Excluded Minors in Cubic Graphs

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Abstract

Let G be cubic, with girth ≥ 5, such that for every partition X, Y of its vertex set with $|X|, |Y| \geq 7$ there are ≥ 6 edges between X and Y. We prove that if there is no homeomorphic embedding of the Petersen graph in G, and G is not one particular 20-vertex graph, then either $G \backslash v$ is planar for some vertex v or G can be drawn with crossings in the plane, but with only two crossings, both on the infinite region.

We also prove several other theorems of the same kind.

1 Introduction

All graphs in this paper are simple and finite. Circuits have no repeated vertices; the *girth* of a graph is the length of the shortest circuit. If G is a graph and $X \subseteq V(G), \delta_G(X)$ or $\delta(X)$ denotes the set of edges with one end in X and the other in $V(G) \setminus X$. We say a cubic graph G is *cyclically k-connected*, for $k \geq 1$ an integer, if G has girth $\geq k$, and $|\delta_G(X)| \geq k$ for every $X \subseteq V(G)$ such that both X and $V(G) \setminus X$ include the vertex set of a circuit of G.

A *homeomorphic embedding* of a graph G in a graph H is a function η such that

- for each $v \in V(G)$, $\eta(v)$ is a vertex of H, and $\eta(v_1) \neq \eta(v_2)$ for all distinct $v_1, v_2 \in V(G)$
- for each $e \in E(G)$, $\eta(e)$ is a path of H with ends $\eta(v_1)$ and $\eta(v_2)$, where e has ends v_1, v_2 in G; and no edge or internal vertex of $\eta(e_1)$ belongs to $\eta(e_2)$, for all distinct $e_1, e_2 \in E(G)$
- for all $v \in V(G)$ and $e \in E(G)$, $\eta(v)$ belongs to $\eta(e)$ if and only if v is an end of e in G.

We denote by $\eta(G)$ the subgraph of H consisting of all the vertices $\eta(v)$ $(v \in V(G))$ and all the paths $\eta(e)$ $(e \in E(G))$. We say that H *contains* G if there is a homeomorphic embedding of G in H.

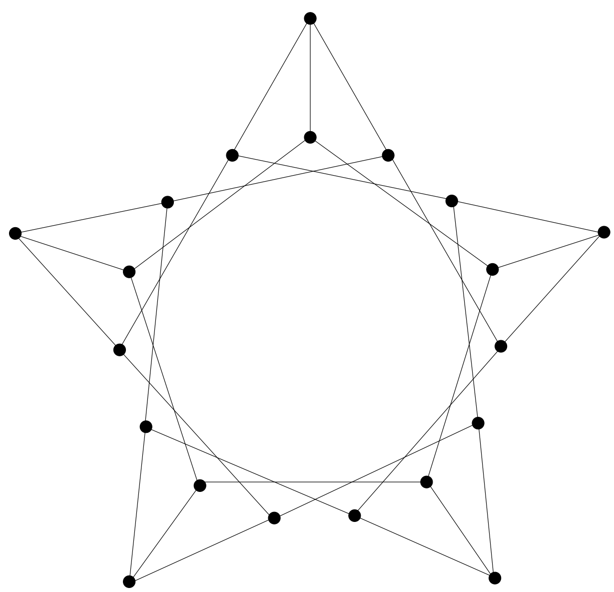


Figure 1: Starfish

This is the second of three papers whose main objective is a structural description of the cyclically five-connected cubic graphs not containing the Petersen graph. Here we show that, if we increase the connectivity requirement a little, all such graphs have a simple description. More precisely, we say that G is *theta-connected* if G is cubic and cyclically five-connected, and $|\delta_G(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$. We say G is *apex* if $G \backslash v$ is planar for some vertex v (we use $\backslash$ to denote deletion); and G is *doublecross* if it can be drawn in the plane with only two crossings, both on the infinite region. The graph *Starfish* is shown in Fig. 1. Henceforth, we define *Petersen* to be the Petersen graph. Our main result is the following.

1.1 *Let G be theta-connected. Then G does not contain Petersen if and only if either G is apex, or G is doublecross, or G is isomorphic to Starfish.*

The "if" part of 1.1 is easy and we omit it. The "only if" part is an immediate consequence of the following three theorems. The graph *Jaws* is defined in Fig. 2.

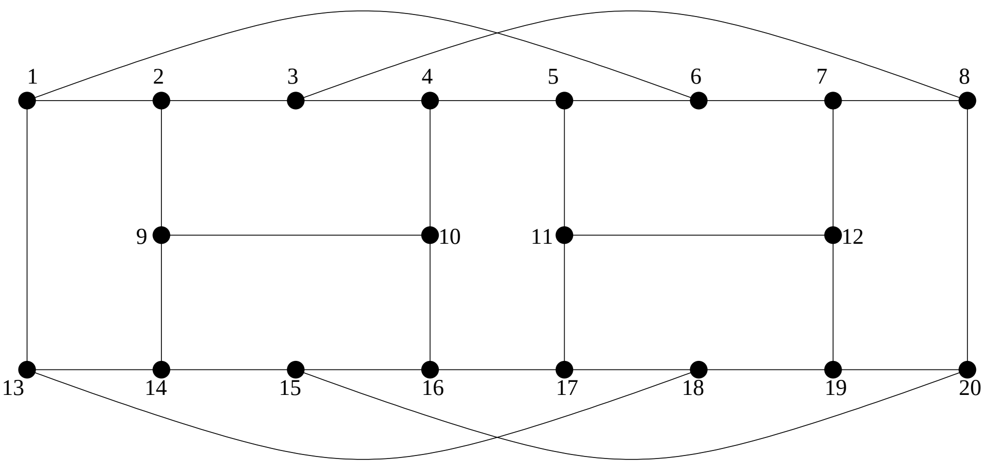


Figure 2: Jaws

1.2 *Let G be theta-connected, and not contain Petersen. If G contains Starfish then G is isomorphic to Starfish.*

1.3 *Let G be theta-connected, and not contain Petersen. If G contains Jaws then G is doublecross.*

1.4 *Let G be theta-connected, and not contain Petersen. If G contains neither Jaws nor Starfish, then G is apex.*

1.2, proved in section 17, is an easy consequence of a theorem of a previous paper [3], and 1.3 is proved in section 18. The main part of the paper is devoted to proving 1.4. Our approach is as follows.

A graph H is *minimal* with property P if there is no graph G with property P such that H contains G and H is not isomorphic to G. In Figure 3 we define four more graphs, namely *Triplex, Box, Ruby* and *Dodecahedron.* A theorem of McCuaig [1] asserts

1.5 *Petersen, Triplex, Box, Ruby and Dodecahedron are the only graphs minimal with the property of being cubic and cyclically five-connected.*

A graph G is *dodecahedrally-connected* if it is cubic and cyclically five-connected, and for every $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$ and $|\delta_G(X)| = 5$, $G|X$ cannot be drawn in a disc Δ so that the five vertices in X with neighbours in $V(G) \setminus X$ are drawn in $bd(\Delta)$. We shall prove the following three theorems.

1.6 *Petersen, Triplex, Box and Ruby are the only graphs minimal with the property of being cyclically* 5*-connected and non-planar.*

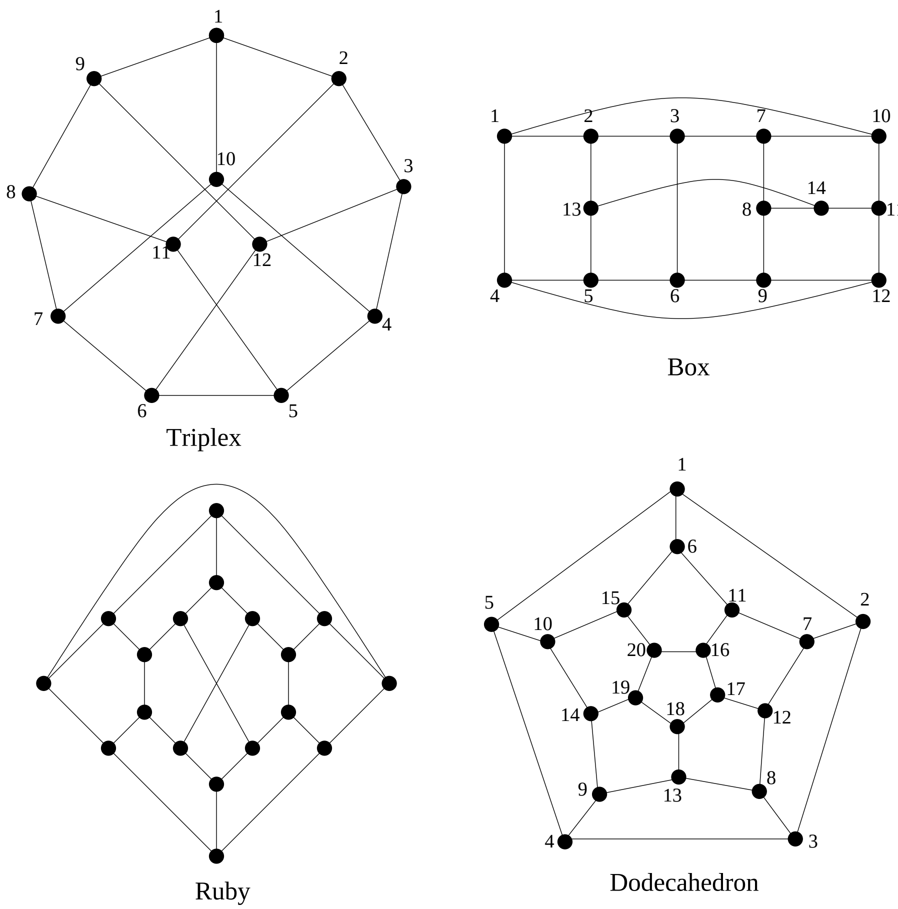


Figure 3: Triplex, Box, Ruby and Dodecahedron

1.7 *Petersen, Triplex and Box are the only graphs minimal with the property of being dodecahedrally-connected and having crossing number* > 1.

We say G is *arched* if $G\backslash e$ is planar for some edge e.

1.8 *Petersen and Triplex are the only graphs minimal with the property of being dodecahedrally-connected and not arched.*

Then we use 1.8 to find all the graphs minimal with the property of being dodecahedrally-connected and non-apex (there are 6). Let us say G is *die-connected* if it is dodecahedrally-connected and $|\delta_G(X)| \geq 6$ for every $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 9$. We use the last result to find all graphs minimal with the property of being die-connected and non-apex (there are 9); and then use that to find the minimal graphs with the property of being theta-connected and non-apex. There are three, namely Petersen, Starfish, and Jaws, and from this 1.3 follows.

2 Extensions

It will be convenient to denote by ab or ba an edge with ends a and b (since we do not permit parallel edges, this is unambiguous). Let ab and cd be distinct edges of a graph G. They are *diverse* if a, b, c, d are all distinct and a, b are not adjacent to c or d. We denote by $G+(ab, cd)$ the graph obtained from G as follows: delete ab and cd, and add two new vertices x and y and five new edges xa, xb, yc, yd, xy. We call x, y (in this order) the *new vertices* of $G+(ab, cd)$. Multiple applications of this operation are denoted in the natural way; for instance, if $e, f, \in E(G)$ are distinct, and $G' = G + (e, f)$, and $g, h \in E(G')$ are distinct, we write $G + (e, f) + (g, h)$ for $G' + (g, h)$.

Similarly, let ab, cd, ef be distinct edges of G. We denote by $G + (ab, cd, ef)$ the graph obtained by deleting ab, cd and ef, and adding four new vertices x, y, z, w, and nine new edges $xa, xb, yc, yd, ze, zf, wx, wy, wz$; and call x, y, z, w (in this order) the *new vertices* of $G+(ab, cd, ef)$.

