show that in this case, $m(\delta(X)) \geq 2m(vw) + d$. To see this, note that

$$\begin{aligned} m(\delta(X \cup \{u, w\})) \leq \; & m(\delta(X)) - m(ux) - m(uv) - m(vw) - m''(xw) \\ & + (d - m(uw) - m(vw) - m''(xw)) \leq m(\delta(X)) - 2m(vw) \end{aligned}$$

since $m''(xw) \geq 0$ and $m(ux) + m(uv) + m(uw) = d$. Since $m(\delta(X \cup \{u, w\})) \geq d$, it follows that $m(\delta(X)) \geq 2m(vw) + d$ as required. This proves that (G', m') is a d-target. Since $m'(uw) > m(ux), m(vw)$ (the first from the hypothesis), it follows that (G', m') is smaller than (G, m), and so is d-edge-colourable; let $F'_1, \ldots, F'_d$ be a d-edge-colouring. Now every perfect matching containing vx also contains uw, since vx is not disjoint from any other edge incident with u. Hence there are at least $m(vw)$ of $F'_1, \ldots, F'_d$ that contain both vx and uw. Choose $m(vw)$ of them, say $F'_1, \ldots, F'_{m(vw)}$; and for $1 \leq i \leq m(vw)$ define $F_i = (F'_i \setminus \{vx, uw\}) \cup \{vw, ux\}$. Define $F_i = F'_i$ for $m(vw) + 1 \leq i \leq d$. Then every edge e of G is in $m(e)$ of $F_1, \ldots, F_d$, a contradiction. Thus there is no such triangle uvw. This proves 4.4. ∎

4.5 *No minimum* 8*-counterexample contains Conf(3) or Conf(4).*

Proof. To handle both cases at once, let us assume that (G, m) is an 8-target, and uvw, uwx are triangles with $m(uv) + m(uw) + m(vw) + m(ux) \geq 8$, (where possibly $m(uw) = 0$); and either (G, m) is a minimum 8-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 8-counterexample (G_0, m_0) say. We must show that $m(uw) = 0$ and $(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2)$. Let (G, m') be obtained by switching (G, m) on u-v-w-x-u.

(1) *(G, m') is not smaller than (G, m).*

Because suppose it is. Then it admits an 8-edge-colouring; because if (G, m) is a minimum 8-counterexample this is clear, and otherwise $m(uw) = 0$, and (G', m') is smaller than (G_0, m_0). Let $F'_1, \dots, F'_8$ be an 8-edge-colouring of (G', m'). Since

$$m'(uv) + m'(uw) + m'(vw) + m'(ux) \geq 9,$$

one of $F'_1, \dots, F'_8$, say F'_1, contains two of uv, uw, vw, ux and hence contains vw, ux. Then

$$(F'_1 \setminus \{vw, ux\}) \cup \{uv, wx\}$$

is a perfect matching, and it together with $F'_2, \dots, F'_8$ provide an 8-edge-colouring of (G, m), a contradiction. This proves (1).

From (1) we deduce that $\max(m(ux), m(vw)) < \max(m(uv), m(wx))$. It follows that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7,$$

by (1) applied with u, w exchanged; and

$$m(uv) + m(ux) + m(wx) + m(uw) \leq 7,$$

by (1) applied with v, x exchanged. Consequently $m(ux) > m(wx)$, and hence $m(ux) \geq 2$; and $m(vw) > m(wx)$, and so $m(vw) \geq 2$. Suppose that $m(uv) \leq 3$. Since

$$\max(m(ux), m(vw)) < \max(m(uv), m(wx)),$$

it follows that $m(uv) = 3$ and $m(vw) = m(ux) = 2$; and therefore $m(wx) = 1$, since $m(ux) > m(wx)$. But this is contrary to (1).

We deduce that $m(uv) \geq 4$. Since $m(vw) \geq 2$ and $m(uv) + m(uw) + m(vw) + m(wx) \leq 7$, it follows that $m(uw) + m(wx) \leq 1$; so $m(uw) = 0$ and $m(wx) = 1$. But then

$$(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2).$$

This proves 4.5. ∎

5 Guenin's cuts

We still have many configurations to handle, to finish the proof of 4.1, but all the others are handled by a method of Guenin [5], which we introduce in this section. In particular, nothing so far has assumed the truth of 1.1 for $d = 7$, but now we will need to use that.

Let (G, m) be a d-target, and let x-u-v-y be a three-edge path of G, where x, y are incident with a common region. Let (G', m') be obtained from $(G, m) + xy$ by switching on the cycle x-u-v-y-x. We say that (G', m') is obtained from (G, m) by *switching on x-u-v-y*. If (G', m') is smaller than (G, m), we say that the path x-u-v-y is *switchable*.

Let G be a three-connected graph drawn in the plane, and let G^* be its dual graph; let us identify $E(G^*)$ with $E(G)$ in the natural way. A *cocycle* means the edge-set of a cycle of the dual graph; thus, $Q \subseteq E(G)$ is a cocycle of G if and only if Q can be numbered $\{e_1, \ldots, e_k\}$ for some $k \geq 3$ and there are distinct regions $r_1, \ldots, r_k$ of G such that for $1 \leq i \leq k$, e_i is incident with r_i and with r_{i+1} (where r_{k+1} means r_1).

Guenin's method is the use of the following:

5.1 *Let $d \geq 1$ be an integer such that every $(d-1)$-regular oddly $(d-1)$-edge-connected planar graph is $(d-1)$-edge-colourable. Let (G, m) be a minimum d-counterexample, and let x-u-v-y be a path of G with x, y on a common region. Let (G', m') be obtained by switching on x-u-v-y, and let $F_1, \ldots, F_d$ be a d-edge-colouring of (G', m'), where $xy \in F_k$. Let $I = \{1, \ldots, d\} \setminus \{k\}$ if $xy \notin E(G)$, and $I = \{1, \ldots, d\}$ if $xy \in E(G)$. Then for each $i \in I$, there is a cocycle Q_i of G' with the following properties:*

- *for $1 \leq j \leq d$ with $j \neq i$, $|F_j \cap Q_i| = 1$;*
- *$|F_i \cap Q_i| \geq 5$;*
- *there is a set $X \subseteq V(G)$ with $|X|$ odd such that $\delta_{G'}(X) = Q_i$; and*
- *$uv, xy \in Q_i$ and $ux, vy \notin Q_i$.*

Proof. Let $i \in I$. If $i \neq k$ and $xy \in F_i$, it follows that $m'(xy) \geq 2$ since $xy \in F_k$; and so $xy \in E(G)$. Thus in either case F_i is a perfect matching of G. For each edge e of G', let $p(e) = 1$ if $e \in F_i$, and $p(e) = 0$ otherwise; and for each edge e of G, let $n(e) = m(e) - p(e)$. Thus (G, n) has the property that for each vertex z, $n(\delta_G(z)) = d - 1$. If there is a list of $d - 1$ perfect matchings of G such that every edge e is in $n(e)$ of them, then adding F_i to this list gives a d-edge-colouring of (G, m), a contradiction. Thus by hypothesis, there exists $Y \subseteq V(G)$ with $|Y|$ odd and with $n(\delta_G(Y)) < d - 1$. Since $|Y|$ and $n(\delta_G(Y))$ have the same parity, it follows that $n(\delta_G(Y)) \leq d - 3$. Since $\delta_G(Y)$ is an edge-cut of the connected graph G, it can be partitioned into "bonds" (edge-cuts $\delta_G(X)$ such that $G|X$, $G \setminus X$ are both connected), and hence one of these bonds $\delta_G(X)$ has $n(\delta_G(X))$ odd, and consequently $|X|$ also odd. Since $\delta_G(X)$ is a bond of G and hence $\delta_{G'}(X)$ is a bond of G', there is a cocycle Q_i of G' with $Q_i = \delta_{G'}(X)$. We claim that Q_i satisfies the theorem. For we have seen the third assertion; we must check the other three.

