The minimal nonplanar strong digraphs

Stephen Bartell
Princeton University, Princeton, NJ 08544

Paul Seymour[1]
Princeton University, Princeton, NJ 08544

February 13, 2025; revised March 9, 2025

[1]Supported by AFOSR grant FA9550-22-1-0234, and NSF grant DMS-2154169.

Abstract

Kuratowski's theorem says that the minimal (under subgraph containment) graphs that are not planar are the subdivisions of K_5 and of $K_{3,3}$. Here we study the minimal (under subdigraph containment) strongly-connected digraphs that are not planar. We also find the minimal strongly-connected non-outerplanar digraphs and the minimal strongly-connected non-series-parallel digraphs.

1 Introduction

What are the minimal digraphs (under subdigraph containment) that are strong and nonplanar? ("Strong" means strongly-connected.) We will prove that for every such digraph, its underlying graph is a subdivision of a graph of one of the seven types shown in Figures 1 and 2. (We will define these types more precisely later.)

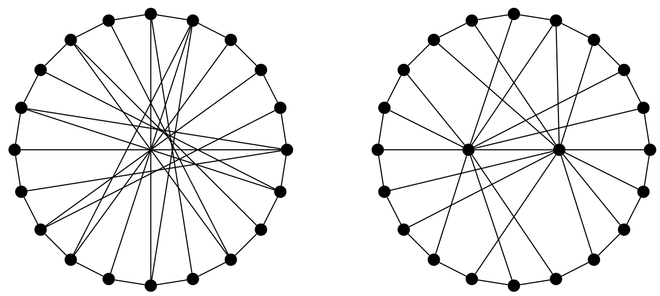

Figure 1: A Möbius chain and a double wheel.

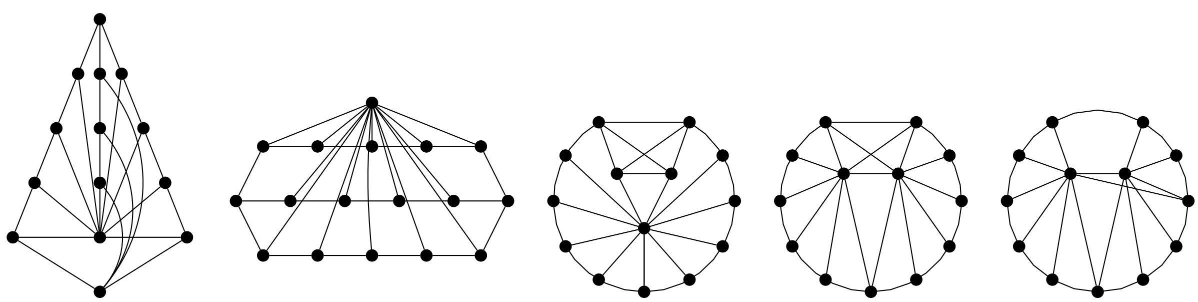

Figure 2: A conch, a mussel, a scallop, a clam and a whelk.

Remarkably, the question reduces, almost exactly, to a question about undirected graphs. Say a nonplanar graph G is *almost-planar* if G is 3-connected and F is the edge-set of a forest, where F is the set of edges e such that $G \setminus e$ is nonplanar. If G is a digraph that is minimal strong and nonplanar, and not the directed subdivision of a smaller digraph, then it is an orientation of an almost-planar graph; and we will find explicitly all the almost-planar graphs. (These are the graphs of the two figures.) Thus we have:

1.1 *A strong digraph is planar if and only if no subdigraph is a directed subdivision of a strong orientation of either a Möbius chain, a double wheel, a conch, a scallop, a mussel, a clam, or a whelk.*

Let us say a *fan* in G is a pair (P, v) where P is a path of G with length at least two, and $v \in V(G) \setminus V(P)$ is adjacent to every vertex of the interior of P (and possibly also adjacent to its ends), and each vertex in the interior of v has degree three in G. If $\{v, x, y\}$ is a triangle of a graph, let us subdivide the edge xy, replacing it by a path P, and make v adjacent to all the new vertices introduced (and possibly delete the edges vx, vy). Then (P, v) is a fan of the new graph, and we call this process *opening a fan.* All the five types of graphs shown in Figure 2 can be obtained from graphs of bounded size (in most cases, from K_5) by opening a fan at most three times, and so in some sense they are "essentially" of bounded size; but this is not so for the types of Figure 1.

1.1 does not quite answer the question of what the minimal strong nonplanar digraphs are. The seven different types of graphs in the figures all give rise to infinitely many instances of minimal strong nonplanar digraphs, but not every graph of these types can be appropriately oriented, and some can be oriented in many ways. This is a non-trivial issue, that we discuss in the final section.

This work grew out of the senior thesis [1] of one of the authors, and a much easier question: what are the minimal digraphs that are strong and not outerplanar? (*Outerplanar* means it can be drawn in a plane with all vertices incident with the infinite region,) For that problem we can list all such digraphs explicitly. Similarly we can find explicitly all minimal digraphs that are strong and not series-parallel (we say a digraph is *series-parallel* if it is an orientation of a graph that contains no subdivision of K_4 as a subgraph).

All graphs and digraphs in this paper are assumed to be finite, and to have no loops or parallel edges, and digraphs have no directed cycles of length two. If G is a digraph and $uv \in E(G)$, *subdividing* uv means making a new digraph by deleting the edge uv and adding a directed path from u to v of new vertices (except for u, v), and we call a digraph obtained from G by iterating this process a *directed subdivision* of G.

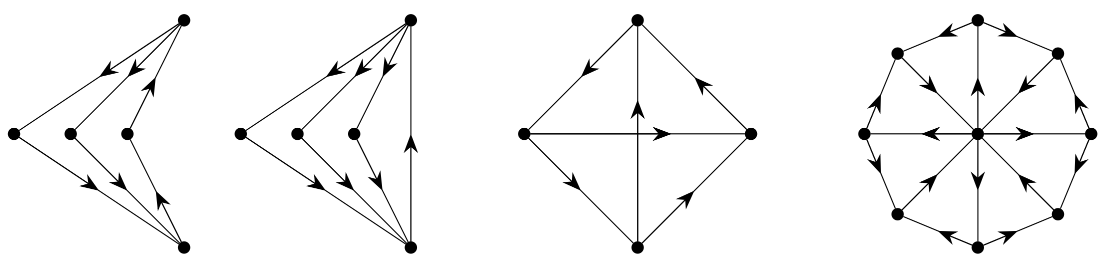

Figure 3: Obstructions for outerplanar.

In Figure 3 we show four digraphs. Any directed subdivision of the first is a *strong theta*, and any directed subdivision of the second is a *reinforced theta*. The third is called the *strong directed* K_4, and the fourth represents a class of digraphs; this one is an *8-diwheel*. In general, if $k \geq 2$ is an integer, a $2k$-*diwheel* is a digraph obtained from a cycle of length $2k$ by adding a new vertex adjacent with every vertex of the cycle, and then directing the edges in such a way that all the triangles become cyclic triangles. We will show:

1.2 *A strong digraph is outerplanar if and only if no subdigraph is a directed subdivision of one of the digraphs in Figure 3; that is, no subdigraph is a strong theta, or a reinforced theta, or a directed subdivision of the strong directed* K_4*, or a directed subdivision of a* $2k$*-wheel for some* $k \geq 2$.

We remark that we could forbid subdividing the rightmost, vertical edge of the second digraph in the figure, because when this is subdivided, the digraph contains a strong theta. Similarly we could forbid subdividing three edges of the strong directed K_4, and we could forbid subdividing the edges of the diwheel incident with the "central" vertex, for the same reason. If we do that, then we have the list of all the minimal digraphs that are strong and not outerplanar.

For series-parallel, the list is the same except we omit the two theta-digraphs. We will show:

1.3 *A strong digraph is series-parallel if and only if no subdigraph is a directed subdivision of the strong directed K_4 or of a $2k$-wheel for some $k \geq 2$.*

The paper is organized as follows. In the next section we prove 1.2 and 1.3, which are both easy. Then we identify all the almost-planar graphs, and this is broken into several steps. First, in section 3 we show that every almost-planar graph that contains a subdivision of the four-rung Möbius ladder V_8 is a Möbius chain. Then in section 4 we introduce two more eight-vertex graphs U_8 and W_8; and show that:

- the only almost-planar graph not containing a subdivision of $K_{3,3}$ is K_5;
- every almost-planar graph that contains a subdivision of $K_{3,3}$ and not of U_8, V_8 or W_8 is either a Möbius chain, scallop or clam;
- every almost-planar graph that contains a subdivision of W_8 and not of V_8 is a double wheel;
- every almost-planar graph that contains a subdivision of U_8 and not of V_8 is either a double wheel, conch, whelk or mussel.

