INDUCED SUBGRAPH DENSITY. IV. NEW GRAPHS WITH THE ERDŐS-HAJNAL PROPERTY

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Abstract. Erdős and Hajnal conjectured that for every graph H, there exists $c > 0$ such that every H-free graph G has a clique or a stable set of size at least $|G|^c$ ("H-free" means with no induced subgraph isomorphic to H). Alon, Pach, and Solymosi reduced the Erdős-Hajnal conjecture to the case when H is *prime* (that is, H cannot be obtained by vertex-substitution from smaller graphs); but until now, it was not shown for any prime graph with more than five vertices.

We will provide infinitely many prime graphs that satisfy the conjecture. Let H be a graph with the property that for every prime induced subgraph G' with $|G'| \geq 3$, G' has a vertex of degree one and a vertex of degree $|G'| - 2$. We will prove that every graph H with this property satisfies the Erdős-Hajnal conjecture, and infinitely many graphs with this property are prime.

More generally, say a graph is *buildable* if every prime induced subgraph with at least three vertices has a vertex of degree one. We prove that if H_1 and $\overline{H_2}$ are buildable, there exists $c > 0$ such that every graph G that is both H_1-free and H_2-free has a clique or a stable set of size at least $|G|^c$.

Our proof method also extends to ordered graphs; and we obtain a theorem which significantly extends a recent result of Pach and Tomon about excluding monotone paths.

Indeed, we prove a stronger result, that we can weaken the "H-free" hypothesis of the Erdős-Hajnal conjecture to one saying that there are not many copies of H; and strengthen its conclusion, deducing a "polynomial" version of Rödl's theorem conjectured by Fox and Sudakov.

We also obtain infinitely many new prime tournaments that satisfy the Erdős-Hajnal conjecture (in tournament form). Say a tournament is *buildable* if it can be grown from nothing by repeatedly either adding a vertex of out-degree ≤ 1 or in-degree ≤ 1, or vertex-substitution. All buildable tournaments satisfy the tournament version of the Erdős-Hajnal conjecture.

1. Introduction

All graphs in this paper are finite and simple. For a graph G, let $|G| := |V(G)|$, and let $\overline{G}$ denote its complement. An *induced subgraph* of G is a graph obtained from G by removing vertices. For a graph H, a *copy* of H in G is an isomorphism from H to an induced subgraph of G; and G is *H-free* if there is no copy of H in G.

A conjecture of Erdős and Hajnal [10, 11] (see [4, 14] for surveys) asserts that:

Conjecture 1.1. *For every graph H, there exists $c > 0$ such that in every H-free graph G there is a clique or a stable set of size at least $|G|^c$.*

A graph H satisfying this conjecture is said to have the *Erdős-Hajnal property.* The simplest version of our main result is the following (we prove stronger versions later):

Theorem 1.2. *Let H be a graph, such that every prime induced subgraph H' with at least three vertices has both a vertex of degree one and a vertex of degree $|H'| - 2$ (and so degree one in the complement). Then H has the Erdős-Hajnal property.*

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Such graphs H exist, and infinitely many of them are prime. This is remarkable, because until now, and despite intensive effort, there were no prime graphs with more than five vertices that were known to have the Erdős-Hajnal property.

We had better define "prime". Given two graphs H_1, H_2 and a vertex $v \in V(H_1)$, the graph obtained *from H_1 by substituting H_2 for v* is formed by taking the disjoint union of $H_1 \setminus \{v\}$ and H_2, and then adding edges to make every vertex of H_2 adjacent to all the neighbours of v in H_1. This operation is *vertex-substitution*; and a graph that cannot be constructed by vertex-substitution from two graphs both with fewer vertices is said to be *prime*. Equivalently, a graph H is prime if there is no $S \subseteq V(H)$ with $1 < |S| < |H|$ such that all vertices in S have the same neighbourhood in $V(H) \setminus S$. (All graphs with at most two vertices are prime; let us say a graph is *non-trivial* if it has at least three vertices. We really only care about non-trivial prime graphs.) The connection with the Erdős-Hajnal property is a theorem of Alon, Pach, and Solymosi [1]:

Theorem 1.3. *Let H_1, H_2 have the Erdős-Hajnal property, and let H be obtained from H_1 by substituting H_2 for $v \in V(H_1)$. Then H has the Erdős-Hajnal property.*

In view of Theorem 1.3, to show that more graphs satisfy the Erdős-Hajnal conjecture, we have to find more prime graphs that satisfy it. However, until now the only non-trivial prime graphs known to have the Erdős-Hajnal property were the bull (obtained from a four-vertex path by adding a new vertex adjacent to the two middle vertices of the path) [7], the five-cycle C_5 [9], and the four-vertex path P_4. It is still open whether the five-vertex path P_5 and its complement have the Erdős-Hajnal property (these are equivalent, because a graph has the Erdős-Hajnal property if and only if its complement does). See [2, 16] for some recent progress on the P_5 question.

In this paper, we will give infinitely many prime graphs that all have the Erdős-Hajnal property, such as the graphs of Fig. 1. Figure 2 shows a more complicated example. All the graphs of the figures are prime.

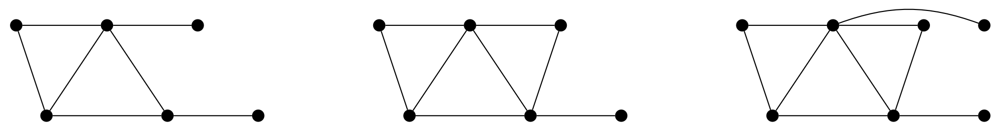

FIGURE 1. The two six-vertex prime graphs in $\mathcal{H}$, and one on seven vertices.

We define $\mathcal{H}$ to be the class of all graphs that can be constructed by a sequence of the following operations, starting with one-vertex graphs:

- choosing a graph G that is already constructed, choosing a vertex $v \in V(G)$ that has degree at least $|G| - 1$, and adding a new vertex adjacent only to v;
- choosing a graph G that is already constructed, choosing a vertex $v \in V(G)$ that has degree at most one, and adding a new vertex with neighbour set $V(G) \setminus \{v\}$ (this is the operation of the first bullet, in the complement graph);
- choosing two graphs H_1, H_2 that are already constructed, and substituting H_2 for a vertex of H_1.

For instance, the graph of Fig. 2 belongs to $\mathcal{H}$. (Add one vertex at a time in the order of the numbers.) It is easy to see that:

- the bull and P_4 belong to $\mathcal{H}$;
- if $H \in \mathcal{H}$, then so is its complement and all its induced subgraphs;

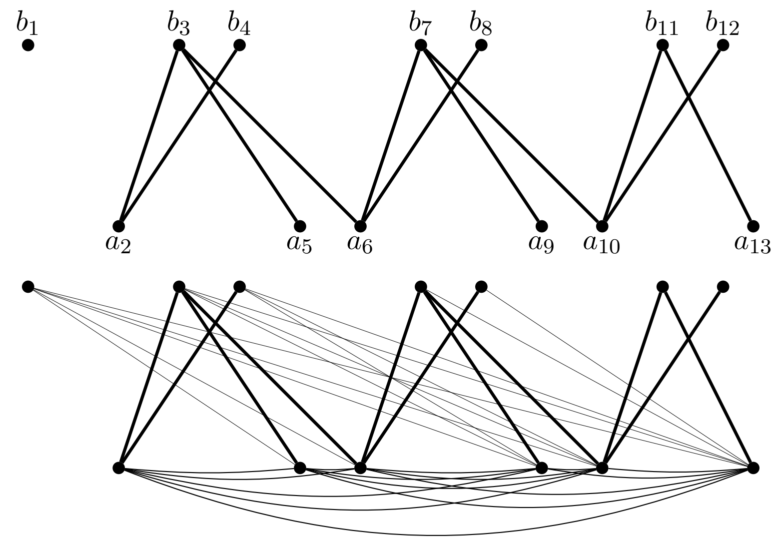

FIGURE 2. Start with a path (a_2-b_3-a_6-b_7-a_{10}-b_{11} in this case), add a leaf at every vertex, add an isolated vertex b_1, and take a bipartition (A, B), numbered as shown. Now make A a clique; and make a_i, b_j adjacent if $i \geq j + 4$.

- each prime graph in $\mathcal{H}$ is a *split graph*, that is, its vertex set is the union of a clique and a stable set; and
- $H \in \mathcal{H}$ if and only if every nontrivial prime induced subgraph H' has a vertex of degree one and a vertex of degree $|H'| - 2$.

Thus, our theorem Theorem 1.2 says:

Theorem 1.4. *Every $H \in \mathcal{H}$ has the Erdős-Hajnal property.*

We claim that $\mathcal{H}$ contains an infinite number of prime graphs (including the bull, but not C_5, and unfortunately not P_5). Indeed, $\mathcal{H}$ contains a prime graph with h vertices for every $h \geq 4$. To see this, observe first that there are prime graphs in $\mathcal{H}$ with four vertices and five vertices (P_4 and the bull). Second, if $H \in \mathcal{H}$ is prime, then there is a prime graph in $\mathcal{H}$ with $|H| + 2$ vertices; because let v be a vertex of H with degree one, adjacent to u say. Add two new vertices, one adjacent to all vertices of H, and the other just adjacent to u; then this enlarged graph is also prime and belongs to $\mathcal{H}$.

We point out that the third bullet in the definition of $\mathcal{H}$ is not really important. If we just want to construct all the prime graphs in $\mathcal{H}$, the first two bullets are enough. Note, however, that having used, say, the first bullet operation on some vertex v, adding a new vertex u, one can then use the first bullet operation again on the same vertex v, adding a "nonadjacent twin" of u: this is the same as substituting a two-vertex stable set for u. At that stage, the graph is not prime, but might still eventually grow into a prime graph, because later steps in the growing process might restore primeness. Moreover, if we want to avoid using the third bullet, then this repetition of the same operation on the same vertex may be necessary (for instance, to grow the graph of Fig. 2). When we come to the "pairs of graphs" extension of the result, vertex-substitution will become important, and it is convenient to retain it here to make that extension simpler.

