# Induced subgraph density. I. A loglog step towards Erdős-Hajnal

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**Abstract**

In 1977, Erdős and Hajnal made the conjecture that, for every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ has a clique or stable set of size at least $|G|^c$; and they proved that this is true with $|G|^c$ replaced by $2^{c\sqrt{\log |G|}}$. Until now, there has been no improvement on this result (for general $H$).

We prove a strengthening: that for every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ with $|G| \geq 2$ has a clique or stable set of size at least

$$2^{c\sqrt{\log |G| \log\log |G|}}.$$

Indeed, we prove the corresponding strengthening of a theorem of Fox and Sudakov, which in turn was a common strengthening of theorems of Rödl, Nikiforov, and the theorem of Erdős and Hajnal mentioned above.

# 1 Introduction

A graph $G$ *contains* a graph $H$ if $H$ is isomorphic to an induced subgraph of $G$, and $G$ is *$H$-free* otherwise. $|G|$ denotes the number of vertices of the graph $G$; and we write $\kappa(G)$ for the largest $t$ such that $G$ has a clique or stable set of cardinality $t$. For most $n$-vertex graphs $G$, $\kappa(G) = O(\log n)$, but this changes dramatically if we forbid some induced subgraph. In 1977, Erdős and Hajnal [7, 8] proposed the following well-known conjecture:

**1.1 Conjecture:** *For every graph $H$ there exists $c > 0$ such that $\kappa(G) \geq |G|^c$ for every $H$-free graph $G$.*

This has attracted a great deal of attention over the years, but despite this, it is only known to be true for a few graphs $H$: the graphs with at most five vertices, except the five-vertex path and its complement, and graphs that can be made from these by vertex-substitution. (Not exactly true any more: in a paper currently being prepared [10], three of us have found infinitely many graphs that satisfy 1.1 and that are not obtainable from smaller graphs by vertex-substitution.) It still remains open when $H$ is the five-vertex path, despite a great deal of study.

But at least, it is known that excluding any fixed induced subgraph will guarantee that $\kappa(G)$ is much bigger than $\log |G|$. Erdős and Hajnal themselves proved the following:

**1.2** *For every graph $H$ there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G|}}$ for every non-null $H$-free graph $G$.*

(All logarithms in this paper are to base 2.) Indeed they proved something slightly stronger, that for all $H$ and all $c > 0$ the same conclusion holds, provided that $|G|$ is sufficiently large. Rather surprisingly, until now there has been no improvement on this, for a general graph $H$. Our result is such an improvement:

**1.3** *For every graph $H$ there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$ for every $H$-free graph $G$ with $|G| \geq 2$.*

Bounds of this form have been considered before. For instance, in [3] the conclusion of 1.3 was proved when $H$ is the five-vertex cycle $C_5$ (it was later proved in [2] that $H = C_5$ satisfies 1.1). For a general graph $H$, Conlon, Fox and Sudakov [6] mention 1.3 as an intermediate goal towards proving 1.1 and pose an appealing conjecture (1.4 below), the resolution of which would establish 1.3. (Our argument takes a different approach and leaves their conjecture open, however.)

In a sense, 1.3 is a natural halfway point between 1.1 and 1.2, as we will explain next, but first we need some definitions. A *cograph* means a $P_4$-free graph, where $P_4$ denotes the path with four vertices; and we denote by $\mu(G)$ the largest $t$ such that some $t$-vertex induced subgraph of $G$ is a cograph. Cliques and stable sets induce cographs, and every cograph $J$ has a clique or stable set of size at least $|J|^{1/2}$; so 1.1, 1.2, and 1.3 are equivalent to the same statements with $\kappa(G)$ replaced by $\mu(G)$, and in that form they are often easier to work with. If $X \subseteq V(G)$, $G[X]$ denotes the induced subgraph with vertex set $X$. A *pure pair* in $G$ is a pair of disjoint subsets $A, B$ of $V(G)$ such that either there are no edges between $A, B$, or all edges between $A, B$ are present; and let us say a pair of disjoint subsets $A, B$ is *almost-pure* if either every vertex in $B$ has at most $|A|/(2\mu(G))$ neighbours in $A$, or every vertex in $B$ has at most $|A|/(2\mu(G))$ non-neighbours in $A$.

If we are trying to prove that $\mu(G) \ge f(|G|)$ for all $H$-free graphs $G$, where $f$ is some function, it is enough to know that all $H$-free graphs $G$ with $|G| > 1$ have pure pairs $A, B$ with $|A|, |B|$ appropriately large in terms of $|G|$. Because then we could deduce by induction on $|G|$ that $G[A]$ contains a cograph $C$ with $|C| \ge f(|A|)$, and similarly $G[B]$ contains a large cograph $D$, and so $V(C) \cup V(D)$ induces a cograph in $G$, and therefore $\mu(G) \ge f(|A|) + f(|B|)$; and if $|A|, |B|$ are large enough, then $f(|A|)+f(|B|) \ge f(|G|)$ and the inductive step is complete. In fact, for this purpose, an almost-pure pair $A, B$ is just as good as a pure pair: first choose $D$ as before; then, since $|D| \le \mu(G)$ and the pair $A, B$ is almost-pure, there exists $A' \subseteq A$ with $|A'| \ge |A|/2$ such that $A', D$ is a pure pair, and we apply the inductive hypothesis to $G[A']$, and reach the same conclusion as before.

How does $f$ depend on the sizes of $A, B$? Suppose that, in every $H$-free graph $G$ with $|G| \ge 2$, there is an almost-pure pair $A, B$ such that:

- $|A|, |B| \ge |G|/\mu(G)^k$ for some fixed $k > 0$; then we could take $f(x) := 2^{c\sqrt{\log x}}$ for some fixed $c > 0$. This is in fact true, and this is how Erdős and Hajnal [8] proved 1.2.
- $|A|, |B| \ge k|G|$ for some fixed $k > 0$; then we could take $f(x) := x^c$ for some fixed $c > 0$. This would prove that $H$ satisfies 1.1. Unfortunately, the only graphs $H$ with this property are subgraphs of the four-vertex path, which we already know satisfy 1.1, so this line is not useful for proving that graphs $H$ satisfy 1.1. Nevertheless, such "linear" pure pairs are of interest; it is shown in [4] that for every forest $H_1$ and forest complement $H_2$, every graph $G$ with $|G| > 1$ that contains neither of $H_1, H_2$ admits a linear pure pair. See also [5] for another theorem that yields linear pure pairs.
- $|A| \ge |G|/\mu(G)^k$ and $|B| \ge k'|G|$ for some fixed $k, k' > 0$; then we could take
$$f(x) := 2^{c\sqrt{\log x \log\log x}}.$$
Consequently, in this case, which is a mix of the hypotheses of the other bullets, we could prove that $H$ satisfies our theorem 1.3. So in this sense, our result is at a natural half-way point between 1.2 and 1.1.

We should clarify a point. Pairs $A, B$ as in the third bullet will be important in the proof of 1.3, but we have not been able to show that every $H$-free graph contains such a pair. But that might be true; indeed, there is a related conjecture of Conlon, Fox and Sudakov [6] (this is the conjecture we mentioned earlier), that:

**1.4 Conjecture:** *For every graph $H$ there exists $c_1, c_2 > 0$ such that for every $H$-free graph $G$ with $|G| \ge 2$, and all $\varepsilon \in (0, 1/2)$, there exist disjoint $A, B \subseteq V(G)$ with $|A| \ge \varepsilon^{c_1}|G|$ and $|B| \ge c_2|G|$, such that the number of edges between $A, B$ is either at most $\varepsilon|A| \cdot |B|$ or at least $(1-\varepsilon)|A| \cdot |B|$.*

Over the years, there have been several theorems discovered that are related to the Erdös-Hajnal conjecture 1.1, and our proof method allows us to strengthen some of them. First, there is a fundamental theorem of Rödl [12]:

**1.5** *For every graph $H$ and all $\varepsilon > 0$, there exists $\delta > 0$ with the following property. For every $H$-free graph $G$, there exists $S \subseteq V(G)$ with $|S| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges.*

How large can we take $\delta$ as a function of $\varepsilon$? Rödl's original proof gave a tower-type bound, because it used the regularity lemma, but Fox and Sudakov [9] made a significant improvement, proving a version of 1.5 that implies 1.2:

**1.6** *There exists $c > 0$ such that for every graph $H$ and all $\varepsilon \in (0, 1/2)$, setting $\delta = 2^{-c|H|(\log \frac{1}{\varepsilon})^2}$ satisfies 1.5.*

(To deduce 1.2, just set $\varepsilon = 2^{-\sqrt{\frac{\log |G|}{2c|H|}}}$, and apply Turán's theorem. The proof is similar to the proof that 1.9 implies 1.3, which we give later.)