This completes the description of all almost-planar graphs. Finally, in section 6 we discuss how to direct the edges of an almost-planar graph to make a minimal strong nonplanar digraph.

2 The outerplanar and series-parallel theorems

We will often speak of the paths and cycles of a digraph G. If we mean directed paths or directed cycles we will say so, and otherwise we mean a digraph P such that P^- is a path or cycle of G^-; it need not be directed. The following lemma is fundamental to the paper (it is closely related to theorem 3.3 of [3]):

2.1 *Let G be a strong digraph, and let C be a cycle of G. Then either there is an edge e of C such that $G \setminus e$ is strong, or there is a partition (A, B) of $V(G)$ such that there is only one edge from A to B and only one from B to A.*

Proof. For each edge e, let $n(e)$ be the number of directed cycles of G that contain e, and choose $e \in E(C)$ with $n(e)$ minimum. Let $e = ab$. If $G \setminus e$ is strong, the theorem holds, so we assume that $G \setminus e$ is not strong; and so there is a partition (A, B) of $V(G)$ with $A, B \neq \emptyset$, such that there are no edges of $G \setminus e$ from A to B. Consequently, $a \in A$ and $b \in B$, since G is strong. Let $e_1, \ldots, e_k$ be the edges of G from B to A. Every directed cycle of G that contains e_i also contains e, and contain no other edge in $\{e_1, \ldots, e_k\}$; and so $n(e) = n(e_1) + \cdots + n(e_k)$. Since C is a cycle, one of $e_1, \ldots, e_k$ is an edge of C, say e_1. From the choice of e, $n(e_1) \geq n(e) = n(e_1) + \cdots + n(e_k)$, and so $n(e_2), \ldots, n(e_k) = 0$. But every edge belongs to a directed cycle, since G is strong, and so $k = 1$, and the theorem holds. This proves 2.1. ∎

2.2 *Let G be a strong digraph, such that G^- is 2-connected, and let $u, v \in V(G)$ such that $G^- \setminus \{u, v\}$ is disconnected. Let $D_1, \ldots, D_k$ be the components of $G^- \setminus \{u, v\}$. Then for $1 \leq i \leq k$ there is a directed path P_i of $G[V(D_i) \cup \{u, v\}]$, from u to v or from v to u, with length at least two.*

Proof. There may be an edge of G joining u, v; let F be the set of all such edges. Let $G_i = G[V(D_i) \cup \{u, v\}] \setminus F$. Since G^- is 2-connected, there is a path Q_i of G_i between u, v. Let $e \in E(Q_i)$; then at least one end of e is in $V(D_i)$. But e belongs to a directed cycle C_e of G, since G is strong, and if this cycle contains both u, v then it contains a directed path between u, v containing e and the theorem holds. So we assume that for each $e \in E(Q_i)$, not both u, v belong to $V(C_e)$, and so C_e is a subdigraph of G_i. But then the union of the C_e ($e \in E(Q_i)$) is strong, and contains both u, v, and none of its edges joins u, v; and so it contains a directed path from u to v and from v to u, both of length at least two. This proves 2.2. ∎

A *k-wheel* is a graph obtained from a cycle of length k by adding a new vertex adjacent to each vertex of the cycle. When G is a graph and $F \subseteq E(G)$, an *F-cycle* of G means a cycle C of G with $E(C) \subseteq F$.

2.3 *Let G be a 2-connected graph with a subgraph that is a subdivision of K_4. Let F be the set of all edges e such that $G \setminus e$ contains a subdivision of K_4 as a subgraph, and suppose there is no F-cycle. Then G is a subdivision of a k-wheel for some $k \geq 3$.*

Proof. Choose $k \geq 3$ maximum such that there is a subgraph H of G that is a subdivision of a k-wheel. Consequently there is a cycle C of G, and a vertex $p_0 \notin V(C)$, and k paths $P_1, \ldots, P_k$ of H, each with one end p_0 and otherwise pairwise vertex-disjoint, and each with an end in $V(C)$ (say p_i) and with no other vertex in $V(C)$.

We may assume (for a contradiction) that $H \neq G$. Since G is 2-connected, there is a path Q of G with distinct ends $u, v \in V(H)$ and with no other vertex or edge in H. Suppose first that $u = p_0$. If $v \in V(P_i)$ for some i, then there is an F-cycle formed by Q and the subpath of P_i between u, v, a contradiction; so $v \in V(C)$, and G contains a subdivision of a $(k+1)$-wheel, again a contradiction. Thus $u, v \neq p_0$.

(1) $k \geq 4$.

Suppose that $k = 3$. Then, from the symmetry between p_0, p_1, p_2, p_3, it follows that $u, v \neq p_0, p_1, p_2, p_3$. If there is a path R of H between u, v containing at most one of p_0, p_1, p_2, p_3, then $E(R) \subseteq F$, and so $Q \cup R$ is an F-cycle, a contradiction. So every path of H between u, v passes through at least two of p_0, p_1, p_2, p_3. But then adding Q to H makes a subdivision of $K_{3,3}$, and so all its edges are in F, a contradiction. This proves (1).

If $u, v \in V(P_1 \cup \cdots \cup P_k)$, let R be the path between them contained in $P_1 \cup \cdots \cup P_k$. Since every edge of $P_1 \cup \cdots \cup P_k$ is in F, it follows that $Q \cup R$ is an F-cycle, a contradiction. So we assume that $u \in V(C) \setminus \{p_1, \ldots, p_k\}$. Hence there is a path R of $H \setminus p_0$ between u, v such that two consecutive vertices in $\{p_1, \ldots, p_k\}$ are not in $V(R)$. Then $E(R) \subseteq F$, and $Q \cup R$ is an F-cycle, a contradiction. Thus $H = G$. This proves 2.3. ∎

Consequently:

2.4 *Let G be a strong digraph, such that G^- is 3-connected. Then either G is a strong directed K_4, or G is a k-diwheel for some even $k \geq 4$, or there is an edge e such that $G \setminus e$ is strong and not series-parallel.*

Proof. Let F be the set of all edges e such that $G^- \setminus e$ contains a subdivision of K_4 as a subgraph. If C is a F-cycle, then by 2.1, $G \setminus e$ is strong for some $e \in E(C)$; and since $G \setminus e$ is strong and not series-parallel, the third outcome holds.

So we assume that there is no F-cycle. By 2.3, G^- is a subdivision of a k-wheel for some $k \geq 3$, and hence is a k-wheel, since it is 3-connected. So there is a cycle C of G of length k, and a vertex $p_0 \notin V(C)$, adjacent with each vertex of $V(C)$.

If $k = 3$, it follows that G is a strong directed K_4, as required, so we assume that $k \geq 4$. We claim that k is even and G is a k-diwheel. Let A be the set of $v \in V(G) \setminus \{p_0\}$ with zero indegree in C, and let B be those with zero out-degree. Thus $|A| = |B|$; if $v \in A$ then $vp_0 \in E(G)$ (because G is strong), and similarly if $v \in B$ then $p_0v \in E(G)$. We need to show that $A \cup B = V(C)$. If $A = \emptyset$ (and hence $B = \emptyset$), C is a directed cycle; choose three edges of G incident with p_0, at least one with head p_0 and at least one with tail p_0. Then C together with these three edges is strong and not series-parallel, and the third outcome holds since $k \geq 4$. Next, suppose that $|A| = 1$ and hence $|B| = 1$, and let $A = \{a\}$ and $B = \{b\}$. So C is the union of two directed paths both from a to b, The digraph formed by the union of C and the edges bp_0, p_0a and a third edge incident with p_0 is strong and not series-parallel, and again the third outcome holds since $k \geq 4$. So $|A| = |B| \geq 2$. The subdigraph of G consisting of C and all edges between p_0 and $A \cup B$ is strong and not series-parallel, and so if $A \cup B \neq V(C)$ then the third outcome holds, and if $A \cup B = V(C)$ then k is even and G is a k-diwheel. This proves 2.4. ∎

Now we prove 1.2, which we restate:

2.5 *A strong digraph is outerplanar if and only if no subdigraph is a strong theta, or a reinforced theta, or a directed subdivision of the strong directed K_4, or of a $2k$-wheel for some $k \geq 2$.*

Proof. Let us say a digraph is *forbidden* if either it is a strong theta, or a reinforced theta, or a directed subdivision of the strong directed K_4, or of a $2k$-wheel for some $k \geq 2$. The "only if" part of the theorem is clear, so we prove "if". Suppose there is a strong digraph G that contains no forbidden digraph as a subdigraph and is not outerplanar; and choose such a digraph G with $|V(G)|$ minimum.