A path has no "repeated" vertices or edges. Its first and last vertices are its *ends*, and its first and last edges are its *end-edges*. A path with ends s and t is called an *(s, t)-path*. If P is a path and $s, t \in V(P)$, the subpath of P with ends s and t is denoted by $P[s, t]$. Let η be a homeomorphic embedding of G in H. An *η-path* in H is a path P with distinct ends both in $V(\eta(G))$, but with no other vertex or edge in $\eta(G)$. Let G, H both be cubic, and let η and P be as above, where P has ends s and t, with $s \in V(\eta(e))$ and $t \in V(\eta(f))$. We can sometimes use P to obtain a new homeomorphic embedding η' of G in H, equal to η except as follows:

- If $e = f$, let $e = uv$, where $\eta(u), s, t, \eta(v)$ lie in $\eta(e)$ in order. Define

$$\eta'(e) = \eta(e)[\eta(u), s] \cup P \cup \eta(e)[t, \eta(v)].$$

- If $e \neq f$ but they have a common end, let $e = uv$ and $f = vw$ say, and let g be the third edge of G incident with v. Define η' by:

$$\begin{aligned}
\eta'(v) &= t, \\
\eta'(e) &= \eta(e)[\eta(u), s] \cup P, \\
\eta'(f) &= \eta(f)[t, \eta(w)], \\
\eta'(g) &= \eta(g) \cup \eta(f)[\eta(v), t].
\end{aligned}$$

- If e, f have no common end, but one end of e is adjacent to one end of f, let $e = uv$, $f = wx$ and $g = vw$ say. Let h, i be the third edges at v, w respectively. Define η' by:

$$\begin{aligned}
\eta'(v) &= s, \\
\eta'(w) &= t, \\
\eta'(e) &= \eta(e)[\eta(u), s], \\
\eta'(f) &= \eta(f)[t, \eta(x)], \\
\eta'(g) &= P, \\
\eta'(h) &= \eta(h) \cup \eta(e)[s, \eta(v)], \\
\eta'(i) &= \eta(i) \cup \eta(f)[\eta(w), t].
\end{aligned}$$

In the first two cases we say that η' is obtained from η by *rerouting* e *along* P, and in the third case by *rerouting* g *along* P. If η is a homeomorphic embedding of G in H, an η*-bridge* is a connected subgraph B of H with $E(B \cap \eta(G)) = \emptyset$, such that either

- $|E(B)| = 1$, $E(B) = \{e\}$ say, and both ends of e are in $V(\eta(G))$, or
- for some component C of $H \backslash V(\eta(G))$, $E(H)$ consists of all edges of H with at least one end in $V(C)$.

It follows that every edge of H not in $\eta(G)$ belongs to a unique η-bridge. We say that an edge e of G is an η*-attachment* of an η-bridge B if $\eta(e) \cap B$ is non-null.

3 Frameworks

We shall often have a cubic graph G, so that G (or sometimes, most of G) is drawn in a surface, possibly with crossings, and also a homeomorphic embedding η of G in another cubic graph H; and we wish to show that the drawing of G can be extended to a drawing of H without introducing any more crossings. For this to be true, one necessary condition is that for each η-bridge B, all its attachments belong to the same "region" of G. Each region of the drawing is bounded either by a circuit (if no crossings involve any edge incident with the region) or by one or more paths, whose first and last edges cross others and no internal edges cross others. For instance, in Figure 2, one region is bounded by the path v_6-v_1-v_2-v_3-v_8; and another by two paths v_6-v_1-v_{13}-v_{18} and v_{15}-v_{20}-v_8-v_3. If we list all these circuits and paths we obtain some set of subgraphs of G, and it is convenient to work with this set rather than explicitly with regions of a drawing of G.

Sometimes, the drawing is just of a subgraph G' of G rather than of all of G; but then we shall always be able to arrange that $\eta(e)$ has only one edge, for every edge e of G not in G'. In this case, all the circuits and paths in the set are subgraphs of G'. This motivates the following definition.

We say $(G, F, \mathcal{C})$ is a *framework* if G is cubic, F is a subgraph of G, and $\mathcal{C}$ is a set of subgraphs of $G \setminus E(F)$, satisfying (F1)–(F7) below. We say distinct edges e, f are *twinned* if there exist distinct $C_1, C_2 \in \mathcal{C}$ with $e, f \in E(C_1 \cap C_2)$.

(F1) Each member of $\mathcal{C}$ is an induced subgraph of $G \setminus E(F)$, with at least three edges, and is either a path or a circuit.

(F2) Every edge of $G \setminus E(F)$ belongs to some member of $\mathcal{C}$, and for every two edges e, f of G with a common end not in $V(F)$, there exists $C \in \mathcal{C}$ with $e, f \in E(C)$.

(F3) If $C_1, C_2 \in \mathcal{C}$ are distinct and $v \in V(C_1 \cap C_2)$, then either $V(C_1 \cap C_2) = \{v\}$, or v is incident with an edge in $C_1 \cap C_2$, or $v \in V(F)$.

(F4) If $C_1 \in \mathcal{C}$ is a path, then every member of $\mathcal{C}$ containing an end-edge of C_1 is a path. Moreover, if also $C_2 \in \mathcal{C} \setminus \{C_1\}$ is a path, then every component of $C_1 \cap C_2$ contains an end of C_1, and every edge of $C_1 \cap C_2$ is an end-edge of C_1.

(F5) For any $C \in \mathcal{C}$, if C is a circuit then $|V(C \cap F)| \leq 1$, and every vertex in $C \cap F$ has degree 1 in F; and if C is a path then every vertex in $C \cap F$ is an end of C and has degree 0 or 2 in F.

(F6) If e, f are twinned and $C \in \mathcal{C}$ with $e \in E(C)$, then $|V(C)| \leq 6$, and either

- $f \in E(C)$, and C is a circuit, and e, f have a common end in $V(F)$, and no path in $\mathcal{C}$ contains any vertex of e or f, or
- $f \in E(C)$, and C is a path with end-edges e, f, and $C \cap F$ is null, or
- $f \notin E(C)$, and C is a path with $|E(C)| = 3$, and e is an end-edge of C, and no end of e belongs to $V(F)$.

(F7) Let $C \in \mathcal{C}$ be a path of length five, with twinned end-edges e, f. Then $|E(C')| \leq 4$ for every path $C' \in \mathcal{C} \setminus \{C\}$ containing e. Moreover, let C have vertices v_0-v_1-$\cdots$-v_5 in order; then there exists $C' \in \mathcal{C}$ with end-edges e and f and with ends v_0 and v_4.

We will prove a theorem that says, roughly, that if we have a framework $(G, F, \mathcal{C})$, and a homeomorphic embedding of G in H, where H is appropriately cyclically connected, then either the drawing of G extends to an drawing of the whole of H, or there is some bounded enlargement of $\eta(G)$ in H to which the drawing does not extend, and this enlargement still has high cyclic connectivity.

These seven axioms are a little hard to digest, and before we go on it may help to see how they will be used. In all our applications of (F1)–(F7) we have some particular graph G in mind and a drawing of it that defines the framework. We could replace (F1)–(F7) just by the hypothesis that $(G, F, \mathcal{C})$ arise from one of these particular cases, but there are nine of them, and it seems clearer to try to abstract the properties that we really use. Here are three examples that might help.

- The simplest application is to prove 1.6; we take G to be Dodecahedron, and F null, and $\mathcal{C}$ to be the set of region-bounding circuits in the drawing of G in Figure 3. Suppose now some H contains G; our result will tell us that either the embedding of G extends to an embedding of H (and hence H is planar), or H contains a non-planar subgraph, a bounded enlargement of $\eta(G)$ with high cyclic connectivity. We enumerate all the possibilities for this enlargement, and check they all contain one of Petersen, Ruby, Box, Triplex. From this, 1.6 will follow.

- When we come to try to understand the graphs that contain Jaws and not Petersen, we take G to be Jaws, and $(G, F, \mathcal{C})$ to be defined by the drawing in Figure 2. Thus, F is null; $\mathcal{C}$ will contain the seven circuits in Figure 2 that bound regions and do not include any of the four edges that cross, together with eight paths (four like 6-1-2-3-8; two like 1-6-5-4-3-8; and two like 6-1-13-18.)

- A last example, one with F non-null; when we prove 1.8, we take G to be Box, and $(G, F, \mathcal{C})$ to be defined by the drawing in Figure 3, where F is the edge $v_{13}v_{14}$. In this case, take the drawing of Box given in Figure 3, and delete the edge $v_{13}v_{14}$, and we get a drawing of $G \setminus v_{13}v_{14}$ without crossings; let $\mathcal{C}$ be the set of circuits that bound regions in this drawing. The only twinned edges are v_2v_{13} with v_5v_{13}, and v_8v_{14} with $v_{11}v_{14}$.

(F1)–(F7) have a number of easy consequences, for instance, the following four results.

3.1 *Let $(G, F, \mathcal{C})$ be a framework.*

- *F is an induced subgraph of G.*
- *Let $e \in E(G) \setminus E(F)$. Then e belongs to at least two members of $\mathcal{C}$, and to more than two if and only if e is an end-edge of a path in $\mathcal{C}$ and neither end of e is in $V(F)$; and in this case e belongs to exactly four members of $\mathcal{C}$, all paths, and it is an end-edge of each of them.*

- *For every two edges e, f of G with a common end with degree three in $G \setminus E(F)$, there is at most one $C \in \mathcal{C}$ with $e, f \in E(C)$.*

Proof. Let $e = uv$ be an edge of $E(G) \setminus E(F)$. We claim that $|\{u, v\} \cap V(F)| \leq 1$. For by (F2) there exists $C \in \mathcal{C}$ with $e \in E(C)$. If C is a circuit the claim follows from (F5), and if C is a path then one of u, v is internal to C, and again it follows from (F5). Thus the first claim holds.

For the second claim, again let $e = uv$ be an edge of $E(G) \setminus E(F)$. We may assume that $u \notin V(F)$. Let u be incident with e, e_1, e_2. By (F2) there exist $C_1, C_2 \in \mathcal{C}$ with $e, e_i \in E(C_i)$ $(i = 1, 2)$. Hence $C_1 \neq C_2$, so e belongs to at least two members of $\mathcal{C}$.

No other member of $\mathcal{C}$ contains e and either e_1 or e_2, by (F6), since $u \notin V(F)$. Hence every other $C \in \mathcal{C}$ containing e is a path with one end u. If e is not an end-edge of any path in $\mathcal{C}$ the second claim is therefore true, so we assume it is. Hence by (F4), C_1 and C_2 are both paths with end-edge e, and both have one end v. If $v \in V(F)$, there is no path in $\mathcal{C}$ containing e with one end u, by (F5), so we may assume that $v \notin V(F)$. Let v be incident with e, e_3, e_4; then by (F2) there exist $C_3, C_4 \in \mathcal{C}$ with $e, e_i \in E(C_i)$ $(i = 3, 4)$; and C_3, C_4 both have one end u. Hence $C_1, \ldots, C_4$ are all distinct, and no other member of $\mathcal{C}$ contains e. This proves the second claim.

For the third claim, let $v \in V(G)$ be incident with edges $e, f, g \in E(G) \setminus E(F)$. Suppose there exist distinct $C, C' \in \mathcal{C}$ both containing e, f. Thus e, f are twinned. If C is a circuit, then by (F6) $v \in V(F)$, and by (F5) v has degree one in F, a contradiction. Thus C is a path. By (F6) both e, f are end-edges of C, and hence C has length two, a contradiction. This proves the third claim, and hence proves 3.1. ∎

3.2 *Let $C_1, C_2 \in \mathcal{C}$ be distinct. Then $|E(C_1 \cap C_2)| \leq 2$, and if equality holds then either*

- *$C_1 \cap C_2$ is a 2-edge path with middle vertex v in $V(F)$, and v has degree one in F, and C_1 and C_2 are both circuits, or*
- *$C_1 \cap C_2$ consists of two disjoint edges e, f and their ends, and C_1, C_2 are both paths with end-edges e, f, and no end of e or f is in $V(F)$.*

Proof. Let $e, f \in E(C_1 \cap C_2)$ be distinct. If C_1 is a path then by (F6) and (F4), so is C_2, and both C_1 and C_2 have end-edges e, f, and no end of e or f is in $V(F)$. But then by (F6) $|E(C_1 \cap C_2)| = 2$ (for any third edge in $E(C_1 \cap C_2)$ would also have to be an end-edge of C_1, which is impossible); and if $v \in V(C_1 \cap C_2)$ is not incident with e or f, then v is internal to both paths and hence is incident with an edge of $C_1 \cap C_2$, a contradiction. Thus in this case (ii) holds. We may assume then that C_1 and C_2 are both circuits. By (F6), e, f have a common end, v say, in $V(F)$. By (F5) no other vertex of C_1 or C_2 is in $V(F)$, and v has degree one in F. By (F6), $E(C_1 \cap C_2) = \{e, f\}$, and hence (i) holds. This proves 3.2. ∎

3.3 *Let $C_1, C_2 \in \mathcal{C}$ be distinct with $|E(C_1 \cap C_2)| \geq 2$. Then $|E(C_1)| \geq 4$.*

Proof. Suppose that C_1 is a circuit. If $|E(C_1)| = 3$, then since C_2 is an induced subgraph of $G \setminus E(F)$ and $|E(C_1 \cap C_2)| \geq 2$ it follows that C_1 is a subgraph of C_2 which is impossible. Hence the result holds if C_1 is a circuit. Now let C_1 be a path. Let $e, f \in E(C_1 \cap C_2)$ be distinct; then by (F6), e and f are end-edges of C_1, and by (F4) C_2 is a path with end-edges e, f. Hence again C_1 is not a subgraph of C_2, and so since C_2 is an induced subgraph of $G \setminus E(F)$ it follows that $|E(C_1)| \geq 4$. This proves 3.3. ∎

3.4 *Let $(G, F, \mathcal{C})$ be a framework, and let $e, f_1, f_2 \in E(G)$ be distinct. If e, f_1 are twinned then e, f_2 are not twinned.*

Proof. Let $C_1, C_1' \in \mathcal{C}$ be distinct with $e, f_1 \in E(C_1 \cap C_1')$, and suppose that there exist $C_2, C_2' \in \mathcal{C}$, distinct, with $e, f_2 \in E(C_2 \cap C_2')$. At least three of C_1, C_1', C_2, C_2' are distinct, and they all contain e, and so by 3.1 all of C_1, C_1', C_2, C_2' are paths and e is an end-edge of each of them. By (F6) C_1 has end-edges e and f_1, and $f_2 \notin E(C_1)$. Since $e, f_1 \in E(C_1)$, by 3.3 $|E(C_1)| \geq 4$; but since $f_2 \notin E(C_1)$, by (F6) $|E(C_1)| \leq 3$, a contradiction. This proves 3.4. ∎

Now let $(G, F, \mathcal{C})$ be a framework, and let η_F be a homeomorphic embedding of F into H. We list a number of conditions on the framework, H and η_F that we shall prove have the following property. Suppose that these conditions are satisfied, and there is a homeomorphic embedding of G in H extending η_F; then the natural drawing of $G \setminus E(F)$ (where the members of $\mathcal{C}$ define the region-boundaries) can be extended to one of $H \setminus E(\eta_F(F))$. They are the following seven conditions (E1)–(E7).

