Cut Coloring and Circuit Covering

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Abstract

We prove that for every 3-edge-connected graph G there exists a partition of $E(G)$ into at most nine sets $\{X_1, X_2, \ldots, X_m\}$ so that $G \setminus X_i$ is 2-edge-connected for every $1 \leq i \leq m$. We then generalize this result, proving that for every (2k+1)-edge-connected graph G there is a partition of $E(G)$ into a bounded number (depending only on k) of sets $\{X_1, X_2, \ldots, X_m\}$ so that $G \setminus X_i$ is $2k$-edge-connected for every $1 \leq i \leq m$.

In the third section of the paper, we apply our theorem to prove that for every 3-edge-connected graph G and every map $p : E(G) \to \{84, 86\}$, there exists a multiset of circuits of G so that every edge $e \in E(G)$ is contained in exactly $p(e)$ of these circuits. This resolves a question of Goddyn. This result is then bootstrapped to a more general theorem on circuit covers.

If $G = (V, E)$ is a directed graph, a flow $\phi : E \to \mathbf{Z}$ of G is called an *antisymmetric* flow if $\phi(E) \cap -\phi(E) = \emptyset$. In the fourth section of the paper, we apply our theorem to show that every directed graph with no directed edge-cuts of size ≤ 2 has an antisymmetric flow of bounded size. This resolves a question of Nešetřil and Raspaud.

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1 Introduction

Throughout the paper, we consider only finite graphs, which may have multiple edges and loops. For any finite set S, we define a *coloring* of S to be a partition of S. The size of a coloring is the size of the partition. A *t-coloring* is a coloring of size $\leq t$. If G is a graph, a *vertex coloring* (*edge coloring*) of G is a coloring of $V(G)$ ($E(G)$). A vertex coloring Ω of G is *proper* if every $X \in \Omega$ is an independent set of G. If G is a directed graph and $A \subseteq V(G)$, we define $\delta^+(A)$ to be the set of all edges with initial vertex in A and terminal vertex in $V(G) \setminus A$. We define $\delta^-(A) = \delta^+(V(G) \setminus A)$. If $A = \{v\}$, we will simplify this notation to $\delta^+(v)$ and $\delta^-(v)$ respectively. We will use $\mathbf{Z}$ to denote the set of integers and $\mathbf{Z}^+$ to denote the set of nonnegative integers. If $X, Y \subseteq \mathbf{Z}$, then we define $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$. For any $k \in \mathbf{Z}^+$, we define $kX = \{0\}$ if $k = 0$, and $kX = X + (k-1)X$ otherwise. A multiset T with ground set S is formally a map $T : S \to \mathbf{Z}^+$. We think of T as a set in which an element may occur many times. For instance, if $P : S \to \{0, 1\}$ is a property, we say that k members of T satisfy P if $\sum_{s \in S} T(s)P(s) = k$.

If G is a graph, we define an edge coloring Ω of G to be *k-courteous* if $G \setminus X$ is k-edge-connected for every $X \in \Omega$. Our main theorem is as follows.

Theorem 1.1 *Every $(2k+1)$-edge-connected graph has a $2k$-courteous edge coloring of size at most 81^{k^2}.*

For graphs of even edge-connectivity, the analogous property does not hold. More precisely, for every pair of positive integers k and t, there is a $2k$-edge-connected graph $G_{k,t}$ which has no $(2k-1)$-courteous edge-coloring of size $\leq t$. The construction is simple: Let $G_{k,t}$ be the graph obtained from a circuit of length $t+1$ by replacing each edge by k parallel edges. This graph is $2k$-edge-connected, but any t-coloring of $E(G_{k,t})$ must have a color class X of size ≥ 2. Since every pair of edges is in an edge-cut of size $2k$, the removal of X will drop the edge-connectivity of the resulting graph to at most $2k-2$.

Planar duality exchanges minimal edge-cuts and circuits. Thus, by Theorem 1.1, the edges of any planar graph of girth $\geq 2k+1$ can be colored with at most 81^{k^2} colors so that for any circuit C and any color class X, we have that $|E(C) \setminus X| \geq 2k$. For arbitrary graphs and fixed $g, h \in \mathbf{Z}^+$, one may ask whether there is a bounded size coloring of every girth

g graph so that $|E(C) \setminus X| \geq h$ for any color class X and any circuit C. However, this question has a negative answer. Indeed, every t-coloring of $E(G)$ where G is a girth g graph with chromatic number $> 2^t$ contains a monochromatic circuit of odd length. To see this, note that if $E(G)$ can partitioned into $\{X_1, X_2, \ldots, X_t\}$ so that $(V(G), X_i)$ is bipartite for $1 \leq i \leq t$, then $\chi(G) \leq 2^t$.

It may be worth remarking that a similar sounding question has been studied for vertex connectivity. Thomassen [22] and independently Szegedy proved that for every $a, b > 0$, there exists an integer $f(a, b)$, so that for every $f(a, b)$-connected graph G there is a partition of $V(G)$ into $\{A, B\}$ with $G[A]$ a-connected and $G[B]$ b-connected. Thomassen has conjectured that the smallest value is $f(a, b) = a+b+1$. Hajnal [9] has proved that $f(a, b) = 4a+4b+13$ is sufficient. These theorems when applied to a line graph $L(G)$ do not give results about courteous edge-colorings of G, since a set of edges $S \subseteq E(G)$ may induce a highly connected subgraph in $L(G)$ while $(V(G), S)$ is disconnected.

Let G be a 2-edge-connected graph and let $p : E(G) \to \mathbf{Z}^+$ be a map. For a subset $S \subseteq E(G)$, we define $p(S)$ to be $\sum_{e \in S} p(e)$. A circuit cover $\mathcal{C}$ of (G, p) is a multiset of circuits of G so that every edge $e \in E(G)$ is contained in exactly $p(e)$ circuits of $\mathcal{C}$. Seymour [18] gave two obviously necessary conditions on p for (G, p) to have a circuit cover.

(i) $p(C)$ is even for every edge cut C of G.