(1) *G^- is 3-connected.*

It follows easily that G^- is 2-connected. Suppose that $u, v \in V(G)$ such that $G^- \setminus \{u, v\}$ is disconnected. Let $D_1, \ldots, D_k$ be the components of $G^- \setminus \{u, v\}$, and for $1 \leq i \leq k$ let P_i be a directed path of $G[V(D_i) \cup \{u, v\}]$, from u to v or from v to u, with length at least two (this exists by 2.2). If each P_i is from u to v, then since G is strong, there is a directed path from v to u; and either it has length at least two, and is contained in one of the graphs $G[V(D_i) \cup \{u, v\}]$ (and so could replace P_i), or it has length one. So we can assume that if $P_1, \ldots, P_k$ are all from u to v, there is an edge vu; and similarly if they are all from v to u, then there is an edge uv. Consequently, if $k \geq 3$ then G contains a strong theta or a reinforced theta as a subdigraph, a contradiction; and so $k = 2$.

Let P_1 be from u to v say. Let $G_1 = G[V(D_i) \cup \{u, v\}]$, and let H_1 be obtained from G_1 by adding a directed edge vu if it is not already present. Thus there is a subdigraph of G isomorphic to a directed subdivision of H_1, and so if H_1 contains a forbidden digraph then so does G, which is impossible. Hence H_1 contains no forbidden subdigraph. But H_1 is strong, and so is outerplanar, since H_1 has size less than that of G. Since D_1 is connected, it follows that $H_1 \setminus \{u, v\}$ is connected, and so there is an outerplanar drawing of H_1, such that the edge vu is incident with the infinite region. Similarly, choose an outerplanar drawing of H_2 (defined in the same way, starting with D_2), with vu incident with the infinite region. By combining these drawings, we obtain an outerplanar drawing of G, a contradiction. This proves (1).

From 2.4, either G is a strong directed K_4, or G is a $2k$-diwheel for some $k \geq 2$, or there is an edge e such that $G \setminus e$ is strong and not series-parallel. In the first two cases the result holds, and in third case is impossible from the minimality of G. This proves that there is no such G, and so proves 2.5. ∎

Now we prove the theorem for series-parallel digraphs, which we restate:

2.6 *A strong digraph is series-parallel if and only if no subdigraph is a directed subdivision of the strong directed K_4 or of a $2k$-wheel for some $k \geq 2$.*

Proof. We only prove the "if" part, since "only if" is clear.

Suppose for a contradiction that there is a strong digraph G that is not series-parallel, and no subdigraph is either a directed subdivision of either the strong directed K_4 or of a $2k$-wheel for some $k \geq 2$. Choose G with $|V(G)|$ minimum. It follows that G^- is 2-connected. If G^- is not 3-connected, choose $u, v \in V(G)$ such that $G^- \setminus \{u, v\}$ is not connected. For each component D of $G \setminus \{u, v\}$, as in the proof of 1.2 we can add one of uv, vu to $G[V(D) \cup \{u, v\}]$ to make it strong, and consequently series-parallel. Since this holds for each such D, it follows that G is series-parallel, a contradiction. Consequently G^- is 3-connected.

By 2.4, either G is a strong directed K_4, or G is a k-diwheel for some $k \geq 3$, or there is an edge e such that $G \setminus e$ is strong and not series-parallel. The first two cases do not hold from the choice of G, and the third case is impossible from the minimality of G. This proves that there is no such G, and so proves 2.6. ∎

3 Almost-planar graphs containing a 4-rung Möbius ladder

Now we turn to the proof of 1.1, which occupies most of the rest of the paper. Let us say G is a *Kuratowski digraph* if G is strong and nonplanar, and minimal (under subdigraph containment) with these properties, and not the directed subdivision of a smaller digraph that is strong and nonplanar. We saw in the previous section that finding the minimal digraphs that are strong and not series-parallel more-or-less reduces to find the 3-connected graphs G with no F-cycle, where F is the set of edges e such that $G \setminus e$ is not series-parallel. Similarly, we will see that all Kuratowski digraphs are orientations of almost-planar graphs. Let us see first that:

3.1 *Let G be a digraph minimal (under subdigraph) with the property that it is strong and nonplanar. Then G is a directed subdivision of a digraph H such that H^- is almost-planar.*

Proof. We proceed by induction on $|V(G)|$. If G is a directed subdivision of a digraph H with $|V(H)| < |V(G)|$, then H is also minimal (under subdigraph) with the property that it is strong and nonplanar; but the theorem holds for H, from the inductive hypothesis, and so also holds for G. Hence we may assume that there is no such H, and so G is a Kuratowski digraph. In this case we will prove that G^- is almost-planar.

Clearly G^- is 2-connected; suppose that $u, v \in V(G)$, and $G^- \setminus \{u, v\}$ is disconnected. Let $D_1, \ldots, D_k$ be the components of $G^- \setminus \{u, v\}$. As in the proof of 1.2, for $1 \le i \le k$ we can add one of the edges uv, vu to $G[D_i \cup \{u, v\}]$ to make it strong, and hence planar from the inductive hypothesis. But then G is planar, a contradiction. This proves that G^- is 3-connected.

Let F be the set of edges e such that $G \setminus e$ is not planar. If there is an F-cycle C in G, then by 2.1 there is an edge $e \in E(C)$ such that $G \setminus e$ is strong; and since $e \in F$, it follows that $G \setminus e$ is strong and nonplanar, a contradiction to the minimality of G. So G^- is almost-planar. This proves 3.1. ∎

The value of this theorem is that there is something close to a converse: each of the seven types of almost-planar graphs gives rise to infinitely many Kuratowski digraphs. We will discuss how to recover a suitable orientation of G later.

In this and the next two sections we find all graphs that are almost-planar.

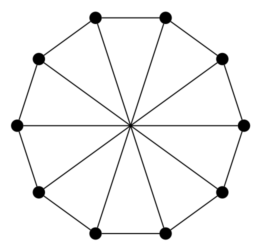

Figure 4: A five-rung Möbius ladder.

For $k \ge 2$, the *Möbius ladder* with k rungs is the graph obtained from a cycle of length $2k$ by making opposite vertices of the cycle adjacent. Thus the 2-rung Möbius ladder is K_4, the 3-rung ladder is $K_{3,3}$, and the 4-rung ladder is the graph known as V_8 or the "Wagner graph". Let us say the *ladder number* of a graph G is the maximum k such that G has a subgraph that is a subdivision of the k-rung Möbius ladder. (This is well-defined for almost-planar graphs since they all contain a subdivision of K_4.) We want to prove a theorem describing the 3-connected almost-planar graphs, but its proof breaks into cases depending on ladder number. The case when the ladder number is two is easy, and in the case when the ladder number is at least four the argument is straightforward; but ladder number three is more tricky, and left until last.

We observe:

3.2 *If G is almost-planar, with ladder number two, then $G = K_5$.*

Proof. Let G be almost-planar, with ladder number two. Consequently G is nonplanar, and contains no subdivision of $K_{3,3}$. By a theorem of D. W. Hall [2], it follows that $G = K_5$. This proves 3.2. ∎

Let C be a cycle of a graph G; then edges in $E(G) \setminus E(C)$ with both ends in $V(C)$ are called *chords* of C. If C has vertices c_1-$\cdots$-c_n-c_1 in order, then a chord with ends c_h, c_i *C-crosses* one with ends c_j, c_k if h, i, j, k are all different and both path of C between c_h, c_i contain one of c_j, c_k. Let G be a nonplanar graph obtained from a cycle C by adding chords of C, subject to the rule that every two chords must either C-cross or share an end, and each vertex of C is an end of at least one of the chords, and there is no cycle composed completely of chords. (See Figure 5.) This graph is more general than a Möbius ladder: let us call it a *Möbius chain*, and C is its *base cycle*.

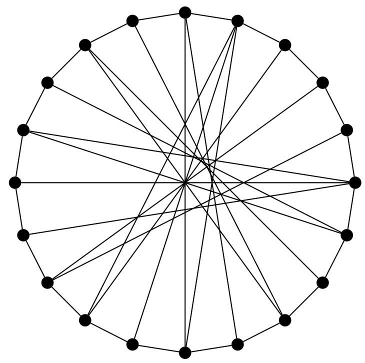

Figure 5: A Möbius chain.

For instance, the graph K_5 does not count as a Möbius chain, because there is a cycle of chords. Möbius chains are almost-planar.