The Erdős-Hajnal property of the bull was first proved by Chudnovsky and Safra [7] using the strong perfect graph theorem [6] and a decomposition theorem for bull-free graphs, and later reproved by Chudnovsky, Scott, Seymour, and Spirkl [9] via a different approach that simultaneously showed the Erdős-Hajnal property of C_5. Our proof of Theorem 1.4 gives a third proof of the Erdős-Hajnal property of the bull.

We have been striving, for the last forty years or so (some of us, anyway) to prove that all five-vertex graphs have the Erdős-Hajnal property. We still can't do the five-vertex path P_5 or $\overline{P_5}$, but, out of curiosity, what about six-vertex graphs? Our result does two of them, and it turns out that there are only four more (two complementary pairs) that do not contain P_5 or $\overline{P_5}$ and that we still cannot prove have the Erdős-Hajnal property, shown in Fig. 3. (We *can* prove that these four graphs have a "near-Erdős-Hajnal" property [17]). We omit the details.

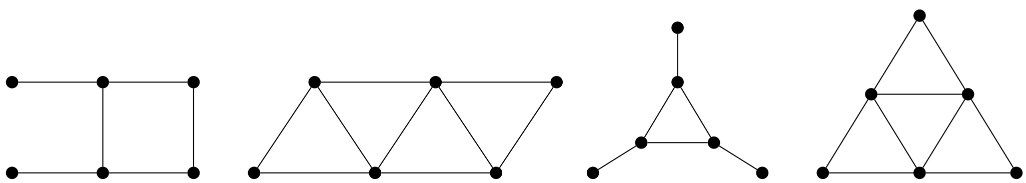

FIGURE 3. The six-vertex graphs not containing P_5 or $\overline{P_5}$ that remain open.

We will in fact prove a statement stronger than Theorem 1.4, that we will explain now. Indeed, our final result is stronger than Theorem 1.4 in four ways:

- We will prove that these graphs satisfy a conjecture of Fox and Sudakov (explained below), not just that they have the Erdős-Hajnal property.
- We can obtain the same conclusion under a weaker hypothesis; we just need that there are not many copies of H in G, rather than none at all.
- Each prime graph in $\mathcal{H}$ has a vertex of degree one and so does its complement, and this is what we need for the inductive proof to work. It is just as good, and gives a stronger theorem, if we exclude two graphs instead of one, with the property that every non-trivial prime induced subgraph of the first has a vertex of degree one, and the same for the complement of the second.
- All this works just as well for ordered graphs; and we obtain consequences for ordered graphs and tournaments.

Let us explain these things in more detail.

For $\varepsilon \in (0, \frac{1}{2})$, a graph G is *ε-restricted* if one of $G, \overline{G}$ has maximum degree at most $\varepsilon|G|$. We say that $S \subseteq V(G)$ is *ε-restricted* in G if $G[S]$ is ε-restricted where $G[S]$ is the subgraph of G induced on S. Let us recall a theorem of Rödl [20]:

Theorem 1.5. *For every graph H and every $\varepsilon \in (0, \frac{1}{2})$, there exists $\delta > 0$ such that for every H-free graph G, there is an ε-restricted subset $S \subseteq V(G)$ with size at least $\delta|G|$.*

If G, H are graphs, $\operatorname{ind}_H(G)$ denotes the number of copies of H in G. Nikiforov [18] extended Theorem 1.5, replacing the condition "H-free" with "few copies of H":

Theorem 1.6. *For every graph H and every $\varepsilon \in (0, \frac{1}{2})$, there exists $\delta > 0$ such that if G is a graph with $\operatorname{ind}_H(G) \le (\delta|G|)^{|H|}$, there is an ε-restricted subset $S \subseteq V(G)$ with size at least $\delta|G|$.*

The proofs of Rödl and Nikiforov used the regularity lemma, and gave bounds for δ^{-1} that were tower-type in terms of ϵ^{-1}. Fox and Sudakov [13] gave much better bounds, using a different proof method; they proved that, in both theorems, δ can be chosen to be $2^{-d(\log\frac{1}{\varepsilon})^2}$ for some $d > 0$ depending only on H. They also conjectured the following "polynomial Rödl" version of Theorem 1.5:

Conjecture 1.7. *For every graph H, there exists $d > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$, and every H-free graph G, there is an ε-restricted subset of $V(G)$ with size at least $\varepsilon^d|G|$.*

This conjecture, if true, would imply the Erdős-Hajnal conjecture. But even more could be true: the following "polynomial Nikiforov" statement was conjectured in [15]. It unifies Theorem 1.6 and Conjecture 1.7.

Conjecture 1.8. *For every graph H, there exists $d > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every graph G with $\operatorname{ind}_H(G) \le (\varepsilon^d|G|)^{|H|}$, there is an ε-restricted subset of $V(G)$ with size at least $\varepsilon^d|G|$.*

It is not known whether the converse implications, that the Erdős-Hajnal conjecture 1.1 implies 1.7, and that 1.7 implies 1.8, are true. Let us say a graph H is *viral* if it satisfies the strongest of these, Conjecture 1.8. Recent developments on Erdős-Hajnal conjecture suggest that viral graphs could be the "right" concept to investigate. For example, it is shown in [15] that all the graphs known to have the Erdős-Hajnal property are indeed viral; and the class of viral graphs is known to be closed under vertex-substitution, while this is not known for the class of graphs satisfying Conjecture 1.7. Moreover, the best general bound known in Conjecture 1.1 is $2^{c\sqrt{\log n \log\log n}}$ (for some c depending on H), and the best quantitative dependence of δ on ε known in Theorem 1.5 is

$$\delta = 2^{-d(\log\frac{1}{\varepsilon})^2/\log\log\frac{1}{\varepsilon}}$$

(for some $d > 0$ depending on H); and both these results were obtained in [3] with a proof relying crucially on counting induced subgraphs. We will prove the following, which immediately implies Theorem 1.4:

Theorem 1.9. *Every $H \in \mathcal{H}$ is viral.*

The proof of this is by induction on $|H|$; and for inductive purposes we must be proving the stronger statement that these graphs are viral, and not just that they have the Erdős-Hajnal property, or even that they satisfy Conjecture 1.7.

Although progress on the Erdős-Hajnal property itself has been slow, there are several papers in the literature showing that graphs that contain neither of two given graphs have polynomial-sized cliques or stable sets. For instance, it is shown in [8] that if $H_1, \overline{H_2}$ are forests, there exists $c > 0$ such that every graph G that is both H_1-free and H_2-free has a clique or stable set of size at least $|G|^c$. The reason this "pair of graphs" approach is comparatively so successful, is that the proof method uses Theorem 1.5 as the first step, and thereafter works inside a subgraph that is either very sparse or very dense. One of the graphs H_1, H_2 is good for the sparse case, and the other for the dense case, while it may be difficult to find a single graph that is good for both cases simultaneously. One could try to derive a graph with the Erdős-Hajnal property by asking that $H_1 = H_2$; but for instance, if H is both a forest and the complement of a forest, then H has at most four vertices, and we already know that such graphs have the Erdős-Hajnal property. The same happens for all the "pair of graphs" results found so far: if we insist that the same graph H fills both roles, we get nothing of interest. But in the present paper, that is not so. There is a "pair of graphs" version, which is perhaps simpler and more natural; and it remains nontrivial (and gives Theorem 1.9) if we insist that the two graphs are the same.

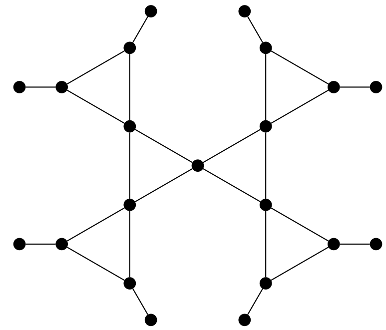

FIGURE 4. With H as shown, $H \in \mathcal{J}$, and so $\{H, \overline{H}\}$ is viral.

Let $\mathcal{J}$ be the family of graphs J with the property that every induced subgraph of J either contains a vertex of degree at most one or is not prime. For instance, $\mathcal{J}$ contains every forest, and all line graphs of forests. The graphs H such that $H \in \mathcal{J}$ and $\overline{H} \in \mathcal{J}$ are precisely the graphs in $\mathcal{H}$, so the following implies that the members of $\mathcal{H}$ have the Erdős-Hajnal property, by taking $H_1 = H_2 \in \mathcal{H}$:

Theorem 1.10. *If $H_1, \overline{H_2} \in \mathcal{J}$, there exists $c > 0$ such that if G is both H_1-free and H_2-free, then G has a clique or stable set of size at least $|G|^c$.*

Consequently, this contains the theorem of [8] mentioned earlier, about excluding a forest and a forest complement. And we will extend Theorem 1.10 to a viral version, in Theorem 7.5.

The proof method extends to ordered graphs. An argument of Alon, Pach and Solymosi [1] shows that the Erdős-Hajnal conjecture is equivalent to the same statement for ordered graphs; and one can define "vertex-substitution" and "prime" for ordered graphs just as for graphs; and again it suffices to consider only prime ordered graphs. But the only prime ordered graphs that (until now) we knew had the ordered Erdős-Hajnal property had at most *three* vertices. We will provide infinitely many. Indeed, each graph in $\mathcal{H}$ can be ordered to make a prime ordered graph with the ordered Erdős-Hajnal property. For instance, the graph of Fig. 2, when ordered such that

$$b_{12} \le b_{11} \le b_8 \le b_7 \le b_4 \le b_3 \le b_1 \le a_2 \le a_5 \le a_6 \le a_9 \le a_{10} \le a_{13}$$

becomes a prime ordered graph that has the ordered Erdős-Hajnal property.

Let $\mathcal{K}$ be the class of ordered graphs G that can be grown from nothing by repeatedly either adding a vertex of degree at most one at one end or the other of its linear order, or vertex-substitution. We will prove:

Theorem 1.11. *If $H_1, \overline{H_2} \in \mathcal{K}$, there exists $c > 0$ such that if G is an ordered graph that is both H_1-free and H_2-free (in the appropriate sense for ordered graphs), then G has a clique or stable set of size at least $|G|^c$.*

This contains both Theorems 1.10 and 1.12. A recent theorem of Pach and Tomon [19] says the same thing, assuming that $H_1, \overline{H_2}$ are both obtained by giving a path its natural ordering, so that is also a special case of Theorem 1.11.