(ii) $p(e) \leq p(C \setminus e)$ for every edge cut C and every edge $e \in C$

We will say that p is *admissable* if p satisfies both of the above criteria. For any set S and integer k, we will let $\mathbf{k}_S : S \to \mathbf{Z}$ be the map given by the rule $\mathbf{k}_S(s) = k$ for every $s \in S$. The celebrated cycle double cover conjecture is precisely the statement that $(G, \mathbf{2}_{E(G)})$ has a circuit cover. If G has no Petersen minor, Alspach, Goddyn, and Zhang [3] proved that (G, p) has a circuit cover for every admissable map p. For general graphs, Bermond, Jackson, and Jaeger [4] proved that $(G, \mathbf{4}_{E(G)})$ has a circuit cover, and Fan [6] proved that $(G, \mathbf{6}_{E(G)})$ has a circuit cover. However, considerably less is known about circuit covers for non-constant weight functions. In this direction, Goddyn put forth the following conjecture.

Conjecture 1.2 (Goddyn [8]) *There exists an integer k so that for every 2-edge-connected graph G and every admissable map $p : E(G) \to \{k, k+2\}$, there is a circuit cover of (G, p).*

In the third section of this paper, we apply a special case of the main theorem to achieve the following result.

Theorem 1.3 *For every 2-edge-connected graph G, every $i, j, k, l \geq 0$, and every admissable map $p : E(G) \to i\{4\} + j\{6\} + k\{32, 36\} + l\{48, 54\} + \{0, 1\}$, there exists a circuit cover of (G, p).*

The $m = 0$ case of the following corollary of Theorem 1.3 resolves Conjecture 1.2 for the value $k = 84$.

Corollary 1.4 *For every 2-edge-connected graph G, every integer $m \geq 0$, and every admissable map $p : E(G) \to [32m + 84, 36m + 87] \cap \mathbf{Z}$, there exists a circuit cover of (G, p).*

Proof: Observe that $\{84, 85, 86, 87\} \subseteq \{32, 36\} + \{48, 54\} + \{0, 1\}$. It follows from this that $[32m + 84, 36m + 87] \cap \mathbf{Z} \subseteq (m + 1)\{32, 36\} + \{48, 54\} + \{0, 1\}$. Thus, the corollary follows by applying theorem 1.3 with $i = j = 0$, $k = m + 1$, and $l = 1$. $\square$

Let G be a directed graph, let Γ be an abelian group, and let $\phi : E(G) \to \Gamma$ be a map. We will say that ϕ is a *flow* (or a Γ-*flow*) if for every $v \in V(G)$ we have that:

$$\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e). \tag{1}$$

It follows that if ϕ is a flow and $A \subseteq V(G)$ then

$$\sum_{e \in \delta^+(A)} \phi(e) = \sum_{e \in \delta^-(A)} \phi(e). \tag{2}$$

A flow ϕ is said to be *nowhere-zero* if $\phi(e) \neq 0$ for every $e \in E(G)$. For an integer k, we call ϕ a k-*flow* if $\Gamma = \mathbf{Z}$ and $|\phi(e)| < k$ for every $e \in E(G)$. Following [16], we define ϕ to be an *antisymmetric* flow (or *antiflow*) if $\phi(E(G)) \cap -\phi(E(G)) = \emptyset$. Note that an antiflow is necessarily a nowhere-zero flow.

Let G be a directed graph, let $\{C_0, C_1, \ldots, C_t\}$ be the set of all circuits of the underlying undirected graph, and let ϕ_i be a 2-flow of G with $supp(\phi_i) = E(C_i)$ for $0 \leq i \leq t$. It follows from equation (2) that if G has a cut-edge, then G does not have a nowhere-zero $\mathbf{Z}$-flow. Conversely, if G has no cut-edge, then $\sum_{i=0}^{t} 2^i \phi_i$ is a nowhere-zero $\mathbf{Z}$-flow of G. Thus, G

has a nowhere-zero $\mathbf{Z}$-flow if and only if G has no cut-edge. Actually, there is a universal upper bound k such that every graph with no cut-edge has a nowhere-zero k-flow. This was first conjectured by Tutte [23] (who also conjectured that $k = 5$ is true). Jaeger [10] and independently Kilpatrick [13] resolved the weak form of Tutte's conjecture, showing that it is true for $k = 8$. This was subsequently improved by Seymour [19] who showed that it is true for $k = 6$. The case $k = 5$ is still open.

Since every antiflow is also a nowhere-zero flow, if G has a one-edge cut, then G does not have a $\mathbf{Z}$-antiflow. It also follows from equation (2) that, if G has a directed two-edge cut, then G does not have a $\mathbf{Z}$-antiflow. Conversely, if G has no directed edge cuts of size ≤ 2, it is not difficult to verify that $\sum_{i=0}^{t} 3^i \phi_i$ is a $\mathbf{Z}$-antiflow of G. Thus, G has a $\mathbf{Z}$-antiflow if and only if G has no directed cut of size ≤ 2. In analogy with the case of nowhere-zero flows, Nešetřil and Raspaud asked the following question.

Problem 1.5 (Nešetřil, Raspaud [16]) *Is there a fixed integer k so that every directed graph with no directed edge-cut of size ≤ 2 has a k-antiflow?*

In the third section of the paper, we apply a special case of our theorem to prove the following theorem, which resolves the above problem.

Theorem 1.6 *Every directed graph with no directed edge-cut of size ≤ 2 has a 10^{12}-antiflow.*

There does not seem to be a natural candidate for the best possible value of k in problem 1.5. Nešetřil and Raspaud [16] have remarked that there are planar graphs which give a lower bound of 16.

In the final section of the paper, we will discuss a couple of related edge-coloring problems. Here we mention two open questions.

Conjecture 1.7 (DeVos) *If a, b are positive integers and $G = (V, E)$ is an $(a + b + 2)$-edge-connected graph, then there exists a partition of E into $\{A, B\}$ so that (V, A) is a-edge-connected and (V, B) is b-edge-connected.*

Problem 1.8 *Does there exist a fixed integer k so that for every k-edge-connected graph G there is a subset $A \subseteq E(G)$ so that for every edge-cut C we have that $1/3 \leq |C \cap A|/|C| \leq 2/3$?*

The above problem is similar in nature to some questions asked by Goddyn. It should be noted that the values $1/3$ and $2/3$ are of no special importance in the above question. Indeed, an affirmative answer to the same problem with $1/3$ and $2/3$ replaced by ϵ and $1-\epsilon$ for any $1 > \epsilon > 0$ would still imply the following conjecture of Jaeger.