From the choice of X we have $n(\delta_G(X)) \leq d - 3$. Since $|X|, |V(G) \setminus X| \geq 3$ (because $n(\delta_G(z)) = d - 1$ for each vertex z), it follows from 2.1 that $m(\delta_G(X)) \geq d + 2$, and so $p(\delta_G(X)) \geq 5$, that is, $|F_i \cap Q_i| \geq 5$. This proves the second assertion. We recall that $F_1, \ldots, F_d$ is a d-edge-colouring of (G', m'); and so for $1 \leq j \leq d$ with $j \neq i$, some edge of $\delta_{G'}(X)$ belongs to F_j, and so

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| \geq d - 1.$$

On the other hand, every edge e of G' belongs to $m'(e)$ of $F_1, \ldots, F_d$, and hence to $m'(e) - p(e)$ of the $d - 1$ perfect matchings in this list without F_i. Consequently

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| = \sum_{e \in Q_i} m'(e) - p(e).$$

It follows that $\sum_{e\in Q_i} m'(e) - p(e) \geq d-1$; but $m'(e)-p(e) = n(e)$ for all edges of G' except xu, uv, vy, xy, and so

$$|\{uv, xy\} \cap Q_i| - |\{ux, vy\} \cap Q_i| + \sum_{e\in Q_i} n(e) \geq d-1.$$

Since $\sum_{e\in Q_i} n(e) \leq d-3$, it follows that $uv, xy \in Q_i$ and $ux, vy \notin Q_i$. This proves the fourth assertion. Moreover, since

$$\sum_{1\leq j\leq d, j\neq i} |F_j \cap Q_i| = d-1,$$

it follows that $|F_j \cap Q_i| = 1$ for all $j \in \{1, \ldots, d\}$ with $j \neq i$. This proves the first assertion, and so proves 5.1. ∎

By the result of [6], every 7-regular oddly 7-edge-connected planar graph is 7-edge-colourable, so we can apply 5.1 when $d = 8$.

5.2 *No minimum* 8*-counterexample contains Conf(5) or Conf(6).*

Proof. To handle both at once, let us assume that (G, m) is an 8-target, and uvw, uwx are two triangles with $m^+(uv) + m(uw) + m^+(wx) \geq 7$; and either (G, m) is a minimum 8-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 8-counterexample. We claim that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7.$$

If $m(uw) > 0$ this follows from 4.5 since we do not have Conf(3); and if $m(uw) = 0$ then one of $m(uv), m(wx) \geq 3$, and since 4.5 implies that we do not have Conf(4), again the claim holds. This proves that $m(uv)+m(uw)+m(vw)+m(wx) \leq 7$. Since $m^+(uv)+m(uw)+m^+(wx) \geq 7$ and hence $m(uv) + m(uw) + m(wx) \geq 5$, it follows that $m(vw) \leq 2$ and similarly $m(ux) \leq 2$.

We claim that u-x-w-v-u is switchable. For suppose not; then we may assume that $m(vw) > \max(m(uv), m(wx))$ and $m(vw) \geq m(ux)$. Yet $m(vw) \leq 2$, and so $m(uv), m(wx) = 1$, and $m(ux) \leq 2$. Since u-x-w-v-u is not switchable, it follows that $m(ux) = 2$; and since $m^+(uv) + m(uw) + m^+(wx) \geq 7$, it follows that $m(uw) \geq 3$, giving Conf(3), contrary to 4.5. This proves that u-x-w-v-u is switchable.

Let r_1, r_2 be the second regions incident with uv, wx respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. Let $k = m(uv) + m(uw) + m(wx) + 2$. Let (G, m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G, m'), where F_i contains one of uv, uw, wx for $1 \leq i \leq k$. For $1 \leq i \leq 8$, let Q_i be as in 5.1.

(1) *For* $1 \leq i \leq 8$*, either* $F_i \cap Q_i \cap D_1 \neq \emptyset$*, or* $F_i \cap Q_i \cap D_2 \neq \emptyset$*; and both are nonempty if either* $k = 8$ *or* $i = 8$*.*

For let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = wx$, $e_2 = uw$, and $e_3 = uv$. Since F_j contains one of e_1, e_2, e_3 for $1 \leq j \leq k$, it follows that none of $e_4, \ldots, e_n$ belongs to any F_j with $j \leq k$ and $j \neq i$, and, if $k = 7$ and $i \neq 8$, that only one of them is in F_8. But since at most one of e_1, e_2, e_3 is in F_i and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 7$; so either e_4, e_5 belong only to F_i, or e_n, e_{n-1}

belong only to F_i, and both if $k = 8$ or $i = 8$. But if e_4, e_5 are only contained in F_i, then they both have multiplicity one, and are disjoint, so e_4 is a door for r_1 and hence $e_4 \in F_i \cap Q_i \cap D_1$. Similarly if e_n, e_{n-1} are only contained in F_i then $e_n \in F_i \cap Q_i \cap D_2$. This proves (1).

Now $k \leq 8$, so one of r_1, r_2 is small since $m^+(uv) + m(uw) + m^+(wx) \geq 7$; and if $k = 8$ then by (1) $|D_1|, |D_2| \geq 8$, a contradiction. Thus $k = 7$, so both r_1, r_2 are small, but from (1) $|D_1| + |D_2| \geq 9$, again a contradiction. This proves 5.2. ∎

5.3 *No minimum* 8*-counterexample contains Conf(7).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uvw is a triangle with $m^+(uv) + m^+(uw) \geq 7$. Let r_1, r_2 be the second regions for uv, uw respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. By 5.2, we do not have Conf(5), so neither of r_1, r_2 is a triangle. Since $m(uv) + m(uw) \geq 5$, one of $m(uv), m(uw) \geq 3$, so we may assume that $m(uv) \geq 3$. Let tu be the edge incident with r_2 different from uw. Since $m(uv) + m(uw) \geq 5$, it follows that $m(tu) \leq 3$, and by 4.2, $m(vw) \leq 2$. Thus the path t-u-v-w is switchable. Note that t, w are non-adjacent in G, since r_2 is not a triangle. Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. Let $k = m(uv) + m(uw) + 2$; thus $k \geq 7$, since $m(uv) + m(uw) \geq 5$, and we may assume that for $1 \leq j < k$, F_j contains one of uv, uw, and $tw \in F_k$.