All these results will be extended, in Theorem 2.1 and Theorem 7.5, to say that the corresponding objects are viral. This extension is critical for inductive reasons.

We can apply Theorem 1.11 to tournaments, and obtain new tournaments with the Erdős-Hajnal property. (See [5] for some related results.) Say a tournament is *buildable* if it can be grown from nothing by repeatedly either adding a vertex of out-degree ≤ 1 or in-degree ≤ 1, or vertex-substitution. We will show:

Theorem 1.12. *For every buildable tournament H, there exists c such that if G is a tournament with no subtournament isomorphic to H, then there is a transitive set in $V(G)$ with size at least $|G|^c$.*

2. Ordered graphs, and a sketch of the proof

The main motivation for our work was unordered graphs and the Erdős-Hajnal conjecture, but the proof works equally well for ordered graphs, and we thereby gain a much more powerful result. We would like to outline the idea of the proof as soon as we can, but we need first to set up more definitions, particularly about ordered graphs.

An *ordered graph* is a pair $G = (F, \le)$ where F is a graph and $\le$ is a linear order of $V(F)$; and we define $G^\natural := F$, and we define $\le_G$ to equal $\le$. A *copy* of an ordered graph H in an ordered graph G is an isomorphism ϕ from $H^\natural$ to an induced subgraph J of $G^\natural$, such that for all distinct $u, v \in V(H)$, $u \le_H v$ if and only if $\phi(u) \le_G \phi(v)$. We extend many definitions for graphs to ordered graphs in the natural way; so for instance, if G is an ordered graph, we write $V(G) := V(G^\natural)$; $|G| := |G^\natural|$; $\overline{G} := (\overline{G^\natural}, \le_G)$; "degree in G" means degree in $G^\natural$; a "blockade in G" means a blockade in $G^\natural$; and so on. We use $[n]$ to denote $\{1, 2, \dots, n\}$ for every integer $n \ge 1$.

For every $x > 0$ and graphs G, H, define

$$\mu_H(x, G) := \frac{\mathrm{ind}_H(G)}{(x|G|)^{|H|}}.$$

Let $\mathcal{F}$ be a finite set of graphs. For a graph G, we define $\mu_{\mathcal{F}}(x, G) := \max_{H\in\mathcal{F}} \mu_H(x, G)$; and we say that $\mathcal{F}$ is *viral* if there exists $d > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$, and for every graph G with $\mu_{\mathcal{F}}(\varepsilon^d, G) \le 1$, there is an ε-restricted $S \subseteq V(G)$ with $|S| \ge \varepsilon^d |G|$. We call such a number d a *viral exponent* for $\mathcal{H}$. Thus a graph H is viral if and only if $\{H\}$ is viral. These definitions extend to ordered graphs in the natural way. Thus, when G, H are ordered graphs, $\mathrm{ind}_H(G)$ denotes the number of copies of H in G, and so on.

We need to define vertex-substitution for ordered graphs. Let H_1, H_2 be ordered graphs, and let $v \in V(H_1)$. The ordered graph H obtained *from H_1 by substituting H_2 for v* is the pair $(H^\natural, \le_H)$, where $H^\natural$ is the graph obtained from $H_1^\natural$ by substituting $H_2^\natural$ for v, and $\le_H$ is defined by:

- if $x, y \in V(H_1) \setminus \{v\}$ then $x \le_H y$ if and only if $x \le_{H_1} y$;
- if $x, y \in V(H_2)$ then $x \le_H y$ if and only if $x \le_{H_2} y$;
- if $x \in V(H_1) \setminus \{v\}$ and $y \in V(H_2)$, then $x \le_H y$ if and only if $x \le_{H_1} v$.

An ordered graph is *prime* if it cannot be obtained by vertex-substitution from two smaller ordered graphs.

We say $v \in V(H)$ is the *first* vertex of an ordered graph H if $v \le_H u$ for all $u \in V(H)$, and the *last* vertex is defined similarly. We say that v is an *end vertex* of H if v is either the first or last vertex of H. Let $\mathcal{K}$ be the class of all ordered graphs K with the property that for every induced ordered subgraph G of K, either G is not prime, or there exists $v \in V(G)$ with degree at most one in $G^\natural$, such that v is an end vertex of H. Our ultimate goal is to prove the following:

Theorem 2.1. *For all $H, J \in \mathcal{K}$, the pair $\{H, \overline{J}\}$ is viral.*

All the other theorems we mentioned will be corollaries of this.

Before we can sketch the proof, we need a few more definitions. For a graph G and disjoint $A, B \subseteq V(G)$, B is *x-sparse* to A if every vertex in B has at most $x|A|$ neighbours in A, and *$(1-x)$-dense* to A if every vertex in B has at most $x|A|$ nonneighbours in A. A *blockade* in a graph or ordered graph G is a finite sequence $(B_1, \dots, B_n)$ of (possibly empty) disjoint subsets of $V(G)$; its *length* is n and its *width* is $\min_{i \in [n]} |B_i|$. For $k, w \geq 0$, $(B_1, \dots, B_n)$ is a *(k, w)-blockade* if its length is at least k and its width is at least w. For $x \in (0, \frac{1}{2})$, this blockade is *x-sparse* if B_j is x-sparse to B_i for all $i, j \in [n]$ with $i < j$, and *$(1-x)$-dense* if B_j is $(1-x)$-dense to B_i for all $i, j \in [n]$ with $i < j$.

Here, finally, is a sketch of the proof of Theorem 2.1. We work by induction on $|H| + |J|$. If one of H, J is not prime, the result follows easily from the inductive hypothesis, by means of Lemma 3.4 below. So we can assume they are both prime, and hence, for each of H, J, some end vertex has degree one. Let H', J' be obtained from H, J respectively, by deleting an end vertex of degree one. Inductively we know that $\{H', \overline{J}\}$ and $\{H, \overline{J'}\}$ are both viral; and this will be enough to imply that $\{H, \overline{J}\}$ is viral (Theorem 6.1). Let d_0 be large enough to be a viral exponent for both $\{H', \overline{J}\}$ and $\{H, \overline{J'}\}$.

There is a key result, Theorem 4.3 below, that deduces the property of being viral from the existence of suitable blockades. It follows that if there exists d such that for every ordered graph G with $\mu_{\mathcal{F}}(x^d, G) \leq 1$, and every $x \in (0, \frac{1}{2})$, there is an x-sparse or $(1-x)$-dense blockade in G with appropriate length and width, then $\mathcal{F}$ is viral. Because of this, we will try to find such a blockade, instead of trying to find directly a large ε-restricted set. Thus, let $d \geq d_0$ be some large number, let $x \in (0, \frac{1}{2})$, and let G be an ordered graph, with $\mu_{\mathcal{F}}(x^d, G) \leq 1$. We assume for a contradiction that the blockade we want does not exist; that is, there is no x-sparse or $(1-x)$-dense blockade in G with length k, where $2 \leq k \leq 1/x$, and width at least $\lfloor |G|/k^d \rfloor$.

Let $h = \max(|H|, |J|)$. We will grow a nested sequence of subsets $V(G) = S_0 \supseteq S_1 \supseteq \cdots$, where for each i, $|S_i| \geq 2^{-6hd^i} |S_{i-1}|$ and G_i is $2^{-4hd^{i-1}}$-restricted. We have a method to define S_{i+1} in terms of S_i, provided S_i is $1/h$-restricted; but it does not work to define S_1, because G is not $1/h$-restricted, so we have to do something else to get S_1. We need a large subset S_1 which is 2^{-4h}-restricted. Nikiforov's theorem would give us such a thing, except we are working with ordered graphs; so we need an ordered graphs version of Nikiforov's theorem (Theorem 3.2 below). This gives $G[S_1]$. The latter will either have small maximum degree, or the same in the complement, and by moving to the complement if necessary we may assume it has small maximum degree.

Now we will describe the general step, to obtain S_{i+1} from S_i when $i \geq 1$. We recall that $H' = H \setminus \{v\}$, where v is an end vertex of H with degree one. Now we will use that $\{H', \overline{J}\}$ is viral.(In the other case, when $G[S_1]$ has small maximum degree in the complement, we would use that $\{H, \overline{J'}\}$ is viral.) Reversing the order if necessary, we may assume that v is the last vertex of H.

Let $y = 2^{-2hd_{i-1}}$. Thus S_i is y^2-restricted. G is an ordered graph; let S be the set of the first $\lceil y|S_i| \rceil$ vertices of S_i. If $G[S]$ does not contain many copies of H' (and we already know that it does not contain many copies of $\overline{J}$, since G itself does not), we can use that $\{H', \overline{J}\}$ is viral to find $S_{i+1} \subseteq S$. So we assume that it does contain many copies of H'. Let u be the neighbour of v in H, and let $H'' = H \setminus \{u, v\}$. It follows that $G[S]$ contains many copies of H'' that each can be extended to many copies of H' in $G[S]$. But they cannot all be extended to many copies of H in G, because there are not that many copies of H. So there is a copy X of H'' in $G[S]$, that extends to many copies of H' in $G[S]$, and yet does not extend to many copies of H in G.

Let A_1 be the set of vertices in S that give an extension of X to a copy of H'; so A_1 is large, all the vertices in A_1 have the same neighbours in $V(X)$, and they are all in the right position in the order $\leq_G$

with respect to the vertices in X. (For simplicity, we are conflating the copy X, which is an isomorphism, with its image, an induced subgraph of $G[S]$.) Since $G[S_i]$ has maximum degree at most $y^2|S_i|$, there are not many vertices in the rest of S_i that have a neighbour in $V(X)$; and for the others, say B, any edge between A_1 and B gives a copy of H. (This is where we use that v is the last vertex of H and S is an initial segment of $G[S_i]$; all the vertices in B are in the right order relative to X and A_1.) Since X does not extend to many copies of H, there are not many edges between A_1 and B; and by throwing away a few outliers, we can choose $B_1 \subseteq B$, x-sparse to A_1, where B_1 still contains almost all of S_i. We have produced an x-sparse blockade (A_1, B_1) in $G[S_i]$ of length two, where $|A_1|$ is at least $\text{poly}(y)|G|$, and $|B_1|$ is only slightly smaller than $|S_i|$. That is the only argument where we use the leaf of H; it is Lemma 5.1.