Conjecture 1.9 (Jaeger [11]) *There is a fixed integer k so that every k-edge-connected graph has a nowhere-zero 3-flow.*

2 Finding Courteous Edge-Colorings

Our main goal in this paper is to provide a simple proof of the existence of bounded size courteous colorings, so we will do fairly little to optimize the size of our bound. At the end of this section we will give some indication of how our bound can be improved. We start the section by proving a lemma concerning a partition problem for a family of subpaths of a tree. We will use a consequence of this lemma, Proposition 2.3, to prove a color splitting lemma, and then the main theorem.

If $\mathcal{S} = \{S_i\}_{i \in I}$ is a family indexed by I and $J \subseteq I$, we let $\mathcal{S}[J]$ denote the family $\{S_j\}_{j \in J}$. If G is a graph and $\mathcal{H} = \{H_i\}_{i \in I}$ is a finite family of subgraphs of G, we define the weight function $w_{\mathcal{H}} : E(G) \to \mathbf{Z}^+$ as follows: $w_{\mathcal{H}}(e) = |\{i \in I \mid e \in E(H_i)\}|$ for every $e \in E(G)$. If $\mu : E(G) \to \mathbf{Z}^+$, we call a partition Ω of I a *μ-resistive* partition (with respect to G and $\mathcal{H}$) if $w_{\mathcal{H}[I \setminus X]}(e) \geq \mu(e)$ for every $X \in \Omega$ and every $e \in E(G)$. If $\mu(e) = k$ for every $e \in E(G)$, a μ-resistive partition is called a *k-resistive* partition.

If T is a tree and $v \in V(T)$ is a vertex of degree one, we call v a *leaf vertex*. If $e \in E(T)$ is incident with a leaf vertex, we call e a *leaf edge*. If G is a graph and $e \in E(G)$, we let G/e denote the graph obtained from G by contracting the edge e. The following lemma is the key tool needed for the proof of our main theorem.

Lemma 2.1 *Let T be a tree and let $\mathcal{P} = \{P_i\}_{i \in I}$ be a finite family of subpaths of T with $w_{\mathcal{P}}(e) \geq 2$ for every $e \in E(G)$. Then there is a 1-resistive partition Ω of I of size ≤ 3.*

Proof: We proceed by induction on $|V(T)| + |I|$. The theorem is trivial if $|V(T)| \leq 2$ so we may assume that $|V(T)| \geq 3$. Further, we may assume by induction that $E(P_i) \neq \emptyset$ for every $i \in I$.

If T contains a non-leaf edge e with $w_{\mathcal{P}}(e) = 2$, then let A_1, A_2 be the components of $T \setminus e$, and let T_1, T_2 be the trees $A_1 \cup e$ and $A_2 \cup e$ respectively. For $k = 1, 2$, let $I_k = \{i \in I \mid E(P_i) \cap E(T_k) \neq \emptyset\}$ and let $\mathcal{P}_k = \{P_i \cap T_k\}_{i \in I_k}$. For $k = 1, 2$, apply the lemma inductively to the tree T_k and the family of paths $\mathcal{P}_k$ to obtain a 1-resistive partition Ω_k of I_k. Let $I_1 \cap I_2 = \{x, y\}$ and for $k = 1, 2$ let $X_k \in \Omega_k$ be the set with $x \in X_k$ and let $Y_k \in \Omega_k$ be the set with $y \in Y_k$ (note that $X_k \neq Y_k$). Let $Z = I \setminus (X_1 \cup X_2 \cup Y_1 \cup Y_2)$. If $Z \neq \emptyset$ then $\{X_1 \cup X_2, Y_1 \cup Y_2, Z\}$ is a 1-resistive partition of I. Otherwise, $\{X_1 \cup X_2, Y_1 \cup Y_2\}$ is a 1-resistive partition of I. Thus, we may assume that $w_{\mathcal{P}}(e) \geq 3$ for every non-leaf edge e of T.

Choose $i \in I$ and let $U \subseteq V(G)$ be the set of ends of P_i which are leaf vertices of T and are incident with an edge $e \in E(T)$ with $w_{\mathcal{P}}(e) = 2$. Let $\mathcal{P}' = \{P_j \setminus U\}_{j \in I \setminus \{i\}}$ and let $T' = T \setminus U$. Applying the lemma inductively to the tree T' and the family of paths $\mathcal{P}'$ we obtain a 1-resistive partition Ω' of $I \setminus \{i\}$ (with respect to T' and $\mathcal{P}'$). If $|\Omega'| < 3$, then $\Omega = \Omega' \cup \{\{i\}\}$ is a 1-resistive partition of I as required. Otherwise, for every $u \in U$, choose $i_u \in I \setminus \{i\}$ so that $u \in V(P_{i_u})$ and choose $Z \in \Omega'$ so that $i_u \notin Z$ for every $u \in U$. Now by construction, $\Omega = (\Omega' \setminus Z) \cup \{Z \cup \{i\}\}$ is a 1-resistive partition of I. This completes the proof. $\square$

A straightforward iteration of this argument gives us the following proposition.

Proposition 2.2 *Let T be a tree with $|E(T)| > 0$, let $\mathcal{P} = \{P_i\}_{i \in I}$ be a finite family of subpaths of T and let $\mu : E(T) \to \mathbf{Z}^+$ be a map with $M = \max_{e \in E(T)} \mu(e)$. If $w_{\mathcal{P}}(e) > \mu(e)$ for every $e \in E(G)$ with $\mu(e) > 0$, then there is a μ-resistive partition Ω of I with size at most 3^M.*

Proof: We proceed inductively on $M + |E(T)|$. If $M = 0$ then $\Omega = \{I\}$ satisfies the proposition. If T contains an edge e with $\mu(e) = 0$, then let

$$P'_i = \begin{cases} P_i/e & \text{if } e \in E(P_i) \\ P_i & \text{otherwise} \end{cases}$$

and apply the lemma inductively to the tree T/e and the family of paths $\{P'_i\}_{i \in I}$. The resulting partition of I satisfies the proposition. Thus, we may assume that $\mu(e) > 0$ for every $e \in E(G)$ and thus that $w_{\mathcal{P}}(e) \geq 2$ for every $e \in E(G)$. By the above lemma, we may choose a 1-resistive partition Υ of I of size ≤ 3. For every $Y \in \Upsilon$, let $\mu_Y : E(T) \to \mathbf{Z}^+$ be given by the rule $\mu_Y(e) = \max\{0, \mu - w_{\mathcal{P}[I \setminus Y]}\}$ and apply the proposition inductively to T for the map μ_Y and the family of paths $\{P_i\}_{i \in Y}$ to obtain a partition Ω_Y of Y (note that $\max_{e \in E(T)} \mu_Y < M$ so this is possible). Let $\Omega = \cup_{Y \in \Upsilon} \Omega_Y$. Now, $|\Omega| \leq 3^M$ by construction. Furthermore, for every $e \in E(T)$ and $X \in \Omega$, if $X \in \Omega_Y$ for $Y \in \Upsilon$, then