Nikiforov [11] gave a different strengthening of 1.5:

**1.7** *For every graph $H$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that if $G$ is a graph containing fewer than $(\delta|G|)^{|H|}$ induced copies of $H$, then there exists $S \subseteq V(G)$ with $|S| \geq \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges.*

Fox and Sudakov [9] were able to incorporate the analogous strengthening of 1.7 into 1.6:

**1.8** *There exists $c > 0$ such that for every graph $H$ and all $\varepsilon \in (0, 1/2)$, setting $\delta = 2^{-c|H|(\log \frac{1}{\varepsilon})^2}$ satisfies 1.7.*

Our main result is:

**1.9** *For every graph $H$ there exists $c$ such that, if $\varepsilon \in (0, 1/2)$ and*

$$\delta = 2^{-c(\log \frac{1}{\varepsilon})^2 / \log \log \frac{1}{\varepsilon}},$$

*and $G$ is a graph containing fewer than $(\delta|G|)^{|H|}$ induced copies of $H$, then there exists $S \subseteq V(G)$ with $|S| \geq \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $\varepsilon\binom{|S|}{2}$ edges.*

This strengthens the result 1.8 of Fox and Sudakov, and improves the best known quantitative bounds in Nikiforov's theorem 1.7 and Rödl's theorem 1.5. It also implies 1.3, as we will show later.

In the final section we discuss analogous results for tournaments and ordered graphs.

## 2 A sketch of the proof

Let us give here a sketch of the proof of 1.3. If $G, H$ are graphs, a *copy* of $H$ in $G$ is an isomorphism from $H$ to an induced subgraph of $G$, and we denote by $\text{ind}_H(G)$ the number of copies of $H$ in $G$. The key step in the proof of 1.3 is the following:

**2.1** *For all $H$, there exist $k_1, k_2 > 0$ such that for every non-null graph $G$ and every $x$ with $0 < x \leq \frac{1}{8|H|}$, if $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$, then there is a sequence $B_1, \ldots, B_k$ of pairwise disjoint subsets of $V(G)$ with $k \geq \log(1/x)$, and each of cardinality at least $\lfloor x^{k_2}|G|\rfloor$, such that for $1 \leq i \leq k$, either every vertex of $B_{i+1} \cup \cdots \cup B_k$ has at most $x|B_i|$ neighbours in $B_i$, or every vertex of $B_{i+1} \cup \cdots \cup B_k$ has at most $x|B_i|$ non-neighbours in $B_i$.*

If we are given 2.1, then 1.3 can be deduced easily, by choosing $c > 0$ sufficiently small, and using induction on $|G|$. We may assume that $|G|$ is at least any constant (by choosing $c$ small enough), and hence $x \leq \frac{1}{8|H|}$ where $x := 1/(2\mu(G))$, since $x \leq (\log |G|)^{-1}$ by a standard Ramsey bound. So we can apply 2.1 to the $H$-free graph $G$, and obtain $B_1, \ldots, B_k$. Then choose subsets $D_i \subseteq B_i$ for $i = k, k-1, \ldots, 1$ in turn, such that $D_i \cup \cdots \cup D_k$ induces a cograph, as follows. Having chosen $D_{i+1}, \ldots, D_k$, since $D_{i+1} \cup \cdots \cup D_k$ induces a cograph, it has cardinality at most $\mu(G)$, and so (assuming every vertex of $B_{i+1} \cup \cdots \cup B_k$ has at most $x|B_i|$ neighbours in $B_i$; the other case is similar), at least half the vertices in $B_i$ have no neighbour in $D_{i+1} \cup \cdots \cup D_k$. By induction, we may choose $D_i$ from this half, inducing a cograph and of cardinality at least $(|B_i|/2)^c$. Then $D_i \cup \cdots \cup D_k$ induces a cograph, completing the inductive definition. Consequently $\mu(G) \geq \sum(|B_i|/2)^c$, and the result follows after some calculation, which we omit. (We will not actually prove 1.3 this way; our proof goes via the stronger theorem 1.9.)

The main issue is how to prove 2.1. We need the following lemma:

**2.2** *Let $H$ be a graph, let $g \in V(H)$, and let $H' := H \setminus \{g\}$. Let $b, c > 0$, and let $a := b + (1+c)|H|$. Let $G$ be a graph, let $A, B$ be disjoint subsets of $V(G)$, and let $0 < x \leq 1/2$. Suppose that every vertex in $A$ has at least $x|B|$ non-neighbours in $B$. Then either:*

- *there exists $B' \subseteq B$ with $|B'| \geq x|B|$ such that $\text{ind}_{H'}(G[B']) < x^b|B'|^{|H'|}$; or*
- *$\text{ind}_H(G) \geq x^a|A| \cdot |B|^{|H|-1}$; or*
- *there exists $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq x^a|A|$ and $|B'| \geq x^a|B|$ such that the number of edges between $A', B'$ is at most $2x^c|A'| \cdot |B'|$.*

The idea of the proof of 2.2 is as follows. Let $g$ have degree $d$ in $H$, let $H_d := H$, and for $i = d-1, d-2, \ldots, 0$ let $H_i$ be obtained from $H_{i+1}$ by deleting one of the $i+1$ edges of $H_{i+1}$ incident with $g$. For each $v \in A$, it has at least $x|B|$ non-neighbours in $B$, and we may assume that this set of non-neighbours ($B'$ say) contains at least $x^b|B'|^{|H|-1}$ copies of $H \setminus \{g\}$, because otherwise the first bullet holds. Since this is true for each $w$, there are "many" copies of $H_0$, and we may assume that there are not "many" copies of $H_d = H$ (where the first "many" is some huge number times the second), because otherwise the second bullet holds. (In fact we find many copies of $H$ with one vertex in $A$ and the other in $B$.) So for some $i$, the number of copies of $H_i$ is many times bigger than the number of copies of $H_{i+1}$. This will lead us to a pair $A', B'$ satisfying the third bullet.

Next, let us explain how to use 2.2 to prove 2.1. Let $g \in V(H)$ and $F = H \setminus \{g\}$. We will assume inductively that 2.1 holds for $F$, with $k_1, k_2$ replaced by $k_1', k_2'$ say. Choose an induced subgraph $J$ of $H$ maximal such that $G$ contains a large "approximate blowup" of $J$; that is, $|J|$ disjoint subsets $B_j$ ($j \in V(J)$) of $V(G)$, each of size at least $x^k|G|$ (where $k$ is an appropriate constant depending on $|J|$), and such that for distinct $i, j \in V(J)$, if $ij \notin E(J)$ then there are at most $x|B_i| \cdot |B_j|$ edges between $B_i$ and $B_j$, and the same in the complement if $ij \in E(J)$. It cannot be that $J = H$ since otherwise there would be $x^{k_1}|G|^{|H|}$ copies of $H$ in $G$, contrary to the hypothesis; let $h \in V(H) \setminus V(J)$. Let $W$ be the set of vertices of $G$ in none of the sets $B_j$ ($j \in V(J)$). We can arrange that $W$ contains almost all vertices of $G$.