Now we look inside the set B_1 above, and repeat the same argument; we obtain either the desired set S_{i+1}, or an x-sparse blockade (A_1, A_2, B_2) in $G[S_i]$ of length three, where $|A_1|, |A_2|$ are both at least $\text{poly}(y)|G|$, and $|B_2|$ is only slightly smaller than $|S_i|$. By repeating this $1/y$ times (which we can, there is enough room), we obtain either the set S_{i+1} that we want, or the blockade that we assumed did not exist. This is Lemma 5.3.

3. Some lemmas about counting subgraphs

We need to extend Nikiforov's theorem to ordered graphs, and to do so, we use the following result of Rödl and Winkler [21]:

Theorem 3.1. *For every ordered graph J there is a graph H such that, for every ordering of $V(H)$, the resultant ordered graph contains J.*

We deduce:

Theorem 3.2. *For every ordered graph J and every $\varepsilon \in (0, \frac{1}{2})$, there exists $\delta > 0$ such that if G is a ordered graph with $\text{ind}_J(G) \le (\delta|G|)^{|J|}$, there is an ε-restricted subset $S \subseteq V(G)$ with size at least $\delta|G|$.*

Proof. Choose H as in Theorem 3.1, and choose δ' such that setting $\delta = \delta'$ satisfies Theorem 1.6. Let $h := |H|$ and $j := |J|$, and let $\delta := (\delta')^{h/j}$. We claim that δ satisfies the theorem. To show this, let G be a ordered graph with $\text{ind}_J(G) \le (\delta|G|)^{|J|}$. We must show that there is an ε-restricted subset $S \subseteq V(G)$ with size at least $\delta|G|$.

Since each copy of J in $G^\natural$ extends to at most $|G|^{h-j}$ copies of H in $G^\natural$, and each copy of H in $G^\natural$ is an extension of some such copy (because of the choice of H), there are at most

$$\text{ind}_J(G)|G|^{h-j} \le (\delta|G|)^j|G|^{h-j} = (\delta'|G|)^h$$

copies of H in $G^\natural$. But then the result follows from the choice of δ'. This proves Theorem 3.2. ■

We observe:

Lemma 3.3. *If $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F}$ is viral, then so is $\mathcal{F}'$.*

Proof. This follows from the fact that $\mu_{\mathcal{F}}(x, G) \le \mu_{\mathcal{F}'}(x, G)$ for every $x > 0$ and every graph (or ordered graph) G. ■

Here is an extension of a result of [12]:

Lemma 3.4. *Let $\mathcal{F}$ be a finite set of graphs, and let H_1, H_2 be graphs such that $\mathcal{F} \cup \{H_1\}$ and $\mathcal{F} \cup \{H_2\}$ are viral. Let H be obtained by substituting H_2 for a vertex v of H_1. Then $\mathcal{F} \cup \{H\}$ is viral. The same is true for ordered graphs in place of graphs.*

Proof. The following proof works for both graphs and ordered graphs. If H_1 or H_2 is in $\mathcal{F}$ then $\mathcal{F}$ is viral, and so $\mathcal{F} \cup \{H\}$ is viral by Lemma 3.3. Thus we may assume $H_1, H_2 \notin \mathcal{F}$. For $i \in \{1,2\}$, let $h_i := |H_i|$, and let $d_i > 0$ be a viral exponent for $\mathcal{F}_i := \mathcal{F} \cup \{H_i\}$. We claim that $d := d_1h_1 + d_2 + 1$ is a viral exponent for $\mathcal{F}' := \mathcal{F} \cup \{H\}$. To see this, we may assume $h_1 \geq 2$. Now, let $\varepsilon \in (0, 1/2)$ and let G be a graph (or ordered graph) with $\mu_{\mathcal{F}'}(\varepsilon^d, G) \leq 1$; and suppose for a contradiction that there is no ε-restricted $S \subseteq V(G)$ with $|S| \geq \varepsilon^d|G|$. Then $\mu_{\mathcal{F}_1}(\varepsilon^{d_1}, G) > 1$ by the choice of d_1; and so $\mu_{\mathcal{F}_1}(\varepsilon^{d_1}, G) > 1$. Thus, since $\mu_{\mathcal{F}}(\varepsilon^{d_1}, G) \leq \mu_{\mathcal{F}}(\varepsilon^d, G) \leq \mu_{\mathcal{F}'}(\varepsilon^d, G) \leq 1$, we deduce that $\mu_{H_1}(\varepsilon^{d_1}, G) > 1$. It follows that

$$\mathrm{ind}_{H_1}(G) > (\varepsilon^{d_1}|G|)^{h_1} = \varepsilon^{d_1h_1}|G|^{h_1}.$$

For every copy φ of $H_1 \setminus v$ in G, let I_φ be the set of copies φ' of H_1 with $\varphi'|_{V(H_1 \setminus v)} = \varphi$. Let T be the set of copies φ of $H_1 \setminus v$ in G with $|I_\varphi| \geq \varepsilon^{d_1h_1+1}|G|$; then

$$\sum_{\varphi \in T} |I_\varphi| \geq \mathrm{ind}_{H_1}(G) - |G|^{h_1-1} \cdot \varepsilon^{d_1h_1+1}|G| > \varepsilon^{d_1h_1}|G|^{h_1} - \varepsilon^{d_1h_1+1}|G|^{h_1} \geq \varepsilon^{d_1h_1+1}|G|^{h_1}.$$

Thus $|T| > \varepsilon^{d_1h_1+1}|G|^{h_1-1} \geq (\varepsilon^d|G|)^{h_1-1}$ since $h_1 \geq 2$ and $|I_\varphi| \leq |G|$ for all $\varphi \in T$.

Claim 3.5. *For every $\varphi \in T$, there are at least $(\varepsilon^d|G|)^{h_2}$ copies φ'' of H in G with $\varphi''|_{V(H_1 \setminus v)} = \varphi$.*

Proof. Let $A := \{\varphi'(v) : \varphi' \in I_\varphi\}$; then $|A| \geq \varepsilon^{d_1h_1+1}|G|$. Thus, since G includes no ε-restricted $S \subseteq V(G)$ with $|S| \geq \varepsilon^d|G|$, $G[A]$ contains no ε-restricted $S \subseteq A$ with $|S| \geq \varepsilon^{d_2}|A| \geq \varepsilon^d|G|$. The choice of d_2 then implies that $\mu_{\mathcal{F}_2}(\varepsilon^{d_2}, G[A]) > 1$. Hence, because

$$\mu_{\mathcal{F}}(\varepsilon^{d_2}, G[A]) \leq \mu_{\mathcal{F}}(\varepsilon^{d_1h_1+d_2+1}, G) = \mu_{\mathcal{F}}(\varepsilon^d, G) \leq 1,$$

we obtain $\mu_{H_2}(\varepsilon^{d_2}, G[A]) > 1$, and so $\mathrm{ind}_{H_2}(G[A]) > (\varepsilon^{d_2}|A|)^{h_2} \geq (\varepsilon^d|G|)^{h_2}$. Since each copy of H_2 in $G[A]$ together with φ forms a copy φ'' of H in G with $\varphi''|_{V(H_1 \setminus v)} = \varphi$ and these copies are distinct, the proof of Claim 3.5 is complete. □

Now, Claim 3.5 yields

$$\mathrm{ind}_H(G) \geq |T|(\varepsilon^d|G|)^{h_2} > (\varepsilon^d|G|)^{h_1-1}(\varepsilon^d|G|)^{h_2} = (\varepsilon^d|G|)^{|H|}$$

and so $\mu_{\mathcal{F}'}(\varepsilon^d, G) \geq \mu_H(\varepsilon^d, G) > 1$, a contradiction. This proves Lemma 3.4. ■

4. Being divisive and being viral

For the next three sections, we will focus on ordered graphs and proving Theorem 2.1. At the end of the paper, we look at its corollaries, for unordered graphs, for tournaments, and for excluding one graph instead of two.

Let us clear up a slight discrepancy between this paper and [15]. In that paper the definition of "viral" was slightly different. Let $\mathcal{F}$ be a finite set of graphs or ordered graphs. We say that $\mathcal{F}$ is *weakly viral* if there exists $d > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$, and for every graph (or ordered graph, appropiately) G with $\mu_{\mathcal{F}}(\varepsilon^d, G) \leq 1$, there is a subset $S \subseteq V(G)$ with $|S| \geq \varepsilon^d|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges. (So we are not restricting the maximum degree in one of $G[S], \overline{G}[S]$, just the average degree.) This is the definition of "viral" in [15].) We call such a number d a *weak viral exponent* for $\mathcal{F}$. It does not really matter which definition we use, becuase of the following.

Lemma 4.1. *Let $\mathcal{F}$ be a finite set of graphs or ordered graphs. If d is a viral exponent for $\mathcal{F}$ then it is a weak viral exponent for $\mathcal{F}$. Conversely, if d is a weak viral exponent for $\mathcal{F}$, then $3d$ is a viral exponent for $\mathcal{F}$. In particular, $\mathcal{F}$ is viral if and only if it is weakly viral.*

Proof. The first assertion is clear. For the second, let d be a weak viral exponent for $\mathcal{F}$, let $\varepsilon \in (0, \frac{1}{2})$, and let G be a graph or ordered graph with $\mu_{\mathcal{H}}(\varepsilon^{3d}, G) \le 1$. Since $\varepsilon/4 \ge \varepsilon^3$ it follows that $\mu_{\mathcal{H}}((\varepsilon/4)^d, G) \le 1$. Since d is a weak viral exponent for $\mathcal{F}$, there is a subset $T \subseteq V(G)$ with $|T| \ge (\varepsilon/4)^d|G|$ such that one of $G[T], \overline{G}[T]$ has at most $(\varepsilon/4)\binom{|T|}{2}$ edges, and we may assume the first. Consequently, at most $|T|/2$ vertices in T have degree in $G[T]$ more than $\varepsilon|T|/2$; and so there exists $S \subseteq T$ with $|S| \ge |T|/2$ such that every vertex in S has degree at most $\varepsilon|T|/2 \le \varepsilon|S|$ in $G[T]$ and hence in $G[S]$. Thus S is ε-restricted. This proves Lemma 4.1. ■

We say that a finite set $\mathcal{F}$ of graphs is "divisive", if every graph G either contains many copies of some member of $\mathcal{F}$, or admits a blockade that is both long and wide, and either sparse or dense. More exactly, $\mathcal{F}$ is *divisive* if there exist $b, c > 0$ such that for every $x \in (0, c)$ and every graph G with $\mu_{\mathcal{F}}(x^b, G) \le 1$, there is an x-sparse or $(1-x)$-dense $(k, \lfloor |G|/k^b \rfloor)$-blockade in G where $k \in [2, 1/x]$. (This property is a variant of the so-called *quasi-Erdős-Hajnal property* employed in [9, 19, 22].) Similarly, a finite set $\mathcal{F}$ of ordered graphs is *divisive* if there exist $b, c > 0$ such that for every $x \in (0, c)$ and every ordered graph G with $\mu_{\mathcal{F}}(x^b, G) \le 1$, there is an x-sparse or $(1-x)$-dense $(k, \lfloor |G|/k^b \rfloor)$-blockade in G where $k \in [2, 1/x]$. We call (b, c) a pair of *divisive sidekicks* for $\mathcal{F}$.