3.3 *If G is an almost-planar graph with ladder number at least four, then G is a Möbius chain.*

Proof. Let k be the ladder number of G, and choose a cycle C of G such that there are vertices $p_1, \dots, p_{2k}$ of C, distinct and in order in C (although possibly C has additional vertices), such that there are paths $P_1, \dots, P_k$ of G, pairwise disjoint, such that for $1 \le i \le k$, P_i has ends p_i and p_{i+k} and has no other vertex or edge in C. Let H be the union of C and the paths $P_1, \dots, P_k$; so H is a subdivision of the k-rung Möbius ladder. For $1 \le i \le 2k$, let C_i be the path of C between p_i, p_{i+1} that does not contain p_{i+2} (reading subscripts modulo $2k$). Let F be the set of all edges e such that $G \setminus e$ is nonplanar. Thus there is no F-cycle in G, since G is almost-planar. Since $k \ge 4$, $E(P_i) \subseteq F$ for $1 \le i \le k$. Let us say an *H-jump* is a triple (Q, u, v) where $u, v \in V(H)$ are distinct and Q is a path of G with ends u, v and with no other vertex of edge in H.

(1) *$u, v \in V(C)$ for every H-jump (Q, u, v).*

Suppose that u belongs to the interior of P_i say. Then $v \notin V(P_i)$, since otherwise the subpath of P_i between u, v makes an F-cycle with Q. There is a path R of H between u, v consisting of a subpath of P_i, a subpath of P_j if v belongs to the interior of some P_j, and a path R' of C such that at most $(k-2)/2$ of $P_1, \dots, P_k$ have an end in the interior of R'. Consequently, the graph obtained from $H \cup Q$ by deleting the interior of R' is nonplanar, and so $E(R') \subseteq F$. It follows that $E(R) \subseteq F$, and therefore $Q \cup R$ is an F-cycle, a contradiction. This proves (1).

(2) *For every H-jump (Q, u, v) and every path R of C between u, v, at least $k-2$ of $P_1, \dots, P_k$ have an end in the interior of R.*

If not, then the graph obtained from $H \cup Q$ by deleting the interior of R is nonplanar, and so $E(R) \subseteq F$, and $Q \cup R$ is an F-cycle, which is impossible. This proves (2).

(3) *For every H-jump (Q, u, v), every path R of C between u, v contains an end of each of $P_1, \dots, P_k$.*

Suppose not; then from (2), not both u, v are ends of $P_1, \dots, P_k$, so we assume that $u \notin \{p_1, \dots, p_{2k}\}$.

We assume that u belongs to the interior of C_{2k}, and $p_1, \dots, p_{k-2}$ belong to the interior of R. The vertex p_{k-1} might belong to the interior of R, or be an end of R (and hence equal v), or not belong to R. But certainly p_1, p_2 belong to the interior of R. The graph obtained from $H \cup Q$ by deleting the interior of C_1 is nonplanar; indeed, if $j \le k-1$ is maximum such that $p_j \in V(R)$, then the graph obtained from $H \cup Q$ by deleting the interior of $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$ is nonplanar. Consequently $E(C_1) \subseteq F$. But the graph obtained from $H \cup Q$ by deleting the interior of C_{k+1} is topologically equivalent to the one obtained by deleting the interior of C_1, and so is also nonplanar; and so $E(C_{k+1}) \subseteq F$. Then $C_1 \cup C_{k+1} \cup P_1 \cup P_2$ is an F-cycle, a contradiction. This proves (3).

(4) *For every H-jump (Q, u, v), exactly one of u, v is in $\{p_1, \dots, p_{2k}\}$.*

If neither of u, v belong to $\{p_1, \dots, p_{2k}\}$, then from (3), $H \cup Q$ is a subdivision of a $(k+1)$-rung Möbius ladder, contradicting that k is the ladder number of G. If both $u, v \in \{p_1, \dots, p_{2k}\}$, then by (3) u, v are ends of the same one of $P_1, \dots, P_k$, which contradicts that there is no F-cycle. This proves (4).

(5) *Each of $P_1, \dots, P_k$ has length one, and $V(H) = V(G)$.*

Suppose that there is a component D of $G \setminus V(H)$. Since G is 3-connected, there are three vertices in $V(H)$ that have a neighbour in $V(D)$, and every two of them can be joined by a path with the properties of the path Q. In each case one of these three paths violates (4). So $V(H) = V(G)$. If some P_i has length more than one, then since G is 3-connected, there is an H-jump (Q, u, v) such that one of u, v is in the interior of P_i, contrary to (1). This proves (5).

There is no cycle with no edges in $E(C)$, since any such cycle would be an F-cycle, a contradiction. From (5), G is obtained from H by adding chords C; and by (3), every chord C-crosses or meets each of the edges $p_1p_{k+1}, \dots, p_kp_{2k}$. To complete the proof, we must show that every two additional chords meet or C-cross each other. Thus, let $p_{2k}a$ and p_ib be edges of G not in $E(H)$. (This is without loss of generality, since each such edge has an end in $\{p_1, \dots, p_{2k}\}$ by (4).) From (5), $a, b \notin \{p_1, \dots, p_{2k}\}$. From (3), $a \in V(C_{k-1} \cup C_k)$ and from (5), $a \notin \{p_1, \dots, p_{2k}\}$, so a belongs to the interior of one of C_{k-1}, C_k. Let H' be obtained from H by deleting the edge p_kp_{2k} and adding $p_{2k}a$; this is another k-rung Möbius ladder with the same base cycle C. From (3) applied to H', p_ib meets or C-crosses $p_{2k}a$. This proves 3.3. ∎

4 Excluding subdivisions of U_8, V_8 and W_8

Next we turn to the case of almost-planar graphs with ladder number three, that is, when G contains a subdivision of $K_{3,3}$ but not of V_8. It turns out that there are several kinds of almost-planar graph with this property, and in this section we look at those that also contain no subdivision of two other graphs U_8, W_8, defined later.

We shall need to look at the structure of G relative to a subdivision of $K_{3,3}$ it contains; let us set up some notation for this. If H is a subgraph of G that is a subdivision of $K_{3,3}$, there are distinct $a_1, a_2, a_3, b_1, b_2, b_3 \in V(G)$ and paths $P_{i,j}$ between a_i and b_j for all $i, j \in \{1, 2, 3\}$, pairwise vertex-disjoint except for their ends, such that H is the union of these nine paths. Let us call this the *standard notation* for H.

4.1 *Let G be an almost-planar graph with ladder number three, and let H be a subgraph that is a subdivision of $K_{3,3}$. Then $V(H) = V(G)$. Moreover, if $e = uv$ is an edge of $G \setminus E(H)$, then (in the standard notation) e is incident with at least one of $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and there do not exist $i, j \in \{1, 2, 3\}$ such that $u, v \in V(P_{i,j})$.*

Proof. Let F be the set of all edges e such that $G \setminus e$ is nonplanar. We use the standard notation for H.

(1) *If Q is a path between distinct $u, v \in V(H)$, with no other vertex or edge in H, then there do not exist $i, j \in \{1, 2, 3\}$ such that $u, v \in V(P_{i,j})$; and at least one of u, v is in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$.*

If $u, v \in V(P_{i,j})$ then the subpath of $P_{i,j}$ joining u, v, together with Q, is an F-cycle, so there is no such (i, j). For the second statement, we may assume that u belongs to the interior of $P_{1,1}$. and v belongs to the interior of $P_{i,j}$ for some $(i, j) \neq (1, 1)$. If $i = 1$, let R be the path of H between u, v consisting of subpaths of $P_{1,1}$ from u to a_1 and a subpath of $P_{1,j}$ from a_1 to v; then all its edges are in F, a contradiction. So $i \neq 1$ and similarly $j \neq 1$. But then $H \cup Q$ is a subdivision of a four-rung Möbius ladder, a contradiction. This proves (1).

Now suppose that $V(H) \neq V(G)$, and let D be a component of $G \setminus V(H)$. Since G is 3-connected, there are at least three vertices in $V(H)$ that have a neighbour in $V(D)$, say u, v, w. By the second statement of (1), at least two of u, v, w belongs to $\{a_1, a_2, a_3, b_1, b_2, b_3\}$; and by the first statement of (1), one of $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ contains none of u, v, w. Thus, from the symmetry and (1), we may assume that $v = a_2, w = a_3$, and either $u = a_1$ or u belongs to the interior of $P_{1,1}$. In either case, adding D and the edges between D, H to H, and then deleting b_2 and the interiors of $P_{1,2}, P_{2,2}, P_{3,2}$ makes a nonplanar graph; so all edges of $P_{1,2}, P_{2,2}, P_{3,2}$ belong to F. Similarly all edges of $P_{1,3}, P_{2,3}, P_{3,3}$ belong to F, and so there is an F-cycle, a contradiction. Thus $V(H) = V(G)$. This proves 4.1. ∎

We need to define three further types of graph. Let us say a *scallop* is a 3-connected graph either equal to K_5, or consisting of a cycle with vertices c_1-$\cdots$-c_n-c_1 in order (with $n \geq 3$), and three further vertices u, v, w; u, v, c_1, c_n are pairwise adjacent, and w is adjacent to u, v and to $c_2, \ldots, c_{n-1}$, and possibly to c_1, c_n.