It might be that for some $j \in V(J)$ there are at least $|W|/(4|H|)$ vertices in $W$ that have fewer than $x|B_j|$ neighbours in $B_j$; if this happens, let $A$ be the set of these vertices, and start the proof over again working within $A$ (so now all vertices in $A$ have very few neighbours in $B_j$, and if this iteration happens enough times we will produce the sequence of subsets in 2.1). Let us assume that this does not happen; and so at least half (say) the vertices in $W$ have at least $x|B_j|$ neighbours in each $B_j$, and at least $x|B_j|$ non-neighbours in each $B_j$ (similarly). Henceforth we work only with such vertices, and we denote the set of such vertices by $W'$. Choose $I \subseteq J$ maximal such that there exists $X \subseteq W'$ with $|X|$ large (under an appropriate definition: "large" shrinks as $|I|$ grows) such that for each $i \in I$ there exists $B'_i \subseteq B_i$, with $B'_i$ still large, such that for each $u \in X$, $u$ has at most $x|B'_i|$ non-neighbours in $B'_i$ if $h, i$ are adjacent, and at most $x|B'_i|$ neighbours in $B'_i$ if $h, i$ are nonadjacent. From the maximality of $J$, $I \neq J$ (because if $I = J$ we could set $B_h = X$, contrary to the maximality of $J$). Now we apply 2.2, setting $A = X$ and $B = B'_j$ for some $j \in V(J) \setminus V(I)$, and deduce that one of its three bullets holds. If the first holds, then because $F = H \setminus \{g\}$ satisfies 2.1, we obtain that 2.1 holds for $F$ and $G[B'_j]$, and hence it also holds for $H$ and $G$. The second bullet of 2.2 does not hold, from the hypothesis of 2.1. Also the third bullet does not hold, from the maximality of $I$. This completes the sketch of the proof of 1.3.

# 3 The proof of 2.2

If $A, B \subseteq V(G)$ are disjoint and $0 \leq x \leq 1$, we say

- $B$ is *$x$-sparse* to $A$ if every vertex in $B$ has at most $x|A|$ neighbours in $A$; and
- $B$ is *$x$-dense* to $A$ if every vertex in $B$ has at least $x|A|$ neighbours in $A$.

In this section we prove 2.2, which we restate:

**3.1** *Let $H$ be a graph, and let $g \in V(H)$. Let $b, c > 0$, and define $a := b + (1 + c)|H|$. Let $G$ be a graph, let $A, B$ be disjoint subsets of $V(G)$, and let $0 < x \leq 1/2$. Suppose that $A$ is $(1 - x)$-sparse to $B$. Then either:*

- *there exists $B' \subseteq B$ with $|B'| \geq x|B|$ such that $\text{ind}_{H\setminus\{g\}}(G[B']) < x^b|B'|^{|H|-1}$; or*
- $\text{ind}_H(G) \geq x^a|A| \cdot |B|^{|H|-1}$*; or*
- *there exists $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq x^a|A|$ and $|B'| \geq x^a|B|$ such that the number of edges between $A', B'$ is at most $2x^c|A'| \cdot |B'|$.*

**Proof.** Let $g$ have degree $d$, let $H_d := H$, and inductively for $t = d - 1, \ldots, 0$, let $H_t$ be obtained from $H_{t+1}$ by deleting one of the $t + 1$ edges of $H_{t+1}$ incident with $g$. Let $k = |H|$.

Let $v \in A$. By hypothesis, the set $B'$ (say) of non-neighbours of $v$ in $B$ has cardinality at least $x|B|$. If $G[B']$ contains fewer than $x^b|B'|^{k-1}$ copies of $H \setminus \{g\} = H_0 \setminus \{g\}$, then the first bullet holds, so we assume not. Consequently there are at least $x^b|B'|^{k-1} \geq x^{k-1+b}|B|^{k-1}$ copies $\phi$ of $H_0$ in $G$ such that $\phi(g) = v$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$. It follows by summing over all $v \in A$ that there are at least $x^{k-1+b}|A| \cdot |B|^{k-1}$ copies $\phi$ of $H_0$ such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$.

For $0 \leq t \leq d$, let $\tau_t$ be the number of copies $\phi$ of $H_t$ in $G$ such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$. Let $s := |A| \cdot |B|^{k-1}$. We have just seen that $\tau_0 \geq sx^{k-1+b}$. We may assume that $\tau_d < sx^a \leq sx^{k-1+b+cd}$, because otherwise the second bullet holds. Consequently, for some $t$ with $1 \leq t \leq d$,

- $\tau_{t-1} \geq sx^{k-1+b+c(t-1)} \geq 2sx^a$, and
- $\tau_t < sx^{k-1+b+ct}$, and therefore $\tau_t < x^c\tau_{t-1}$.

Let $\Phi$ be the set of all copies $\phi$ of $H_{t-1}$ in $G$ such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H)\setminus\{g\}$. There is one edge of $H_t$ that is not an edge of $H_{t-1}$, say $gi$. Let $J$ be the graph obtained from $H$ by deleting both $g$ and $i$, and let $\Psi$ be the set of all copies of $J$ in $G[B]$. Each member of $\Phi$ is an extension of some member of $\Psi$. For each $\psi \in \Psi$, let $n(\psi)$ be the number of $\phi \in \Phi$ that are extensions of $\psi$. Thus the sum of $n(\psi)$ over all $\psi \in \Psi$ equals $\tau_{t-1}$. Let $\Psi'$ be the set of all $\psi \in \Psi$ such that $n(\psi) \geq \tau_{t-1}/(2|B|^{k-2})$, and $\Psi'' = \Psi \setminus \Psi'$. Thus

$$\sum_{\phi \in \Psi''} n(\psi) \leq |B|^{k-2}\left(\frac{\tau_{t-1}}{2|B|^{k-2}}\right) = \tau_{t-1}/2$$

since $|\Psi''| \leq |B|^{k-2}$ and $n(\psi) \leq \tau_{t-1}/(2|B|^{k-2})$ for each $\phi \in \Psi''$. Since

$$\sum_{\psi \in \Psi''} n(\psi) + \sum_{\psi \in \Psi'} n(\psi) = \tau_{t-1},$$

it follows that

$$\sum_{\psi \in \Psi'} n(\psi) \geq \tau_{t-1}/2.$$

For each $\psi \in \Psi'$, let $U(\psi)$ be the set of all $u \in B$ such that mapping $i$ to $u$ extends $\psi$ to a copy of $H \setminus \{g\}$, and let $V(\psi)$ be the set of all $v \in A$ such that mapping $g$ to $v$ extends $\psi$ to a copy of $H \setminus \{i\}$. Thus $n(\psi)$ is the number of pairs $(u, v)$ with $u \in U(\psi)$ and $v \in V(\psi)$ such that $u, v$ are nonadjacent. Let $p(\psi)$ be the number of edges between $U(\psi)$ and $V(\psi)$. Thus $\tau_t$ is at least the sum of $p(\psi)$ over all $\psi \in \Psi'$. Since $\tau_t < x^c\tau_{t-1}$, it follows that

$$\sum_{\psi \in \Psi'} p(\psi) \leq \tau_t < x^c\tau_{t-1} \leq 2x^c \sum_{\psi \in \Psi'} n(\psi);$$

and consequently there exists $\psi \in \Psi'$ such that $p(\psi) \leq 2x^c n(\psi)$.