$$\begin{aligned} w_{\mathcal{P}[I \setminus X]}(e) &= w_{\mathcal{P}[I \setminus Y]}(e) + w_{\mathcal{P}[Y \setminus X]}(e) \\ &\geq w_{\mathcal{P}[I \setminus Y]}(e) + \mu_Y(e) \\ &\geq w_{\mathcal{P}[I \setminus Y]}(e) + \mu - w_{\mathcal{P}[I \setminus Y]}(e) \\ &= \mu(e) \end{aligned}$$

This completes the proof. $\square$

If T is a spanning tree of G and Ω is a coloring of $E(G) \setminus E(T)$, we will say that Ω is *k-courteous for T* if for every fundamental cocircuit C of T and for every $X \in \Omega$, we have that $|C \setminus X| \geq k$. Note that $C \setminus X$ will always contain exactly one edge of T.

Proposition 2.3 *Let T be a spanning tree of a graph G. If every fundamental cocircuit of T has size $\geq k+1$, then there is a coloring Ω of $E(G) \setminus E(T)$ of size at most 3^{k-1} which is k-courteous for T.*

Proof: For every edge $e \in E(G) \setminus E(T)$, let $P_e \subseteq T$ be the unique subpath of T so that $P_e \cup e$ is a circuit. Let $\mathcal{P} = \{P_e\}_{e \in E(G) \setminus E(T)}$ and let $\mu : E(T) \to \mathbf{Z}$ be the constant function $k-1$. Now, apply the previous proposition to T, $\mathcal{P}$ and μ. This gives us a $(k-1)$-resistive partition Ω of $E(G) \setminus E(T)$ of size at most 3^{k-1}. Now, an edge $e \in E(G) \setminus E(T)$ is in the fundamental cocircuit of an edge $f \in E(T)$ if and only if $f \in E(P(e))$. Since every fundamental cocircuit of T also contains an edge of T, it follows that Ω is a k-courteous coloring for T. $\square$

Let G be a graph and let T be a tree. For every vertex $v \in V(T)$, let $S(v)$ be a subset of $V(G)$ so that $\{S(v)\}_{v \in V(T)}$ is a family of pairwise disjoint subsets of $V(G)$ with union equal to $V(G)$ (note that $S(v) = \emptyset$ is possible for one or more vertices $v \in V(T)$). If $H \subseteq T$, we will let $S(H) = \cup_{v \in V(H)} S(v)$. For every edge $e \in E(T)$, if the components of $T \setminus e$ are A, B, then let $C(e) = \{xy \in E(G) \mid x \in S(A) \text{ and } y \in S(B)\}$. Thus, each edge of T is associated with an edge-cut of G. We will say that (T, S) is a *k-edge-cut tree* for G if $\{C(e) \mid e \in E(T)\}$

is precisely the set of all edge-cuts of G of size k. The following well-known proposition follows easily from the fact that minimum edge-cuts of odd size cannot cross.

Proposition 2.4 *If G is a $(2k+1)$-edge-connected graph, then G has a $(2k+1)$-edge-cut tree.*

With this proposition, we are ready to prove the following lemma.

Lemma 2.5 *Let G be a k-edge-connected graph and let Ω be a j-courteous edge-coloring of G of size t. If j is odd and $k \geq j+2$, then we may refine Ω to obtain a $(j+1)$-courteous edge-coloring Ω' of size at most $3t$.*

Proof: Let $X \in \Omega$ be given. It will suffice to show that there exists a 3-coloring Υ of X so that $G \setminus Y$ is $(j+1)$-edge-connected for every $Y \in \Upsilon$. Consider the graph $G' = G \setminus X$. This graph is j-edge-connected, and j is odd, so by the previous proposition we may choose a j-edge-cut tree (T, S) for G'. Now, starting from T, we will construct a new graph H as follows. For every edge $e = xy \in X$, add a new edge $e' = x'y'$ to H so that $x \in S(x')$ and $y \in S(y')$. Let $X' = E(H) \setminus E(T)$. The fundamental cocircuits of T in the graph H are in one-to-one correspondence with the edge-cuts of size j in G'. Since G is k-edge-connected, and $k \geq j+2$, it follows that every fundamental cocircuit of T in the graph H contains ≥ 3 edges (including the one in T). Therefore, by proposition 2.3 applied to the graph H, we may choose a 3-coloring Υ' of X' which is 2-courteous for the tree T. Let Υ be the corresponding 3-coloring of X. We claim that $G \setminus Y$ is $(j+1)$-edge-connected for every $Y \in \Upsilon$. To prove this, let C be a cocircuit of G. If C is an edge-cut corresponding to a fundamental cocircuit of T, then $(C \cap X) \setminus Y \neq \emptyset$ by construction, so $|C \setminus Y| = |C \setminus X| + |(C \cap X) \setminus Y| \geq j+1$ as desired. Otherwise, $|C \setminus Y| \geq |C \setminus X| \geq j+1$. This completes the proof. $\square$

The following proposition is a simple but useful consequence of the Tutte/Nash-Williams disjoint trees theorem.

Proposition 2.6 *If G is a $(2k+1)$-edge-connected graph, then G has $2k+1$ spanning trees $T_1, T_2, \ldots, T_{2k+1}$ so that every edge of G is in at most two of these trees.*

Proof: Let G' be obtained from G by taking an additional copy of each edge. Now, G' is $(4k+2)$-edge-connected, so by a theorem of Tutte [24] and Nash-Williams [15], we may choose $2k+1$ edge-disjoint spanning trees of G'. In the original graph, this gives us $2k+1$ spanning trees using every edge at most twice, as desired. $\square$

With the help of Proposition 2.6 and Lemma 2.5, we can already show that 3-edge-connected graphs have 2-courteous edge-colorings of size ≤ 9. We include this proposition here since this bound is better than the one achieved by the main theorem.