A *clam* is a 3-connected graph consisting of a cycle C with vertices c_1-$\cdots$-c_n-c_1 in order, and two further vertices u, v, adjacent, with the following properties:

- there exists $j \in \{2, \ldots, n-1\}$ such that u, v are each adjacent to each of c_1, c_j, c_n;
- the set of neighbours of u in $V(C)$ is $\{c_n, c_1, c_2, \ldots, c_j\}$, and the set of neighbours of v in $V(C)$ is $\{c_j, c_{j+1}, \ldots, c_n, c_1\}$.

A *whelk* is similar. There is a cycle $C = c_1$-$\cdots$-c_n-c_1 and two extra vertices u, v, adjacent to each other. There exist i, j with $2 \leq i < j \leq n$ such that the set of neighbours of u in $V(C)$ is $\{c_1, c_2, \ldots, c_j\}$, and the set of neighbours of v in $V(C)$ is $\{c_j, c_{j+1}, \ldots, c_n, c_i\}$. (See Figure 2.) Clams and whelks are both almost-planar.

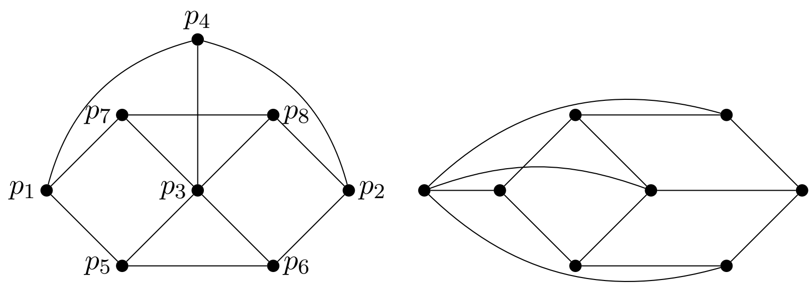

Figure 6: The graph U_8 and W_8.

The graphs U_8, W_8 are shown in Figure 6. They are both almost-planar, with ladder number three, and not Möbius chains, and they are the only minimal such graphs:

4.2 *Let G be an almost-planar graph with ladder number three that contains no subdivision of U_8 or W_8. Then G is either a Möbius chain or a scallop or clam.*

Proof. Choose a subgraph H of G that is a subdivision of $K_{3,3}$, using the standard notation. By 4.1, $V(H) = V(G)$, and every edge of G not in $E(H)$ has at least one end in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. We need to look at such edges with *exactly* one end in this set; we will worry about those with both ends in the set later. Let us say that (a_i, b_j) is *active* if a_i is adjacent to a vertex in the interior of $P_{i',j}$ for some $i' \in \{1, 2, 3\}$ different from i. We define that (b_j, a_i) is active similarly; that is, if b_j is adjacent to a vertex in the interior of $P_{i,j'}$ for some $j' \in \{1, 2, 3\}$ different from j.

(1) *If (a_i, b_j) is active then $P_{i,j}$ has length one.*

Without loss of generality, let a_1 have a neighbour v in the interior of $P_{2,1}$. Let H' be be the subdivision of $K_{3,3}$ obtained from H by adding the edge a_1v and deleting the interior of $P_{1,1}$; then $V(H') = V(G)$ by 4.1, and so $P_{1,1}$ has length one. This proves (1).

(2) *For all $i, j \in \{1, 2, 3\}$, not both (a_i, b_j) and (b_j, a_i) are active.*

Suppose (without loss of generality) that a_1 has a neighbour in the interior of $P_{2,1}$, and b_1 has a neighbour in the interior of $P_{1,2}$. The graph obtained from H by adding these two edges and deleting the interior of $P_{1,1}$ is a subdivision of V_8, a contradiction. This proves (2).

(3) *If $i, j, j' \in \{1, 2, 3\}$ with $j \neq j'$, then not both (a_i, b_j) and $(a_i, b_{j'})$ are active. The same holds with the a_i's and b_j's exchanged.*

If both pairs are active, the graph obtained from H by adding the corresponding two edges is a subdivision of U_8, a contradiction. This proves (3).

(4) *If $i, j, j' \in \{1, 2, 3\}$ with $j \neq j'$, then not both (a_i, b_j) and $(b_{j'}, a_i)$ are active. The same holds with the a_i's and b_j's exchanged.*

Suppose (without loss of generality) that a_1 has a neighbour u in the interior of $P_{2,1}$, and b_2 has a neighbour v in the interior of one of $P_{1,1}, P_{1,3}$. By (1), $P_{1,1}$ has length one, and so v is in the interior of $P_{1,3}$. Let H' be the subdivision of $K_{3,3}$ obtained from H by adding the edge b_2v and deleting the interior of $P_{1,2}$; then the edge a_1u violates 4.1 applied to H'. This proves (4).

(5) *If $i, j, j' \in \{1, 2, 3\}$ with $j \neq j'$, then not both (b_j, a_i) and $(b_{j'}, a_i)$ are active. The same holds with the a_i's and b_j's exchanged.*

Suppose (without loss of generality) that (b_1, a_1) and (b_2, a_1) are active. By (1), $P_{1,1}$ and $P_{1,2}$ both have length one, and so b_1, b_2 both have a neighour in the interior of $P_{1,3}$. Let u, v be neighbours of b_1, b_2 respectively in the interior of $P_{1,3}$. By exchanging b_1, b_2 if necessary, we may assume that u lies in the subpath of $P_{1,3}$ between a_1 and v. But then the edge b_1u violates 4.1 applied to the subdivision of $K_{3,3}$ obtained from H by adding the edge b_2v and deleting the edge a_1b_2. This proves (5).

From (2)–(5), there are at most three active pairs, and they are vertex-disjoint in the natural sense. Thus we may assume that there are no active pairs except possibly one of $(a_1, b_1), (b_1, a_1)$, one of $(a_2, b_2), (b_2, a_2)$ and one of $(a_3, b_3), (b_3, a_3)$.

(6) *$P_{1,1}, P_{2,2}, P_{3,3}$ each have length one.*

If the interior of $P_{1,1}$ is nonempty, then since G is 3-connected, there is a path Q of G with one end in this interior, and the other in $V(H) \setminus V(P_{1,1})$, and with no other vertices or edges in H. By 4.1, Q has length one and its end outside $P_{1,1}$ is one of a_2, a_3, b_2, b_3, contradicting that no such pair is active. This proves (6).

Let C be the cycle

$$P_{1,3} \cup P_{2,3} \cup P_{2,1} \cup P_{3,1} \cup P_{3,2} \cup P_{1,2}.$$

Thus, $V(C) = V(G)$, from (6) and since $V(H) = V(G)$. Let $\mathcal{A}$ be the set of edges of G with both ends in $\{a_1, a_2, a_3\}$, let $\mathcal{B}$ be the set with both ends in $\{b_1, b_2, b_3\}$, and let $\mathcal{E}$ be the set of edges in $E(G) \setminus E(H)$ that are not in $\mathcal{A} \cup \mathcal{B}$.

(7) *Every two edges in $\mathcal{E}$ either meet or C-cross.*

Let a_1 have a neighbour u in the interior of $P_{2,1}$; it suffices to show that a_1u meets or C-crosses every other edge in $\mathcal{E}$. Certainly a_1u meets or crosses every edge in $\mathcal{E}$ incident with a_1, a_2, a_3 or b_3, and so it remains to handle those incident with b_1 and those incident with b_2. By (2), there are no edges of $\mathcal{E}$ incident with b_1. Suppose that $b_2v \in \mathcal{E}$ does not meet or cross a_1u. Then v belongs to the subpath of $P_{2,1}$ between b_1 and u, and $v \neq u$. But then the graph obtained from H by adding the edges a_1u and b_2v to H is a subdivision of W_8, which is impossible. This proves (7).

So far, these results apply to any choice of H, but now let us choose H to maximize the number of active pairs. Since now we will need to compare the number of active pairs for different choices of H, from now on we will write "H-active" for active.