Since $\psi \in \Psi'$, it follows that

$$|U(\psi)| \cdot |V(\psi)| \geq n(\psi) \geq \frac{\tau_{t-1}}{2|B|^{k-2}} \geq \frac{2sx^a}{2|B|^{k-2}} = x^a|A| \cdot |B|;$$

and since $|U(\psi)| \leq |A|$ and $|V(\psi)| \leq |B|$, it follows that $|U(\psi)| \geq x^a|A|$ and similarly $|V(\psi)| \geq x^a|B|$. Since

$$p(\psi) \leq 2x^c n(\psi) \leq 2x^c|U(\psi)| \cdot |V(\psi)|,$$

there are at most $2x^c|U(\psi)| \cdot |V(\psi)|$ edges between $U(\psi)$ and $V(\psi)$. Hence the third bullet holds, setting $A' = U(\psi)$ and $B' = V(\psi)$. This proves 3.1. ∎

# 4 The proof of 2.1

Now we turn to the proof of 2.1. We will need:

**4.1** *Let $A, B$ be disjoint subsets of $V(G)$, such that there are at most $c|A| \cdot |B|$ edges between $A$ and $B$. Then there exists $A' \subseteq A$ with $|A'| \geq |A|/2$ such that $A'$ is $2c$-sparse to $B$.*

**Proof.** There are at most $c|A| \cdot |B|$ edges between $A$ and $B$, and so at most $|A|/2$ vertices in $A$ have more than $2c|B|$ neighbours in $B$. This proves 4.1. ∎

Let $J$ be a graph, and $t > 0$ an integer, and $q \leq 1$ a real number. Let $G$ be a graph, and let $A_j$ $(j \in V(J))$ be pairwise disjoint subsets of $V(G)$. We say that the family $(A_j : j \in V(J))$ is a $(t, q)$-*blowup of $J$* if

- each set $A_j$ $(j \in V(J))$ has cardinality $t$;
- for all distinct $i, j \in V(J)$, if $ij \notin E(J)$ then $A_i, A_j$ are $q$-sparse to each other, and if $ij \in E(J)$ then $A_i, A_j$ are $(1 - q)$-dense to each other.

We observe:

**4.2** *Let $J$ be a graph, and $t > 0$ an integer. If there is a $(t, 1/|J|)$-blowup of $J$ in $G$, then* $\mathrm{ind}_J(G) \geq (t/|J|)^{|J|}$.

**Proof.** Let $A_j$ $(j \in V(J))$ be a $(t, 1/|J|)$-blowup of $J$ in $G$. If $I$ is an induced subgraph of $J$, a copy $\phi$ of $I$ is *good* if $\phi(i) \in A_i$ for each $i \in I$.

(1) *Let $I$ be an induced subgraph of $J$, and suppose that $\phi$ is a good copy of $I$. Then there are at least $(t/|J|)^{|J|-|I|}$ good copies of $J$ that extend $\phi$.*

The proof is by induction on $|V(J)| - |V(I)|$. If this is zero then the claim is true, so we may assume that there exists $j \in V(J) \setminus V(I)$. Let $I'$ be the induced subgraph of $J$ with vertex set $V(I) \cup \{j\}$. For each $i \in V(I)$, let us say that $v \in A_j$ is *$i$-conforming* if either $ij \in E(J)$ and $\phi(i), v$ are adjacent in $G$, or $ij \notin E(J)$ and $\phi(i), v$ are nonadjacent in $G$. From the definition of a $(t, q)$-blowup, for each $i \in V(I)$ there are at most $t/|J|$ vertices in $A_j$ that are not $i$-conforming; and so there are at least $t - t|I|/|J| \geq t/|J|$ vertices $v \in A_j$ such that $v$ is $i$-conforming for each $i \in V(I)$. For each such $v$, let $\phi'$ be the extension of $\phi$ obtained by mapping $j$ to $v$; then $\phi'$ is a good copy of $I'$. From the inductive hypothesis, there are at least $(t/|J|)^{|J|-|I|-1}$ good copies of $J$ that extend $\phi'$; and since there are at least $t/|J|$ choices of $v$ and hence of $\phi'$, the claim follows. This proves (1).

But then the theorem follows from (1) by setting $I$ to be the null graph. This proves 4.2. ∎

The bulk of the proof of 2.1 consists of the following lemma:

**4.3** *For all graphs $H$, all $g \in V(H)$, and all $\alpha > 0$, there exist $\beta, \gamma > 0$ such that for every graph $G$ with $|G| \geq 2$ and all $x$ with $0 < x \leq 1/(8|H|)$, either:*

- *there exists $A \subseteq V(G)$ with $|A| \geq x^{\beta}|G|$ such that* $\mathrm{ind}_{H \setminus \{g\}}(G[A]) < x^{\alpha}|A|^{|H|-1}$; *or*

- $\mathrm{ind}_H(G) \ge x^\gamma |G|^{|H|}$; *or*
- *there are disjoint subsets* $A, B \subseteq V(G)$ *with* $|A| \ge x^\beta |G|$ *and* $|B| \ge |G|/(2|H|)$, *such that* $B$ *is either* $x$*-sparse or* $(1-x)$*-dense to* $A$.

**Proof.** We may assume that $|H| \ge 2$, because otherwise the theorem holds taking $\gamma = 1$. It suffices to prove the result assuming that $1/x$ is an integer. Indeed, suppose that $H, g, \alpha$ are given, and setting $\beta = \beta'$ and $\gamma = \gamma'$ satisfies the theorem for all $G$ and $x$ with $1/x$ an integer. Then setting $\beta = 2\beta'$ and $\gamma = 2\gamma'$ satisfies the theorem for all $G$ and $x$. To see this, let $0 < x \le 1/(8|H|)$, and let $x' = 1/\left(\lceil 1/x \rceil\right)$. Then $1/x'$ is an integer, and $1/x' = \lceil 1/x \rceil \le \frac{8|H|+1}{8|H|x}$, and so

$$x^2 \le \frac{x}{8|H|} \le \frac{8|H|x}{8|H|+1} \le x' \le x.$$

Consequently $(x')^{\beta'} \ge (x^2)^{\beta'} = x^\beta$ and similarly $(x')^{\gamma'} \ge x^\gamma$ and hence, whichever bullet of the theorem holds for $x', \beta', \gamma'$, the same bullet holds for $x, 2\beta', 2\gamma'$. So to prove the theorem, we just need to exhibit values of $\beta, \gamma$ that work when $1/x$ is an integer.

By increasing $\alpha$ if necessary, we may assume that $\alpha$ is an integer and $\alpha \ge |H|(|H|+1)$. Define $r_{|H|} = 0$, and inductively for $i = |H|-1, \dots, 1$ define

$$r_i = \alpha + 2|H| + 1 + (|H|+1)r_{i+1}.$$

Let $\beta := r_1 + 3$ and $\gamma := 2r_1 + \beta|H|$. we claim that $\beta, \gamma$ satisfy the theorem (when $1/x$ is an integer).

Thus, let $G$ be a graph with $|G| \ge 2$, and let $x > 0$ such that $0 < x \le 1/(8|H|)$, where $1/x$ is an integer. If $x^\beta |G| \le 1$, the third bullet is true taking $|A| = 1$ (unless $|G| - 1 < |G|/|H|$, which is impossible since $|G|, |H| \ge 2$), so we may assume that $x^\beta|G| > 1$. We assume the first two bullets of the theorem are false, and we will show that the third holds.

Let $t := \lfloor x^{\beta-1}|G| \rfloor$; thus $t \ge x^\beta |G|$, because $x^{\beta-1}|G| \ge 1$ and $x \le 1/2$. Let $t_i := x^{-r_i} t$ and $q_i := x^{r_i}/|H|$ for $1 \le i \le |H|$. Thus $t_1, \dots, t_{|H|}$ are integers.

Since $\gamma/|H| \ge \beta + 1$, it follows that

$$t \ge x^\beta |G| \ge (1/x) x^{\gamma/|H|} |G| \ge |H| x^{\gamma/|H|} |G|,$$

and consequently $(t/|H|)^{|H|} \ge x^\gamma |G|^{|H|}$. Hence by 4.2, there is no $(t, 1/|H|)$-blowup (that is, no $(t_{|H|}, q_{|H|})$-blowup) of $H$ in $G$. Let $J$ be a maximal induced subgraph of $H$ such that there is a $(t_{|J|}, q_{|J|})$-blowup $A_j$ ($j \in V(J)$) of $J$ in $G$, and let $k := |J|$.