(E1) H is cubic and cyclically 4-connected, and if $(G, F, \mathcal{C})$ has any twinned edges, then H is cyclically 5-connected. Also, $\eta_F(e)$ has only one edge for every $e \in E(F)$.

Let J be the subgraph of F obtained by deleting all vertices with degree one in F. Let G' be a cubic graph with J a subgraph of G'. A homeomorphic embedding η of G' in H is said to *respect* η_F if $\eta(e) = \eta_F(e)$ for every $e \in E(J)$, and $\eta(v) = \eta_F(v)$ for every $v \in V(J)$.

(E2) Let $e, f \in E(G)$ be distinct, and let each of them either belong to $E(G) \setminus E(F)$ or have an end with degree one in F. If there is a homeomorphic embedding of $G + (e, f)$ in H respecting η_F, then

- (i) if $e \in E(F)$ then there exists $C \in \mathcal{C}$ with $f \in E(C)$ so that one end of e is in $V(C)$ (and in particular, $f \notin E(F)$)
- (ii) if $e, f \notin E(F)$ then there exists $C \in \mathcal{C}$ with $e, f \in E(C)$.

If e, f, g are distinct edges of $E(G)$ such that no member of $\mathcal{C}$ contains all of e, f, g, but one contains e, f, one contains e, g and one contains f, g, we call $\{e, f, g\}$ a *trinity.* A trinity is *diverse* if every two edges in it are diverse in $G \setminus E(F)$.

(E3) For every diverse trinity $\{e, f, g\}$ there is no homeomorphic embedding of $G + (e, f, g)$ in H respecting η_F.

(E4) Let v have degree one in F, incident with $g \in E(F)$. Let C_1, C_2 be the two members of $\mathcal{C}$ containing v. For all $e_1 \in E(C_1) \setminus E(C_2)$ and $e_2 \in E(C_2) \setminus E(C_1)$ such that e_1 and e_2 have no common end, there is no homeomorphic embedding of $G + (e_1, g) + (e_2, vy)$ in H respecting η_F, where $G + (e_1, g)$ has new vertices x, y.

(E5) Let v have degree one in F, incident with $g \in E(F)$. Let u be a neighbour of v in $G \setminus E(F)$, and let C_0 be the (unique, by 3.1) member of $\mathcal{C}$ that contains u and not v. Let u have neighbours v, w_1, w_2. Let $G' = G + (uw_1, g)$ with new vertices x_1, y_1; and let $G'' = G' + (uw_2, vy_1)$ with new vertices x_2, y_2. Let $i = 1$ or 2, and let $e = ux_i$.

(i) Let f be an edge of C_0 not incident with w_1 or w_2, and with no end adjacent to w_i. (This is vacuous unless $|E(C_0)| \geq 6$).) There is no homeomorphic embedding of $G'' + (e, f)$ in H respecting η_F.

(ii) Let $f \in E(F) \setminus \{g\}$ incident in G with a vertex $r \in V(C_0) \setminus \{w_1, w_2, u\}$. There is no homeomorphic embedding of $G'' + (e, f)$ in H respecting η_F.

Two edges of $G \setminus E(F)$ are *distant* if they are diverse in G and not twinned. Let $C \in \mathcal{C}$. We shall speak of a sequence of vertices and/or edges of C as being *in order* in C, with the natural meaning (that is, if C is a path, in order as C is traversed from one end, and if C is a circuit, in order as C is traversed from some suitable starting point).

- If e, f, g, h are distinct edges of C, in order, and e, g are distant and so are f, h, we call $G + (e, g) + (f, h)$ a *cross extension (of G, over C) of the first kind.*

- If e, uv, f are distinct edges of C, and either e, u, v, f are in order, or f, e, u, v are in order, and e, uv are distant and so are uv, f, we call $G + (e, uv) + (uy, f)$ a *cross extension of the second kind*, where $G + (e, uv)$ has new vertices x, y.

- If u_1v_1 and u_2v_2 are distant edges of C and u_1, v_1, u_2, v_2 are in order, we call $G + (u_1v_1, u_2v_2) + (xv_1, yv_2)$ a *cross extension of the third kind*, where $G + (u_1v_1, u_2v_2)$ has new vertices x, y.

(E6) For each $C \in \mathcal{C}$ and every cross extension G' of G over C of the first, second or third kinds, there is no homeomorphic embedding of G' in H respecting η_F.

(E7) Let $C \in \mathcal{C}$ be a path with $|E(C)| = 5$, with vertices v_0-$\cdots$-v_5 in order, and let v_0v_1 and v_4v_5 be twinned. Let $G_1 = G + (v_0v_1, v_4v_5)$ with new vertices x_1, y_1; let $G_2 = G_1 + (v_1v_2, y_1v_5)$ with new vertices x_2, y_2; and let $G_3 = G_2 + (v_0x_1, y_2v_5)$. There is no homeomorphic embedding of G_3 in H respecting η_F.

4 Degenerate trinities

Now (E3) was a statement about diverse trinities; our first objective is to prove the same statement about non-diverse trinities.

A trinity is a *Y-trinity* if some two edges in it (say e and f) have a common end u, the third edge in it (g say) is not incident with u, and if h denotes the third edge incident with u then there exist $C_1, C_2 \in \mathcal{C}$ with $e, g, h \in E(C_1)$ and $f, g, h \in E(C_2)$. It is *circuit-type* or *path-type* depending whether g and h have a common end or not.

4.1 *Let $(G, F, \mathcal{C})$ be a framework and let H, η_F satisfy (E1)–(E7). For every path-type Y-trinity $\{e, f, g\}$ there is no homeomorphic embedding of $G + (e, f, g)$ in H respecting η_F.*

Proof. Let u, h, C_1, C_2 be as above. Since g, h have no common end, it follows from (F6) that C_1 and C_2 are both paths with end-edges g, h. Let $e = uw_1, f = uw_2$. Now $u \notin V(F)$ since it is an internal vertex of the path C_1. Also, since h is an end-edge of C_1 and $g \in E(C_1)$, it follows that w_1 is an internal vertex of C_1, and so $w_1 \notin V(F)$, and similarly $w_2 \notin V(F)$. Suppose that η is a homeomorphic embedding of G into H respecting η_F, so that e, f, g are all η-attachments of some η-bridge B.

By 3.3, $|E(C_1)| \geq 4$, and so g is not incident with w_1, and similarly not with w_2. By (F7), at least one of C_1, C_2 has length at most four, and so we may assume that the edges of C_1 in order are h, e, w_1y, g. It follows that y is an internal vertex of C_1, and so $y \notin V(F)$. Let η' be obtained from η by rerouting w_1y along an η-path in B from $\eta(g)$ to $\eta(e)$. Then η' respects η_F, and w_1y and f are both η'-attachments of an η'-bridge. By (E2) there exists $C \in \mathcal{C}$ with $w_1y, f \in E(C)$, and hence with $e \in E(C)$ since C is an induced subgraph of $G \setminus E(F)$. But then $e, w_1y \in E(C \cap C_1)$, and $C_1 \neq C$, so e, w_1y are adjacent twinned edges, and yet their common end w_1 is not in $V(F)$, contrary to (F6). There is therefore no such η. This proves 4.1. ∎

Let $\{e, f, g\}$ be a circuit-type Y-trinity, where $e = xw_1$, $f = xw_2$ and $g = vw_3$, where $v, w_3 \neq x$ and v, x are adjacent in G. Let $h = vx$, and let w_4 be the third neighbour of v. Since g, h are twinned and share an end, 3.1 implies that $vw_4 \in E(F)$. Hence $w_4 \neq w_1, w_2$, since no member of $\mathcal{C}$ contains both v, w_4. (See Figure 4.) We wish to consider three rather similar graphs G_1, G_2, G_3 called *expansions* of the Y-trinity $\{e, f, g\}$. Let G' be obtained from G by deleting x and v, and adding five new vertices x_1, x_2, x_3, y_1, y_2 and nine new edges x_1w_1, x_2w_2, x_3w_3 and x_iy_j for $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Let G_1, G_2, G_3 be obtained from G' by deleting the edge y_2a (where a is x_1, x_2 and x_3 respectively), and adding a new vertex x_4 and three new edges x_4w_4, x_4y_2, x_4a. (See Figure 4.)

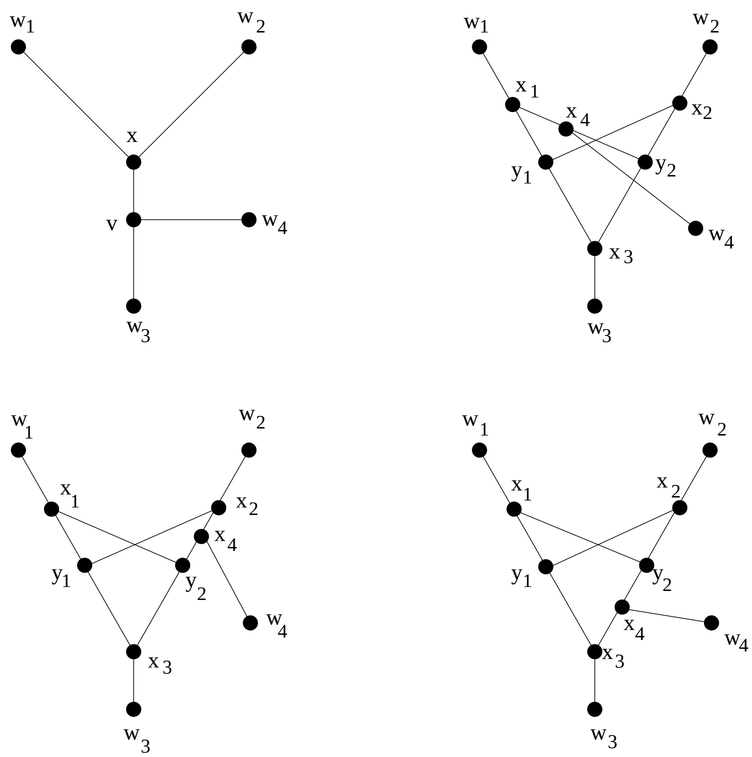


Figure 4: A circuit-type Y-trinity, and its three expansions.

4.2 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let $\{e, f, g\}$ be a circuit-type Y-trinity, and G_1, G_2, G_3 its three expansions. Then for $k = 1, 2, 3$ there is no homeomorphic embedding of G_k in H respecting η_F. In particular, there is no homeomorphic embedding of $G + \{e, f, g\}$ in H respecting η_F.*

Proof. Let $v, x, w_1, \ldots, w_4$ be as in Figure 4 and let G_1, G_2, G_3 be labelled as in Figure 4. Suppose that η is a homeomorphic embedding of some G_k in H respecting η_F. Let A be the subgraph of G_k induced on $\{x_1, x_2, x_3, x_4, y_1, y_2\}$, and B the subgraph of G_k induced on the complementary set of vertices. (B is a subgraph of G_1, G_2 and G_3.) Since H is cyclically five-connected by (E1) since there are twinned edges in G, there are five disjoint paths $P_1, \ldots, P_5$ of H from $\eta(A)$ to $\eta(B)$, each with no vertex in $\eta(A) \cup \eta(B)$ except its ends. Let P_i have ends $a_i \in V(\eta(A))$ and $b_i \in V(\eta(B))$. Let us choose k, η and $P_1, \ldots, P_5$ such that

(1) $P_1 \cup \cdots \cup P_5 \cup \eta(A) \cup \eta(x_1w_1) \cup \eta(x_2w_2) \cup \eta(x_3w_3) \cup \eta(x_4w_4)$ *is minimal.*

Now one of $a_1, \ldots, a_5$ is different from $\eta(x_1), \eta(x_2), \eta(x_3), \eta(x_4)$, say a_5. Let p be the first vertex (that is, closest to a_5) in P_5 that belongs to

$$\eta(B) \cup \eta(x_1w_1) \cup \eta(x_2w_2) \cup \eta(x_3w_3) \cup \eta(x_4w_4)$$

(this exists since $b_5 \in V(\eta(B))$), and let $P = P_5[a_5, p]$. Moreover, from the theory of network flows, it follows easily that $\{a_1, \ldots, a_4\} = \{\eta(x_1), \ldots, \eta(x_4)\}$, and we may assume that $a_i = \eta(x_i) (1 \leq i \leq 4)$.