Proposition 2.7 *If G is a 3-edge-connected graph, then G has a 2-courteous edge-coloring of size ≤ 9.*

Proof: By Lemma 2.5, it will suffice to prove that G has a 1-courteous edge-coloring of size 3. By proposition 2.6 we may choose $T_1, T_2, T_3 \subseteq G$ so that $E(T_1) \cap E(T_2) \cap E(T_3) = \emptyset$. Let $A_i = E(G) \setminus E(T_i)$ for $1 \leq i \leq 3$, and let $\mathcal{A} = \{A'_1, A'_2, A'_3\}$ be a partition of $E(G)$ so that $A'_i \subseteq A_i$ for every $1 \leq i \leq 3$. Now, $\mathcal{A}$ is a 1-courteous coloring of $E(G)$ as required. $\square$

We are now ready to prove Theorem 1.1.

Theorem 1.1 *Every $(2k+1)$-edge-connected graph has a $2k$-courteous edge coloring of size at most 81^{k^2}.*

Proof: Let G be a $(2k+1)$-edge-connected graph. If $k=0$, then the theorem is trivial. Otherwise, $3(3^{2k-2}+1)^{2k+1} \leq 81^{k^2}$, so by Lemma 2.5 it will suffice to prove that G has a $(2k-1)$-courteous edge-coloring of size at most $(3^{2k-2}+1)^{2k+1}$. By proposition 2.6 we may choose $2k+1$ spanning trees $T_1, T_2, \ldots, T_{2k+1}$ of G so that every edge $e \in E(G)$ is in at most two of the trees. By Proposition 2.3, for every $1 \leq i \leq 2k+1$, we may choose a $(2k-1)$-courteous coloring Ω_i of $E(G) \setminus E(T_i)$ for the tree T_i of size at most 3^{2k-2} (we could get a $2k$-courteous coloring, but this is unnecessary). Let $\Omega'_i = \Omega_i \cup \{E(T_i)\}$ and let Ω be the coloring obtained by taking the common refinement of the colorings $\Omega'_1, \Omega'_2, \ldots, \Omega'_{2k+1}$. Note that for every $X \in \Omega$, all of the edges in X are contained in exactly the same set of trees $T_1, T_2, \ldots, T_{2k+1}$. Now, Ω has size at most $(3^{2k-2}+1)^{2k+1}$. We claim that Ω is $(2k-1)$-courteous. Let $X \in \Omega$ and let C be an edge-cut of G. By construction, X is disjoint from at least $2k-1$ of the trees $T_1, T_2, \ldots, T_{2k+1}$. If every tree T_i disjoint from X contains ≥ 2 edges of C, then $|C \setminus X| \geq 2k-1$ as desired. Otherwise, X is disjoint from a tree T_i that has C as a fundamental cocircuit. Since $X \subseteq X'$ for some $X' \in \Omega_i$, we have that $|C \setminus X| \geq |C \setminus X'| \geq 2k-1$. Since X and C were arbitrary, it follows that Ω is $(2k-1)$-courteous. $\square$

We now define a function $g : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z} \cup \{\infty\}$ given by the rule $g(a,b)$ is the smallest integer t so that every a-edge-connected graph has a b-courteous edge-coloring of size t (or ∞ if no such integer exists). We have proved that $g(a,b) = \infty$ if and only if $b \geq a$, or a is even and $b = a-1$. The existence of k-regular k-edge-connected graphs which are not k-edge-colorable implies that $g(k, k-1) \geq k+1$. Our proof shows that $g(2k+1, 2k) \leq 81^{k^2}$. Actually, one may improve the bounds in our construction significantly. We can prove by means of a considerably more complicated argument that if Ω is a j-courteous t-edge-coloring of a k-edge-connected graph and j is even and $k \geq j+3$, then it is possible to refine Ω to obtain a $30t$-edge-coloring which is $j+1$-courteous. With the help of this fact and Lemma 2.5 one may easily show that $g(2k+1, 2k) \leq C^k$ for some constant C. These are the tightest bounds we know. In addition to the argument earlier in this section that $g(3,2) \leq 9$, we have some more specialized arguments which prove that $g(4,2) \leq 4$ and $g(5,4) \leq 30$.

3 Circuit Covers

If A, B are multisets with ground set S, we define $A+B$ to be the multiset with ground set S given by the map $(A+B)(s) = A(s) + B(s)$ for every $s \in S$. We define a finite set $S \subseteq \mathbf{Z}$ to be *coverable* if for every graph G and every admissable map $p : E(G) \to S$, (G,p) has a circuit cover. If in addition, $S \subseteq 2\mathbf{Z}$, and $2\min(S) \geq \max(S)$, we will say that S is *conveniently coverable*. Note that every map $p : E(G) \to S$ is admissable (and thus (G,p) has a circuit cover) for every 3-edge-connected graph G and every conveniently coverable set S.

We define a set $C \subseteq E(G)$ to be a *cycle* if every vertex of $(V(G), C)$ has even degree. We say that a list $\mathcal{C} = C_1, C_2, \ldots, C_k$ is a *k-cycle cover* of (G,p) if every edge $e \in E(G)$ appears in exactly $p(e)$ cycles of $\mathcal{C}$ (for convenience, we have used lists instead of multisets in the definition of cycle covers). The goal of this section is to prove the following two theorems on circuit and cycle covers. Theorem 1.3 is restated here in terms of coverable sets.

Theorem 1.3 *For every $i, j, k, l \geq 0$, the set $i\{4\} + j\{6\} + k\{32, 36\} + l\{48, 54\} + \{0, 1\}$ is coverable.*

Theorem 3.1 *If G is $4k$-edge-connected, and $p : E(G) \to \{1, 2, \ldots, 2k\}$ is admissable, then (G, p) has a $(2k+1)$-cycle cover.*

Theorem 1.3 will require three simple propositions and one lemma.