In this section we give a key result, which allows us to deduce the viral property from divisiveness, and will be crucial in the proof of Theorem 1.9. We will need the following, which is proved in [15]:

Theorem 4.2. *Let G be a graph, and let $\varepsilon \in (0, \frac{1}{2})$ and $d \ge 1$. Let $x = \varepsilon^{12d}$. Suppose that for every induced subgraph F of G with $|F| \ge \varepsilon^{4d}|G|$, there is an x-sparse or $(1-x)$-dense blockade in F of length $k \in [2, 1/x]$ and width at least $|F|/k^d$. Then there exists $S \subseteq V(G)$ with $|S| \ge x^{d+1}|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges.*

This is used to prove the following (the "only if" part below is the same as the proof of the corresponding result for unordered graphs in [15]):

Theorem 4.3. *If $\mathcal{F}$ is a finite set of ordered graphs, then it is divisive if and only if it is viral.*

Proof. We will only need the "only if" direction, but the "if" direction of this theorem is simple. To see this, we assume that $\mathcal{F}$ is a viral set of graphs, or of ordered graphs. Let d be a viral exponent for $\mathcal{F}$. Let $x > 0$ with $x \le \min(1/16, 1/d)$; and let G be a graph or ordered graph with $\mu_{\mathcal{F}}(x^{2d}, G) \le 1$. There exists an x^2-restricted $S \subseteq V(G)$ with $|S| \ge x^{2d}|G|$. Let $k := \lceil x^{-1/2} \rceil \in [2, 1/x]$, and choose a sequence $(B_1, \ldots, B_k)$ of disjoint subsets of S, all of size $\lfloor 2x|S| \rfloor$ (such subsets exist since $k\lfloor 2x|S| \rfloor \le 4x^{1/2}|S| < |S|$). Then $(B_1, \ldots, B_k)$ is an x-sparse or $(1-x)$-dense $(k, \lfloor |G|/k^{4d+2} \rfloor)$-blockade in G (since $\lfloor x^2|S| \rfloor \le x\lfloor 2x|S| \rfloor$). This proves the "if" direction.

For the "only if" direction, let (b_0, c_0) be a pair of divisive sidekicks for $\mathcal{H}$, and let $d = \max(b_0, 1/c_0)$ (so $(d, 1/d)$ is also a pair of divisive sidekicks for $\mathcal{H}$). Let $c := 12(d+1)(d+2)$. We claim that c is a weak viral exponent for $\mathcal{H}$.

To show this, let $\varepsilon \in (0, \frac{1}{2})$ and let G be a graph with $\mu_{\mathcal{H}}(\varepsilon^c, G) \le 1$. We must show that there exists $S \subseteq V(G)$ with $|S| \ge \varepsilon^c|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges. We may assume that $|G| > \varepsilon^{-c}$, since otherwise we may take $|S| \le 1$. Let $d' = d + 1$, and $x = \varepsilon^{12d'}$. We claim that:

Claim 4.4. *For every induced subgraph F of G with $|F| \ge \varepsilon^{4d'}|G|$, there is an x-sparse or $(1-x)$-dense blockade in F of length $k \in [2, 1/x]$ and width at least $|F|/k^{d'}$.*

Proof. We observe that $x^d\varepsilon^{4d'} = \varepsilon^{12dd'+4d'} \geq \varepsilon^c$, and so $x^d|F| \geq \varepsilon^c|G|$. It follows that

$$\mu_{\mathcal{F}}(x^d, F) \leq \mu_{\mathcal{F}}(\varepsilon^c, G) \leq 1.$$

Since d is an exponent for the divisiveness of $\mathcal{H}$, there exists $k \in [2, 1/x]$ such that there is an x-sparse or $(1-x)$-dense blockade in F of length at least k and width at least $\lfloor |F|/k^d \rfloor$. But

$$|F|/k^d \geq x^d|F| \geq \varepsilon^c|G| > 1,$$

and so

$$\lfloor |F|/k^d \rfloor \geq |F|/(2k^d) \geq |F|/k^{d+1} = |F|/k^{d'}.$$

This proves Claim 4.4. □

From Theorem 4.2, with d replaced by d', we deduce that there exists $S \subseteq V(G)$ with $|S| \geq x^{d'+1}|G| = \varepsilon^c|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges. This proves that $\mathcal{H}$ is weakly viral, and hence viral by Lemma 4.1, and so proves Theorem 4.3. ■

5. Using the leaf

In this section, we are given an ordered graph H with a vertex v of degree one that is an end vertex of H, and a sparse "host" ordered graph G, and we would like to show that either:

- G contains many copies of H; or
- we can locate a subset of $V(G)$ of decent size that induces an ordered subgraph containing not too many copies of $H \setminus \{v\}$; or
- there is a sparse blockade in G with length and width polynomially related to the sparsity parameter.

To do this, we shall first prove this with the third outcome replaced by

- there are disjoint subsets A, B of $V(G)$ where B is sparse to A, and A has decent size, and $|B|$ contains almost all vertices of G.

Then, by iterating the procedure, we will convert the two subsets of this last outcome into a sparse blockade of the desired length and width.

For sets X, Y, Z with $Z \subseteq X$ and a map $f\colon X \to Y$, let $f|_Z$ denote the restriction of f to Z. If H is an ordered graph, and $X \subseteq V(G)$, $H \setminus X$ and $H[X]$ are defined in the natural way.

We obtain sparse pairs by means of the following lemma:

Lemma 5.1. *Let H be an ordered graph with an end vertex v of degree one. Let $h := |H| \geq 2$, and let $H' := H \setminus \{v\}$. Let $x, y > 0$ with $x \leq y \leq \frac{1}{2h}$, and let G be an ordered graph with maximum degree at most $y|G|$. Then for every $a \geq 2$, one of the following outcomes hold:*

- $\operatorname{ind}_H(G) > x^{2a+h}|G|^h$;
- *there exists $S \subseteq V(G)$ with $|S| \geq y|G|$ such that $\operatorname{ind}_{H'}(G[S]) \leq y^{a-2}|S|^{h-1}$; or*
- *there are disjoint $A, B \subseteq V(G)$ such that $|A| \geq \lfloor y^a|G| \rfloor$, $|B| \geq (1-hy)|G|$, and B is x-sparse to A.*

Proof. We assume that the last two outcomes do not hold; then $y^a|G| \geq 1$ for otherwise the third outcome trivially holds. From the symmetry, we may assume that v is the last vertex of H. Let u be the neighbour of v in H, and let $J := H \setminus \{u, v\}$. Let S be the set of the first $\lceil y|G| \rceil$ vertices of G; that is, S is the subset of $V(G)$ with $|S| = \lceil y|G| \rceil$ such that $p \leq_G q$ for all $p \in S$ and $q \in V(G) \setminus S$. For every copy φ of J in $G[S]$, let I_φ be the set of copies φ' of H' in $G[S]$ with $\varphi'|_{V(J)} = \varphi$. Let T be the set of copies φ

of J in $G[S]$ with $|I_\varphi| \ge y^a|G|$. Since $y^a|G| \le y^{a-1}|S|$ and there are at most $|S|^{h-2}$ copies of J in $G[S]$, we have (note that $y \in (0, \frac{1}{2})$)

$$\sum_{\varphi \in T} |I_\varphi| \ge \mathrm{ind}_{H'}(G[S]) - |S|^{h-2} \cdot y^a|G| > y^{a-2}|S|^{h-1} - y^{a-1}|S|^{h-1} \ge y^{a-1}|S|^{h-1},$$

and so $|T| > y^{a-1}|S|^{h-2}$, since $|I_\varphi| \le |S|$ for all $\varphi \in T$.

Claim 5.2. *For every $\varphi \in T$ there are at least $x^{a+3}|G|^2$ copies φ'' of H in G with $\varphi''|_{V(J)} = \varphi$.*

Proof. Let $P := \varphi(V(J))$; then $|P| = h-2$. Let $A' := \{\varphi'(v) : \varphi' \in I_\varphi\}$; then $A' \subseteq S$ and $|A'| = |I_{\varphi'}| \ge y^a|G|$. Let $A \subseteq A'$ with $|A| = \lceil y^a|G| \rceil$. Now $|V(G) \setminus S| = |G| - \lceil y|G| \rceil \ge |G|(1-y) - 1$; and at most $y(h-2)|G|$ vertices in $V(G) \setminus S$ have a neighbour in P (note that there could be no edges between A, P). Let B be the set of vertices in $V(G) \setminus S$ with no neighbours in P; then

$$|B| \ge (1 - (h-1)y)|G| - 1.$$

Since the third outcome does not hold, there are at most $(1-hy)|G|$ vertices in B that have fewer than $x|A|$ neighbours in A. Thus the number of vertices in B with at least $x|A|$ neighbours in A is at least

$$|B| - (1-hy)|G| \ge y|G| - 1 \ge 2y^2|G| - 1 \ge y^2|G|$$

(note that $y|G| \ge 2y^2|G| \ge 2y^a|G| \ge 2$). Consequently there are at least $xy^2|A| \cdot |G| \ge x^{a+3}|G|^2$ edges between A, B. Since the endpoints of each such edge together with P form a copy φ' of H in G with $\varphi'|_{V(J)} = \varphi$, there are at least $x^{a+3}|G|^2$ copies φ' of H in G with $\varphi'|_{V(J)} = \varphi$, as claimed. □

Therefore, Claim 5.2 implies that

$$\mathrm{ind}_H(G) \ge |T| \cdot x^{a+3}|G|^2 > y^{a-1}|S|^{h-2} \cdot x^{a+3}|G|^2 \ge y^{a+h-3}x^{a+3}|G|^h \ge x^{2a+h}|G|^h$$

which is the first outcome. This proves Lemma 5.1. ■

Now we turn sparse pairs into sparse, long, wide blockades.