(2) $p \in V(\eta(B))$.

Subproof. Suppose not; then $p \in V(\eta(x_iw_i))$ for some i. Let $a_5 \in V(\eta(h))$, where $h \in E(A)$. Suppose first that h is incident with x_i. By rerouting h along P we obtain a contradiction to (1), since not every (in fact, no) edge of $\eta(h)[a_5, \eta(x_i)]$ belongs to $P_1, \ldots, P_5$. Now suppose that $h = ab$ where a is adjacent to x_i. By rerouting ax_i along P to give η', we again obtain a contradiction to (1), for the same reason. Note, however, that a may be one of $a_1, \ldots, a_4$ (a_j say) and if so then the corresponding P_j has no vertex in $\eta'(A)$; but if so, let $P'_j = P_j \cup \eta(h)[a_j, a_5]$, and then $(\{P_1, P_2, P_3, P_4, P_5\} \setminus \{P_j\}) \cup \{P'_j\}$ is the desired set of disjoint paths contradicting (1).

Thus, neither end of h is adjacent to x_i. Consequently, $h \neq y_2x_4$, and h is not incident with y_1. If $i = 4$, then by replacing $\eta(x_4w_4)[\eta(x_4), p]$ by P we obtain a homeomorphic embedding of some $G_{k'}$ (where possibly $k' \neq k$), contradicting (1). Thus, $1 \leq i \leq 3$.

The only remaining possibility is that there is a four-vertex path x_i, a, b, x_j for some $j \neq i$, where $\{a, b\} = \{y_2, x_4\}$, and $h = bx_j$. If $b = x_4$, we replace $\eta(x_iy_2)$ by P, and if $a = x_4$ we replace $\eta(x_iy_2)[\eta(x_i), \eta(x_4)]$ by P, and add $\eta(x_iy_2)[\eta(x_4), \eta(y_2)]$ to P_4. In both cases we obtain a homeomorphic embedding of some $G_{k'}$ contradicting (1). This proves (2).

Hence $p \in V(\eta(h_2))$ for some $h_2 \in E(B)$. Let $a_5 \in \eta(h_1)$ where $h_1 \in E(A)$. Now we examine the possibilities for h_1 and h_2. Let $C_1, C_2 \in \mathcal{C}$ be the two members of $\mathcal{C}$ that contain v, and let $C_0 \in \mathcal{C}$ contain e and f. Thus C_1, C_2 are circuits, and v is the only vertex of F in $V(C_1 cup C_2)$.

(3) $h_2 \notin E(F)$.

Subproof. Suppose that $h_2 \in E(F)$. Then one end r of h_2 has degree one in F. Since C_1, C_2 contain no vertex of F except v, it follows that $r \notin V(C_1 \cup C_2)$. Suppose that $k = 3$. By (E2)(i) applied to $\eta(G_3 \backslash y_2)$, h_1 is not incident with y_2 and $h_1 \neq x_3x_4$. By (E2)(i) applied to $\eta(G_3 \backslash y_1)$ there exists $C \in \mathcal{C}$ containing h_2 and v, a contradiction. Thus, $k \neq 3$, and from the symmetry between G_1 and G_2 (exchanging w_1 and w_2) we may assume that $k = 1$. By (E2)(i) applied to $\eta(G_1\backslash\{x_1x_4, x_2y_2\})$ it follows that h_1 is not incident with y_2 and $h_1 \neq x_1x_4, y_1x_3$, so h_1 is x_1y_1 or x_2y_1; and from (E2)(i) applied to the same graph, $r \in V(C_0)$. But $w_1, w_2, x \notin V(F)$ (because they are in $V(C_1 \cup C_2)$) contrary to (E5)(ii). This proves (3).

(4) $k = 1$ *or* 2.

Subproof. Suppose that $k = 3$. First, suppose that h_1 is incident with y_1. By restricting η to $G_3\backslash y_1$ we obtain a homeomorphic embedding η' of G in H respecting η_F, so that e, f, g and h_2 are all η'-attachments in $E(G) \setminus E(F)$ of some η'-bridge. Since C_1, C_2 have no vertices in F except v, it follows from (E2) that $h_2 \notin E(F)$, and $h_2 \in E(C_1 \cup C_2)$. Since C_1 and C_0 are the only members of $\mathcal{C}$ containing e it follows that $h_2 \in E(C_0 \cup C_1)$, and similarly $h_2 \in E(C_0 \cup C_2)$ and $h_2 \in E(C_1 \cup C_2)$. Thus h_2 belongs to two of C_0, C_1, C_2. But $E(C_0 \cap C_1) = \{e\}$, $E(C_0 \cap C_2) = \{f\}$, $E(C_1 \cap C_2) = \{g, xv\}$, and none of these four edges is h_2 since $h_2 \in E(B)$. This proves that h_1 is not incident with y_1. If h_1 is incident with y_2 then by restricting η to $G_3\backslash y_2$ we obtain a homeomorphic embedding η' of G in H respecting η_F so that e, f, vw_4 and h_2 are all η'-attachments of some η'-bridge. Hence $h_2 \in E(C_0 \cup C_1)$ and $h_2 \in E(C_0 \cup C_2)$ by (E2)(ii), and $h_2 \in E(C_1 \cup C_2)$ by (E2)(i); and again we obtain a contradiction. Thus, $h_1 = x_3x_4$. From (E2)(ii) applied to the restriction of η to $G_3\backslash y_1, h_2 \in E(C_1 \cup C_2)$. From (E4) applied to the restriction of η to $G_3\backslash y_2$, we obtain from the paths $\eta(x_1y_2) \cup \eta(x_4y_2)$ and P that $h_2 \notin E(C_2)$ (because h_2 and e have no common end if $h_2 \in E(C_2)$); and from the paths $\eta(x_2y_2) \cup \eta(x_4y_2)$ and P that $h_2 \notin E(C_1)$, similarly, a contradiction. This proves (4).

From (4) and the symmetry between w_1 and w_2 (exchanging G_1 and G_2) we may therefore assume that $k = 1$. Suppose first that h_1 is one of $x_1x_4, x_4y_2, x_2y_2, x_3y_2$. From (E2)(i) applied to the restriction of η to $G_1\backslash\{x_1x_4, x_2y_2\}$ we deduce that $h_2 \in E(C_1 \cup C_2)$; but from (E2)(ii) and (E4) applied to the restriction of η to $G_1\backslash\{x_1x_4\}$, $h_2 \notin E(C_1)$, and from the restriction to $G_1\backslash\{x_2y_2\}$, $h_2 \notin E(C_2)$, a contradiction. Hence h_1 is incident with y_1. From (E2)(ii) applied to the restriction to $G_1\backslash\{x_3y_1, x_2y_2\}$, $h_2 \in E(C_0 \cup C_2)$; and from (E2)(ii) applied to the restriction to $G_1\backslash\{x_3y_1, x_1x_4\}$, $h_2 \in E(C_0 \cup C_1)$. Since $h_2 \notin E(C_1 \cap C_2)$ it follows that $h_2 \in E(C_0)$, and hence $h_2 \notin E(C_1 \cup C_2)$. From the same restriction, (E2)(ii) implies that $h_1 \neq x_3y_1$ (since $h_2 \notin E(C_1 \cup C_2)$).

Hence h_1 is x_1y_1 or x_2y_1. From (E4) applied to $\eta(G\backslash x_1) \cup P$, h_2 is not incident with w_1, and from (E4) applied to $\eta(G\backslash x_2) \cup P, h_2$ is not incident with w_2. By (E5), one end of h_2, say a, is adjacent to one of w_1, w_2, say w_i, and $h_1 = x_iy_1$; but then rerouting aw_i along P contradicts (E4).

There is therefore no such η, and the first statement of the theorem holds. The second statement of the theorem follows from the first, since $G + (e, f, g)$ is isomorphic to G_3 (and the isomorphism fixes F.) This proves 4.2. ∎

4.3 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let $\{e_1, e_2, e_3\}$ be a trinity so that no vertex is incident with all of e_1, e_2, e_3. Then there is no homeomorphic embedding of $G + (e_1, e_2, e_3)$ in H respecting η_F.*

Proof. For $i = 1, 2, 3$ there exists $C_i \in \mathcal{C}$ with $\{e_1, e_2, e_3\} \setminus \{e_i\} \subseteq E(C_i)$ and $e_i \notin E(C_i)$, since $\{e_1, e_2, e_3\}$ is a trinity. Suppose first that e_1, e_2 have a common end v say; and let h be the third edge incident with v. By hypothesis $h \neq e_3$. If $v \in V(F)$ then since v has degree two in C_3, C_3 is a circuit, and hence e_1 is not an end-edge of C_2; and if $v \notin V(F)$ then by (F3) either e_1 is not an end-edge of C_2, or e_2 is not an end-edge of C_1, and we may assume the first. Hence in either case e_1 is not an end-edge of C_2. Since $e_1 \in E(C_2)$ and $e_2 \notin E(C_2)$, it follows that $h \in E(C_2)$. By (F3), since $e_3 \in E(C_1 \cap C_2)$, it follows that $h \in E(C_1)$, since since v not in $V(F)$ by (F5); and so $\{e_1, e_2, e_3\}$ is a Y-trinity, contrary to 4.1 and 4.2.

Thus, no two of e_1, e_2, e_3 have a common end. Suppose that there is a homeomorphic embedding of $G+(e_1, e_2, e_3)$ in H respecting η_F. Then there is a homeomorphic embedding of G in H respecting η_F, such that e_1, e_2, e_3 are all η-attachments of the same η-bridge B say. By (E3), $\{e_1, e_2, e_3\}$ is not diverse in $G \setminus E(F)$, so we may assume that $e_1 = a_1b_1$ and $e_2 = a_2b_2$, where a_1, a_2 are adjacent in $G \setminus E(F)$. Let $a_1a_2 = e_0$.

Since $e_1, e_2 \in E(C_3)$ and C_3 is an induced subgraph of $G \setminus E(F)$, it follows that $e_0 \in E(C_3)$, and in particular $e_0 \notin E(F)$. Let a_1 have neighbours b_1, a_2, c_1 and a_2 have neighbours a_1, b_2, c_2 in G.

Since e_0 is not an end-edge of C_3, it is not an end-edge of C_1 or C_2, by (F4). Since e_0 and e_3 have no common end, and $e_3 \in E(C_1 \cap C_2)$, it follows from (F6) that $e_0 \notin E(C_1 \cap C_2)$; we assume that $e_0 \notin E(C_1)$ without loss of generality. Suppose that $e_0 \in E(C_2)$. Since $e_0, e_1 \in E(C_2 \cap C_3)$, it follows from (F6) that C_2, C_3 are both circuits, $a_1 \in V(F)$ and $a_1c_1 \in E(F)$. Hence $a_2c_2 \in E(C_2)$ (since $e_2 \notin E(C_2)$). Moreover by (F3), a_2 is incident with an edge in $C_1 \cap C_2$, since $E(C_1 \cap C_2) \neq \emptyset$ and $a_2 \in V(C_1 \cap C_2)$. Since $e_0 \notin E(C_1)$ and $e_2 \notin E(C_2)$ it follows that $a_2c_2 \in E(C_1)$. Since $E(C_1 \cap C_2)$ contains e_3 and a_2c_2 and C_2 is a circuit, it follows from (F6) that $c_2 \in V(F)$, and so $a_1, c_2 \in V(C_2 \cap F)$ contrary to (F5). This proves that $e_0 \notin E(C_2)$.

If $a_1 \in V(F)$ then $a_1c_1 \notin E(C_2)$ by (F5), and so e_1 is an end-edge of C_2. By (F4), C_3 is a path, and a_1 is an internal vertex of it, contrary to (F5). Hence $a_1 \notin V(F)$, and similarly $a_2 \notin V(F)$.