(8) *Every edge in $\mathcal{E}$ meets or C-crosses every edge in $\mathcal{A} \cup \mathcal{B}$.*

Let a_1 have a neighbour u in the interior of $P_{2,1}$; it suffices to show that a_1u meets or C-crosses every edge in $\mathcal{A} \cup \mathcal{B}$. This is clear for all such edges except possibly b_1b_2, so we assume that $b_1b_2 \in E(G)$. Let H' be obtained from H by deleting a_1b_1 and adding a_1u. Then (b_2, a_3) is H'-active (because of b_1b_2), and (a_1, u) is H'-active (because of a_1b_1). The pair (b_1, a_1) is not H-active, by (2); if either of $(a_3, b_3), (b_3, a_3)$ is H-active then it is also H'-active; if (a_2, b_2) is H-active then it is H'-active; so from the choice of H (maximizing the number of active pairs), it follows that (b_2, a_2) is H-active and not H'-active. Hence there is a neighbour v of b_2 that belongs to the subpath of $P_{2,1}$ between u, b_1, with $v \neq b_1$. Since $b_2v \in \mathcal{E}$, from (7) it follows that $v = u$. But then a_1-u-b_2-b_1-a_1 is an F-cycle, a contradiction. This proves (8).

(9) *If some two edges in $\mathcal{A} \cup \mathcal{B}$ do not meet or C-cross, then G is a scallop.*

Certanly every two edges in $\mathcal{A}$ meet, and the same for $\mathcal{B}$; so we may assume that $a_1a_2, b_1b_2 \in E(G)$. Suppose that (a_1, b_1) is H-active, and let a_1 have a neighbour u in the interior of one of $P_{2,1}, P_{3,1}$. By (8), $u \notin V(P_{2,1})$ and so u is in the interior of $P_{3,1}$. But then a_1-b_1-b_2-a_2-a_1 is an F-cycle, a contradiction. So (a_1, b_1) is not H-active, and similarly none of $(b_1.a_1), (a_2, b_2), (b_2, a_2)$ are H-active. By (2) and the symmetry, we may assume that (a_3, b_3) is not H-active, so all edges in $\mathcal{E}$ are incident with b_3. If neither of a_1a_3, a_2a_3 exists, then G is a scallop, so we assume that a_1a_3 exists. If $\mathcal{E} \neq \emptyset$, let $b_1v \in \mathcal{E}$; so v belongs to the interior of one of $P_{3,1}, P_{3,2}$. By (8), since b_1v meets or C-crosses a_1a_3, it follows that $v \in V(P_{3,2})$; but then a_1-a_2-b_2-b_1-a_1 is an F-cycle, a contradiction. So $\mathcal{E} = \emptyset$, and so $|V(G)| = 6$. If neither of the edges b_1b_3, b_2b_3 exists, then G is a scallop; and since there is symmetry exchanging b_1, b_2, we may assume that $b_1b_3 \in E(G)$. But then every edge between $\{a_1, b_1\}$ and $\{a_2, a_3, b_2, b_3\}$ is in F, and so there is an F-cycle, a contradiction. This proves (9).

In view of (7), (8) and (9), we assume that every two edges in $E(G) \setminus E(C)$ meet or C-cross. If there is no cycle of chords for C, then G is a Möbius chain with base cycle C; so we assume that there is a cycle D edge-dsjoint from C. In this case we will prove that G is a clam. Since D is not an F-cycle, there is an edge $f \in E(D) \setminus F$. Since $f \notin F$, it follows that $f = a_ib_i$ for some $i \in \{1, 2, 3\}$, so we may assume that $f = a_1b_1$. Since $f \notin F$, (a_1, b_1) and (b_1, a_1) are not H-active; and since $f \in E(D)$, we may assume that $a_1a_3, b_1b_2 \in E(D)$. Let J be the graph obtained from H by adding a_1a_3, b_1b_2, and let us examine the edges in $G \setminus E(J)$.

- If (a_2, b_2) is H-active, a_2 has a neighbour in the interior of $P_{3,2}$ (it cannot be in the interior of $P_{1,2}$ since it must C-cross b_1b_2). But then $G \setminus f$ is nonplanar, a contradiction. So (a_2, b_2) is not H-active. Since $a_2a_1 \notin E(G)$ (because it does not C-cross b_1b_2), it follows that the only possible edge in $E(G) \setminus E(J)$ incident with a_2 is a_2a_3. Similarly no edge of $G \setminus E(J)$ is incident with b_3 except possible b_2b_3.
- (a_1, b_1) and (b_1, a_1) are not H-active, since $a_1b_1 \notin F$. So there are no edges of $G \setminus E(J)$ incident with a_1 or b_1.

Consequently, every edge of $G \setminus E(J)$ is incident with one of a_3, b_2, and so D has length five, and there exists $c \in V(P_{2,3})$ such that D is the cycle a_1-a_3-c-b_2-b_1-a_1. Since all edges of $G \setminus E(C)$ incident with b_2 meet or C-cross both a_1b_1 and a_3c, it follows that all remaining neighbours of b_2 belong to

$P_{2,1}$ or to the subpath of $P_{2,3}$ between a_2 and c, and a similar statement holds for a_2. Consequently G is a clam. This proves 4.2. ∎

5 Double wheels

That concludes the "Möbius"-type results. Now we look at the almost-planar graphs with ladder number three that contain a subdivision of one of U_8, W_8. Let us say a graph G is a *double wheel* if it is 3-connected and nonplanar, and consists of a cycle C and two further vertices u, v, adjacent to each other, such that at most one vertex of C is adjacent to both u, v. (Thus clams and whelks are not double wheels.) We call u, v its *wheel centres* and C its *base cycle*. Double wheels are almost-planar.

5.1 *Let G be an almost-planar graph with ladder number three, that contains a subdivision of W_8. Then G is a double wheel.*

Proof. Let C be a cycle of G, and let $a, b \in V(G) \setminus V(C)$; let $p_1, p_2, p_3, p_4, p_5, p_6 \in V(C)$ be in cyclic order; and let $P_1, \ldots, P_6$ paths between $\{a, b\}$ and $V(C)$, with no internal vertex in $\{a, b\} \cup V(C)$, and pairwise vertex-disjoint except for their ends, where P_i is between a and p_i for $i = 1, 3, 5$ and P_i is between b and p_i for $i = 2, 4, 6$. Let P_0 be a path between a, b, vertex-disjoint from C and with no internal vertex in $P_1 \cup \cdots \cup P_6$. For $1 \leq i \leq 6$ let C_i be the path of C between p_i and p_{i+1} (reading subscripts modulo 6). Let H be the union of C and $P_0, P_1, \ldots, P_6$. Since the graph obtained from H by deleting the interior of P_i contains a subdivision of $K_{3,3}$, 4.1 implies that P_i has length one, for $1 \leq i \leq 6$, and $V(H) = V(G)$. As usual, let F be the set of edges e such that $G \setminus e$ is nonplanar; so $ap_1, ap_3, ap_5, bp_2, bp_4, bp_6 \in F$.

If every edge of $G \setminus E(H)$ is incident with a or b, then P_0 has length one (since G is 3-connected) and so G is a double wheel. Hence we suppose (for a contradiction), that $uv \in E(G) \setminus E(H)$, and $u, v \neq a, b$.

(1) *Every edge of $G \setminus E(H)$ is incident with a or b, and P_0 has length one.*

Suppose first that P_0 has legth more than one. Then there is an edge $uv \in E(G) \setminus E(H)$ where u belongs to the interior of P_0. Thus $v \notin V(P_0)$ (since otherwise there would be an F-cycle, as usual), so $v \in V(C)$. From the symmetry we may assume that either $v = p_1$ or v belongs to the interior of C_1. In either case, adding uv to H and deleting ap_1 and bp_2 gives a subdivision of V_8, a contradiction. This proves that P_0 has length one. Now suppose that there is an edge $uv \in E(G) \setminus E(H)$, and $u, v \neq a, b$. It follows that $u, v \in V(C)$. From 4.1, one of u, v equals one of $p_1, \ldots, p_6$, say $u = p_1$ without loss of generality. By 4.1 again, applied to the subdivision of $K_{3,3}$ obtained from H by deleting ap_1 and bp_6, v is one of p_2, p_3, p_4, p_5. Since there is no F-cycle, $v \neq p_2$.If $v = p_4$ then $ab \in F$, and a-p_1-p_4-b-a is an F-cyce, a contradiction. So $v \in \{p_3, p_5\}$ and we may assume that $v = p_3$ from the symmetry. But the graph obtained from H by adding p_1p_3 and deleting p_2 is nonplanar, and so $E(C_1 \cup C_2) \subseteq F$, and since $p_1p_3 \in F$ this is impossible. This proves (1).