Lemma 5.3. *Let H be an ordered graph with $h := |H| \ge 2$, and let $v \in V(H)$ be an end vertex of H, and have degree one. Let $H' := H \setminus \{v\}$. Let $x, y > 0$ with $x \le y \le 4^{-h}$, and let G be an ordered graph with maximum degree at most $y^2|G|$. Then for every $a \ge 2$, one of the following holds:*

- $\mathrm{ind}_H(G) > x^{2a+2h}|G|^h$;
- $\mathrm{ind}_{H'}(G[S]) \le y^{a-2}|S|^{h-1}$ *for some $S \subseteq V(G)$ with $|S| \ge y^2|G|$; or*
- *there is an x-sparse $(y^{-1}, \lfloor y^{a+1}|G| \rfloor)$-blockade in G.*

Proof. Suppose that none of the outcomes holds; then $y^{a+1}|G| \ge 1$. Choose an x-sparse blockade $(B_0, B_1, \dots, B_k)$ in G such that $|B_{i-1}| \ge y^{a+1}|G|$ for all $i \in [k]$, and $|B_k| \ge (1-hy)^k|G|$, with k maximum. Since the third outcome does not hold, we have $k < y^{-1}$, which implies (since $1 - t \ge 4^{-t}$ for all $t \in [0, \frac{1}{2}]$) that

$$|B_k| \ge (1-hy)^k|G| \ge 4^{-hyk}|G| > 4^{-h}|G| \ge y|G| \ge x|G|.$$

It follows that $G[B_k]$ has maximum degree at most $y^2|G| \le y|B_k|$. Since the first two outcomes do not hold, we have

- $\mathrm{ind}_H(G[B_k]) \le \mathrm{ind}_H(G) \le x^{2a+2h}|G|^h \le x^{2a+h}|B_k|^h$; and
- $\mathrm{ind}_{H'}(G[B_k]) > y^{a-2}|B_k|^{h-1}$.

By Lemma 5.1, there are disjoint $A, B \subseteq B_k$ with

$$|A| \geq \lfloor y^a |B_k| \rfloor \geq \lfloor y^{a+1}|G| \rfloor \quad \text{and} \quad |B| \geq (1-hy)|B_k| \geq (1-hy)^{k+1}|G|$$

such that B is x-sparse to A. Therefore $(B_0, \dots, B_{k-1}, A, B)$ is an x-sparse blockade, and this contradicts the maximality of k. This proves Lemma 5.3. ■

6. Producing the blockade

To finish the proof of Theorem 2.1, we need to write out the argument sketched in Section 2.

Theorem 6.1. *Let $\mathcal{F}$ be a finite set of ordered graphs, and let $H, \overline{J} \in \mathcal{F}$. Let v be an end vertex of H with degree one, and let w be an end vertex of J with degree one. Let $H' = H \setminus \{v\}$ and $J' = J \setminus \{w\}$. If $\{H'\} \cup (\mathcal{F} \setminus \{H\})$ and $\{\overline{J'}\} \cup (\mathcal{F} \setminus \{\overline{J}\})$ are both viral then $\mathcal{F}$ is viral.*

Proof. Define $h := \max(|H|, |J|, 4)$, and $c := 4^{-h}$. By Theorem 3.2, there exists $c' > 0$ such that for every ordered graph G with $\mu_{\mathcal{F}}(c', G) \leq 1$, there is a c^2-restricted $S \subseteq V(G)$ with $|S| \geq c'|G|$. Since $\mathcal{F}_1 := \{H'\} \cup (\mathcal{F} \setminus \{H\})$ and $\mathcal{F}_2 := \{\overline{J'}\} \cup (\mathcal{F} \setminus \{\overline{J}\})$ are viral, there exists $d > 0$ that is a viral exponent for them both; and we may increase d so that $d \geq \max(4, \log_c(c'))$. Let $a := 2d^2h$ and $b := a + 6d + 1$. We need the following claim.

Claim 6.2. *Let $x, y > 0$ with $x \leq y \leq c$, and let G be a y^2-restricted ordered graph. Then either*

- *$\mu_{\mathcal{F}}(x^{b-4d}, G) > 1$; or*
- *$\min(\mu_{H'}(y^{2d^2}, G[S]), \mu_{\overline{J'}}(y^{2d^2}, G[S])) \leq 1$ for some $S \subseteq V(G)$ with $|S| \geq y^2|G|$; or*
- *there is an x-sparse or $(1-x)$-dense $(y^{-1}, \lfloor y^{a+1}|G| \rfloor)$-blockade in G.*

Proof. If G has maximum degree at most $y^2|G|$, then Lemma 5.3 implies that either:

- $\mathrm{ind}_H(G) > x^{2a+2|H|}|G|^{|H|} \geq (x^{b-4d}|G|)^{|H|}$, where the second inequality is by the choice of b (and so $\mu_{\mathcal{F}}(x^{b-4d}, G) > 1$); or
- $\mathrm{ind}_{H'}(G[S]) \leq y^{a-2}|S|^{|H'|} \leq (y^{2d^2}|S|)^{|H'|}$ for some $S \subseteq V(G)$ with $|S| \geq y^2|G|$, where the second inequality is because $a - 2 = 2d^2h - 2 \geq 2d^2(h-1)$ by the choice of a (and so $\mu_{H'}(y^{2d^2}, G[S]) \leq 1$); or
- there is an x-sparse $(y^{-1}, \lfloor y^{a+1}|G| \rfloor)$-blockade in G.

If $\overline{G}$ has maximum degree at most $y^2|G|$, then similarly, also by Lemma 5.3, either:

- $\mathrm{ind}_{\overline{J}}(G) = \mathrm{ind}_J(\overline{G}) > x^{2a+2|J|}|G|^{|J|} \geq (x^{b-4d}|G|)^{|J|}$; or
- $\mathrm{ind}_{\overline{J'}}(G[S]) = \mathrm{ind}_{J'}(\overline{G}[S]) \leq y^{a-2}|S|^{|J'|} \leq (y^{2d^2}|S|)^{|J'|}$ for some $S \subseteq V(G)$ with $|S| \geq y^2|G|$; or
- there is a $(1-x)$-dense $(y^{-1}, \lfloor y^{a+1}|G| \rfloor)$-blockade in G.

This proves Claim 6.2. □

We claim that (b, c) is a pair of divisive sidekicks for $\mathcal{F}$; and hence $\mathcal{F}$ is divisive, and consequently viral by Theorem 4.3. Thus, let $x \in (0, c)$, and let G be a graph with $\mu_{\mathcal{F}}(x^b, G) \leq 1$. Suppose for a contradiction that there is no x-sparse or $(1-x)$-dense $(k, \lfloor |G|/k^b \rfloor)$-blockade in G with $k \in [2, 1/x]$; then $|G| \geq x^{-b/2}$, because otherwise the blockade with $\lfloor 1/x \rfloor \geq x^{-1/2}$ empty blocks contradicts our supposition. Let $m \geq 2$ be the least integer with $c^{d^{m-1}} \leq x$; then $c^{d^{m-2}} \geq x$.

Claim 6.3. *There is a nested sequence $V(G) = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m$ such that $|S_i| \geq c^{3d^i}|S_{i-1}|$ and S_i is $c^{2d^{i-1}}$-restricted in G for all $i \in [m]$.*

Proof. Since $\mu_{\mathcal{F}}(c',G) \le \mu_{\mathcal{F}}(c^d,G) \le \mu_{\mathcal{F}}(x^b,G) \le 1$ by the choice of b,d, there exists a c^2-restricted $S_1 \subseteq V(G)$ with $|S_1| \ge c'|G| \ge c^d|G|$. This proves the base case.

Now, for $i \in [m]$ with $i < m$, assuming that $S_0, S_1, \dots, S_i$ have been constructed, we shall construct S_{i+1}. Let $y := c^{d^{i-1}} \ge c^{d^{m-2}} \ge x$. Since $d^{i-1} + d^{i-2} + \cdots + 1 = \frac{d^i-1}{d-1} < \frac{1}{3}d^i$ (as $d \ge 4$), and consequently $3d^i + 3d^{i-1} + \cdots + 3d + 3 < 4d^i$, we have

$$|S_i| \ge c^{3d^i}|S_{i-1}| \ge c^{3d^i+3d^{i-1}}|S_{i-2}| \ge \cdots \ge c^{3d^i+3d^{i-1}+\cdots+3d}|S_0| \ge c^{4d^i}|G| = y^{4d}|G| \ge x^{4d}|G|.$$

It follows that

$$\mu_{\mathcal{F}}(x^{b-4d}, G[S_i]) \le \mu_{\mathcal{F}}(x^b, G) \le 1.$$

Thus, since S_i is y^2-restricted in G and $0 < x \le y \le c$, Claim 6.2 implies that either

- $\min(\mu_{H'}(y^{2d^2}, G[S]), \mu_{\overline{J'}}(y^{2d^2}, G[S])) \le 1$ for some $S \subseteq S_i$ with $|S| \ge y^2|S_i|$; or
- $G[S_i]$ contains an x-sparse or $(1-x)$-dense $(y^{-1}, \lfloor y^{a+1}|S_i| \rfloor)$-blockade.