Now e_1, e_2, e_3 are all η-attachments of B. Let P be an η-path in B with ends in $\eta(e_1)$ and $\eta(e_2)$, and let η' be obtained by rerouting e_0 along P. Then η' is a homeomorphic embedding of G in H respecting η_F. Since e_3 is an η-attachment of B, it follows that e_0 and e_3 are η'-attachments of some η'-bridge. By (E2) there exists $C_4 \in \mathcal{C}$ with $e_0, e_3 \in E(C_4)$. Since $a_1 \notin V(F)$ it follows from (F6) that $e_1 \notin E(C_4)$. But from (F4) applied to C_3 and C_4, e_0 is not an end-edge of C_4. By (F3) applied to C_2 and C_4, $a_1c_1 \in E(C_2 \cap C_4)$. Since $E(C_2 \cap C_4)$ contains both a_1c_1 and e_3, it follows that e_3, a_1c_1 are twinned, and similarly so are e_3, a_2c_2, contrary to 3.4. Thus there is no such η. This proves 4.3. ∎

Next we need the following lemma.

4.4 *Let η be a homeomorphic embedding of a cubic graph G in a cyclically 4-connected cubic graph H. Let $v \in V(G)$, incident with edges e_1, e_2, e_3, and suppose that e_1, e_2, e_3 are η-attachments of some η-bridge. Then there is a homeomorphic embedding η' of G in H, such that $\eta'(u) = \eta(u)$ for all $u \in V(G) \setminus \{v\}$, and $\eta'(e) = \eta(e)$ for all $e \in E(G) \setminus \{e_1, e_2, e_3\}$, and such that for some edge $e_4 \neq e_1, e_2, e_3$ of G, e_1, e_2, e_3, e_4 are η'-attachments of some η'-bridge.*

Proof. For $1 \leq i \leq 3$, let e_i have ends v and v_i. Let $G' = G + (e_1, e_2, e_3)$, with new vertices x_1, x_2, x_3, w. By hypothesis, there is a homeomorphic embedding η' of G' in H such that $\eta'(u) = \eta(u)$ for all $u \in V(G) \setminus \{v\}$, and $\eta'(e) = \eta(e)$ for all $e \in E(G) \setminus \{e_1, e_2, e_3\}$. Choose η' so that

$$\eta'(v_1x_1) \cup \eta'(v_2x_2) \cup \eta'(v_3x_3)$$

is minimal. Since H is cyclically four-connected, there is an η'-path with one end in

$$\bigcup (V(\eta'(vx_i)) \cup V(\eta'(wx_i)) : 1 \leq i \leq 3)$$

and the other end, t, in

$$V(\eta(G\backslash v) \cup \eta'(v_1x_1) \cup \eta'(v_2x_2) \cup \eta'(v_3x_3)).$$

From the choice of η' it follows that t belongs to none of $\eta'(v_1x_1), \eta'(v_2x_2), \eta'(v_3x_3)$, and so it belongs to $\eta'(e_4) = \eta(e_4)$ for some $e_4 \in E(G\backslash v)$. This proves 4.4. ∎

4.5 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let $\{e_1, e_2, e_3\}$ be a trinity. There is no homeomorphic embedding of $G + (e_1, e_2, e_3)$ in H respecting η_F.*

Proof. By 4.3 we may assume that $v \in V(G)$ is incident with e_1, e_2 and e_3. Suppose η is a homeomorphic embedding of $G + (e_1, e_2, e_3)$ in H respecting η_F. By 4.4 there is an edge $e_4 \neq e_1, e_2, e_3$ of G so that there are homeomorphic embeddings of each of $G + (e_2, e_3, e_4)$, $G + (e_1, e_3, e_4)$, $G + (e_1, e_2, e_4)$ in H respecting η_F. First suppose that $e_4 \notin E(F)$. Since no vertex is incident with all of e_2, e_3, e_4, it follows from 4.3 that $\{e_2, e_3, e_4\}$ is not a trinity; and so from (E2) there exists $C_1 \in \mathcal{C}$ with $e_2, e_3, e_4 \in E(C_1)$. Similarly there exist $C_2, C_3 \in \mathcal{C}$ with $e_1, e_3, e_4 \in E(C_2)$ and $e_1, e_2, e_4 \in E(C_3)$. Since $\{e_1, e_2, e_3\}$ is a trinity, $e_i \notin E(C_i)(1 \leq i \leq 3)$, and so C_1, C_2, C_3 are all distinct.

Now if e_4 is not the end-edge of any path in $\mathcal{C}$, then since $C_2 \cap C_3$ contains e_1 and e_4 it follows from (F6) that e_1 and e_4 have a common end, and similarly so do e_i and e_4 for $i = 1, 2, 3$, which is impossible. Hence e_4 is an end-edge of some path in $\mathcal{C}$. By (F4) C_1, C_2 and C_3 are all paths. Since $e_3, e_4 \in E(C_1 \cap C_2), C_1$ has end-edges e_3 and e_4; and since $e_2, e_4 \in E(C_1 \cap C_3), C_1$ has end-edges e_2 and e_4, a contradiction. This proves that $e_4 \in E(F)$.

Since η respects η_F, it follows (since $\eta(e_4)$ has an internal vertex) that e_4 has an end x with degree one in F. Let $C_1, C_2 \in \mathcal{C}$ be the only members of $\mathcal{C}$ containing x. By (E2), $e_1, e_2, e_3 \in E(C_1 \cup C_2)$; but this is impossible since e_1, e_2, e_3 have a common end v. This proves 4.5. ∎

4.6 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let η be a homeomorphic embedding of G in H respecting η_F. For every η-bridge B there exists $C \in \mathcal{C}$ such that for every η-attachment e of B, if $e \notin E(F)$ then $e \in E(C)$, and if $e \in E(F)$ then e has an end in $V(C)$.*

Proof. By (E2)(i), at most one η-attachment of B is in $E(F)$. Suppose there is such an η-attachment $g \in E(F)$. Hence $\eta(g)$ has an internal vertex, and so $\eta(g) \neq \eta_0(g)$, by (E1). It follows that some end v of g has degree one in F, since η respects η_F. Let C_1, C_2 be the two members of $\mathcal{C}$ containing v. By (E2)(i), every η-attachment of B except g belongs to $E(C_1)$ or to $E(C_2)$. Suppose that one, e_1 say, is in $E(C_1) \setminus E(C_2)$, and another, e_2 say, is in $E(C_2) \setminus E(C_1)$. By (E4), e_1 and e_2 have a common end; but this contradicts 4.2 (applied to the third type of expansion). Hence one of $E(C_1), E(C_2)$ contains every η-attachment except g, as required.

We may assume, therefore, that $Z \subseteq E(G) \setminus E(F)$, where Z is the set of all η-attachments of B. Suppose, for a contradiction, that there is no $C \in \mathcal{C}$ with $Z \subseteq E(C)$, and choose $X \subseteq Z$ minimal so that there is no $C \in \mathcal{C}$ with $X \subseteq E(C)$. By (F2), $|X| \geq 2$; by (E2)(ii), $|X| \neq 2$; and by 4.5, $|X| \neq 3$. Hence $|X| \geq 4$. Let $X = \{e_1, \ldots, e_k\}$ say, where $k \geq 4$. For each $i \in \{1, \ldots, k\}$, there exists $C \in \mathcal{C}$ including $X \setminus \{x_i\}$, from the minimality of X. All these members of $\mathcal{C}$ are different, and so every two members of X are twinned, contrary to 3.4. This proves 4.6. ∎

5 Crossings on a region

With $(G, F, \mathcal{C})$ and H, η_F as usual, let G' be a cubic graph with F a subgraph of G'. A homeomorphic embedding η of G' of H is said to *extend* η_F if $\eta(e) = \eta_F(e)$ for all $e \in E(F)$ and $\eta(v) = \eta_F(v)$ for all $v \in V(F)$. (Thus, if η extends η_F then it respects η_F.)

Let η extend η_F, and let B be an η-bridge. Since η extends η_F, it follows that no η-attachment of B is in $E(F)$, and so by 4.6, there exists $C \in \mathcal{C}$ such that every η-attachment of B belongs to C. If C is unique, we say that B *sits on* C.

Our objective in this section is to show that if η extends η_F, then for every $C \in \mathcal{C}$ all the bridges that sit on C can be simultaneously drawn within the "region" that C bounds. There may be some bridges that sit on no member of $\mathcal{C}$, but we shall worry about them later.

Let C be a path or circuit in a graph J. We say paths P, Q of J *cross* with respect to C, if P, Q are disjoint, and P has distinct ends $p_1, p_2 \in V(C)$, and Q has distinct ends $q_1, q_2 \in V(C)$, and no other vertex of P or Q belongs to C, and these ends can be numbered such that either p_1, q_1, p_2, q_2 are in order in C, or q_1, p_1, q_2, p_2 are in order in C. We say that J is *C-planar* if J can be drawn in a closed disc Δ so that every vertex and edge of C is drawn in the boundary of Δ. We shall prove:

5.1 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let η be a homeomorphic embedding of G in H that extends η_F, let $C \in \mathcal{C}$, and let $\mathcal{A}$ be a set of η-bridges that sit on C. Let $J = \eta(C) \cup \bigcup(B : B \in \mathcal{A})$. Then J is $\eta(C)$-planar.*

5.1 is a consequence of the following.

5.2 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Let η be a homeomorphic embedding of G in H that extends η_F, and let $C \in \mathcal{C}$. Let P, Q be η-paths that cross with respect to $\eta(C)$. Then for one of P, Q, the η-bridge that contains it does not sit on C.*

Proof of 5.1, assuming 5.2.

Suppose that $X, Y \subseteq V(J)$ with $X \cup Y = V(J)$ and $V(C) \subseteq Y$, so that $|X \setminus Y| \geq 2$ and no edge of J has one end in $X \setminus Y$ and the other in $Y \setminus X$. We claim that $|X \cap Y| \geq 4$. For let $Y' = Y \cup (V(H) \setminus X)$; then no edge of H has one end in $X \setminus Y'$ and the other in $Y' \setminus X$, and $X \cup Y' = V(H)$, and $|X \setminus Y'| \geq 2$, and so X and Y' both includes the vertex set of a circuit of H. Since H is cyclically four-connected, it follows that $|X \cap Y'| \geq 4$, and so $|X \cap Y| \geq 4$ as claimed.

From this and theorems 2.3 and 2.4 of [2], it follows, assuming for a contradiction that J is not $\eta(C)$-planar, that there are η-paths P, Q in J that cross with respect to $\eta(C)$. By 5.2 the η-bridge containing one of P, Q does not sit on C and hence does not belong to $\mathcal{A}$, a contradiction. This proves 5.1. ∎

Proof of 5.2.

We remark, first, that

(1) *If B is an η-bridge that sits on C, and $e \in E(C)$ is an η-attachment of B, then there is an η-attachment $g \in E(C)$ such that $g \neq e$ and g is not twinned with e.*

Subproof. By 3.1 it follows that B has at least two η-attachments. Suppose that every η-attachment different from e is twinned with e; then by 3.4 there is only one other, say f, and e, f are twinned, and therefore there exists $C' \neq C$ in $\mathcal{C}$ containing all η-attachments of B, contradicting that iB sits on C. This proves (1).

For $e, f \in E(C)$, let

$$\epsilon(e,f) = \begin{cases} 3 & \text{if } e = f, \\ 2 & \text{if } e \neq f, \text{ and } e,f \text{ are twinned} \\ 0 & \text{if } e \neq f, \text{ and } e,f \text{ are not twinned.} \end{cases}$$

Let P have ends p_1, p_2, and let Q have ends q_1, q_2; and let B_1, B_2 be the η-bridges containing P, Q respectively. Let $p_i \in V(\eta(e_i))$ and $q_i \in V(\eta(f_i))$ for $i = 1, 2$, and let $N = \epsilon(e_1, e_2) + \epsilon(f_1, f_2)$. We prove by induction on N that one of B_1, B_2 does not sit on C. We may assume they both sit on C, for a contradiction.

(2) *Either e_1, e_2 are different and not twinned, or f_1, f_2 are different and not twinned.*

Subproof. Suppose that e_1 and e_2 are equal or twinned, and so are f_1, f_2. We claim that $|\{e_1, e_2, f_1, f_2\}| \leq 2$, and if this set has two members then they are twinned. For suppose that $e_1 = e_2$. Since P, Q cross, it follows that one of $f_1, f_2 = e_1$, say $f_1 = e_1$; and since either $f_2 = f_1$ or f is twinned with f_1, the claim follows. So we may assume that e_1, e_2 are twinned, and similarly so are f_1, f_2. But by (F5) and (F6), only one pair of edges of C are twinned, and so again the claim holds.

Since B_1 sits on C, by (1) it has an η-attachment $g \neq e_1$ that is not twinned with e_1; and so $g \neq e_1, e_2, f_1, f_2$. Take a minimal path R in B_1 between $V(P \cup Q)$ and $V(\eta(g))$. Then by replacing P by the union of R and a subpath of $P \cup Q$ from an end of R to an appropriate end of P or Q, we contradict the inductive hypothesis on N. This proves (2).