Thus $J \ne H$; let $h \in V(H) \setminus V(J)$, and $L := \bigcup_{j \in V(J)} A_j$. For each $j \in V(J)$, let $M_j$ be the set of vertices $v \in V(G) \setminus L$ such that

- if $hj \in E(H)$, then $v$ has at most $x|A_j|$ neighbours in $A_j$;
- if $hj \notin E(H)$, then $v$ has at most $x|A_j|$ non-neighbours in $A_j$.

For each $j \in V(J)$, since $t_k \ge t \ge x^\beta |G|$, we may assume that $|M_j| < |G|/(2|H|)$, since otherwise the third bullet of the theorem holds. Since

$$|L| = kt_k \le kx^{\beta-1-r_k}|G| \le x^2|H| \cdot |G| \le |G|/(2|H|),$$

it follows that the union of $L$ and the sets $M_j$ ($j \in V(J)$) has cardinality at most $|G|/2$. Let $Z$ be the set of vertices of $G$ that do not belong to $L$ or to any of the sets $M_j$ ($j \in V(J)$). Thus $|Z| \geq |G|/2$; and for each $j \in V(J)$, if $hj \in E(H)$ then $Z$ is $x$-dense to $A_j$, and if $hj \notin E(H)$ then $Z$ is $(1-x)$-sparse to $A_j$. Let $s := x^{r_k - r_{k+1}}$.

(1) *Let $j \in V(J)$, and let $Y \subseteq Z$ with $|Y| \geq |Z|s^{k-1}$. Then there exist $C \subseteq A_j$ with $|C| = 2t_{k+1}$, and $X \subseteq Y$ with $|X| \geq s|Y|$, such that $X$ is $\frac{1}{2}q_{k+1}$-sparse to $C$ if $hj \notin E(H)$, and $X$ is $(1-\frac{1}{2}q_{k+1})$-dense to $C$ if $hj \in E(H)$.*

By taking complements if necessary, we may assume that $hj \notin E(H)$, and so $Y$ is $(1-x)$-sparse to $A_j$. We will apply 2.2 with $b, c, A, B$ replaced by $\alpha, r_{k+1}+1, Y, A_j$; note that the expression $b + (1+c)|H|$ of 2.2 becomes $\alpha + (r_{k+1}+2)|H| = r_k - r_{k+1} - 1$. By 2.2, we deduce that either:

- there exists $A' \subseteq A_j$ with $|A'| \geq x|A_j|$ such that $\text{ind}_{H\setminus\{g\}} H(G[A']) < x^{\alpha}|A'|^{|H|-1}$; or
- $\text{ind}_H(G) \geq x^{r_k - r_{k+1}-1}|Y| \cdot |A_j|^{|H|-1}$; or
- there exist $A' \subseteq A_j$ and $D \subseteq Y$ with $|A'| \geq x^{r_k - r_{k+1}-1}|A_j|$ and $|D| \geq x^{r_k - r_{k+1}-1}|Y|$ such that the number of edges between $A', D$ is at most $2x^{r_{k+1}+1}|A'| \cdot |D|$.

If the first bullet above holds, then the first bullet of the theorem holds, since $|A'| \geq x|A_j| = xt_{|J|} = x^{1-r_{|J|}}t \geq t \geq x^{\beta}|G|$ (because $|J| < |H|$ and so $r_{|J|} \geq 1$), a contradiction. If the second holds, then the second bullet of the theorem holds, also a contradiction, since

$$x^{r_k}|Y| \cdot |A_j|^{|H|-1} \geq x^{r_k + r_1 + \beta|H|}|G|^{|H|} \geq x^{2r_1 + \beta|H|}|G|^{|H|} = x^{\gamma}|G|^{|H|}$$

(because

$$|A_j|^{|H|-1} \geq t^{|H|-1} \geq x^{\beta(|H|-1)}|G|^{|H|-1} \geq x^{\beta|H|}|G|^{|H|-1}.$$

and $|Y| \geq |Z|s^{k-1} \geq x^{(k-1)r_k}|G| \geq x^{r_1}|G|$.)

Thus the third bullet above holds. Let $A', D$ be the corresponding subsets. Since

$$|A'| \geq x^{r_k - r_{k+1}-1}t_k = x^{r_k - r_{k+1}-1}x^{-r_k}t = t_{k+1}/x \geq 2t_{k+1},$$

it follows by averaging that we may choose $C \subseteq A'$ with $|C| = 2t_{k+1}$ such that the number of edges between $C, D$ is at most $2x^{r_{k+1}+1}|C| \cdot |D|$. By 4.1, there exists $X \subseteq D$ with

$$|X| \geq \frac{|D|}{2} \geq \frac{1}{2}x^{r_k - r_{k+1}-1}|Y| \geq s|Y|$$

such that $X$ is $4x^{r_{k+1}+1}$-sparse to $C$, and hence $\frac{1}{2}q_{k+1}$-sparse to $C$, since $4x^{r_{k+1}+1} \leq \frac{1}{2|H|}x^{r_{k+1}}$ (because $x \leq 1/(8|H|)$). This proves (1).

Starting with $Y = Z$, and applying (1) recursively to each $j \in V(J)$, we obtain a subset $X \subseteq Z$ with $|X| \geq |Z|s^k$, and a subset $C_j \subseteq A_j$ with $|C_j| = 2t_{k+1}$ for each $j \in V(J)$, such that for each $j \in V(J)$, $X$ is $\frac{1}{2}q_{k+1}$-sparse to $C_j$ if $hj \notin E(H)$, and $X$ is $(1-\frac{1}{2}q_{k+1})$-dense to $C_j$ if $hj \in E(H)$. Since (from the choice of $\beta$)

$$|X| \geq s^k|Z| \geq \frac{1}{2}x^{kr_k - kr_{k+1}}|G| \geq x^{\beta - 1 - r_{k+1}}|G| \geq tx^{-r_{k+1}} = t_{k+1}$$

we may choose $D_h \subseteq X$ with $|D_h| = t_{k+1}$. By 4.1, for each $j \in V(J)$ there exists $D_j \subseteq C_j$ with $|D_j| = t_{k+1}$ such that $D_j, D_h$ are $q_{k+1}$-sparse to each other if $hj \notin E(H)$, and $D_j, D_h$ are $(1 - q_{k+1})$-dense to each other if $hj \in E(H)$. Since $A_j$ $(j \in V(J))$ is a $(t_k, q_k)$-blowup of $J$, it follows that $D_j$ $(j \in V(J'))$ is a $(t_{k+1}, q_{k+1})$-blowup of $J'$, where $J'$ is the induced subgraph of $H$ with vertex set $V(J) \cup \{h\}$, contrary to the maximality of $J$. This proves 4.3. ∎

Let us say a *blockade* in $G$ is a sequence $\mathcal{B} = (B_1, \ldots, B_k)$ of pairwise disjoint subsets of $V(G)$, and we call $B_1, \ldots, B_k$ its *blocks*. (In some earlier papers, the blocks of a blockade must be nonempty, but here it is convenienet to allow empty blocks.) The *length* of the blockade $\mathcal{B} = (B_1, \ldots, B_k)$ is $k$, and its *width* is the minimum of the cardinalities of its blocks. For $\varepsilon > 0$, the blockade $\mathcal{B} = (B_1, \ldots, B_k)$ is *$\varepsilon$-restricted* if for all $i$ with $1 \le i \le k$, either $B_{i+1} \cup \cdots \cup B_k$ is $\varepsilon$-sparse to $B_i$ or $B_{i+1} \cup \cdots \cup B_k$ is $(1 - \varepsilon)$-dense to $B_i$. Now we use 4.3 to prove 2.1, which we restate:

**4.4** *For all $H$, there exist $k_1, k_2 > 0$ such that for every non-null graph $G$ and every $x$ with $0 < x \le \frac{1}{8|H|}$, if $\operatorname{ind}_H(G) < x^{k_1}|G|^{|H|}$, there is an $x$-restricted blockade in $G$ with length at least $\log(1/x)$ and width at least $\lfloor x^{k_2}|G| \rfloor$.*

**Proof.** The result is trivially true when $|H| \le 1$, and we proceed by induction on $|H|$. So we may assume that $|H| \ge 2$, and $g \in V(H)$, and the theorem holds for $H \setminus \{g\}$; let $k'_1, k'_2$ be the corresponding constants (using $H \setminus \{g\}$ instead of $H$). Let $d := \log(2|H|)$. Choose $\beta, \gamma$ satisfying 4.3, taking $\alpha = k'_1$. Let $k_1 := \gamma + d|H|$ and $k_2 := k'_2 + \beta + d$. We will show that $k_1, k_2$ satisfy the theorem.