Let $I = \{1, \ldots, 8\} \setminus \{k\}$, and for each $i \in I$, let Q_i be as in 5.1. Now let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$, $e_2 = uw$, and $e_3 = tw$. Since F_j contains one of e_1, e_2, e_3 for $1 \leq j \leq k$ it follows that none of $e_4, \ldots, e_n$ belongs to any F_j with $j \leq k$ and $j \neq i$; and if $k = 7$ and $i \neq 8$, only one of them belongs to F_8. Since F_i contains at most one of e_1, e_2, e_3 and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 7$, and so either e_4, e_5 are only contained in F_i, or e_n, e_{n-1} are only contained in F_i; and both if either $k = 8$ or $i = 8$. Thus either $e_4 \in F_i \cap Q_i \cap D_2$ or $e_n \in F_i \cap Q_i \cap D_1$, and both if $k = 8$ or $i = 8$. Since $k \leq 8$, one of r_1, r_2 is small since $m^+(uv) + m^+(uw) \geq 7$; and yet if $k = 8$ then $|D_1|, |D_2| \geq |I| = 7$, a contradiction. Thus $k = 7$, so r_1, r_2 are both small, and yet $|D_1| + |D_2| \geq 8$, a contradiction. This proves 5.3. ∎

5.4 *No minimum* 8*-counterexample contains Conf(8).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uvw is a triangle, and its edges have multiplicities $3, 2, 2$ (in some order). We will show that the second region r for uw has a door disjoint from uw. By 4.5, we do not have Conf(3), so r is not a triangle. By exchanging u, w if necessary we may assume that $m(vw) = 2$. Let tu be the edge incident with r different from uw. We claim that the path t-u-v-w is switchable. For certainly $m(uv) \geq m(vw)$, so it suffices to check that $m(uv) \geq m(tu)$. If not, then since $m(uv) \geq 2$ and $m(uv) + m(uw) \geq 5$, it follows that $m(uv) = 2$, $m(tu) = 3$ and $m(uw) = 3$, and we have Conf(2), contrary to 4.4. Thus t-u-v-w is switchable. Let (G', m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G', m'). Since $m'(uv) + m'(uw) = 6$, we may assume that $F_1, \ldots, F_6$ each contain one of uv, uw; and $tw \in F_7$, and therefore $vw \in F_8$. Let $I = \{1, \ldots, 6, 8\}$; and for $i \in I$, let Q_i be as in 5.1. Since Q_8 contains uv, uw, tw and $F_1, \ldots, F_7$ each contain one of uv, uw, tw, it follows that no other edge of Q_8 belongs to any of $F_1, \ldots, F_7$, and so $Q_8 \cap F_8$ contains a door for r, say e. Moreover $e \neq tu$ since $tu \notin Q_8$; and e is not incident with w since $vw \in F_8$. Consequently e is disjoint from uw. This proves 5.4. ∎

5.5 *No minimum* 8*-counterexample contains Conf(9).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uv_1v_2 is a triangle, with $m(uv_1), m(uv_2), m(v_1v_2) = 2$, such that the second regions r_1, r_2 for uv_1, uv_2 respectively both have at most one door, and no door that is disjoint from uv_1v_2. For $i = 1, 2$, let D_i be the set of doors for r_i. For $i = 1, 2$, let ux_i and v_iy_i be edges incident with r_i different from uv_i.

Now $x_1 \neq x_2$ since u has degree at least four; and so $m(ux_1) + m(ux_2) \leq 4$ and we may assume that $m(ux_1) \leq 2$. Consequently the path x_1-u-v_2-v_1 is switchable. Note that v_1, x_1 may be adjacent, but if so then $m(v_1x_1) = 1$ from 4.5. Let (G', m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring, where $uv_2 \in F_1, F_2, F_3$, and $uv_1 \in F_4, F_5$ and $v_1x_1 \in F_6$, and $v_1x_1 \in F_7$ if $v_1x_1 \in E(G)$. Since v_1v_2 belongs to some F_i, and v_1v_2 meets all of uv_2, uv_1, v_1x_1, we may assume that $v_1v_2 \in F_8$. Let $I = \{1, \ldots, 5, 7, 8\}$ if $x_1v_1 \notin E(G)$, and $I = \{1, \ldots, 8\}$ otherwise. For $i \in I$, let Q_i be as in 5.1.

We claim that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ for $i = 7, 8$. First suppose that $v_1x_1 \notin E(G)$. Then for $1 \leq j \leq 6$ and for $i = 7, 8$, $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$, and so no other edges of Q_i belong to any F_j with $j \in \{1, \ldots, 6\}$. Since only one edge of $Q_i \setminus \{uv_2, uv_1, v_1x_1\}$ belongs to the F_j with $j \in \{7, 8\} \setminus \{i\}$, it follows that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ as required. Now suppose that $v_1x_1 \in E(G)$. Then for $1 \leq j \leq 7$ and for $i = 7, 8$, $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$, and so no other edges of Q_i belong to any F_j with $j \in \{1, \ldots, 7\}$ and $j \neq i$. For $i = 7$, as before it follows that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$; for $i = 8$ we find that $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2 \neq \emptyset$. Thus in any case, we have $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ for $j = 7, 8$.

Now by hypothesis, $D_1 \cup D_2 \subseteq \{ux_1, ux_2, v_1y_1, v_2y_2\}$; and $ux_1 \notin Q_7, Q_8$ from the choice of switchable path, and $v_1y_1, v_2y_2 \notin F_8$ since $v_1v_2 \in F_8$. Thus $ux_2 \in F_8 \cap D_2$. Since $|D_2| \leq 1$ by hypothesis, it follows that $v_2y_2 \notin D_2$, and $ux_2 \notin F_7$ since $ux_2 \in F_8$ and $m(ux_2) = 1$. Thus $v_1y_1 \in D_1$. Now $m(ux_2) = 1$, and so the path x_2-u-v_1-v_2 is switchable; so by the same argument with v_1, v_2 exchanged, it follows that $ux_1 \in D_1$ and $v_2y_2 \in D_2$, contrary to the hypothesis. This proves 5.5.

5.6 *No minimum* 8*-counterexample contains Conf(10).*

Proof. For suppose that (G, m) is a minimum counterexample, with a square $uvwx$ and a triangle wxy, where $m(uv) = m(wx) = m(xy) = 2$, and $m(vw) = 4$. By 4.5, we do not have Conf(4), and it follows that $m(ux) = 1$. Since $m(\delta(w)) = 8$ it follows that $m(wy) \leq 2$, and so u-x-y-w is switchable. Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. We may assume that $xy \in F_1, F_2, F_3$, and $xw \in F_4, F_5$, and $uw \in F_6$. Let $I = \{1, \ldots, 8\} \setminus \{6\}$, and let Q_i $(i \in I)$ be as in 5.1. Now $vw \notin F_4, F_5, F_6$, so there are four values of $i \in \{1, 2, 3, 7, 8\}$ such that $vw \in F_i$, and from the symmetry we may assume that F_1, F_2, F_7 contain vw (and so does one of F_3, F_8). It follows that $vw \notin Q_i$ for $i \in I$, and so $uv \in Q_i$ for each $i \in I$. Since uv belongs to two of $F_1, \ldots, F_8$, there exists $j \neq 8$ with $uv \in F_j$. Moreover, F_j does not contain vw, and so $j \neq 1, 2, 7$; so $j \in \{3, 4, 5, 6\}$. But $|Q_1 \cap F_j| \geq 2$, since one of $xy, xw, vw \in Q_1 \cap F_j$, a contradiction. This proves 5.6. ∎