Proposition 3.2 *Let $S \subseteq \mathbf{Z}$ be a finite set. If there is a circuit cover of (G, p) for every 3-edge-connected graph G and every admissable map $p : E(G) \to S$, then S is coverable.*

Proof: Let G be a 2-edge-connected graph and let $p : E(G) \to S$ be an admissable map. We will prove by induction on $|E(G)|$ that (G, p) has a circuit cover. If G is 3-edge-connected, then we are finished by our assumption. Thus, we may assume that G has a 2-edge-cut $\{e, f\}$. Now, $p|_{E(G)\setminus e}$ is an admissable map for G/e, so inductively we may choose a circuit cover $\mathcal{C}'$ of $(G/e, p|_{E(G)\setminus e})$. Since $p(e) = p(f)$, the corresponding multiset $\mathcal{C}$ of circuits of G is now a circuit cover of (G, p). This completes the proof. $\square$

Proposition 3.3 *If S, T are conveniently coverable, then $S + T$ is conveniently coverable.*

Proof: Let G be a 3-edge-connected graph and let $p : E(G) \to S + T$. It will suffice to show that (G, p) has a circuit cover. Choose $p_s : E(G) \to S$ and $p_t : E(G) \to T$ so that $p_s + p_t = p$. Next, choose a circuit cover $\mathcal{C}_s$ of (G, p_s) and a circuit cover $\mathcal{C}_t$ of (G, p_t). Now $\mathcal{C}_s + \mathcal{C}_t$ is a circuit cover of (G, p) as required. $\square$

Proposition 3.4 *If S is conveniently coverable, then $S + \{0, 1\}$ is coverable.*

Proof: Let G be a 3-edge-connected graph and let $p : E(G) \to S + \{0, 1\}$ be an admissable map. By Proposition 3.2, it will suffice to show that (G, p) has a circuit cover. Let $X = \{e \in E(G) \mid p(e) \text{ is odd}\}$. Since p is admissable, (V, X) is eulerian, so we may choose a set of circuits $\mathcal{C}_1$ of the graph (V, X) so that every edge in X is contained in exactly one circuit of $\mathcal{C}_1$. We define $p_2 : E(G) \to \mathbf{Z}$ as follows:

$$p_2(e) = \begin{cases} p(e) - 1 & \text{if } p(e) \text{ is odd} \\ p(e) & \text{if } p(e) \text{ is even} \end{cases}$$

Since $p_2(E(G)) \subseteq S$, we may choose a circuit cover $\mathcal{C}_2$ of (G, p_2). Now $\mathcal{C}_1 + \mathcal{C}_2$ is a circuit cover of (G, p) as required. $\square$

Lemma 3.5 *If $\{2t\}$ is coverable, then $\{16t, 18t\}$ is conveniently coverable.*

Proof: Let G be a 3-edge-connected graph and let $p : E(G) \to \{16t, 18t\}$. By proposition 3.2, it will suffice to show that (G, p) has a circuit cover. By proposition 2.7, we may choose a 2-courteous edge-coloring $\{X_1, X_2, \ldots, X_9\}$ of G. For every $1 \leq i \leq 9$, let $A_i = \{e \in X_i \mid p(e) = 16t\}$. Now, for every $1 \leq i \leq 9$, the graph $G \setminus A_i$ is 2-edge-connected, so we may choose a circuit cover $\mathcal{C}_i$ of $(G \setminus A_i, 2t)$. Then $\mathcal{C}_1 + \mathcal{C}_2 + \ldots + \mathcal{C}_9$ is a circuit cover of (G, p) as required. $\square$

Finally, we are ready to prove theorem 1.3.

Proof of Theorem 1.3: Bermond, Jackson, and Jaeger [4] proved that $\{4\}$ is coverable, and Fan [6] proved that $\{6\}$ is coverable. Thus, by Lemma 3.5 we have that $\{32, 36\}$ and $\{48, 54\}$ are conveniently coverable. Since $\{4\}$ and $\{6\}$ are also conveniently coverable, by Proposition 3.3 we have that for every $i, j, k, l \geq 0$ the set $i\{4\} + j\{6\} + k\{32, 36\} + l\{48, 54\}$ is conveniently coverable. By Proposition 3.4 we have that for every $i, j, k, l \geq 0$ the set $i\{4\} + j\{6\} + k\{32, 36\} + l\{48, 54\} + \{0, 1\}$ is coverable as required. $\square$

The proof of Theorem 3.1 will require two propositions.

Proposition 3.6 *If T is a spanning tree of G, and $Y \subseteq E(G) \setminus E(T)$, then there exists a cycle $C \subseteq G$ with $E(C) \setminus E(T) = Y$.*

Proof: For every edge $e \in E(G) \setminus E(T)$, let $C(e)$ be the edge set of the fundamental circuit of e with respect to T. Then $C = \triangle_{e \in Y} C(e)$ is a cycle of G with $E(C) \setminus E(T) = Y$ as required. $\square$

Proposition 3.7 *Let G be a $4k$-edge-connected graph, and let $p : E(G) \to \{2, 4, 6, \ldots, 2k\}$. Then there is a list $\mathcal{C} = C_1, C_2, \ldots, C_{2k}$ of cycles of G so that every edge $e \in E(G)$ appears in either $p(e)$ or $p(e) - 1$ cycles of $\mathcal{C}$.*

Proof: By the Tutte/Nash-Williams spanning trees theorem, we may partition $E(G)$ into $\{X_1, X_2, \ldots, X_{2k}\}$ so that each set X_i contains the edge set of a spanning tree T_i of G. Now, for every $1 \leq i \leq 2k$ and every $e \in X_i$, choose a subset $U(e) \subseteq \{1, 2, \ldots, 2k\}$ so that $|U(e)| = p(e) - 1$ and so that $i \notin U(e)$. For $1 \leq i \leq 2k$, let $Y_i = \{e \in E(G) \mid i \in U(e)\}$. Now, by the previous proposition, for every $1 \leq i \leq 2k$ we may choose a cycle C_i of G so that $E(G) \setminus E(T_i) = Y_i$. By construction, $\mathcal{C} = C_1, C_2, \ldots, C_{2k}$ satisfies the proposition. $\square$

Proof of Theorem 3.1 Let G be a $4k$-edge-connected graph, let $p : E(G) \to \{1, 2, \ldots, 2k\}$ be an admissable map, and let $S = \{e \in E(G) \mid\ p(e) \text{ is odd }\}$. Define $p' : E(G) \to \mathbf{Z}$ by the following rule:

$$p'(e) = \begin{cases} 2k + 1 - p(e) & \text{if } e \in S \\ p(e) & \text{otherwise} \end{cases}$$

By the above proposition, we may choose a list $\mathcal{C} = C_1, C_2, \ldots, C_{2k}$ of $2k$ cycles of G so that every edge $e \in E(G)$ is in either $p'(e)$ or $p'(e) - 1$ cycles of $\mathcal{C}$. Let $T = \{e \in E(G) \mid$ e is in $p'(e) - 1$ cycles of $\mathcal{C}\}$ Since T is the set of edges covered an odd number of times by $\mathcal{C}$, we find that T is a cycle. Thus, setting $C_{2k+1} = T$, we have that $C_1, C_2, \ldots, C_{2k}, C_{2k+1}$ is a (2k+1)-cycle cover of (G, p'). Since p is an admissable map, S is also a cycle of G. Thus, the list $S\triangle C_1, S\triangle C_2, \ldots, S\triangle C_{2k+1}$ is a (2k+1)-cycle cover of (G, p) as required. □

4 Antisymmetric Flows

The study of nowhere-zero flows began with Tutte's observation that for a planar graph G and the geometric dual G^* of G, G has a nowhere-zero k-flow if and only if G^* has a proper k-coloring. In a similar (but less tight) sense (see [16]), antisymmetric flows are dual to oriented graph colorings. The main theorem of this section is as follows.