(3) *$e_1 \neq e_2$ and $f_1 \neq f_2$.*

Subproof. Suppose that $e_1 = e_2$, say. Since P, Q cross, one of f_1, f_2 equals e_1, say $f_1 = e_1 = e_2$; and by (2), $f_2 \neq f_1$, and f_1, f_2 are not twinned. By (1), B_1 has an η-attachment $g \in E(C)$ not twinned with e_1. Hence there is a minimal path R of B from $V(P)$ to $V(Q) \cup \eta(g)$. If it meets $\eta(g)$, we contradict the inductive hypothesis as before, so we assume R has one end in $V(P)$ and the other in $V(Q)$.

Let $f_1 = uv$, and let $G' = G + (f_1, f_2)$ with new vertices x, y. By adding Q to $\eta(G)$ we see that there is a homeomorphic embedding η'' of G' in H extending η_F so that ux, vx and xy are all η''-attachments of some η''-bridge (including $P \cup R$). From 4.4, we may choose η'' extending η_F so that ux, vx, xy and some fourth edge g are all η''-attachments of some η''-bridge. In other words, we may choose a homeomorphic embedding η' of G in H extending η_F such that there exist

- an η'-path P' with ends p'_1, p'_2 in $V(\eta'(f_1))$,
- an η'-path Q' with ends q'_1, q'_2 disjoint from P', where q'_1 lies in $\eta'(f_1)$ between p'_1 and p'_2, and $q'_2 \in V(\eta'(f_2))$,
- a path R' with one end in P', the other end in Q', and with no other vertex or edge in $\eta'(G) \cup P' \cup Q'$, and

- a path S' with one end in $P' \cup R'$, the other end in $\eta'(g)$ where $g \neq f_1$, and with no other vertex or edge in $\eta'(G) \cup P' \cup Q' \cup R'$.

Let B' be the η'-bridge containing $P' \cup Q' \cup R' \cup S'$. By 4.6, there exists $C' \in \mathcal{C}$ so that all η'-attachments of B' are in $E(C')$. Now $f_1 \neq f_2$ and they are not twinned, so $C' = C$, and hence B' sits on C. Let T be an η'-path in $P' \cup R' \cup S'$ with one end in $\eta'(f_1)$ and the other in $\eta'(g)$, chosen such that Q', T cross with respect to $\eta'(C)$. Then both Q', T are contained in B', and yet B' sits on C, and $\epsilon(f_1, g) < \epsilon(f_1, f_1)$, contrary to the inductive hypothesis. This proves (3).

(4) *e_1, e_2 are not twinned, and f_1, f_2 are not twinned.*

Subproof. Suppose that f_1, f_2 are twinned, say. Let $f_1 = v_1x_1$ and $f_2 = v_2x_2$ where either $v_1 = v_2 \in V(F)$ and C is a circuit, or C is a path with ends v_1, v_2. By (1), there is an η-attachment of B_2 different from f_1, f_2; and so there is a minimal η-path R in B_2 from $V(Q)$ to $V(P) \cup V(\eta(C \backslash \{f_1, f_2\}))$. From the inductive hypothesis, R does not meet $\eta(C \backslash \{f_1, f_2\})$, and so it meets P. Let R have ends $r_1 \in V(P)$ and $r_2 \in V(Q)$, and for $i = 1, 2$, let $P_i = P[p_i, r_1]$ and $Q_i = Q[q_i, r_2]$.

Now for $i = 1, 2$, $x_i \notin V(F)$ by (F5) (since if C is a circuit then $v_1 \in V(F)$ by (F6)). For $i = 1, 2$, let g_i be the edge of G not in C_i incident with x_i, and let h_i be the edge of C different from f_i that is incident with x_i.

Now since either C is a path and f_1, f_2 are end-edges of C, or C is a circuit and f_1, f_2 have a common end, and since P, Q cross, we may assume that $e_1 = f_1$, and p_1 lies in $\eta(f_1)$ between q_1 and $\eta(v_1)$. It follows that $e_2 \neq f_1, f_2$ by (2).

Suppose first that either $e_2 = h_1$ or x_1 is adjacent to an end of e_2. By rerouting h_1 along P, we obtain a homeomorphic embedding η' of G in H extending η_F, so that g_1, h_1 and f_2 are all η'-attachments of some η'-bridge (containing $Q \cup R$). Since no member of $\mathcal{C}$ contains all of g_1, h_1 and f_2, this contradicts 4.6. Hence $e_2 \neq h_1$ and x_1 is not adjacent to any end of e_2.

By (F6), $|V(C)| \leq 6$, so either $e_2 = h_2$, or C is a circuit and x_2 is adjacent to an end of e_2. Suppose first that C is a path; so $e_2 = h_2$. By rerouting h_2 along $P_2 \cup R \cup Q_2$ and adding P_1 and Q_1, we obtain a homeomorphic embedding (in H, respecting η_F) of a cross extension of G over C of the third kind, contrary to (E6). Thus C is a circuit, and so $g_1 \in E(F)$, and therefore $\{f_1, g_2, h_2\}$ is a circuit-type Y-trinity. But then by rerouting h_2 along $P_2 \cup R \cup Q_2$ and adding P_1 and Q_1 we contradict 4.2 (the first or second type of expansion). This proves (4).

(5) *e_1, e_2 have no common end, and f_1, f_2 have no common end.*

Subproof. Suppose that e_1, e_2 have a common end, v say. Since e_1, e_2 are not twinned by (4), it follows from 3.1 that v has degree three in $G \setminus E(F)$; and so by (F5), $v \notin V(F)$. Since P, Q cross, we may assume that $f_1 = e_1$ and $p_1, q_1, \eta(v)$ are in order in $\eta(e_1)$. Let f, e_1, e_2 be the three edges of G incident with v. Let η' be obtained from η by rerouting e_1 along P. Then η' is a homeomorphic embedding of G in H extending η_F, and f, f_1 and f_2 are η'-attachments in $E(G) \setminus E(F)$ of the η'-bridge containing Q. From 4.6, there exists $C' \in \mathcal{C}$ with $f, f_1, f_2 \in E(C')$. Since $f \notin E(C)$ it follows that $C' \neq C$, and so f_1, f_2 are twinned, contrary to (4). This proves (5).

Thus e_1, e_2 have no common end, and nor do f_1, f_2. By (E6), we may assume that one end of e_1

is adjacent to one end of e_2. Since P, Q cross, we may therefore assume that for some edge $g = uv$ of C, u is an end of e_1, v is an end of e_2, $f_1 \in \{e_1, g\}$, and if $f_1 = e_1$ then $p_1, q_1, \eta(u)$ are in order in $\eta(e_1)$. Let u be incident with g, e_1, g_1 and v with g, e_2, g_2, and let η' be obtained by rerouting g along P. Then η' is a homeomorphic embedding of G in H respecting η_F. (It may not extend η_F, since one of g_1, g_2 may be in $E(F)$.) Let $X = \{g_1, f_2\}$ if $f_1 = e_1$, and let $X = \{g_1, g_2, f_2\}$ if $f_1 = g$. Then every member of X is an η'-attachment of the η'-bridge containing Q. By 4.6 there exists $C' \in \mathcal{C}$ so that for each $h \in X$, $h \in E(C')$ if $h \notin E(F)$ and h has an end in $V(C')$ if $h \in E(F)$.

Suppose first that $f_1 = g$. Hence $f_2 \neq e_1, g, e_2$. Now (F5) implies that one of $u, v \notin V(F)$, and from the symmetry we may assume that $v \notin V(F)$. Since $g_1, g_2 \in X$, it follows that $g_2 \in E(C')$, and in particular $C' \neq C$. Since f_1, f_2 are not twinned, it follows that $g \notin E(C')$. Since C' is an induced subgraph of $G \setminus E(F)$, we deduce that $u \notin V(C')$, and so $g_1 \notin E(C')$, and therefore $g_1 \in E(F)$. Since $v \in V(C \cap C') \setminus V(F)$, (F3) implies that $e_2 \in E(C')$. Thus e_2, f_2 are twinned. But this contradicts (F6).

We deduce that $f_1 \neq g$, and so $f_1 = e_1$. Suppose next that $u \notin V(F)$. Thus $g_1 \notin E(F)$, and so $g_1 \in E(C')$. Hence $C' \neq C$, and so $e_1 \notin E(C')$ since f_1, f_2 are not twinned. By (F3) we deduce that $g \in E(C')$, and so f_2, g are twinned. Since g is not an end-edge of C, (F6) implies that $f_2 = e_2$ and $g_2 \in E(F)$. But then $\{e_1, g_1, e_2\}$ is a Y-type trinity, and adding P and Q provides a homeomorphic embedding (in H, respecting η_F) of an expansion of this trinity, contrary to 4.2.

This proves that $u \in V(F)$. Consequently $v \notin V(F)$, and it follows (by exchanging P, Q, and exchanging e_1, e_2) that $f_2 \neq e_2$. Since C contains e_1, g, it follows that u is not an end of C, and so by (F5), $g_1 \in E(F)$. By (F2) there exists $C_2 \in \mathcal{C}$ containing g, g_2, since $v \notin V(F)$. Since $g_1 \in E(F)$, we deduce that $e_1 \in E(C_2)$. Since f_1, f_2 are not twinned, it follows that $f_2 \notin E(C_2)$. Thus $g_2 \in E(C_2) \setminus E(C)$, and $f_2 \in E(C) \setminus E(C_2)$, and f_2, g_2 have no common end, since $f_2 \neq e_2$. But rerouting g along P gives a homeomorphic embedding of G in H respecting η_F, and adding $\eta(g)$ and Q to it contradict (E4). This proves 5.2. ∎

6 The bridges between twins

To apply these results about frameworks, we have to choose a homeomorphic embedding η of G in H, and there is some freedom in how we do so. If we choose it carefully we can make several problems disappear simultaneously. The most important consideration is to ensure that each η-bridge has at least two η-attachments, but that is rather easy. With more care, we can also discourage η-bridges from having η-attachments in certain difficult places. To do so, we proceed as follows.

Let $(G, F, \mathcal{C})$ be a framework, and let η be a homeomorphic embedding of G in H extending η_F, as usual. An edge e of G is a *twin* if there exists f such that e, f are twinned. (Thus, stating that "e, f are twins" does not imply that they are twinned with each other.) An edge $e \in E(G) \setminus E(F)$ is

- *central* if it does not belong to any path in $\mathcal{C}$ and is not a twin;
- *peripheral* if there is a path $C \in \mathcal{C}$ such that e is an edge of C but not an end-edge of C (and hence e is not a twin)
- *critical* if either e is a twin or e is an end-edge of some path in $\mathcal{C}$.

By (F4), no edge is both peripheral and critical, so every edge of $E(G) \setminus E(F)$ is of exactly one of these three kinds.

An edge $f \in E(H)$ is said to *η-attach* to $e \in E(G)$ if there is a path P of H with no internal vertex in $V(\eta(G))$ with $f \in E(P)$ and with one end a vertex of $\eta(e)$. (Thus f η-attaches to e if and only if either $f \in E(\eta(e))$ or f belongs to an η-bridge for which e is an η-attachment.) Let

- $L_1(\eta)$ be the set of edges in $E(H)$ that η-attach to some central edge of G;
- $L_2(\eta)$ be the set of edges in $E(H)$ that η-attach to an edge of G which is either peripheral or central;
- $L_3(\eta)$ be the set of edges in $E(H)$ that attach to two edges of G that are not twinned; and
- $L_4(\eta)$ be the set of edges in $E(H)$ that attach to two edges of G.

We say that η is *optimal* if it is chosen (among all homeometric embeddings of G in H extending η_F) with the four-tuple of cardinalities of these sets lexicographically maximum; that is, for every homeomorphic embedding η' extending η_F, some $j \in \{1, \ldots, 5\}$, $|L_i(\eta)| = |L_i(\eta)|$ for $1 \leq i < j$, and $|L_j(\eta)| > |L_j(\eta)|$ if $j \leq 4$. In this section we study the properties of optimal embeddings.