5.7 *No minimum* 8*-counterexample contains Conf(11), Conf(12) or Conf(13).*

Proof. To handle all these cases simultaneously, let us assume that (G, m) is a 8-target, and v_1-v_2-v_3-v_4-v_5-v_1 are the vertices in order of some cycle of G, and this cycle bounds a disc which is the union of three triangles of G, namely $v_1v_2v_3$, $v_1v_3v_5$ and $v_3v_4v_5$. Moreover, there is a subset $Z \subseteq \{v_1v_3, v_3v_5\}$ such that $m(e) = 0$ for all $e \in Z$ and deleting the edges in Z gives a minimum 8-counterexample. Finally, we assume that

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and

$$m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \geq 7.$$

To obtain the subcases Conf(11), Conf(12) and Conf(13), we set, respectively,

- $Z = \{v_1v_3\}$, $m(v_1v_2) \geq 3$, $m(v_3v_4) \geq 3$, $m(v_3v_5) = 1$, $m^+(v_4v_5) \geq 3$, and $m(v_1v_5) \leq 3$
- $Z = \{v_3v_5\}$, $m^+(v_1v_2) \geq 3$, $m(v_2v_3) = 2$, $m(v_3v_4) \geq 2$, $m(v_1v_3) = 2$, $m(v_1v_5) \leq 3$ and $m^+(v_4v_5) \geq 2$
- $Z = \{v_1v_3, v_3v_5\}$, $m(v_1v_2) \geq \max(m(v_2v_3), m(v_1v_5))$.

(Edges not mentioned are unrestricted.) Let (G, m') be obtained by switching on the sequence v_2-v_3-v_5-v_1-v_2. (We postpone for the moment the question of whether this sequence is switchable.) Let us suppose (for a contradiction) that (G, m') admits an 8-edge-colouring $F_1, \dots, F_8$. Let $k = m(v_1v_2) + m(v_1v_3) + m(v_3v_5) + 2$; then we may assume that $F_1, \dots, F_k$ each contain exactly one of v_1v_2, v_1v_3, v_3v_5, and $v_3v_5 \in F_k$. Hence $k \leq 8$. Let $I = \{1, \dots, 8\}$ if $m(v_3v_5) \geq 1$, and $I = \{1, \dots, 8\} \setminus \{k\}$ otherwise. Since v_2v_3 meets all the edges v_1v_2, v_1v_3, v_3v_5, it follows that none of $F_1, \dots, F_k$ contain v_2v_3, and so $k + m(v_2v_3) - 1 \leq 8$ and we may assume that $v_2v_3 \in F_j$ for $k + 1 \leq j \leq k + m(v_2v_3) - 1$. Thus there are exactly $9 - k - m(v_2v_3)$ values of $j \in \{1, \dots, 8\}$ such that F_j contains none of $v_1v_2, v_1v_3, v_3v_5, v_2v_3$. Since by hypothesis

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and so $m(v_3v_4) > 9 - k - m(v_2v_3)$, there exists $h \leq k + m(v_2v_3) - 1$ such that $v_3v_4 \in F_h$; since v_3v_4 meets each of v_1v_3, v_2v_3 and v_3v_5, it follows that $v_1v_2 \in F_h$, and so $h < k$; and from the symmetry we may assume that $h = 1$.

For each $i \in I$ let Q_i as in 5.1. Now $|F_j \cap Q_i| = 1$ for $1 \leq j \leq 8$ with $j \neq i$; and since F_1 contains v_1v_2, v_3v_4 it follows that for $i \neq 1$ $v_3v_4 \notin Q_i$. Consequently $v_4v_5 \in Q_i$ for all $i \in I \setminus \{1\}$. Let r_1, r_2 be the second regions for v_1v_2, v_4v_5 respectively, and let their sets of doors be D_1, D_2. Hence for each $j \in \{1, \dots, 8\}$, since there exists $i \in I \setminus \{1\}$ with $i \neq j$, it follows that F_j contains at most one of $v_1v_2, v_1v_3, v_3v_5, v_4v_5$, and so we may assume that $v_4v_5 \in F_j$ for $k+1 \leq j \leq k'$ where $k' = k + m(v_4v_5)$, and in particular $k' \leq 8$. From the hypothesis, $k' \geq 7$.

(1) *For $i \in I \setminus \{1\}$, one of $F_i \cap D_1$, $F_i \cap D_2$ is non-empty, and both if $k' = 8$ or $i = 8$.*

Let $e_1, \dots, e_n, e_1$ be the edges of Q_i in order, where $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_3v_5$ and $e_4 = v_4v_5$. Thus for $1 \leq j \leq k'$, F_j contains one of e_1, e_2, e_3, e_4, and hence contains none of $e_5, \dots, e_n$ if $j \neq i$. Now since F_i contains at most one of e_1, e_2, e_3, e_4 and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 8$. Hence $e_5, \dots, e_n$ belong only to F_i, except that one belongs to F_8 if $i, k' < 8$. This proves (1) as usual.

Since $k' \leq 8$, one of r_1, r_2 is small since $m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \geq 7$. Consequently, (1) implies that $k' = 7$; and so r_1, r_2 are both small, again a contradiction to (1).

This proves that (G, m') is not 8-edge-colourable, and in particular the sequence v_2-v_3-v_5-v_1-v_2 is not switchable. Let us look at the subcases for Conf(11), Conf(12), Conf(13) listed above. In the Conf(11) subcase, $m(v_1v_2) \geq 3 \geq m(v_1v_5)$, so we only need to check that $m(v_1v_2) \geq m(v_2v_3)$. If not, then $m(v_2v_3) = 4$, contrary to Conf(2). In the Conf(13) subcase, the condition that $m(v_1v_2) \geq \max(m(v_2v_3), m(v_1v_5))$ is explicitly given. In the Conf(12) subcase, $m(v_1v_2) \geq 2 \geq m(v_2v_3)$, so we only need to check that $m(v_1v_2) \geq m(v_1v_5)$. Suppose not; then $m(v_1v_5) = 3$ and $m(v_1v_2) = 2$. In this case the sequence v_2-v_3-v_5-v_1-v_2 is not switchable, so we need a different approach.