If the second bullet holds, then since $y^{a+1}|S_i| \ge y^{a+6d+1}|G| \ge y^b|G|$ by the choice of b, $G[S_i]$ (and thus G) contains an x-sparse or $(1-x)$-dense $(k, \lfloor |G|/k^b \rfloor)$-blockade where $k = 1/y \in [2, 1/x]$, contradicting our supposition. Thus the first bullet holds. Let $G' := G[S]$. The choice of a,b implies that $b = a + 6d + 1 \ge 2d^2 + 5d$. Thus, since $|S| \ge y^2|S_i| \ge y^{4d+2}|G| \ge y^{5d}|G|$, we obtain

$$\mu_H(y^{2d^2}, G') \le \mu_H(y^{2d^2+5d}, G) \le \mu_H(x^b, G) \le 1,$$

and similarly $\mu_{\overline{J}}(y^{2d^2}, G') \le 1$. Therefore, recalling that $\mathcal{F}_1 = \{H', \overline{J}\}$ and $\mathcal{F}_2 = \{H, \overline{J'}\}$, we deduce that $\mu_{\mathcal{F}_i}(y^{2d^2}, G') \le 1$ for some $i \in \{1,2\}$. Hence, the choice of d implies that G' (and so G) contains a y^{2d}-restricted $S_{i+1} \subseteq S$ with $|S_{i+1}| \ge y^{2d^2}|S| \ge y^{2d^2+2}|S_i| \ge c^{3d^{i+1}}|S_i|$. Since $y^{2d} = c^{2d^i}$, this completes the induction step and proves Claim 6.3. □

Now, we have $x^{2d} \le c^{2d^{m-1}} \le x^2$, which implies that S_m is x^2-restricted in G. Furthermore,

$$|S_m| \ge c^{3d^m}|S_{m-1}| \ge \ldots \ge c^{3d^m+3d^{m-1}+\cdots+3d}|S_0| \ge c^{4d^m}|G| \ge x^{4d^2}|G| \ge x^{4d^2-b/2} \ge x^{-1}.$$

Let $k := \lceil x^{-1/2} \rceil$; then $k \ge x^{-1/2} \ge 2$. Because $k\lfloor 2x|S_m| \rfloor \le 2x^{-1/2} \cdot 2x|S_m| = 4x^{1/2}|S_m| \le |S_m|$, there are disjoint subsets $B_1, \dots, B_k$ of S_m with $|B_i| = \lfloor 2x|S_m| \rfloor$ for all $i \in [k]$. Since

$$\lfloor 2x|S_m| \rfloor \ge x|S_m| \ge x^{4d^2+1}|G| \ge |G|/k^{8d^2+2} \ge |G|/k^b$$

by the choice of b, and since S_m is x^2-restricted in G, $(B_1, \dots, B_k)$ is an x-sparse or $(1-x)$-dense $(k, |G|/k^b)$-blockade in G, a contradiction.

This proves our claim that (b,c) is a pair of divisive sidekicks for $\mathcal{F}$. Consequently $\mathcal{F}$ is divisive, and therefore viral by Theorem 4.3. This proves Theorem 6.1. ■

Finally, we have:

Proof of Theorem 2.1. We proceed by induction on $|H| + |J|$. If $\min(|H|, |J|) \le 2$ then we are done by Lemma 3.3, since every ordered graph on at most two vertices is viral. Let $|H|, |J| \ge 3$; we assume that the theorem is true for every pair $\{H', \overline{J'}\}$ with $|H'| + |J'| < |H| + |J|$, and we shall prove it for $\mathcal{F} := \{H, \overline{J}\}$.

If one of H, J is not prime, then $\{H, \overline{J}\}$ is viral by Lemma 3.4 and the induction hypothesis. Thus we may assume they are both prime; and so there is a vertex v of H with degree one, and v is either the first or last vertex of H. Choose $u \in V(J)$ similarly, and let $H' := H \setminus \{v\}$ and $J' := J \setminus \{u\}$. From the inductive hypothesis, $\mathcal{F}_1 := \{H', \overline{J}\}$ and $\mathcal{F}_2 := \{H, \overline{J'}\}$ are viral; and so Theorem 6.1 implies that $\{H, \overline{J}\}$ is viral. This proves Theorem 2.1. ■

7. Corollaries

There are several corollaries of these results. First, let us say a *monotone path* is an ordered graph H where $H^\natural$ is a path with vertex set $\{v_1, \dots, v_k\}$, where v_i, v_{i+1} are adjacent for $1 \le i < k$, and $v_i \le_H v_j$ if $i < j$. As we mentioned earlier, Pach and Tomon [19] proved:

Theorem 7.1. *If H_1, H_2 are both monotone paths, then there exists $c > 0$ such that if G is an ordered graph that is both H_1-free and $\overline{H_2}$-free then G has a clique or stable set of size at least $|G|^c$.*

Since monotone paths belong to $\mathcal{K}$, this theorem is a special case of Theorem 2.1. Indeed, it follows that if H_1, H_2 are monotone paths then $\{H_1, \overline{H_2}\}$ is viral.

Second, what happens if we insist $H_1 = \overline{H_2}$ in Theorem 2.1? That gives us a class of ordered graphs with the Erdős-Hajnal property, as follows. Let $\mathcal{L}$ be the class of ordered graphs G minimal such that:

- all one-vertex ordered graphs are in $\mathcal{L}$;
- if $H_1, H_2 \in \mathcal{L}$ and H is obtained from H_2 by substituting H_1 for one of its vertices, then $H \in \mathcal{L}$;
- if $H \in \mathcal{L}$, and the first vertex a of H is adjacent to only the last vertex of H, then we can add a new last vertex adjacent to all vertices in $V(H) \setminus \{a\}$, and this enlarged ordered graph is also in $\mathcal{L}$;
- also three variants of the bullet above, exchanging "first" with "last", and/or exchanging "adjacent" with "nonadjacent", which we do not write out explicitly.

If $H \in \mathcal{L}$, then evidently $H, \overline{H} \in \mathcal{K}$ (in fact $H \in \mathcal{L}$ if and only if $H, \overline{H} \in \mathcal{K}$; this is proved like Lemma 7.6 below). We deduce from Theorem 2.1 that:

Theorem 7.2. *All ordered graphs in $\mathcal{L}$ are viral.*

Let us say that an ordered graph H has the *Erdős-Hajnal property* if there exists $c > 0$ such that if G is an H-free ordered graph, then there is a clique or stable set in G with size at least $|G|^c$. And as with unordered graphs, if an ordered graph H is obtained by vertex-substitution from two smaller ordered graphs with the Erdős-Hajnal property, then H has the Erdős-Hajnal property. So we would like to find prime ordered graphs that have the Erdős-Hajnal property. It is striking that, until now, there were none known with more than three vertices. In Fig. 5 we show all the prime ordered graphs on four vertices (up to taking complements and reversing the order); as we said, none of them were previously known to have the Erdős-Hajnal property. Our result shows that the first, the fifth and the seventh have the property, and indeed are viral, because they belong to $\mathcal{L}$.

Next, let us look at corollaries of Theorem 2.1 for unordered graphs. We will use a nice correspondence between the classes $\mathcal{L}, \mathcal{K}$ of ordered graphs and the classes $\mathcal{H}, \mathcal{J}$ of (unordered) graphs:

Lemma 7.3. *If $G \in \mathcal{K}$ then $G^\natural \in \mathcal{J}$; and if $F \in \mathcal{J}$, there is a linear order $\le$ of $V(F)$ such that $(F, \le) \in \mathcal{K}$. Similarly, if $G \in \mathcal{L}$ then $G^\natural \in \mathcal{H}$; and if $F \in \mathcal{H}$, there is a linear order $\le$ of $V(F)$ such that $(F, \le) \in \mathcal{L}$.*

The proofs are straightforward arguments by induction and we omit them. We will also need:

Lemma 7.4. *Let $\mathcal{F}$ be a finite set of ordered graphs, and let $\mathcal{F}^\natural$ be the set $\{F^\natural : F \in \mathcal{F}\}$. If $\mathcal{F}$ is viral then $\mathcal{F}^\natural$ is viral.*

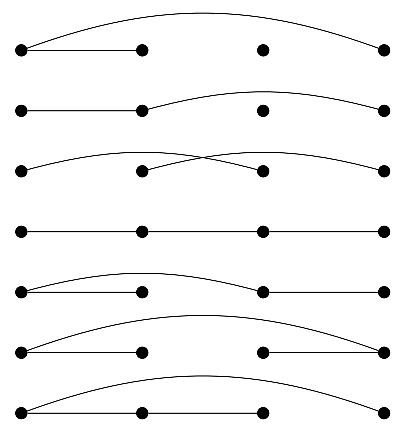

FIGURE 5. The four-vertex prime ordered graphs (up to complements and reversal).

Proof. Let d be a viral exponent for $\mathcal{F}$; we claim d is also a viral exponent for $\mathcal{F}^\natural$. Let $\varepsilon \in (0, \frac{1}{2})$, and let G be a graph with $\mu_{\mathcal{F}^\natural}(\varepsilon^d, G) \le 1$. choose a linear order of $V(G)$, to obtain an ordered graph H with $H^\natural = G$. Thus $\mu_{\mathcal{F}}(\varepsilon^d, G) \le \mu_{\mathcal{F}^\natural}(\varepsilon^d, G) \le 1$, and so there is an ε-restricted $S \subseteq V(G)$ with $|S| \ge \varepsilon^d|G|$. This proves Lemma 7.4. ■

From Lemma 7.3 and Lemma 7.4, we see that Theorem 2.1 implies:

Theorem 7.5. *If $H_1, H_2 \in \mathcal{J}$, then $\{H_1, \overline{H_2}\}$ is viral.*

This implies Theorem 1.10; and by taking $H_1 = H_2$, we deduce from Lemma 7.6 below that Theorem 7.5 implies Theorem 1.9.

Lemma 7.6. *$H \in \mathcal{H}$ if and only if $H, \overline{H} \in \mathcal{J}$.*

Proof. We first prove the "only if" direction by induction. Let $H \in \mathcal{H}$. If $|H| \le 3$ then obviously $H, \overline{H} \in \mathcal{J}$, so we may assume $|H| \ge 4$, and every induced subgraph J of H with $|J| < |H|$ satisfies $J, \overline{J} \in \mathcal{J}$. If H is not prime then $H, \overline{H} \in \mathcal{J}$ by definition, so we assume it is prime. From the definition of $\mathcal{H}$, we may assume (replacing H by $\overline{H}$ if necessary) that there is a vertex v of H with degree one, such that its neighbour u is adjacent to all except one vertex of $H \setminus \{v\}$. So $H \in \mathcal{J}$; and also $\overline{H} \in \mathcal{J}$, since u has degree one in $\overline{H}$. This completes the inductive step, and so prove the "only if" part.