6.1 *Let η be an optimal homeomorphic embedding of G in H extending η_F. Then every η-bridge has at least two η-attachments.*

Proof. Let $e \in E(G) \setminus E(F)$. Let us say an η-bridge is *singular* if e is its only η-attachment, and *nonsingular* otherwise. Suppose that there is a singular η-bridge. Let $e = uv$, let $p_1, \ldots, p_r$ be the set of vertices of $\eta(e)$ that belong to nonsingular η-bridges, and let $p_0 = \eta(u)$ and $p_{r+1} = \eta(v)$, numbered so that $p_0, p_1, \ldots, p_{r+1}$ are in order in $\eta(e)$. For $0 \leq i \leq r$ let $P_i = \eta(e)[p_i, p_{i+1}]$. Choose j with $0 \leq j \leq t$ such that some singular η-bridge contains a vertex of P_j. Since H is 3-connected, there is an η-bridge B containing a vertex b of the interior of P_j and containing a vertex a of $\eta(G)$ not in P_j. From the definition of $p_1, \ldots, p_r$, it follows that B is singular. Hence there exists $i \neq j$ with $0 \leq i \leq r$ such that b belongs to P_i, and from the symmetry we may assume that $i < j$. Let P be an η-path in B between a, b. Let η' be obtained from η by rerouting e along P. For every edge f of $E(H)$, every η-attachment of f is also an η'-attachment. Consequently $L_i(\eta) \subseteq L_i(\eta')$ for $1 \leq i \leq 4$. But the edge of P_j incident with p_j belongs to $L_4(\eta') \setminus L_4(\eta)$, contrary to the optimality of η. This proves 6.1. ▮

6.2 *Let η be an optimal homeomorphic embedding of G in H extending η_F. Let $C \in \mathcal{C}$ be a path, and suppose that B is an η-bridge and all its η-attachments are edges of C. Then its η-attachments are pairwise diverse in C.*

Proof. We claim first

(1) *If e, f are edges of C with a common end v, and g is the third edge of G incident with v, then $v \notin V(F)$, and either g is central, or g is peripheral and one of e, f is an end-edge of C.*

Subproof. Certainly $v \notin V(F)$ by (F5). If g does not belong to any path of $\mathcal{C}$ then it is not a twin by (F6), and so it is central. Thus we may assume that there is a path $C' \in \mathcal{C}$ containing g. By (F4), C' contains one of e, f, say e, and e is an end-edge of both C, C'. Thus e is critical; and we may therefore assume that g is critical, for otherwise our claim holds. Since g belongs to C', it follows from (F4) that g is an end-edge of some path in $\mathcal{C}$, and hence (from (F4) again), that g is an end-edge of C'. But then C' has length two, contrary to (F1). This proves (1).

(2) *No two η-attachments of B in C have a common end.*

Subproof. Suppose that e, f are η-attachments of B, and they have a common end v. Let g be the third edge of G incident with v. Choose a path P in B from a vertex a of $\eta(e)$ to a vertex b of $\eta(f)$. Let η' be obtained from η by rerouting f along P. Then η' is a homeomorphic embedding of G in H extending η_F (note that $g \notin E(F)$ since $v \notin V(F)$ by (1)). Moreover, since no η-attachment of B is central, it follows that $L_1(\eta) \subseteq L_1(\eta')$, and therefore equality holds. In particular, the edge of $\eta(e)$ incident with $\eta(v)$ therefore does not belong to $L_1(\eta')$, and so g is not central. We deduce that g is peripheral and e is an end-edge of C. Thus f is peripheral, and it follows that $L_2(\eta) \subseteq L_2(\eta')$, and therefore equality holds. But the edge of $\eta(e)$ incident with $\eta(v)$ belongs to $L_2(\eta')$, a contradiction. This proves (2).

To complete the proof, suppose that some two η-attachments e, f of B in C are not diverse in C. Then by (2), there are consecutive vertices u, v, w, x of C, such that $e = uv$ and $f = wx$. Let the third edge of G at v be g and at w be h. Choose a path P in B from a vertex a of $\eta(e)$ to a vertex b of $\eta(f)$. Let η' be obtained from η by rerouting vw along P. Then η' is a homeomorphic embedding of G in H extending η_F. Since no η-attachment of B is central, it follows that $L_1(\eta) \subseteq L_1(\eta')$, and therefore equality holds. In particular, the edge of $\eta(e)$ incident with $\eta(v)$ does not belong to $L_1(\eta')$, and so g is not central. From (1), it follows that g is peripheral and e is an end-edge of C. Similarly h is peripheral and f is an end-edge of C. Hence $L_2(\eta) \subseteq L_2(\eta')$, and therefore equality holds. But the edge of $\eta(e)$ incident with $\eta(v)$ belongs to $L_2(\eta')$, a contradiction. This proves 6.2. ∎

If e, f are twinned edges of G, we denote by $\mathcal{A}(e, f)$ the set of all η-bridges that have no attachments different from e, f. Thus every bridge belongs to $\mathcal{A}(C)$ for some C or to $\mathcal{A}(e, f)$ for some e, f, and to only one (except that $\mathcal{A}(e, f) = \mathcal{A}(f, e)$). The next four theorems are all about a pair of twinned edges e, f, and it is convenient first to set up some notation. Thus, let e, f be twinned edges of G. Let there be r vertices $p_1, \ldots, p_r$ of $\eta(e)$ that belong to an η-bridge with an η-attachment different from e and f, and let $\eta(e)$ have ends p_0 and p_{r+1}, numbered so that $p_0, \ldots, p_{r+1}$ are in order in $\eta(e)$. For $0 \leq i \leq r$, let $P_i = \eta(e)[p_i, p_{i+1}]$. Let $q_0, \ldots, q_{s+1} \in V(\eta(f))$ and $Q_0, \ldots, Q_s$ be defined similarly.

6.3 *Let η be an optimal homeomorphic embedding of G in H extending η_F, and let e, f be twinned edges of G. With notation as above, for every $B \in \mathcal{A}(e, f)$ there exist i and j with $0 \leq i \leq r$ and $0 \leq j \leq s$ such that $B \cap \eta(e) \subseteq P_i$ and $B \cap \eta(f) \subseteq Q_j$.*

Proof. Suppose that some member B of $\mathcal{A}(e, f)$ meets both P_i and P_j, where $0 \leq i < j \leq r$. Let P be an η-path in B between some $a \in V(P_i)$ and some $b \in V(P_j)$. Let η' be obtained from η by rerouting e along P. Since no η-attachment of B is central or peripheral, and no edge of B is in $L_3(\eta)$, it follows that $L_i(\eta) \subseteq L_i(\eta')$ for $i = 1, 2, 3$, and so equality holds in all three. Let B' be an η-bridge containing p_i; then B' has an η-attachment different from e, f, say g. Consequently e, g are not twinned, and in particular, the edge of P_j incident with p_j is in $L_3(\eta')$, a contradiction. This proves 6.3. ∎

6.4 *Let η be an optimal homeomorphic embedding of G in H extending η_F, and let e, f be twinned edges of G. Suppose that e, f have a common end v, and let $e = uv$ and $f = vw$. Then $\mathcal{A}(e, f)$ can be numbered as $\{B_1, \ldots, B_k\}$, such that*

- *B_i has only one edge $c_i d_i$ for $1 \leq i \leq k$;*
- *$\eta(u), c_1, \ldots, c_k, \eta(v)$ are in order in $\eta(e)$, and $\eta(w), d_1, \ldots, d_k, \eta(v)$ are in order in $\eta(f)$; and*
- *for $1 \leq i < k$, one of $\eta(e)[c_i, c_{i+1}]$, $\eta(f)[d_i, d_{i+1}]$ contains a vertex of some η-bridge not in $\mathcal{A}(e, f)$.*

Proof. Using the notation established earlier, we may assume that $\eta(v) = p_0 = q_0$.

(1) *Suppose that M, N are disjoint η-paths, from m to m' and from n to n' respectively, such that*

- *$\eta(u), m, n, \eta(v), m', n', \eta(w)$ are in order in the path $\eta(e) \cup \eta(f)$; and*
- *no edge of $M \cup N$ belongs to $L_2(\eta)$.*

Then there exist i, j with $0 \leq i \leq r$ and $0 \leq j \leq s$ such that m, n belongs to P_i and m', n' belong to P_j.

Subproof. Let m be in P_i and n be in P_h where $0 \leq h < i \leq r$. Let

$$\eta'(e) = \eta(f)[\eta(u), m] \cup M \cup \eta(f)[m', \eta(v)].$$

and

$$\eta'(f) = \eta(e)[\eta(v), n] \cup N \cup \eta(f)[n', \eta(w)].$$

Then η' is a homeomorphic embedding of G in H extending η_F. Since no edge of the η-bridges containing M or N belong to $L_1(\eta)$ or to $L_2(\eta)$, and e, f are critical, it follows that $L_i(\eta) \subseteq L_i(\eta')$ for $i = 1, 2$, and so equality holds in both. Let B be the η-bridge containing p_i. Then there is an η-attachment $g \neq e, f$ of B. Choose $C \in \mathcal{C}$ containing e, g (this is possible by 4.6). From (F6), C is a circuit, and so g is not critical from (F5). Hence g is either central or peripheral, and so the edges of $\eta(e)$ incident with p_i belongs to $L_2(\eta')$, a contradiction. This proves (1).

To complete the proof, for $0 \leq i \leq r$ and $0 \leq j \leq s$ let $\mathcal{A}_{ij}$ be the set of all $B \in \mathcal{A}(e, f)$ with $B \cap \eta(e) \subseteq P_i$ and $B \cap \eta(f) \subseteq Q_j$. From (1), $\mathcal{A}(e, f) = \bigcup \mathcal{A}_{ij}$. For each i, j let J_{ij} be the union of all members of $\mathcal{A}_{ij}$. Suppose that some $|E(J_{ij})| \geq 2$. Since H is cyclically five-connected by (E1), we may assume (by exchanging e and f if necessary) that there are b_1, b', b_2 in P_i, in order, so that b_1 and b_2 both belong to J_{ij}, and b' belongs to some η-bridge $B' \notin \mathcal{A}_{ij}$. Since $b' \neq p_1, \ldots, p_r$ it follows that $B' \in \mathcal{A}(e, f)$, and so $B' \in \mathcal{A}_{ij'}$, for some $j' \neq j$. In particular, J_{ij} and $J_{ij'}$ are disjoint. By 6.1 it follows that there is a path M in J_{ij} and a path N in $J_{ij'}$ violating (1) (possibly with M, N exchanged). This proves that each J_{ij} has at most one edge, and in particular from 6.3, each η-bridge in $\mathcal{A}(e, f)$ has only one edge. The result follows from (2) applied to the paths of length one formed by these η-bridges. This proves 6.4. ∎

6.5 *Let η be an optimal homeomorphic embedding of G in H extending η_F, and let e, f be twinned edges of G. Suppose that e, f are disjoint, and there is no $C \in \mathcal{C}$ of length five with end-edges e, f. Then*

- *there is at most one η-bridge in $\mathcal{A}(e,f)$, and any such η-bridge has only one edge;*
- *no other η-bridge contains any vertex of $\eta(e) \cup \eta(f)$; and*
- *$\mathcal{A}(C) = \emptyset$ for every member of $\mathcal{C}$ containing e or f.*

Proof. Now there is a path in $\mathcal{C}$ with end-edges e, f, and so every member C of $\mathcal{C}$ containing e or f is a path, by (F4). Moreover, if $e, f \in E(C)$ then C has length at most four by hypothesis and (F6), and C has end-edges e, f, and therefore every member of $\mathcal{A}(C)$ has an η-attachment some edge of C different from e, f. By 6.2, this implies that $\mathcal{A}(C) = \emptyset$. On the other hand, if $C \in \mathcal{C}$ contains just one of e, f then C has length three by (F6), and again $\mathcal{A}(C) = \emptyset$ by 6.2. This proves the third assertion. Consequently, $r = s = 0$ (in our previous notation). Since H is cyclically five-connected by (E1), it follows that the union of all η-bridges in $\mathcal{A}(e,f)$ and the paths $\eta(e), \eta(f)$ contains no circuit; and so there is at most one η-bridge in $\mathcal{A}(e,f)$ and any such η-bridge has only one edge. This proves 6.5. ∎

6.6 *Let η be an optimal homeomorphic embedding of G in H extending η_F, and let e, f be twinned edges of G. Suppose that e, f are disjoint, and there exists $C \in \mathcal{C}$ of length five with end-edges e, f. Possibly after exchanging e, f, let C have vertices v_0-v_1-$\cdots$-v_5 in order, where $e = v_0v_1$ and $f = v_4v_5$. Then, $\mathcal{A}(e,f)$ can be numbered as $\{B_1, \dots, B_k\}$ such that*

- *$\mathcal{A}(C')$ is empty for every $C' \neq C$ in $\mathcal{C}$ containing e or f;*
- *for each $B \in \mathcal{A}(C)$, its only η-attachments are v_1v_2 and v_4v_5;*
- *B_i has exactly one edge c_id_i for $1 \leq i \leq k$, where $c_i \in V(\eta(e))$ and $d_i \in V(\eta(f))$;*
- *$\eta(v_0), c_1, \dots, c_k, \eta(v_1)$ are in order in $\eta(e)$, and $\eta(v_4), d_1, \dots, d_k, \eta(v_5)$ are in order in $\eta(f)$.*

Proof. Let $C \in \mathcal{C}$ of length five with end-edges e, f.

(1) *The first assertion of the theorem is true.*

Subproof. By (F7), every other path in $\mathcal{C}$ containing e or f has length at most four. If $C' \in \mathcal{C}$ contains both e, f, then $\mathcal{A}(C') = \emptyset$ by 6.2, since each member of $\mathcal{A}(C')$ has an η-attachment in C different from e, f; and if $C' \in \mathcal{C}$ contains just one of e, f, then it has length three by (F6), and again $\mathcal{A}(C') = \emptyset$ by 6.2. This proves (1).