Theorem 1.6 *Every directed graph with no directed cut of size ≤ 2 has a 10^{12}-antiflow.*

We define a directed graph to be *k-edge-connected* if the underlying undirected graph is k-edge-connected. Now, we will proceed with the proofs of this section. After a simple proposition, we will prove that every graph with no directed cut of size ≤ 2 has a Γ-antiflow in an appropriate fixed group Γ. We will then use this to establish Theorem 1.6.

Proposition 4.1 *Let Γ be an abelian group. If every 3-edge-connected directed graph has Γ-antiflow, then every graph with no directed cuts of size ≤ 2 has a Γ-antiflow.*

Proof: We will proceed by induction on $|E(G)|$. If G is 3-edge-connected, we are finished by assumption. Otherwise, we may choose a 2-edge-cut $\{e_1, e_2\}$ of G. Let $G' = G/e_1$. Inductively, we may choose a Γ-antiflow f' of G'. Let $f : E(G) \to \Gamma$ be given by the rule.

$$f(e) = \begin{cases} f'(e_2) & \text{if } e = e_1 \\ f'(e) & \text{otherwise} \end{cases}$$

Now, f is a Γ-antiflow of G as required. $\square$

We are now ready to prove the existence of bounded size group-valued antiflows. For a positive integer k, we will use $\mathbf{Z}_k$ to denote the group $\mathbf{Z}/k\mathbf{Z}$. The following theorem is an improvement on the authors' original argument suggested by Nešetřil and Raspaud.

Theorem 4.2 *If G is a directed graph with no directed edge-cuts of size ≤ 2, then G has a $(\mathbf{Z}_6)^8 \times (\mathbf{Z}_3)^9$-antiflow.*

Proof: By proposition 4.1, it will suffice to prove the theorem in the case when G is 3-edge-connected. By Proposition 2.7, we may choose a 2-courteous 9-coloring $\Omega = \{X_1, X_2, \ldots, X_9\}$ of $E(G)$. By Seymour's 6-flow theorem, we may choose for every $1 \leq i \leq 8$ a nowhere-zero $\mathbf{Z}_6$-flow of $G \setminus X_i$. Let $\phi_1 : E(G) \to \mathbf{Z}_6^8$ be the direct product of these flows. Now, for every $1 \leq i \leq 9$, since $G \setminus X_i$ is spanning, we may choose a $\mathbf{Z}_3$-flow which takes the value 1 on every edge in X_i. Let $\phi_2 : E(G) \to \mathbf{Z}_3^9$ be the direct product of these flows. We claim that $\phi_1 \times \phi_2$ is an antiflow. Let $e, f \in E(G)$. If e and f are in different color classes of Ω, then there is a coordinate of ϕ_1 in which exactly one of e, f has the value zero. Otherwise, there is a coordinate of ϕ_2 in which both e and f have the value 1. In either case, $\phi(e) \neq -\phi(f)$.
$\square$

To convert this group-valued antiflow into an integer antiflow, we will need one easy proposition and a well known theorem of Tutte.

Proposition 4.3 *Let $(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n) \in \mathbf{Z}^{n+1}$, and assume that $|a_i| \leq k$ and $|b_i| \leq k$ for every $0 \leq i \leq n-1$. If $\sum_{i=0}^{n}(2k+1)^i a_i = -\sum_{i=0}^{n}(2k+1)^i b_i$, then $a_i = -b_i$ for every $0 \leq i \leq n$.*

Proof: Since $\sum_{i=0}^{n}(2k+1)^i(a_i + b_i) = 0$, then since $|a_i + b_i| \leq 2k$ for $0 \leq i \leq n-1$, it follows that $a_i = -b_i$ for every $0 \leq i \leq n$. $\square$

Theorem 4.4 (Tutte) *If $f : E(G) \to \mathbf{Z}_k$ is a flow, then there is a k-flow $f' : E(G) \to \mathbf{Z}$ so that $f'(e) \cong f(e)$ (mod k) for every $e \in E(G)$.*

Now we are ready to prove the main theorem of this section.

Proof of Theorem 1.6: By Theorem 4.2, we may choose $\mathbf{Z}_2$-flows $f_0, f_1, \ldots, f_7$ and $\mathbf{Z}_3$-flows $f_8, f_9, \ldots, f_{24}$ of G so that $f_0 \times f_1 \times \ldots \times f_{24}$ is an antiflow. By Theorem 4.4 we may choose 2-flows $f'_0, f'_1, \ldots, f'_7$ so that $f'_i(e) \cong f_i(e)$ (mod 2) and 3-flows $f'_8, f'_9, \ldots, f'_{24}$ so that $f'_i(e) \cong f_i(e)$ (mod 3). By construction, $f'_0 \times f'_1 \times \ldots \times f'_{24}$ is a $\mathbf{Z}^{25}$-antiflow. Let $h_1 = \sum_{i=0}^{7} 3^i f'_i$ and let $h_2 = \sum_{i=8}^{24} 5^{i-8} f'_i$. By Proposition 4.3 we have that $h_1 \times h_2$ is a $\mathbf{Z}^2$-antiflow. Since $|h_1(e)| \leq (3^8 - 1)/2$ for every $e \in E(G)$, it follows from Proposition 4.3 that $h_1 + 3^8 h_2$ is a 10^{12}-antiflow of G as required. $\square$

5 Related Questions

One way to produce a k-courteous edge-coloring is to find an edge-coloring so that every cut contains edges of at least $k+1$ different colors. One may ask if every a-edge-connected graph has a bounded size edge-coloring so that every cut contains edges of at least b distinct colors for a pair of fixed positive integers a, b. As in the case of courteous edge-coloring, we will define a function $h : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z} \cup \{\infty\}$ given by the rule $h(a,b)$ is the smallest integer t so that every a-edge-connected graph has a t-edge-coloring so that every edge-cut contains edges of at least b different colors (or ∞ if no such integer exists). If $b \leq a/2$, then $h(a,b) = b$ follows immediately from the Tutte/Nash-Williams disjoint spanning trees theorem. We will show that $h(a,b) < \infty$ if and only if $b \leq \lceil a/2 \rceil$.