Thus, let $G$ be a graph and $x$ with $0 < x \le \frac{1}{8|H|}$ such that $\operatorname{ind}_H(G) < x^{k_1}|G|^{|H|}$. Choose an $x$-restricted blockade $\mathcal{B} = (B_1, \ldots, B_k)$ in $G$ with $k$ maximum such that $B_1, \ldots, B_{k-1}$ have cardinality at least $x^{k_2}|G|$ and $|B_k| \ge (2|H|)^{1-k}|G|$.

(1) *We may assume that $|B_k| > x^d|G|$.*

We may assume that $k - 1 < \log(1/x)$ and so

$$|B_k| \ge (2|H|)^{1-k}|G| = 2^{(1-k)d}|G| > 2^{-d\log(1/x)}|G| = x^d|G|.$$

This proves (1).

If $x^{k_2}|G| < 1$, the result holds trivially (because $\lfloor x^{k_2}|G| \rfloor = 0$ and blockades may contain empty blocks), so we may assume that $|G| \ge x^{-k_2}$. By (1), $|B_k| \ge x^d|G| \ge x^{d-k_2} > 1$ and so $|B_k| \ge 2$. Let us apply 4.3 to $G[B_k]$, taking $\alpha = k'_1$. We deduce that either:

- there exists $A \subseteq B_k$ with $|A| \ge x^\beta|B_k|$ such that $\operatorname{ind}_{H\setminus\{g\}}(G[A]) < x^\alpha|A|^{|H|-1}$; or
- $\operatorname{ind}_H(G[B_k]) \ge x^\gamma|B_k|^{|H|}$; or
- there are disjoint subsets $A, B \subseteq B_k$ with $|A| \ge x^\beta|B_k|$ and $|B| \ge |B_k|/(2|H|)$, such that $B$ is either $x$-sparse or $(1 - x)$-dense to $A$.

The second is impossible, since by (1), $x^\gamma|B_k|^{|H|} \ge x^\gamma x^{d|H|}|G|^{|H|} = x^{k_1}|G|^{|H|}$. Also, the third is impossible, from the maximality of $k$, because $x^\beta|B_k| \ge x^\beta x^d|G| \ge x^{k_2}|G|$. Thus the first holds. Let $A$ be the corresponding subset. Since $|A| \ge x^\beta|B_k| \ge x^{\beta+d}|G|$, the inductive hypothesis gives an $x$-restricted blockade in $G[A]$ with length at least $\log(1/x)$ and width at least $\lfloor x^{k_2'}|A|\rfloor \ge \lfloor x^{k_2'}x^{\beta+d}|G|\rfloor = \lfloor x^{k_2}|G|\rfloor$. This proves 4.4. ∎

# 5 Deriving the main theorem

It remains to show that 2.1 implies 1.9, and that 1.9 implies 1.3, and we do so in this section. 2.1 says that graphs that do not contain many copies of $H$ admit blockades with certain properties, but the length of this blockade is critical. 2.1 gives blockades of length $\log(1/x)$, but one might hope that for some graphs $H$, we could obtain a version of 2.1 that gave longer blockades; and then there would be corresponding improvements in 1.9 and 1.3. With that in mind, we have written the argument of this section in greater generality than is needed for this paper. Let us say the *edge-density* of a graph $J$ is the number of edges of $J$ divided by $\binom{|J|}{2}$.

A function $\ell\colon (0,\frac{1}{2}) \to \mathbb{R}^+$ is *subreciprocal* if it is non-increasing and $1 < \ell(x) \le 1/x$ for all $x \in (0,\frac{1}{2})$. (To prove 1.9 we will only need the subreciprocal function $\ell(x) := \log(1/x)$.) If $\ell$ is a subreciprocal function, a graph $H$ is *$\ell$-divisive* if there exist $c \in (0,\frac{1}{2})$ and $d > 1$ such that for every $x \in (0,c)$ and every graph $G$ with $\operatorname{ind}_H(G) \le x^d|G|^{|H|}$, there is a blockade $(B_1,\dots,B_k)$ in $G$ and a cograph $J$, such that

- $(B_1,\dots,B_k)$ has length at least $\ell(x)$ and width at least $\lfloor x^d|G|\rfloor$; and
- $J$ has vertex set $\{1,\dots,k\}$, and for all $i,j \in V(J)$ with $i<j$, $B_j$ is $x$-sparse to $B_i$ if $ij \notin E(J)$, and $B_j$ is $(1-x)$-dense to $B_i$ if $ij \in E(J)$.

Erdős and Hajnal [8] proved that every graph is $\ell$-divisive where $\ell(x) = 2$ for $x \in (0,1/2)$; and 2.1 implies the following:

**5.1** *Every graph is $\ell$-divisive where $\ell(x) := \log(1/x)$ for $0 < x < 1/2$.*

The next theorem implies our main result 1.9, by defining $\ell$ as in 5.1. The proof is an adaptation of an argument of Fox and Sudakov [9].

**5.2** *Let $H$ be an $\ell$-divisive graph for some subreciprocal function $\ell$. Then there exists $C > 0$ such that for every $\varepsilon \in (0,\frac{1}{2})$, there exists $\delta$ with*

$$\delta > 2^{-C\log^2(1/\varepsilon)/\log(\ell(\varepsilon))}$$

*and the following property. For every graph $G$ with $\operatorname{ind}_H(G) \le (\delta|G|)^{|H|}$, there exists $S \subseteq V(G)$ with $|S| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has edge-density at most $\varepsilon$.*

**Proof.** Let $h := |H|$, and let $c \in (0,\frac{1}{2})$ and $d > 1$, as in the definition of $\ell$-divisive. Let $z := \ell(c)^{-1/4} \in (0,1)$, and let $b > 2$ be such that $2^{2-b} = 1-z$. We will show first that setting $C = 64bd$ satisfies the theorem when $\varepsilon \in (0,c)$. Thus, let $\varepsilon \in (0,c)$. Let $x := \frac{1-z}{2}\varepsilon = 2^{1-b}\varepsilon$ and $p := z\sqrt{\ell(x)}$;

then $p \geq \sqrt[4]{\ell(x)} > 1$ since $\ell(x) \geq \ell(\varepsilon) \geq \ell(c) = z^{-4}$. Let $t$ be the least integer such that $p^t \geq \varepsilon^{-2}$; then since $\ell(\varepsilon) \leq \min(\ell(x), 1/\varepsilon) \leq \min(p^4, 1/\varepsilon)$, we obtain

$$1 \leq t = \left\lceil \frac{2\log(1/\varepsilon)}{\log p} \right\rceil \leq \left\lceil \frac{8\log(1/\varepsilon)}{\log(\ell(\varepsilon))} \right\rceil \leq \frac{16\log(1/\varepsilon)}{\log(\ell(\varepsilon))}.$$

Let $\eta := \frac{1}{4}x^d$ and $\delta := x^d\eta^t$; then $x = 2^{1-b}\varepsilon > \varepsilon^b$ since $\varepsilon < \frac{1}{2}$, and so

$$\delta = x^d\eta^t = 4^{-t}x^{d+dt} > \varepsilon^{2t}\varepsilon^{bd(t+1)} > \varepsilon^{4bdt} = 2^{-4bdt\log(1/\varepsilon)}.$$

Hence, taking $C = 64bd$, the bound on $\delta$ in the theorem statement is true.