Since (G, m') given above is not 8-colourable, it follows from 2.1 that $m'(\delta(X)) \geq 10$ for every subset $X \subseteq V(G)$ with $|X|$ odd and $|X|, |V(G) \setminus X| \geq 3$. Let (G, m'') be obtained from (G, m') by switching again on the same sequence. Now (G, m'') is a 8-target, since $m(v_2v_3), m(v_1v_5) \geq 2$; and it is smaller than (G, m), and therefore admits an 8-edge-colouring, say $F_1, \ldots, F_8$. Since $m''(v_1v_2) + m''(v_1v_3) + m''(v_3v_5) + m''(v_1v_5) > 8$, some F_i contains two of $v_1v_2, v_1v_3, v_3v_5, v_1v_5$, and therefore contains v_1v_2 and v_3v_5. By replacing F_i by $(F_i \setminus \{v_1v_2, v_3v_5\}) \cup \{v_2v_3, v_1v_5\}$ we therefore obtain an 8-edge-colouring of (G, m'), a contradiction. This proves 5.7. ▮

5.8 *No minimum 8-counterexample contains Conf(14).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that some edge uv is incident with regions r_1, r_2 where r_1 has at most six doors disjoint from uv, and $m(uv) \geq 5$, and either $m(uv) \geq 6$ or r_2 is small. By exchanging r_1, r_2 if necessary, we may assume that if r_1, r_2 are both small, then the length of r_1 is at least the length of r_2. By 4.5, we do not have Conf(3), so not both r_1, r_2 are triangles, and by 4.2, if $m(uv) \geq 6$ then neither of r_1, r_2 is a triangle; so r_1 is not a triangle. Let x-u-v-y be a path of C_{r_1}. Since $m(e) \geq 5$, this path is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G', m'). Let $k = m'(uv) + m'(xy) \geq 7$. Let $I = \{1, \ldots, 8\} \setminus \{k\}$ if x, y are non-adjacent in G, and $I = \{1, \ldots, 8\}$ if $xy \in E(G)$. For $i \in I$, let Q_i be as in 5.1. Since Q_i contains both uv, xy for each $i \in I$, it follows that for $1 \leq j \leq 8$, F_j contains at most one of uv, xy. Thus we may assume that $uv \in F_i$ for $1 \leq i \leq m'(uv)$, and $xy \in F_i$ for $m'(uv) < i \leq k$. Thus $k \leq 8$. Let D_1 be the set of doors for r_1 that are disjoint from e, and let D_2 be the set of doors for r_2.

(1) *For each $i \in I$, one of $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2$ is nonempty, and if $k = 8$ or $i > k$ then both are nonempty.*

Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = xy$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. Suppose that $k = 8$. Then for $1 \leq j \leq 8$, F_j contains one of e_1, e_2; and hence for all $j \in \{1, \ldots, 8\}$ with $j \neq i$, $e_3, \ldots, e_n \notin F_j$. It follows that e_n, e_{n-1} belong only to F_i and hence $e_n \in F_i \cap Q_i \cap D_2$. Since this holds for all $i \in I$, it follows that $|D_2| \geq |I| \geq 7$. Hence r_2 is big, and so by hypothesis, $m(uv) \geq 6$. Since $k = 8$ it follows that $xy \notin E(G)$. Consequently e_3 is an edge of C_{r_1}, and since e_3, e_4 belong only to F_i, it follows that e_3 is a door for r_1. But $e_3 \neq ux, vy$ from the choice of the switchable path, and so $e_3 \in F_i \cap Q_i \cap D_1$. Hence in this case (1) holds.

Thus we may assume that $k = 7$; and so $m(e) = 5$, and r_2 is small, and $xy \notin E(G)$, and $uv \in F_1, \ldots, F_6$, and $xy \in F_7$. Thus $I = \{1, \ldots, 6, 8\}$. If $i = 8$, then since $uv, xy \in Q_i$ and

F_j contains one of e_1, e_2 for all $j \in \{1, \dots, 7\}$, it follows as before that $e_3 \in F_i \cap Q_i \cap D_1$ and $e_n \in F_i \cap Q_i \cap D_2$. Thus we may assume that $i \leq 6$. For $1 \leq j \leq 8$ with $j \neq i$, $|F_j \cap Q_i| = 1$, and for $1 \leq j \leq 7$, F_j contains one of e_1, e_2. Hence $e_3, \dots, e_n$ belong only to F_i and to F_8, and only one of them belongs to F_8. If neither of e_n, e_{n-1} belong to F_8 then $e_n \in F_i \cap Q_i \cap D_2$ as required; so we assume that F_8 contains one of e_n, e_{n-1}; and so $e_3, \dots, e_{n-2}$ belong only to F_i. Since $n \geq 6$, it follows that $e_3 \in F_i \cap Q_i \cap D_1$ as required. This proves (1).

If $k = 8$, then (1) implies that $|D_1| \geq 7$ as required. So we may assume that $k = 7$ and hence $m(e) = 5$ and $xy \notin E(G)$; and r_2 is small. Suppose that there are three values of $i \in \{1, \dots, 6\}$ such that $|F_i \cap D_1| = 1$ and $F_i \cap D_2 = \emptyset$, say $i = 1, 2, 3$. Let $f_i \in F_i \cap D_1$ for $i = 1, 2, 3$, and we may assume that f_3 is between f_1 and f_2 in the path $C_{r_1} \setminus \{uv\}$. Choose $X \subseteq V(G')$ such that $\delta_{G'}(X) = Q_3$. Since only one edge of $C_{r_1} \setminus \{e\}$ belongs to Q_3, one of f_1, f_2 has both ends in X and the other has both ends in $V(G') \setminus X$; say f_1 has both ends in X. Let Z be the set of edges of G' with both ends in X. Thus $(F_1 \cap Z) \cup (F_2 \setminus Z)$ is a perfect matching, since $e \in F_1 \cap F_2$, and no other edge of $\delta_{G'}(X)$ belongs to $F_1 \cup F_2$; and similarly $(F_2 \cap Z) \cup (F_1 \setminus Z)$ is a perfect matching. Call them F_1', F_2' respectively. Then $F_1', F_2', F_3, F_4, \dots, F_8$ form an 8-edge-colouring of (G', m'), yet f_1, f_2 are the only edges of $D_1 \cup D_2$ included in $F_1' \cup F_2'$, and neither of them is in F_2', contrary to (1). Thus there are no three such values of i; and similarly there are at most two such that $|F_i \cap D_2| = 1$ and $F_i \cap D_1 = \emptyset$. Thus there are at least three values of $i \in I$ such that $|F_i \cap D_1| + |F_i \cap D_2| \geq 2$ (counting $i = 8$), and so $|D_1| + |D_2| \geq 10$. But $|D_1| \leq 6$ by hypothesis and $|D_2| \leq 3$ since r_2 is small, a contradiction. This proves 5.8. ∎

5.9 *No minimum 8-counterexample contains Conf(15) or Conf(16).*

Proof. To handle both at once, we assume that (G, m) is an 8-target with a region r, and $uv \in E(C_r)$, and uvw is another region, satisfying:

- either (G, m) is a minimum 8-counterexample, or $m(uv) = 0$ and deleting uv gives a minimum 8-counterexample
- $m(uv) + m^+(uw) \geq 4$
- every edge of C_r not incident with u is 3-heavy
- let tu be the second edge of C_r incident with u; then the path t-u-w-v is switchable.