Essentially the same example as given in the introduction can be used to prove that there is no bounded size coloring of every $2k$-edge-connected graph so that every edge-cut contains edges of at least $k+1$ distinct colors. To show this, we will exhibit for any pair of positive integers k, t, a $2k$-edge-connected graph $G_{k,t}$ so that for every t-edge-coloring of G, there is an edge-cut of G containing at most k colors. We let $G_{k,t}$ be the graph obtained from a circuit of length $t^k + 1$ by replacing each edge by k parallel edges. Now, any t-coloring of the edges of $G_{k,t}$ must give the same set of colors to two different parallel classes of edges. There is an edge-cut X containing only the edges in these two parallel classes, and it follows that X contains edges of at most k distinct colors.

The positive direction is given by the following proposition.

Proposition 5.1 *If G is $2k+1$-edge-connected, then there is a $\binom{2k+1}{2}$-coloring of $E(G)$ so that every edge-cut of G has edges of at least $k+1$ distinct colors.*

Proof: By proposition 2.6 we may choose $2k+1$ spanning trees $T_1, T_2, \ldots T_{2k+1}$ of G so that every edge is in at most two of these trees. Now, we may choose a partition $\Omega = \{X_{i,j} \mid 1 \leq i < j \leq 2k+1\}$ of $E(G)$ so that for every edge $e \in E(G)$, if $e \in X_{i,j}$, then $e \notin E(T_h)$ for every $i \neq h \neq j$. Since every color class contains edges from at most two of the trees $T_1, T_2, \ldots T_{2k+1}$ and every tree contains an edge from every edge-cut, it follows that every edge-cut of G has edges of at least $k+1$ colors. $\square$

As mentioned in the introduction, for any $g > 0$, there is no fixed integer t so that every graph of girth g has a t-edge-coloring so that no circuit is monochromatic. However, there are other questions (based on different graph parameters) concerning edge colorings of graphs with requirements on the number of colors appearing in each circuit. The following conjecture of Alon is particularly interesting.

Conjecture 5.2 (Alon) *Let G be a graph with maximum degree d. Then G has a proper $(d+2)$-edge-coloring so that every circuit contains edges of ≥ 3 different colors.*

We will close this paper with an edge-coloring theorem of a flavor similar to our previous results. Although the proof is a very simple application of a matroid packing theorem, we find the result somewhat surprising.

Proposition 5.3 *If G is a $(2k+1)$-edge-connected graph, then there is a $(2k+1)$-coloring Ω of $E(G)$ so that $G \setminus X$ contains k edge-disjoint spanning trees for every $X \in \Omega$.*

In a sense, graphs which contain k edge disjoint spanning trees are similar to graphs which are $2k$-edge-connected. Indeed a graph G is $2k$-edge-connected if and only if $G \setminus X$ contains k edge-disjoint spanning trees for any set $X \subseteq E(G)$ with $|X| \leq k$. The above proposition is interesting because the corresponding statement with "contains k edge-disjoint spanning trees" replaced by "is $2k$-edge-connected" is false. In particular, this statement would imply that every $(2k+1)$-edge-connected $(2k+1)$-regular graph has a proper $(2k+1)$-edge coloring.

The proof of proposition 5.3 will follow easily from the following proposition on matroids. We will assume that the reader is familiar with Edmonds' matroid union theorem (see [25] or [17] for a good introduction).

Proposition 5.4 *Let M be a matroid on the ground set E, let a, b be positive integers, and assume that M contains a bases using every element at most b times. Then for any integer i such that $a > bi$, there is an a-coloring Ω of E such that $M \setminus X$ contains i disjoint bases for every $X \in \Omega$.*

Proof: Let the matroid M^i be obtained by taking the union of M with itself i times. We will use ρ to denote the rank function of M and ρ^i to denote the rank function of M^i. By the assumption, for every $S \subseteq E$, we have that $a\rho(S) + b|E \setminus S| \geq a\rho(E)$. It follows from this that $|E \setminus S| \geq a/b(\rho(E) - \rho(S)) \geq i(\rho(E) - \rho(S))$. Thus, $\rho^i(E) = i\rho(E)$, and every base of M^i can be partitioned into i pairwise disjoint bases of M.

If M^i contains a bases $B_1, B_2, \ldots, B_a$ using every element at most $a - 1$ times, then we may choose a coloring $\Omega = \{X_1, X_2, \ldots, X_a\}$ of E so that $X_i \cap B_i = \emptyset$ for every $1 \leq i \leq a$. Thus, by Edmonds' matroid partitioning theorem ([5]), it will suffice to prove that for every $A \subseteq E$ we have that $a\rho^i(A) + (a-1)|E \setminus A| \geq a\rho^i(E)$.

$$\begin{aligned}
a\rho^i(A) + (a-1)|E \setminus A| &= a \min_{B \subseteq A}(i\rho(B) + |A \setminus B|) + (a-1)|E \setminus A| \\
&\geq i \min_{B \subseteq A}(a\rho(B) + b|E \setminus B|) \\
&\geq ia\rho(E) \\
&= a\rho^i(E)
\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 5.3: Apply proposition 5.4 to $M(G)$ with $a = 2k + 1$, $b = 2$, and $i = k$. $\square$

As a final note, we observe that Proposition 5.3 implies that for every 5-edge-connected graph G and every map $p : E(G) \to \{8, 10\}$, there is a circuit cover of (G, p). Simply apply the above proposition to partition $E(G)$ into $\{X_1, \ldots, X_5\}$ and let $A_i = \{e \in X_i \mid p(e) = 8\}$. For $1 \leq i \leq 5$, the graph $G \setminus A_i$ contains two edge-disjoint spanning trees, so it has a double cover by circuits. Together these five circuit covers give a circuit cover of (G, p).

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