From (1), to see that G is a double wheel, note that every edge between $\{a, b\}$ and $V(C)$ belongs to F, and so a, b have at most one common neighbour in C since there is no F-cycle. This proves 5.1. ∎

A *conch* is a 3-connected graph G with the following properties. There are three vertices p_1, p_2, p_3, and three paths P, Q, R, vertex-disjoint except for their ends. The paths P, Q are both between p_1, p_2, and do not contain p_3, and R is between p_2, p_3, and does not contain p_1. The vertex p_3 has at least two neighbours in the interior of each of P, Q, and p_1 has at least two neighbours in the interior of R. There are no other edges, except possibly p_1 is adjacent to p_2 or p_3.

A *mussel* is a 3-connected graph with the following properties. There are three vertices p_1, p_2, p_3 and three paths P, Q, R between p_1, p_2, pairwise vertex-disjoint except for their ends, and each of length at least three. The vertex p_3 belongs to none of these paths, but is adjacent to every vertex in their interiors, and may also be adjacent to p_1 or to p_2. Moreover, p_1, p_2 may be adjacent, although at most two of the edges p_1p_2, p_2p_3, p_1p_3 are present. Let us call p_1, p_2 its *tips* and p_3 its *hinge*. (See Figure 2.) It is easy to check that all conches and mussels are almost-planar.

5.2 *Let G be an almost-planar graph with ladder number three, that contains a subdivision of U_8. Then G is a double wheel, conch, whelk, or mussel.*

Proof. Since G contains a subdivision H of U_8, we may choose vertices $p_1, \ldots, p_8$, and paths $P_{i,j}$ joining p_i, p_j for each edge p_ip_j of U_8, labelled as in Figure 6, where H is the union of all these paths. There is a subdivision of $K_{3,3}$ obtained by deleting from H the interiors of $P_{3,5}$ and $P_{3,7}$, and so by 4.1, $V(H) = V(G)$, and $P_{3,5}$ and $P_{3,7}$ have length one, and similarly $P_{3,6}$ and $P_{3,8}$ have length one. Let F be the set of edges e such that $G \setminus e$ is nonplanar. Thus $p_3p_5, p_3p_6, p_3p_7, p_3p_8 \in F$.

(1) *For each edge $uv \in E(G) \setminus E(H)$, one of $u, v \in \{p_1, p_2, p_3, p_4\}$.*

By 4.1 applied to the same subdivision of $K_{3,3}$, it follows that one of $u, v \in \{p_1, p_2, p_3, p_4, p_6, p_8\}$, and so we may assume that $u = p_6$ without loss of generality. From 4.1 applied to the subdivision of $K_{3,3}$ obtained from H by deleting the edges p_3p_6 and p_3p_7 , it follows that $v \in \{p_1, p_2, p_3, p_4, p_5, p_8\}$, so we may assume that $v \in \{p_5, p_8\}$; and $v \neq p_8$ by 4.1 applied to the subdivision of $K_{3,3}$ obtained by deleting from H the interiors of $P_{3,6}$ and $P_{3,8}$, Hence $v = p_5$, contradicting that there is no F-cycle. This proves (1).

(2) *If no edge of $G \setminus E(H)$ is incident with p_1 or with p_2 then G is either a double wheel, a whelk, or a mussel.*

In this case, $P_{3,4}$ has length one, since no edge of $G \setminus E(H)$ has an end in its interior. Suppose first that p_3 has no neighbours in the interior of $P_{1,4}$ or $P_{2,4}$. We claim that either G is a double wheel, or a whelk. To show this, we assume it is not a double wheel, and so there are at least two vertices in $V(C)$ adjacent to both p_3, p_4, say c, d. All edges between $\{p_3, p_4\}$ and $V(C)$ belong to F except possibly p_1p_4, p_2p_4; so, since there is no F-cycle, we may assume that $d = p_1$ and $P_{1,4}$ has length one, and $p_1p_4 \notin F$. Since $G \setminus p_1p_4$ is planar, there is a subpath Q of $P_{2,8} \cup P_{2,6}$ containing all neighbours of p_4 in $V(C)$ except p_1, and p_3 has no neighbour in the interior of Q. Thus c is one end of Q, and hence $c \in V(P_{2,8} \cup P_{2,6})$. If there is a third vertex e say of C that is adjacent to both p_3, p_4, then e is necessarily the other end of Q, and so also belongs to $V(P_{2,8} \cup P_{2,6})$. By the same argument, applied to c, e, it follows that one of c, e equals p_1, a contradiction. So there is no such e, and so G is a whelk.

Now we assume that p_3 has a neighbour u in the interior of one of $P_{1,4}$, $P_{2,4}$. If p_4 is incident with no edge of $G \setminus E(H)$ then G is a mussel, so we assume that $p_4v \in E(G) \setminus E(H)$. From the symmetry we may assume that v is in the interior of the path $P_{1,5} \cup P_{5,6}$. But then adding the edge p_3u to H and deleting p_3p_4, p_3p_5 and p_3p_7 gives a subdivision of $K_{3,3}$ in which the edge p_4v violates 4.1. This proves (2).

From (2) we may assume that there is an edge $p_1u \in E(G) \setminus E(H)$.

(3) *$u \in V(P_{3,4} \cup P_{2,4})$, and $P_{1,4}$ has length one.*

If v belongs to $P_{1,4} \cup P_{1,5} \cup P_{5,6} \cup P_{1,7} \cup P_{7,8}$, there is an F-cycle using e, a contradiction. So we may assume that v belongs to the interior of $P_{2,6}$. But then $P_{5,6} \cup P_{3,5} \cup P_{3,6}$ is an F-cycle, a contradiction. This proves (3).

(4) *There is no edge in $E(G) \setminus E(H)$ incident with p_4.*

Suppose that $p_4v \in E(G) \setminus E(H)$. From the symmetry, we may assume that v belongs to the interior of $P_{1,5} \cup P_{5,6} \cup P_{2,6}$. But then adding the edges p_4v and p_1u to H, and deleting the interior of $P_{7,8}$ and the edge p_3p_5, gives a nonplanar graph, and so $E(P_{7,8}) \subseteq F$. But $p_3p_7, p_3p_8 \in F$, and so there is an F-cycle, a contradiction. This proves (4).

(5) *If $u \neq p_2, p_3$, then G is a conch.*

To show this, we must check that p_3 has no neighbour in the interiors of $P_{1,4}, P_{2,4}$, and there are no edges in $E(G) \setminus E(H)$ incident with p_2 except possibly p_1p_2, p_2p_3. Since $u \neq p_2, p_3$, it follows that $P_{1,4}$ has length one and the edge $p_1p_4 \in F$. So p_3 has no neighbour in the interior of $P_{1,4}$. Suppose it has a neighbour v in the interior of $P_{2,4}$. Thus $P_{3,4}$ has length one and $p_3p_4, p_3v \in F$. Since $u \in V(P_{3,4} \cup P_{2,4})$, it follows that $u \in V(P_{2,4})$. Let Q be the minimal subpath of $P_{2,4}$ from p_4 to $\{u, v\}$. Then $E(Q) \subseteq F$, and so there is an F-cycle (because $(p_1p_4, p_1u, p_3p_4, p_3v$ are all in F), a contradiction. So p_3 has no neighbour in the interiors of $P_{1,4}, P_{2,4}$. Next suppose that $p_2v \in E(G) \setminus E(H)$ where $v \neq p_1, p_3$. By (3) (with p_1, p_2 exchanged) it follows that $v \in V(P_{3,4} \cup P_{1,4})$. Moreover, $P_{1,4}, P_{2,4}$ have length one; so both u, v belong to $P_{3,4}$. Let Q be the minimal path of $P_{3,4}$ between p_4 and $\{u, v\}$. Then $E(Q) \subseteq F$, and so there is an F-cycle, since $p_1p_4, p_1u, p_2p_4, p_2v \in F$, a contradiction. This proves (5).

From (5) we may assume that no edge in $E(G) \setminus E(H)$ is incident with p_1 except possible p_1p_2, p_1p_3, and similarly none are incident with p_2 except possibly p_1p_2, p_2p_3. But then G is a mussel. This proves 5.2. ∎

In summary, then, by combining 3.2, 3.3, 4.2, 5.1, and 5.2, we have proved the following, which implies 1.1 in view of 3.1:

5.3 *Let G be 3-connected and nonplanar. Then it is almost-planar if and only if G is a Möbius chain, or a double wheel, or a conch, mussel, scallop, clam, or whelk.*

6 Back to digraphs

5.3 is satisfying, but does not really answer the original question: what are the Kuratowski digraphs? By 3.1 and 5.3, every Kuratowski digraph is an orientation of one of the graphs of 5.3. But not all the graphs of 5.3 can be oriented to make Kuratowski digraphs, and some can be oriented to do so in many ways. In this section we explore this issue.