Note that while Conf(16) fits these conditions, some instances of Conf(15) may not, and we will handle them later. Let (G', m') be obtained by switching on the path t-u-w-v, and let $F_1, \dots, F_8$ be an 8-edge-colouring of it. Let $k = m(uw) + m(uv) + 2 \geq 5$; then we may assume that $F_1, \dots, F_{k-1}$ contain one of uw, uv, and $tv \in F_k$. Let $I = \{1, \dots, 8\}$ if $tv \in E(G)$, and $I = \{1, \dots, 8\} \setminus \{k\}$ otherwise. For each $i \in I$ let Q_i be as in 5.1. Thus each Q_i contains all of uw, uv, tv, and so no edge of $Q_i \setminus \{uw, uv, tv\}$ belongs to F_j for any $j \neq i$ with $j \leq k$.

(1) $k = 5$.

For suppose that $k \geq 6$. Choose $i \in I \cap \{7, 8\}$. Since Q_i contains uv, uw, tv, it follows that $F_1, \dots, F_6$

all contain an edge in $\{uv, uw, tv\} \cap Q_i$; and hence no edge of $Q_i \setminus \{uv, uw, tv\}$ belongs to any of $F_1, \ldots, F_6$. Choose an edge f of $C_r \setminus \{u, v\}$ with $f \in Q_i$. Now $f \neq tu$ by the choice of switchable path, and so f is 3-heavy (with respect to (G, m)), and if $f = tv$ then $m'(f) > m(f)$. Consequently there are three values of $j \in \{1, \ldots, 8\} \setminus \{k\}$ such that $F_j \cap Q_i$ contains an edge different from uv, uw, and hence some such j belongs to $\{1, \ldots, 5\}$, a contradiction. This proves (1).

Let r_1 be the second region for uw, and let D_1 be the set of doors for r_1. From (1) it follows that r_1 is small, and so $|D_1| \leq 3$.

(2) *For $i = 6, 7, 8$, $|F_i \cap D_1| = 1$; and the edges of F_6 and F_8 in Q_7 have a common end (they may be the same).*

For let $i \in \{6, 7, 8\}$; then $i \in I$. Let the edges of Q_i be $e_1, \ldots, e_n, e_1$ in order, where $e_1 = uw$, $e_2 = uv$ and $e_3 = tv$. Then $n \geq 7$, since $|F_i \cap Q_i| \geq 5$. Let $h = 3$ if $tv \in E(G)$, and $h = 4$ otherwise. Then e_h is an edge of C_r not incident with u, and so it is 3-heavy; and hence either $m(e_h) \geq 3$, or the second region for e_h is a triangle and e_{h+1} is an edge of it, and $m(e_h) + m(e_{h+1}) \geq 3$. Moreover, if $e_h = tv$ then $m'(e_h) > m(e_h)$. Thus in all cases it follows that there are three values of $j \neq 5$ with $1 \leq j \leq 8$ such that $F_j \cap Q_i$ contains one of e_h, e_{h+1}. We deduce that these three values of j are $6, 7, 8$, since $F_j \cap Q_i \subseteq \{uv, uw\}$ for $1 \leq j \leq 4$. Consequently for $1 \leq j \leq 8$, $F_j \cap Q_i$ includes one of e_1, e_2, e_3, e_4, e_5. It follows that only F_i contains e_n, e_{n-1}, and consequently $e_n \in F_i \cap D_1$. Since $|D_1| \leq 3$, this proves the first assertion of (2). The second follows since, taking $i = 7$ and defining e_h as before, F_6 and F_8 each contain one of e_h, e_{h+1}, and these edges have a common end. This proves (2).

Let $F_i \cap D_1 = \{f_i\}$ for $i = 6, 7, 8$. Thus f_6, f_7, f_8 are distinct, and we may assume that f_6, f_7, f_8 are in order in the path $C_{r_1} \setminus \{uw\}$. Choose $X \subseteq V(G)$ with $\delta_{G'}(X) = Q_7$. Let H be the subgraph of G' with vertex set $V(G)$ and edge set $(F_6 \setminus F_8) \cup (F_8 \setminus F_6)$. Thus each component of H is either a single vertex or a cycle of even length. Now there are either no edges, or two edges, of H that belong to $\delta_{G'}(X)$; and if there are two then they have a common end by (2). It follows that the component of H, say C, that contains f_6 does not contain f_8. Let $F_6' = (F_8 \cap E(C)) \cup (F_6 \setminus E(C))$ and $F_8' = (F_6 \cap E(C)) \cup (F_8 \setminus E(C))$; then F_6', F_8' are perfect matchings of G', and $F_1, \ldots, F_5, F_6', F_7, F_8'$ is an 8-edge-colouring of (G, m'). On the other hand both f_6, f_8 belong to F_8', so this 8-edge-colouring does not satisfy (2), a contradiction.

This completes the argument for Conf(16), and also for Conf(15) when (with notation as in the definition of Conf(15)) the two edges of C_r consecutive with e both have multiplicity at most $m(e)$ (to see this, let u-w-v be a subpath of C_r where $e = uw$, and add a new edge uv with multiplicty zero). Now we handle the remaining case of Conf(15); we assume that

- (G, m) is a minimum 8-counterexample
- r is a region of length at least four, and e is an edge of C_r
- $m^+(e) \geq 4$, and every edge of C_r disjoint from e is 3-heavy
- one of the edges of C_r incident with e has multiplicity more than $m(e)$.

Let C_r have vertices $v_1, \dots, v_p$ in order, where $p \geq 4$, $e = v_1v_2$, and $m(v_2v_3) > m(e)$. It follows that $m(v_1v_2) = 3$ and $m(v_2v_3) = 4$. From 4.5, we do not have Conf(4) so $p \geq 5$. The path v_1-v_2-v_3-v_4 is switchable; let (G, m') be obtained by switching on it. We may assume that $v_2v_3 \in F_i$ for $1 \leq i \leq 5$ and $v_1v_4 \in F_6$. Since $m'(v_1v_2) = 2$ and v_1v_2 meets both v_2v_3 and v_1v_4, it follows that $v_1v_2 \in F_7, F_8$. Consequently $v_pv_1 \in F_h$ for some h with $1 \leq h \leq 5$. Let $I = \{1, \dots, 8\} \setminus \{6\}$. For each $i \in I$ let Q_i be as in 5.1. Now Q_7 contains v_2v_3, v_1v_4, and so for $1 \leq j \leq 6$, $F_j \cap Q_7 \subseteq \{v_2v_3, v_1v_4\}$. In particular $v_pv_1 \notin Q_7$. But Q_7 contains an edge f of C_r, different from v_1v_2, and this edge is 3-heavy, since it is different from v_pv_1 and hence disjoint from e; and so $F_j \cap Q_i \setminus \{v_2v_3, v_1v_4\} \neq \emptyset$ for three values of $j \in \{1, \dots, 8\}$, a contradiction. This proves 5.9.

5.10 *No minimum* 8*-counterexample contains Conf(17) or Conf(18).*

Proof. To handle both at once, we assume that (G, m) is an 8-target with a region r with length at least four, and $uv \in E(C_r)$, and uvw is another region, satisfying:

- either (G, m) is a minimum 8-counterexample, or $m(uv) = 0$ and deleting uv gives a minimum 8-counterexample
- $m(uv) + m^+(uw) \geq 5$
- let t, x be the second neighbours of u, v in C_r respectively; if $m(uv) = 3$ and uv is 5-heavy let $P = C_r \setminus \{u, v\}$, and otherwise let $P = C_r \setminus \{u\}$; then every edge f of P satisfies $m^+(f) \geq 2$, and at most one edge of P is not 3-heavy
- $m(tu), m(vw) \leq m(uw)$.