Now, let $G$ be such that $\operatorname{ind}_H(G) \leq (\delta|G|)^h$. We will show that there exists $S$ as in the theorem. If $\delta|G| \leq 1$ then we are done, so we may assume $\delta|G| > 1$, and hence $|G| > \delta^{-1} > \eta^{-t}$. For all $\varepsilon_1, \varepsilon_2 \geq \varepsilon$ and every integer $s$ with $0 \leq s \leq t$, let $\beta_s(\varepsilon_1, \varepsilon_2)$ be the maximum $\beta > 0$ such that for every induced subgraph $F$ of $G$ with $|F| \geq \eta^s|G|$, there exists $T \subseteq V(F)$ such that $|T| \geq \beta|F|$ and $F[T]$ has edge-density either at most $\varepsilon_1$ or at least $1 - \varepsilon_2$. It suffices to show that $\beta_0(\varepsilon, \varepsilon) \geq \delta$. We claim the following.

(1) *For every integer $s$ with $1 \leq s \leq t$, and for all $\varepsilon_1, \varepsilon_2 \geq \varepsilon$, we have*

$$\beta_{s-1}(\varepsilon_1, \varepsilon_2) \geq \eta \cdot \min(\beta_s(p\varepsilon_1, \varepsilon_2), \beta_s(\varepsilon_1, p\varepsilon_2)).$$

Put $\gamma_1 := \beta_s(p\varepsilon_1, \varepsilon_2)$ and $\gamma_2 := \beta_s(\varepsilon_1, p\varepsilon_2)$, and let $\gamma = \min(\gamma_1, \gamma_2)$. Let $F$ be an induced subgraph of $G$ with $|F| \geq \eta^{s-1}|G|$; we will prove that there exists $T \subseteq V(F)$ such that $|T| \geq \eta\gamma|F|$ and $F[T]$ has edge-density either at most $\varepsilon_1$ or at least $1 - \varepsilon_2$. Since $|F| \geq \eta^{s-1}|G|$, it follows that $|F| \geq \eta^{s-1-t} \geq \eta^{-1}$, since $|G| \geq \eta^{-t}$. Since

$$\operatorname{ind}_H(F) \leq \operatorname{ind}_H(G) \leq (\delta|G|)^h \leq (\eta^{-(s-1)}\delta|F|)^h = (\eta^{t-(s-1)}x^d|F|)^h \leq (x^d|F|)^h \leq x^d|F|^h,$$

there is a blockade $(B_1, \ldots, B_k)$ in $F$ of length $k \geq \ell(x)$ and width at least

$$\lfloor x^d|F| \rfloor = \lfloor 4\eta|F| \rfloor \geq 2\eta|F|$$

(since $|F| \geq \eta^{-1}$), and a cograph $J$ as in the definition of $\ell$-divisive. Let $I$ be a clique or a stable set of $J$ with $q := |I| \geq \sqrt{|J|} \geq \sqrt{\ell(x)} = z^{-1}p$. By the symmetry, we may assume that $I$ is stable in $J$. Let $m := \lceil \eta\gamma_1|F| \rceil$.

Let us renumber $\{B_i : i \in I\}$ as $\{A_1, \ldots, A_q\}$ where for all $i, j$ with $1 \leq i < j \leq q$, $A_j$ is $x$-sparse to $A_i$. Inductively for $i = q, q-1, \ldots, 1$, define $C_i \subseteq A_i$ as follows. Assume $C_q, C_{q-1}, \ldots, C_{i+1}$ have been defined, and let $D_i$ be their union. $D_i$ is $x$-sparse to $A_i$, and so by 4.1 there exists $A'_i \subseteq A_i$ with $|A'_i| \geq \frac{1}{2}|A_i|$ such that $A'_i$ is $2x$-sparse to $D_i$; and in particular

$$|A'_i| \geq \frac{1}{2}|A_i| \geq \eta|F| \geq \eta^s|G|.$$

Thus, by the definition of $\beta_s$, there exists $C_i \subseteq A'_i$ with $|C_i| \geq \gamma_1|A'_i| \geq \eta\gamma_1|F|$ such that $F[C_i]$ has edge-density either at most $p\varepsilon_1$ or at least $1 - \varepsilon_2$. If its edge-density is at least $1 - \varepsilon_2$ then we may set $T = C_i$ and be done; so we may assume that $F[C_i]$ has edge-density at most $p\varepsilon_1$. By averaging, we may assume $|C_i| = \lceil \eta\gamma_1|F| \rceil = m$. This completes the inductive definition of $C_q, C_{q-1}, \ldots, C_1$.

For $1 \le i \le q$, $C_i$ is $2x$-sparse to $D_i = C_q \cup C_{q-1} \cup \dots \cup C_{i+1}$, and so there are at most $2xm^2\binom{q}{2}$ edges between $C_1, \dots, C_q$. Therefore, setting $T := \bigcup_{i=1}^q C_i$, we have $|T| = qm \ge n\gamma_1|F| \ge n\gamma|F|$; and since $q \ge z^{-1}p$ and $2x = (1-z)\varepsilon \le (1-z)\varepsilon_1$, the number of edges of $G[T]$ is at most

$$qp\varepsilon_1\binom{m}{2} + 2xm^2\binom{q}{2} \le z\varepsilon_1 q^2\binom{m}{2} + (1-z)\varepsilon_1 m^2\binom{q}{2} \le \varepsilon_1\binom{qm}{2} = \varepsilon_1\binom{|T|}{2}$$

and so $T$ is a subset of $V(F)$ with the desired property. This proves (1).

By applying (1) for $s = 1, 2, \dots, t$, we obtain

$$\beta_0(\varepsilon, \varepsilon) \ge \eta^t \cdot \min_{0 \le i \le t} \beta_t(p^i\varepsilon, p^{t-i}\varepsilon).$$

Since $p^t\varepsilon^2 \ge 1$ by the choice of $t$, and so $\max\left(p^i\varepsilon, p^{t-i}\varepsilon\right) \ge 1$ if $0 \le i \le t$, we deduce that $\beta_t(p^i\varepsilon, p^{t-i}\varepsilon) = 1$ for all $i$ with $0 \le i \le t$; and hence $\beta_0(\varepsilon, \varepsilon) \ge \eta^t \ge \delta$, as claimed. Thus, setting $C = 64bd$ satisfies the theorem for all $\varepsilon \in (0, c)$.