If G is a digraph, then for each $X \subseteq V(G)$, let $\delta^+(X)$ be the set of all edges of G with tail in X and head in $V(G) \setminus X$, and let $\delta^-(X) = \delta^+(V(G) \setminus X)$. Let $\delta(X)$ be the set of all edges between X and $V(G)\setminus X$. If G is a Kuratowshi digraph, then with F defined as usual, for each $e = uv \in F$, since $G \setminus e$ is nonplanar and hence not strong (from the minimality property of G), there exists $A \subseteq V(G)$ with $\delta^+(A) = \{e\}$. We call such a set A a *back-cut* for e. If in addition $\delta^-(A) \cap F = \emptyset$, we say A is *clean*.

6.1 *Let G be a Kuratowski digraph, and let F be the set of edges e of G such that $G \setminus e$ is nonplanar. Then there is a clean back-cut for each $e \in F$.*

Proof. Choose $P \subseteq F$ maximal such that for each $e \in F$, there is a back-cut A for e with $P \cap \delta^-(A) = \emptyset$. We suppose for a contradiction that $P \neq F$; let $f = xy \in F \setminus P$. Let B be a back-cut for f with $P \cap \delta^-(B) = \emptyset$. From the maximality of P, there exists $e = uv \in F$ such that every back-cut for e contains an edge in $P \cup \{f\}$. Let A be a back-cut for e with $P \cap \delta^-(A) = \emptyset$, and consequently with $f \in \delta^-(A)$. Hence $x \in B \setminus A$ and $y \in A \setminus B$.

Suppose that $v \notin A \cup B$. Then $u \in B \setminus A$, since $e \notin \delta^+(B)$; and so $A \cup B$ is also a back-cut for e. Moreover, $\delta^-(A \cup B) \subseteq \delta^-(A) \cup \delta^-(B)$, and so contains no edges in P; and $e \notin \delta^-(A \cup B)$, contradicting that every back-cut for e contains an edge in $P \cup \{f\}$. This proves that $v \in A \cup B$, and so $v \in B \setminus A$ since $v \notin A$.

Suppose that $u \in A \setminus B$. Then $\delta^+(A \cap B) = \emptyset$, and so $A \cap B = \emptyset$, since G is strong; and $\delta^+(A \cup B) = \emptyset$, and so $A \cup B = V(G)$, since G is strong. Thus $B = V(G) \setminus A$. But e, f are the only edges between A, B, since any other edge would belong to $\delta^+(A) \cup \delta^+(B)$, which is impossible. Hence G^- is not 3-connected, contrary to 3.1.

This proves that $u \in A \cap B$. But then $A \cap B$ is a back-cut for e, and $\delta^-(A \cap B)$ is disjoint from $P \cup \{f\}$ as before, a contradiction. Hence $P = F$. This proves 6.1. ∎

Let G be an almost-planar graph, and let us say an orientation H of G is *good* if it is a Kuratowski digraph. As usual, let F be the set of edges e such that $G \setminus e$ is nonplanar. Let us say a cycle D of G is *fundamental* if it has only one edge in $E(G) \setminus F$. By 6.1, in every good orientation, every fundamental cycle becomes a directed cycle.

The result 6.1 is most helpful when F is the edge-set of a spanning tree T of G. In that case, for each $e \in F$, every clean back-cut for e must be the vertex set of one of the two components of $T \setminus e$; and so an orientation H of G is good if and only if for every $e \in E(G) \setminus F$, the fundamental cycle containing e is a directed cycle of H. It is easy to check that for any spanning tree in any 2-connected graph, there are at most two orientations of G to make all the fundamental cycles directed, and they are reverses of each other. (Once we fix the direction of one edge, we know the direction of all edges in fundamental cycles containing this edge, and so on.) Deciding whether there is a good orientation reduces in this case to a 2-SAT problem in which all clauses are equivalences, and this can be used to show that there is no good orientation if and only if for some odd $k \geq 3$, there are $k+1$ vertex-disjoint subtrees $T_0, T_1, \ldots, T_k$ of T, such that for $1 \le i \le k$ there is an edge of T between $V(T_i)$ and $V(T_0)$, and for $1 \le i \le k$ there is an edge of $E(G) \setminus E(T)$ between $V(T_i)$ and $V(T_{i+1})$ (reading subscripts modulo k). We omit the proof.

So for instance, the double wheel in Figure 1 has no good orientation. To see this, let a, b be the two wheel centres, and let C be the base cycle. Since a, b have a common neighbour in C, F is indeed the edge-set of a spanning tree. Moreover, there is a path P of C with both ends adjacent to a and with a positive even number of its vertices adjacent to b, and it is straightforward to check that this already precludes there being a good orientation. Let us say a double wheel is *proper* if $G \setminus e$ is nonplanar for every edge e between a wheel centre and the base cycle. For proper double wheels in which the wheel centres have a common neighbour, one can show that it is necessary and sufficient that there is no path P as above.

There are other types of almost-planar graphs in which F makes a spanning tree; for instance, a mussel in which the hinge is adjacent to both tips. Then one can show that there is a good orientation if and only if the three paths of the mussel from tip to tip that avoid the hinge either all have odd length or all have even length.

However, when F does not make a spanning tree, it becomes harder to analyze when a good orientation exists, although it seems to become more likely that one does in fact exist. Let T be the forest with vertex set $V(G)$ and edge set F. There are instances in which T has two components and no good orientation exists; for instance, if G is a proper double wheel, and the two wheel centres have no common neighbour, then a good orientation exists if and only if the base cycle has even length. (Again, we omit the proof, which is another application of 6.1.)

Let $e \in F$, and let J be the minimal subgraph of G containing e such that every fundamental cycle with an edge in J is a subgraph of J. It follows that the edges of J in F make a spanning tree of J; and J is an induced subgraph of G. We call such a graph J a *cell* of G. Every two cells share at most one vertex, and so are edge-disjoint. (We skip these proofs, which are all easy.) For a cell J, there are only two orientations of J in which each fundamental cycle included in J becomes directed, reverses of each other; and H is a good orientation of G, and J is a cell of G, then the restriction of H to J must be one of these two orientations.

Thus, for a conch G say, to decide whether it admits a good orientation, we need to know only a few things. Let P, Q, R, p_1, p_2, p_3 be as in the definition of a conch. We need to know whether p_1 is adjacent to p_2 or to p_3, and we need to know the parities of the lengths of P, Q, R. Nothing else is relevant; this information is enough to determine whether a good orientation exists, and we can easily decide each of the 2^5 cases this yields. Similar, for each of the five types of graphs in Figure 2, the problem breaks into a small number of cases, each of which can be decided. (This is related to the "fans" mentioned in the introduction.) We have not bothered to figure out each case.

There remain Möbius chains, and these are much more intractable (and much more interesting).

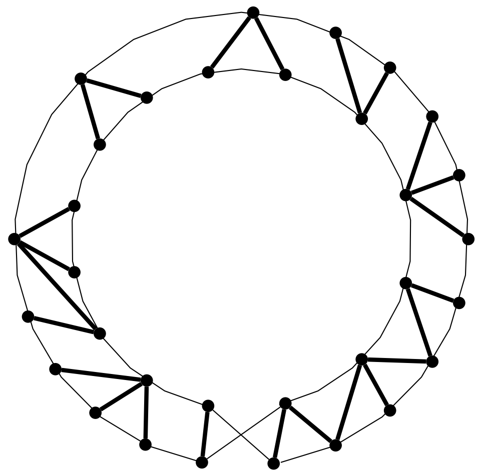

Figure 7: A Möbius chain that does not admit a good orientation. The edges of F are thick.

We have been able to give a pair of necessary and sufficient parity-type conditions for when a Möbius chain admits a good orientation. But they are complicated, and the proof that they are correct is long and messy (about eight pages), and in the end we decided not to inflict it all on the reader. For instance, here is the simpler of the necessary conditions: that if $u, v, w \in V(G)$ are distinct, and $uv, vw \in F$, and u, v, w all have degree at least two in T, then v has even degree in T. The other necessary condition involves sequences (of arbitrary length) of components of T, and we do not give it here (as a clue, the graph in Figure 7 violates it). It follows from that result that in a Möbius chain:

- every Möbius chain in which T contains no four-vertex path admits a good orientation; and
- if T is a spanning tree, then there is a good orientation if and only if every vertex of the spine of T has even degree in T (where the *spine* is the unique minimal path of T that meets all edges of T).

References

[1] S. Bartell, *Obstructions to graph and digraph embedding*, senior thesis, Princeton, 2025.

[2] D. W. Hall, "A note on primitive skew curves", *Bull. Amer. Math. Soc.* **49** (1943), 935–937.

[3] N. Robertson, P. Seymour and R. Thomas, "Permanents, Pfaffian orientations, and even directed circuits", *Annals of Math.*, **150** (1999), 929–975.