The path t-u-w-v is switchable; let (G', m') be obtained by switching on it, and let $F_1, \dots, F_8$ be an 8-edge-colouring of (G', m'). Since r has length at least four, $tv \notin E(G)$. Let $k = m(uw) + (uv) + 2 \geq 6$; we may assume that F_i contains one of uv, uw for $1 \leq i < k$, and F_k contains tv. Let $I = \{1, \dots, 8\} \setminus \{k\}$; and for each $i \in I$ let Q_i be as in 5.1.

(1) *There is at most one value of $i \in I$ such that $Q_i \cap E(P) = \emptyset$, and if i is such a value then $k = 7$ and $m(uv) = 3$ and $m(uw), m(vw) = 2$ and $uw \in F_i$.*

For suppose that $i \in I$ and $Q_i \cap E(P) = \emptyset$. It follows that $P = C_r \setminus \{u, v\}$, and so $m(uv) = 3$ and uv is 5-heavy. Hence $m(uw), m(vw) \geq 2$, and so $m(uw), m(vw) = 2$ by 4.2, and $k = 7$. Now for $1 \leq i \leq 7$, F_i contains one of uw, uv, tv, and since vw meets all of these edges it follows that $vw \in F_8$. But vx belongs to some F_j such that F_j contains none of tv, uv, vw, and so $uw \in F_j$. Then $|F_j \cap Q_i| \geq 2$, so $j = i$ and hence $uw \in F_i$. This proves (1).

Let I' be the set of $i \in I$ such that $Q_i \cap E(P) \neq \emptyset$. By (1), $|I'| \geq 6$. Let r_1 be the second region for uw, and let its set of doors be D_1. Thus $|D_1| \leq 3$ if $k = 6$, since $m(uv) + m^+(uw) \geq 5$. Let I'' be the set of $i \in I'$ such that the edge in $Q_i \cap E(P)$ is not 3-heavy.

(2) *There is a unique edge $f \in E(P)$ that is not 3-heavy, and it belongs to none of $F_1, \dots, F_k$. Moreover, if $i \in I' \setminus I''$ then $k = 6$ and $i \leq 5$ and $F_i \cap Q_i \cap D_1 \neq \emptyset$.*

Suppose that $i \in I' \setminus I''$. There are therefore three values of $j \in \{1, \ldots, 8\}$ such that $F_j \cap Q_i \not\subseteq \{uw, uv, tv\}$, and so at least two that are also different from i. Consequently, for those two values of j, it follows that $uw, uv, yv \notin F_j$ and hence $k = 6$ and $j \in \{7, 8\}$. Thus $i \leq 5$. Let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uw$, $e_2 = uv$ and $e_3 = tv$; then $n \geq 7$, since $|F_i \cap Q_i| \geq 5$. But $F_1, \ldots, F_8$ each contain one of $e_1, \ldots, e_5$, so

$e_n \in F_i \cap Q_i \cap D_1$. This proves the second assertion of (2). For the first assertion, since $|D_1| \leq 3$, it follows that $|I' \setminus I''| \leq 3$. Since $|I'| \geq 6$, it follows that $|I''| \geq 3$. But by hypothesis, there is at most one edge in P that is not 3-heavy, and so this edge exists, say f. It follows that $f \in Q_i$, for all $i \in I''$. Now let $j \in \{1, \ldots, k\}$. Choose $i \in I''$ with $i \neq j$; then $F_j \cap Q_i \subseteq \{uw, uv, tv\}$, and so F_j does not contain f. This proves (2).

By (2) we may assume that $f \in F_{k+1}$. Let r_2 be the second region at f, and let D_2 be its set of doors. By hypothesis, if $m(f) = 1$ then $|D_2| \leq 3$.

Suppose that $k \geq 7$. By (2), $I'' = I'$ and $m(f) = 1$. Let $i \in I'$, and let the edges of Q_i in order be $e_1, \ldots, e_n$, where $e_1 = uw$, $e_2 = uv$, $e_3 = tv$, and $e_4 = f$. Since only one of $e_1, \ldots, e_4$ belongs to F_i, and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 8$. But $F_1, \ldots, F_8$ each contain one of $e_1, \ldots, e_4$, and so $e_5, \ldots, e_n$ only belong to F_i; and hence $e_5 \in F_i \cap Q_i \cap D_2$. Consequently $|D_2| \geq |I'| \geq 6$, a contradiction.

This proves that $k = 6$, and hence $|D_1| \leq 3$, and $I' = I$ by (1), and $7, 8 \in I''$ by (2). Now let $i \in I''$. Let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uw$, $e_2 = uv$, $e_3 = tv$, and $e_4 = f$. Again $n \geq 8$.

Suppose that $m(f) \geq 2$; then $m(f) = 2$ by (2), and $f \in F_7, F_8$, and so $F_1, \ldots, F_8$ each contain one of $e_1, \ldots, e_4$, and therefore $e_5, \ldots, e_n$ belong to no F_j with $j \neq i$. Since $n \geq 8$, it follows that $e_n \in D_1$, and so $F_i \cap Q_i \cap D_1 \neq \emptyset$. By (2), it follows that $F_i \cap Q_i \cap D_1 \neq \emptyset$ for all $i \in I'$, and so $|D_1| \geq |I'| = 7$, a contradiction. Thus $m(f) = 1$, and so $|D_2| \leq 3$.

Again, let $i \in I''$, and let $e_1, \ldots, e_n, e_1$ be as before. Now $F_1, \ldots, F_7$ each contain one of $e_1, \ldots, e_4$, and so $e_5, \ldots, e_n$ belong to no F_j with $1 \leq j \leq 7$ and $j \neq i$, and only one of them belongs to F_8 if $i \neq 8$. We assume first that $i \neq 8$. Since $n \geq 8$, either $e_5, e_6 \notin F_8$, or $e_n, e_{n-1} \notin F_8$, and so either $e_5 \in D_2$ or $e_n \in D_1$. Now we assume $i = 8$. Then $e_5, \ldots, e_n$ belong to no F_j with $1 \leq j \leq 7$, and so $e_5 \in D_2$ and $e_n \in D_1$.

In summary, we have shown that for each $i \in I''$, either $F_i \cap D_1 \neq \emptyset$, or $F_i \cap D_2 \neq \emptyset$ (both if $i = 8$); and $8 \in I''$. By (2), if $i \in I' \setminus I''$ then either $F_i \cap D_1 \neq \emptyset$, or $F_i \cap D_2 \neq \emptyset$; and so $|D_1| + |D_2| \geq |I'| + 1 \geq 7$, a contradiction. This proves 5.10. ∎

5.11 *No minimum* 8*-counterexample contains Conf(19).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that r is a region with length at least five, and e is an edge of C_r, such that $m^+(e) \geq 5$, and every edge of C_r disjoint from e is 2-heavy, and at most two of them are not 3-heavy. By 5.10, we do not have Conf(17), so there are at least two edges in C_r disjoint from e that are not 3-heavy, and so by hypothesis, there are exactly two, say g_1, g_2. Thus $m(g_1), m(g_2) \leq 2$. By hypothesis, g_1, g_2 are 2-heavy.