Let $a > 1$ be such that $2^{-a} = c$; we shall prove that setting $C = 64a^2bd$ works for all $\varepsilon \in (0, \frac{1}{2})$. Let $\varepsilon' := \varepsilon^a$; then $\varepsilon' < 2^{-a} = c$ and $\varepsilon' < \varepsilon$. Consequently, there exists $\delta$ such that for every graph $G$ with $\text{ind}_H(G) \le (\delta|G|)^{|H|}$, there exists $S \subseteq V(G)$ with $|S| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has edge-density at most $\varepsilon'$, and

$$\log\delta > -\frac{64bd\log^2(1/\varepsilon')}{\log(\ell(\varepsilon'))} \ge -\frac{64a^2bd\log^2(1/\varepsilon)}{\log(\ell(\varepsilon))}$$

since $\ell$ is non-increasing. This proves 5.2. ∎

Finally, let us deduce 1.3, which we restate:

**5.3** *For every graph $H$ there exists $c > 0$ such that $\mu(G) \ge 2^{c\sqrt{\log|G|\log\log|G|}}$ for every $H$-free graph $G$ with $|G| \ge 2$.*

**Proof.** Let $c$ be as in 1.9, and let $d = 1/(8c)^{1/2}$. Choose $n_0$ such that for all $n \ge n_0$,

$$\frac{1}{4}\log\log n + \frac{1}{2}\log\log\log n \ge \log(1/d), \text{ and}$$
$$n^{1/2} \ge 2^{d\sqrt{\log n\log\log n}} \ge 4.$$

Choose $c' \le d/2$ with $c' > 0$, such that $n_0^{c'} \le 2$. We will show that $\mu(G) \ge 2^{c'\sqrt{\log|G|\log\log|G|}}$ for every $H$-free graph $G$ with $|G| \ge 2$.

Thus, let $G$ be $H$-free. If $|G| \le n_0$, then $|G|^{c'} \le n_0^{c'} \le 2$, and so $\mu(G) \ge |G|^{c'}$. Hence we may assume that $|G| > n_0$. Let $\varepsilon = 2^{-d\sqrt{\log|G|\log\log|G|}}$, and let $\delta = 2^{-c(\log\frac{1}{\varepsilon})^2/\log\log\frac{1}{\varepsilon}}$; then

$$\log\delta = -\frac{c(\log\frac{1}{\varepsilon})^2}{\log\log\frac{1}{\varepsilon}} = -\frac{cd^2\log|G|\log\log|G|}{\log\log\frac{1}{\varepsilon}}.$$

Since

$$\log\log\frac{1}{\varepsilon} = \frac{1}{2}\log\log|G| + \frac{1}{2}\log\log\log|G| - \log(1/d) \ge \frac{1}{4}\log\log|G|$$

(because $\frac{1}{4}\log\log|G| + \frac{1}{2}\log\log\log|G| \geq \log(1/d)$), it follows that

$$\log\delta \geq -\frac{cd^2\log|G|\log\log|G|}{\frac{1}{4}\log\log|G|} = -4cd^2\log|G| = -\frac{1}{2}\log|G|,$$

and so $\delta \geq |G|^{-1/2}$. By 1.9 and the choice of $c$, there exists $S \subseteq V(G)$ with $|S| \geq |G|^{1/2}$ such that one of $G[S], \overline{G}[S]$ has edge-density at most $\varepsilon$. Since $|G| \geq n_0$ it follows that $|S| > 1/\varepsilon$. By Turán's theorem, $G[S]$ has a clique or stable set of size at least

$$\frac{|S|}{1+\varepsilon|S|} \geq \frac{1}{2\varepsilon} = \frac{1}{2}2^{d\sqrt{\log|G|\log\log|G|}} \geq 2^{(d/2)\sqrt{\log|G|\log\log|G|}} \geq 2^{c'\sqrt{\log|G|\log\log|G|}}.$$

This proves 1.3. ∎

# 6 Ordered graphs

An influential paper of Alon, Pach and Solymosi [1] showed that the Erdos-Hajnal conjecture has equivalent statements for ordered graphs and for tournaments. In this section we observe that our result 1.3 also extends to ordered graphs and tournaments. An *ordered graph* is a pair $(G,<)$, where $G$ is a graph and $<$ is a linear order of its vertex set. If $(G,<)$ and $(H,<')$ are ordered graphs, we say $(G,<)$ is *$(H,<')$-free* if no induced subgraph of $G$ (made into an ordered graph with the order inherited from $<$ in the natural way) is isomorphic to $(H,<')$. Alon, Pach and Solymosi [1] showed that the Erdős-Hajnal conjecture 1.1 is equivalent to the following analogous conjecture for ordered graphs:

**6.1 Conjecture:** *For every ordered graph $(H,<')$ there exists $\tau > 0$ such that $\kappa(G) \geq |G|^\tau$ for every $(H,<')$-free ordered graph $(G,<)$.*

Our theorem 1.3 translates to:

**6.2** *For every ordered graph $(H,<')$ there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log|G|\log\log|G|}}$ for every $(H,<')$-free ordered graph $(G,<)$ with $|G| \geq 2$.*

To prove this, we use a theorem of Rödl and Winkler [13], that says:

**6.3** *For every ordered graph $(H,<')$, there exists a graph $P$ such that, for every linear ordering of $V(P)$, the resulting ordered graph is not $(H,<')$-free.*

**Proof of 6.2.** Let $(H,<')$ be an ordered graph, and choose $P$ as in 6.3. Choose $c$ satisfying 1.3 with $H$ replaced by $P$. Now let $(G,<)$ be an $(H,<')$-free ordered graph. It follows from the property of $P$ that $G$ is $P$-free, and so by 1.3, $\kappa(G) \geq 2^{c\sqrt{\log|G|\log\log|G|}}$. This proves 6.2. ∎

These results also have analogues for tournaments. If $G$ is a tournament, define $\kappa(G)$ to be the size of the largest transitive subset of $V(G)$. Our result becomes:

**6.4** *For every tournament $H$ there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log|G|\log\log|G|}}$ for every $H$-free tournament $G$ with $|G| \geq 2$.*

**Proof.** Fix a linear order $<'$ of $V(H)$, and let $H'$ be the graph with vertex set $V(H)$, in which $uv$ is an edge if $u$ is earlier than $v$ in the linear order $<'$ and $v$ is adjacent from $u$ in $H$. Thus $(H', <')$ is an ordered graph. Choose $c$ as in 6.2 (with $H$ replaced by $H'$). Now let $G$ be an $H$-free tournament. Derive an ordered graph $(G', <)$ from $G$ similarly. Since $G$ is $H$-free, we deduce that $(G', <)$ is $(H', <')$-free, and the result follows from 6.2, since every clique or stable set of $G'$ is a transitive set of $G$. This proves 6.4. ∎

## References


[1] N. Alon, J. Pach and J. Solymosi, "Ramsey-type theorems with forbidden subgraphs", *Combinatorica* **21** (2001), 155–170.

[2] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl, "Erdős-Hajnal for graphs with no five-hole", submitted for publication, `arXiv:2102.04994`.

[3] M. Chudnovsky, J. Fox, A. Scott, P. Seymour and S. Spirkl, "Towards Erdős-Hajnal for graphs with no 5-hole", *Combinatorica* **39** (2019), 983–991, `arXiv:1803.03588`.

[4] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs", *Advances in Math.*, **375** (2020), 107396, `arXiv:1809.00919`.

[5] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. II. Excluding all subdivisions of a graph", *Combinatorica* **41** (2021), 279–405, `arXiv:1804.01060`.

[6] D. Conlon, J. Fox and B. Sudakov, "Recent developments in graph Ramsey theory", in: *Surveys in Combinatorics 2015*, Cambridge University Press, 2015, 49–118.

[7] P. Erdős and A. Hajnal, "On spanned subgraphs of graphs", *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977).

[8] P. Erdős and A. Hajnal, "Ramsey-type theorems", *Discrete Applied Mathematics* **25** (1989), 37–52.

[9] J. Fox and B. Sudakov, "Induced Ramsey-type theorems", *Advances in Mathematics* **219** (2008), 1771–1800.

[10] T. Nguyen, A. Scott and P. Seymour, "Induced subgraph density. IV. New graphs with the Erdős-Hajnal property", in preparation.

[11] V. Nikiforov, "Edge distribution of graphs with few copies of a given graph", *Combin. Probab. Comput.* **15** (2006), 895–902.

[12] V. Rödl, "On universality of graphs with uniformly distributed edges", *Discrete Math.* **59** (1986), 125–134.

[13] V. Rödl and P. Winkler, "A Ramsey-type theorem for orderings of a graph", *SIAM Journal of Discrete Mathematics* **2** (1989), 402–406.