Now, we prove the "if" direction, again by induction. Let H be such that $H, \overline{H} \in \mathcal{J}$. If $|H| \le 3$ then $H \in \mathcal{H}$ and we are done; so we may assume $|H| \ge 4$. Thus, since $H \in \mathcal{J}$, either H contains a vertex of degree one or H is not prime. If H is not prime, then it can be obtained by substituting some graph H_2 for a vertex of some graph H_1, where H_1, H_2 are induced subgraphs of H with $|H_1|, |H_2| < |H|$. Hence $H_i, \overline{H_i} \in \mathcal{J}$ for all $i \in \{1, 2\}$; and so the induction hypothesis implies that $H_1, H_2 \in \mathcal{H}$, which yields $H \in \mathcal{H}$. Hence, we may assume that H has a vertex u of degree one; and similarly, we may assume that $\overline{H}$ has a vertex v of degree one. Since $|H| \ge 4$, $u \ne v$. If $uv \in E(H)$ then v is the unique neighbour of u in H, and so $H \in \mathcal{H}$ since $H \setminus u \in \mathcal{H}$ by induction. If $uv \notin E(H)$ then u is the unique neighbour of v in $\overline{H}$, and so $\overline{H} \in \mathcal{H}$ since $\overline{H} \setminus v = \overline{H \setminus v} \in \mathcal{H}$ by induction. Therefore $H \in \mathcal{H}$, and this proves Lemma 7.6. ■

There is a version of Theorem 6.1 for unordered graphs:

Theorem 7.7. *Let $\mathcal{F}$ be a finite set of graphs, and let $H_1, \overline{H_2} \in \mathcal{F}$. For $i = 1, 2$, let v_i be a vertex of H_i with degree one, and let $H_i' = H_i \setminus \{v_i\}$. If $\mathcal{F}_1 := \{H_1'\} \cup (\mathcal{F} \setminus \{H_1\})$ and $\mathcal{F}_2 := \{\overline{H_2'}\} \cup (\mathcal{F} \setminus \{\overline{H_2}\})$ are both viral then $\mathcal{F}$ is viral.*

Proof. This can be proved directly, in the same way that we proved Theorem 6.1, but it can also be derived from Theorem 6.1, as follows. Let $\mathcal{P}$ be the set of all ordered graphs J such that $J^\natural \in \mathcal{F}$, and define $\mathcal{P}_1$ and $\mathcal{P}_2$ similarly. Thus $\mathcal{P}_1$ and $\mathcal{P}_2$ are viral. Let $\mathcal{Q}_1$ be the set of all J in $\mathcal{P}$ such that $J^\natural = H_1$ and v_1 is an end vertex of J; and let $\mathcal{Q}_1'$ be the set of all ordered graphs $J \setminus \{v_1\}$ where $J \in \mathcal{Q}_1$. Let $\mathcal{Q}_2$ be the set of all $J \in \mathcal{P}$ such that $J^\natural = \overline{H_2}$ and v_2 is an end vertex of J; and let $\mathcal{Q}_2'$ be the set of all ordered graphs $J \setminus \{v_2\}$ where $J \in \mathcal{Q}_2$. Thus $\mathcal{P}_i \subseteq (\mathcal{P} \setminus Q_i) \cup Q_i'$ for $i = 1, 2$. Since $\mathcal{P}_i$ is viral, it follows that $(\mathcal{P} \setminus Q_i) \cup Q_i'$ is viral, for $i = 1, 2$. By repeated application of Theorem 6.1, it follows that $\mathcal{P}$ is viral; and hence $\mathcal{F}$ is viral, by Lemma 7.4. This proves Theorem 7.7. ■

Since it is shown in [15] that C_5 is viral, we can use Theorem 7.7 to obtain viral pairs of prime graphs that are not given by Theorem 7.5. For instance, say a graph H is *five-unicyclic* if all its cycles have length five, and every component has at most one cycle. Then Theorem 7.7 implies that if H_1, H_2 are both five-unicyclic, then $\{H_1, \overline{H_2}\}$ is viral. More complicated constructions are also possible, but we omit the details.

There are also implications for tournaments. A *tournament* is a digraph such that for all distinct uv, exactly one of uv, vu is an edge. For tournaments T, Q, a *copy* of Q in T is an isomorphism from Q to an subtournament of T, and T is *Q-free* if there is no copy of Q in T. Let $\mathrm{ind}_Q(T)$ denote the number of copies of Q in T. We say that Q has the *Erdős-Hajnal property* if there exists $c > 0$ such that every tournament T admitting no copy of Q contains a transitive subtournament on at least $|T|^c$ vertices; and Alon, Pach, and Solymosi [1] proved that the Erdős-Hajnal conjecture is equivalent to the statement that every tournament has the Erdős-Hajnal property. Let $\mathcal{Q}$ be the family of tournaments defined as follows:

- $\mathcal{Q}$ is closed under taking vertex-substitutions; and
- if Q is a tournament, and $v \in V(Q)$ has indegree at most one or out-degree at most one, and $Q \setminus \{v\} \in \mathcal{Q}$, then $Q \in \mathcal{Q}$.

Let Q be a tournament, and take a numbering of its vertex set, say $\{v_1, \dots, v_n\}$. Let G be the graph with vertex set $V(Q)$ in which for $1 \le i < j \le n$, v_i, v_j are adjacent in G if and only if v_i is adjacent from v_j in Q. We call G the *backedge graph* of Q, and if we order its vertex set by $v_i \le v_j$ if $i \le j$, we obtain an ordered graph called the *backedge ordered graph*. It is easy to prove by induction (again, we omit the details) that:

Lemma 7.8. *$Q \in \mathcal{Q}$ if and only there is a numbering of its vertex set such that the resulting backedge ordered graph is in $\mathcal{K}$.*

Then we can prove the tournament analogue of Theorem 1.4, which says:

Theorem 7.9. *Every tournament in $\mathcal{Q}$ has the Erdős-Hajnal property.*

Proof. Let $Q \in \mathcal{Q}$, and take a numbering of $V(Q)$ as in Lemma 7.8. Let $H \in \mathcal{K}$ be the resulting backedge ordered graph. Let H' be the ordered graph obtained from H by reversing the linear order. It follows that $H' \in \mathcal{K}$, since $\mathcal{K}$ is closed under taking reversals. Moreover, $\overline{H'}$ is also a backedge ordered graph of Q, obtained from the reversed ordering.

By Theorem 2.1, there exists $c > 0$ such that if an ordered graph G is both H-free and $\overline{H'}$-free, then there is a clique or stable set of size at least $|G|^c$ in G. Let T be a Q-free tournament, take a numbering of its vertex set, and let P be the resulting backedge ordered graph. Then P is J-free for every backedge ordered graph of Q, because T is Q-free; and in particular, P is both H-free and $\overline{H'}$-free. Hence there is a clique or stable set S in P of size at least $|Q|^c$. But then S is a transitive set of T. This proves Theorem 7.9. ■

One can also extend Theorem 7.9 to a viral version, in terms of the backedge graph, but we omit the details.

References

[1] N. Alon, J. Pach, and J. Solymosi. Ramsey-type theorems with forbidden subgraphs. *Combinatorica*, 21(2):155–170, 2001. 2, 6, 18

[2] P. Blanco and M. Bucić. Towards the Erdős-Hajnal conjecture for P_5-free graphs, *submitted*, arXiv:2210.10755, 2022. 2

[3] M. Bucić, T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. I. A loglog step towards Erdős-Hajnal, *submitted*, arXiv:2301.10147, 2023. 5

[4] M. Chudnovsky. The Erdős-Hajnal conjecture–a survey. *J. Graph Theory*, 75(2):178–190, 2014. 1

[5] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl. Pure pairs. X. Tournaments and the strong Erdős-Hajnal property. *, European J. Combinatorics*, to appear, arXiv:2202.13977. 6

[6] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006. 4

[7] M. Chudnovsky and S. Safra. The Erdős-Hajnal conjecture for bull-free graphs. *J. Combin. Theory Ser. B*, 98(6):1301–1310, 2008. 2, 4

[8] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl. Pure pairs. I. Trees and linear anticomplete pairs. *Adv. Math.*, 375:107396, 20, 2020. 5, 6

[9] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl. Erdős-Hajnal for graphs with no 5-hole. *Proc. Lond. Math. Soc. (3)*, 126(3):997–1014, 2023. 2, 4, 11

[10] P. Erdős and A. Hajnal. On spanned subgraphs of graphs. In *Contributions To Graph Theory And Its Applications (Internat. Colloq., Oberhof, 1977) (German)*, pages 80–96. Tech. Hochschule Ilmenau, Ilmenau, 1977. 1

[11] P. Erdős and A. Hajnal. Ramsey-type theorems. *Discrete Applied Mathematics*, 25:37–52, 1989. 1

[12] J. Fox, T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. II. Sparse and dense sets in cographs, *manuscript*, 2023. 9

[13] J. Fox and B. Sudakov. Induced Ramsey-type theorems. *Adv. Math.*, 219(6):1771–1800, 2008. 5

[14] A. Gyárfás. Reflections on a problem of Erdős and Hajnal. In *The Mathematics Of Paul Erdős, II*, pages 93–98. Springer, Berlin, 1997. 1

[15] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. III. The pentagon and the bull, *manuscript*, 2023. 5, 10, 11, 18

[16] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. V. All paths approach Erdős-Hajnal, *manuscript*, 2023. 2

[17] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. VI. Graphs that approach Erdős-Hajnal, *manuscript*, 2023. 4

[18] V. Nikiforov. Edge distribution of graphs with few copies of a given graph. *Combin. Probab. Comput.*, 15(6):895–902, 2006. 4

[19] J. Pach and I. Tomon. Erdős-Hajnal-type results for monotone paths. *J. Combin. Theory Ser. B*, 151:21–37, 2021. 6, 11, 16

[20] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Math.*, 59(1-2):125–134, 1986. 4

[21] V. Rödl and P. Winkler. A Ramsey-type theorem for orderings of a graph. *SIAM J. Discrete Math.*, 2:402–406, 1989. 9

[22] I. Tomon. String graphs have the Erdős-Hajnal property. *J. Eur. Math. Soc. (JEMS)*, to appear, arXiv:2002.10350, 2023. 11

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