Let $e = uv$, and let the second neighbours of u, v in C_r be t, w respectively. Since $m(e) \geq 4$, it follows that $m(tu), m(vw) \leq m(uv)$ and so the path t-u-v-w is switchable. Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. Let $k = m(e) + 2$. We may assume that $tw \in F_k$. Let $I = \{1, \ldots, 8\} \setminus \{k\}$, and for each $i \in I$ let Q_i be as in 5.1. Let I_1, I_2, I_3 be the sets of $i \in I$ such that $g_1 \in Q_i$, $g_2 \in Q_i$, and $g_1, g_2 \notin Q_i$ respectively.

(1) $k = 6$.

For suppose that $k > 6$. Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = tw$. Thus e_3 is an edge of C_r disjoint from e. Since $|F_i \cap Q_i| \geq 5$ and $|F_i \cap \{e_1, e_2\}| \leq 1$, it follows that $n \geq 6$. Now there are $k \geq 7$ values of $j \in \{1, \ldots, 8\}$ such that F_j contains one of e_1, e_2; and so there is at most value of $j \neq i$ such that F_j contains one of e_3, e_4. It follows that e_3 is not 3-heavy and so $i \in I_1 \cup I_2$. Since this holds for all $i \in I$, we may assume that $|I_1| \geq 4$. Let $i \in I_1$; as before, there is at most one value of $j \neq i$ such that F_j contains one of e_3, e_4. Now $m(g_1) \leq 2$. If $m(g_1) = 2$, then $g_1 \in F_i$, and since this holds for all $i \in I_1$ it follows that g_1 is contained in F_i for four different values of i, a contradiction. Thus $m(g_1) = 1$. Since g_1 is 2-heavy, the second region for g_1 is a triangle with edge set $\{g_1, p, q\}$ say, where $e_4 = p$. Hence one of g_1, p_q has multiplicity one and is contained in F_i. Since this holds for all $i \in I_1$ and $|I_1| \geq 4$, this is impossible. This proves (1).

We may therefore assume that $uv \in F_i$ for $1 \leq i \leq 5$ and $tw \in F_6$. Since $k = 6$, it follows that $m(e) = 4$ and since $m^+(e) \geq 5$, the second region r_1 for uv is small. Let D_1 be its set of doors.

(2) *If $i \in I_3$ then $i \leq 5$ and $F_i \cap Q_i \cap D_1 \neq \emptyset$.*

For let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = tw$. Then $F_1, \ldots, F_6$ each contain an edge in $\{e_1, e_2\}$, and so for $1 \leq j \leq 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belongs to F_j. Now e_3 is 3-heavy, and so there are three values of j such that F_j contains one of e_3, e_4; and so these three values are $i, 7, 8$, and $i \neq 7, 8$. (Thus $i \leq 5$ since $6 \notin I$.) Hence for $1 \leq j \leq 8$, F_j contains one of $e_1, \ldots, e_4$; and so e_n, e_{n-1} belong only to F_i. Hence $e_n \in D_1$. This proves (2).

For $h = 1, 2$, let I'_h be the set of all $i \in I_h$ such that $F_i \cap Q_i \cap D_1 = \emptyset$.

(3) *For $h = 1, 2$, $|I'_h| \leq 2$, and $7, 8 \notin I'_h$, and if $|I'_h| = 2$ then $7, 8 \notin I_h$.*

For let $h = 1$ say. Suppose first that $m(g_1) = 2$, and let $g_1 \in F_a, F_b$ where $1 \leq a < b \leq 8$. Let $i \in I'_1$, and let $e_1, \ldots, e_n$ be as before; then $e_3 = g_1$. Again, for $1 \leq j \leq 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belongs to F_j, and consequently $a, b \in \{i, 7, 8\}$. In particular, $b \geq 7$, and $a \in \{i, 7\}$. Thus if $a \leq 6$ then $i = a$ and so $|I'_1| = 1$ and the claim holds. We assume then that $(a, b) = (7, 8)$. But then $F_1, \ldots, F_8$ each contain one of e_1, e_2, e_3, and so $e_n \in D_1$, contradicting that $i \in I'_1$. So the claim holds if $m(g_1) = 2$.

Next we assume that $m(g_1) = 1$. Since g_1 is 2-heavy, the second region at g_1 is a triangle with edge set $\{g_1, p, q\}$ say. Let $g_1 \in F_a$. Let $i \in I'_1$, and let $e_1, \ldots, e_n$ be as before; then $e_3 = g_1$. Again, for $1 \leq j \leq 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belong to F_j, and consequently $a \in \{i, 7, 8\}$. Thus if $a \neq 7, 8$ then $i = a$ and $|I'_1| = 1$ and the claim holds. We assume then that $a = 7$. Thus each of $F_1, \ldots, F_7$ contains one of e_1, e_2, e_3, and for $1 \leq j \leq 7$ with $j \neq i$, F_j contains none of $e_4, \ldots, e_n$. Since $F_i \cap Q_i \cap D_1 = \emptyset$, there exists $j \in \{1, \ldots, 8\}$ with $j \neq i$ such that F_j contains one of e_n, e_{n-1}; and hence $j = 8$, and so $i \neq 8$. (Also, $i \neq 7$ since $g_1 \in F_7$ and g_1 meets e_4. Consequently, $7, 8 \notin I'_1$.) Thus $F_1, \ldots, F_8$ each contain one of $e_1, e_2, e_3, e_{n-1}, e_n$, and so e_4 is only contained in F_i. Consequently, i

has the property that one of p, q has multiplicity one, and F_i contains it. Thus there are at most two such values of i, and so $|I'_1| \leq 2$. Moreover, if there are two such values, say c, d, then $c, d \leq 5$ and F_c contains one of p, q and F_d contains the other. Consequently if $7 \in I_1$, then one of F_c, F_d contains two edges of Q_7, a contradiction. So if $|I'_1| = 2$ then $7, 8 \notin I_1$. This proves (3).

From (2), we may assume that $7 \in I_1$, and so $|I'_1| + |I'_2| \leq 3$ by (3). Consequently there are at least four values of $i \in I$ such that $F_i \cap Q_i \cap D_1 \neq \emptyset$, and so $|D_1| \geq 4$, a contradiction. This proves 5.11. ∎

This completes the proof of 4.1 and hence of 1.2. Perhaps despite appearances, there was some system to our choice of the β- and γ-rules. We started with the idea that we would normally pass a charge of one from each small region to each big region sharing an edge with it, and made the minimum modifications we could to the β-rules so that the proof of 3.4 worked. Then we experimented with the γ-rules to make 3.5, 3.6 and 3.7 work out.

It is to be hoped that solving these special cases of the main conjecture 1.1 will lead us to a proof of the general case, but that seems far away at the moment. The same approach does indeed work (more simply) for seven-regular planar graphs, and this gives an alternative proof of the result of [4], to appear in [6]. We tried the same again for nine-regular graphs, but there appeared to be some serious difficulties. Maybe more perseverance will bring it through, but it seems much harder than the eight-regular case.

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