(2) *The second assertion is true.*

Subproof. Let C have vertices v_0-v_1-$\cdots$-v_5 in order, where $e = v_0v_1$ and $f = v_4v_5$. Let $B \in \mathcal{A}(C)$. By 6.2, one of e, f is an η-attachment of B, say f; and since B has two η-attachments in C and they are diverse in C by 6.2, and e, f are twinned, it follows that the only other η-attachment of B is v_1v_2. Let $B' \in \mathcal{A}(C)$ with $B' \neq B$; we claim that v_1v_2 and v_4v_5 are the η-attachments of B'. For if not, then by the previous argument v_0v_1 and v_3v_4 are η-attachments of B', contrary to (E6). This proves (2).

In our earlier notation, we may assume that $p_0 = \eta(v_0)$ and $q_0 = \eta(v_4)$. Suppose that B is an η-bridge not in $\mathcal{A}(e,f)$ that meets $\eta(e)$. Then from 6.1 and 4.6, $B \in \mathcal{A}(C')$ for some $C' \in \mathcal{C}$ containing e, and hence $B \in \mathcal{A}(C)$ from (1); but this contradicts (2). Consequently $r = 0$.

(3) *Suppose that M, N are disjoint η-paths, from m to m' and from n to n' respectively, where $\eta(v_0), m, n, \eta(v_1)$ are in order in $\eta(e)$, and $\eta(v_4), n', m', \eta(v_5)$ are in order in $\eta(f)$. Then there exists j with $0 \leq j \leq s$ such that m', n' belong to P_j.*

Subproof. Suppose that $m' \in V(P_j)$ and $n' \in V(P_{j'})$ with $j \neq j'$; then $j < j'$. Let B be the η-bridge containing $q_{j'}$; then $B \notin \mathcal{A}(e,f)$ from the definition of $q_1, \ldots, q_s$, and so B has an η-attachment $g \neq e, f$. From 4.6, and (1) it follows that $B \in \mathcal{A}(C)$, and $g = v_1v_2$. In particular, B is disjoint from M, N. Choose an η-path P in B from $q_{j'}$ to $V(\eta(v_1v_2))$; then M, N, P contradict (E7). This proves (3).

For $0 \leq j \leq s$ let $\mathcal{A}_j$ be the set of all $B \in \mathcal{A}(e,f)$ with $B \cap \eta(f) \subseteq Q_j$. From (1), $\mathcal{A}(e,f) = \bigcup \mathcal{A}_j$. For each j let J_j be the union of all members of $\mathcal{A}_j$. Suppose that some $|E(J_j)| \geq 2$. Since H is cyclically five-connected by (E1), there are distinct b_1, b', b_2 in $\eta(e)$, in order, so that b_1 and b_2 both belong to J_j, and b' belongs to some η-bridge $B' \notin \mathcal{A}_j$. Since $b' \neq p_1, \ldots, p_r$ it follows that $B' \in \mathcal{A}(e,f)$, and so $B' \in \mathcal{A}_{j'}$, for some $j' \neq j$. In particular, J_j and $J_{j'}$ are disjoint. By 6.1 it follows that there is a path M in J_j and a path N in $J_{j'}$ violating (1) (possibly with M, N exchanged). The result follows from (3) applied to the paths of length one formed by these η-bridges. This proves 6.6. ∎

7 Flattenable graphs

Let $(G, F, \mathcal{C})$ be a framework and let H, η_F satisfy (E1). We say that H is *flattenable onto* $(G, F, \mathcal{C})$ *via* η_F if there is

- a homeomorphic embedding η of G in H extending η_F
- a set of η-bridges $\mathcal{B}(C)$, for each $C \in \mathcal{C}$, and
- an edge $N(e)$ of $\eta(e)$, for each edge e of $G \setminus E(F)$ such that for some edge $f \neq e$, e and f are twinned and have no common end

with the following properties. For each $C \in \mathcal{C}$, let $P(C)$ be $\eta(C)$ if C is a circuit, and if C is a path let $P(C)$ be the maximal subpath of $\eta(C)$ that contains $\eta(g)$ for every $g \in E(C)$ that is not an end-edge of C, and does not contain any edge $N(e)$. Then we require:

- every η-bridge belongs to exactly one set $\mathcal{B}(C)$
- if $B \in \mathcal{B}(C)$ then $B \cap \eta(G) \subseteq P(C)$
- for $C \in \mathcal{C}, P(C) \cup \bigcup(B : B \in \mathcal{B}(C))$ is $P(C)$-planar.

The main result, that everything so far has been directed towards, and of which all the other results in the paper will be a consequence, is the following.

7.1 *Let $(G, F, \mathcal{C})$ be a framework, and let H, η_F satisfy (E1)–(E7). Suppose that there is a homeomorphic embedding of G in H extending η_F. Then H is flattenable onto $(G, F, \mathcal{C})$ via η_F.*

Proof. Since there is a homeomorphic embedding of G in H extending η_F, there is an optimal one, say η. We will prove that η is flattenable onto $(G, F, \mathcal{C})$ via η_F. We begin with

(1) *If $e, f \in E(G)$ are twinned and have a common end, there exists $C \in \mathcal{C}$ containing e, f such that*

$$\eta(C) \cup \bigcup (B : \ B \in \mathcal{A}(C) \cup \mathcal{A}(e, f))$$

is $\eta(C)$-planar.

Subproof. Let the two members of $\mathcal{C}$ that contain v be C_1, C_2, where v is the common end of e and f. Let $e = uv$ and $f = vw$, and let $c_1d_1, \ldots, c_kd_k$ be the edges of H with one end in $\eta(e)$ and the other in $\eta(f)$ (these are the edges of the bridges in $\mathcal{A}(e, f)$) numbered as in 6.5). By 5.1 we may assume that $k \geq 1$. Now

$$\eta(C_i) \cup \bigcup (B : \ B \in \mathcal{A}(e, f))$$

is $\eta(C_i)$-planar for $i = 1, 2$. We claim that for either $i = 1$ or $i = 2$, no member of $\mathcal{A}(C_i)$ meets $\eta(e) \cup \eta(f)$ between c_1 and d_1. For if not, there are disjoint η-paths R_1, R_2 so that for $i = 1, 2$, R_i has one end r_i in $\eta(e) \cup \eta(f)$ between c_1 and d_1, and its other end s_i is in $\eta(C_i)$ and not in $\eta(e) \cup \eta(f)$. Let $s_i \in V(\eta(g_i))$ $(i = 1, 2)$. If g_1, g_2 have no common end, this contradicts (E4), and if they have a common end, this contradicts 4.2. (To see this, in each case delete an appropriate end-edge of the subpath of $\eta(e) \cup \eta(f)$ between c_1, d_1.) We may therefore assume that no member of $\mathcal{A}(C_1)$ meets $\eta(e) \cup \eta(f)$ between c_1 and d_1. But then by 5.1, the claim holds. This proves (1).

(2) *Let e, f be twinned, with no common end. Then there are edges $N(e)$ of $\eta(e)$ and $N(f)$ of $\eta(f)$ with the following properties. For each path $C \in \mathcal{C}$ with end-edges e and f, let $P(C)$ be the component of $\eta(C) \setminus \{N(e), N(f)\}$ containing $\eta(C \setminus \{e, f\})$. Then there exists $C_1 \in \mathcal{C}$ with end-edges e, f with $\mathcal{A}(C_1) = \emptyset$, such that $B \cap \eta(G) \subseteq P(C_1)$ for all $B \in \mathcal{A}(e, f)$, and*

$$P(C_1) \cup \bigcup (B : \ B \in \mathcal{A}(e, f))$$

is $P(C_1)$-planar; and for every other $C \in \mathcal{C}$ with end-edges e and f, $B \cap \eta(G) \subseteq P(C)$ for all $B \in \mathcal{A}(C)$ and

$$P(C) \cup \bigcup (B : \ B \in \mathcal{A}(C))$$

is $P(C)$-planar.

Subproof. If no member of $\mathcal{C}$ containing e, f has length five, this follows from 6.5, so we assume that such a member C exists. Then by 6.6, possibly after exchanging e, f, we can number C as v_0-v_1-$\cdots$-v_5, with $e = v_0v_1$ and $f = v_4v_5$, and number $\mathcal{A}(e, f)$ as $\{B_1, \ldots, B_k\}$, such that

- $\mathcal{A}(C')$ is empty for every $C' \neq C$ in $\mathcal{C}$ containing e or f;
- for each $B \in \mathcal{A}(C)$, its only η-attachments are v_1v_2 and v_4v_5;
- B_i has exactly one edge c_id_i for $1 \leq i \leq k$, where $c_i \in V(\eta(e))$ and $d_i \in V(\eta(f))$; and

- $\eta(v_0), c_1, \ldots, c_k, \eta(v_1)$ are in order in $\eta(e)$, and $\eta(v_4), d_1, \ldots, d_k, \eta(v_5)$ are in order in $\eta(f)$.

Let $N(e)$ be the edge of $\eta(e)$ incident with $\eta(v_1)$, and $N(f)$ be the edge of $\eta(f)$ incident with $\eta(v_5)$. Then $B \cap \eta(G) \subseteq P(C)$ for all $B \in \mathcal{A}(C)$, and

$$P(C) \cup \bigcup(B : \ B \in \mathcal{A}(C))$$

is $P(C)$-planar, by 5.1. Now $\mathcal{A}(C') = \emptyset$ for all other paths $C' \in \mathcal{C}$ with end-edges e and f. By (F7) there exists $C_1 \in \mathcal{C}$ with end-edges e and f and with ends v_1 and v_5. Then $B \cap \eta(G) \subseteq P(C_1)$ for all $B \in \mathcal{A}(e, f)$. From the third and fourth bullets above,

$$P(C_1) \cup \bigcup(B : B \in \mathcal{A}(e, f))$$

is $P(C_1)$-planar. This proves (2).

Since no edge of e is twinned with more than one other edge, by 3.4, defining $N(e)$ via (2) is well-defined. Since $\mathcal{A}(C) = \emptyset$ for every path $C \in \mathcal{C}$ with one end-edge twinned with an edge not in C, by (F6) and 6.2, the result follows from 5.1. This proves 7.1. ∎

8 Augmenting paths

We need three more techniques for the second half of the paper, all developed in [3], and in this section we describe the first. If F is a subgraph of G and of H, and η is a homeomorphic embedding of G in H, we say it *fixes* F if $\eta(e) = e$ for all $e \in E(F)$ and $\eta(v) = v$ for all $v \in V(F)$.

Let G be cubic, and let F be a subgraph of G with minimum degree ≥ 2 (possibly null). Let $X \subseteq V(G)$, so that $\delta_G(X) \cap E(F) = \emptyset$. Let $n \geq 1$, let $G_0 = G$, and inductively for $1 \leq i \leq n$ let $G_i = G_{i-1} + (e_i, f_i)$ with new vertices u_i, v_i, where e_i, f_i are edges of G_{i-1} not in $E(F)$. Let η_F be the identity homeomorphic embedding of G_0 to itself; and for $1 \leq i \leq n$, let η_i be obtained from η_{i-1} by replacing e_i and f_i by the corresponding two-edge paths of G_i. Thus η_i is a homeomorphic embedding of G in G_i; it fixes F, and $\eta(v) = v$ for all $v \in V(G)$, and $\eta(e) = e$ for all $e \in E(G)$ except edges of G in $\{e_1, f_1, \ldots, e_i, f_i\}$.

Let $\delta_G(X) = \{x_1y_1, \ldots, x_ky_k\}$, where $x_1, \ldots, x_k \in X$ are all distinct, and $y_1, \ldots, y_k \in V(G) \setminus X$ are all distinct. Suppose in addition:

- $e_1 \in E(G)$ has both ends in X, and $f_n \in E(G)$ with both ends in $V(G) \setminus X$
- for $1 \leq i < n$ there exists $j \in \{1, \ldots, k\}$ such that f_i is the edge of $\eta_{i-1}(x_jy_j)$ incident with y_j, and e_{i+1} is the edge of $\eta_i(x_jy_j)$ incident with v_i and not with y_j
- if $f_1 \in E(\eta(x_jy_j))$ (that is, $f_1 = x_jy_j$) where $1 \leq j \leq k$, then e_1 is not incident with x_j in G, and no end of e_1 is adjacent in $G \setminus E(F)$ to x_j; similarly, if $e_n \in E(\eta(x_jy_j))$ then e_n is not incident with y_j in G, and no end of e_n is adjacent in $G \setminus E(F)$ to y_j
- for $2 \leq i \leq n-1$, let $e_i \in E(\eta_{i-1}(x_jy_j))$ and $f_i \in E(\eta_{i-1}(x_{j'}y_{j'}))$; then $j' \neq j$, and x_j is not adjacent to $x_{j'}$ in $G \setminus E(F)$, and y_j is not adjacent to $y_{j'}$ in $G \setminus E(F)$.

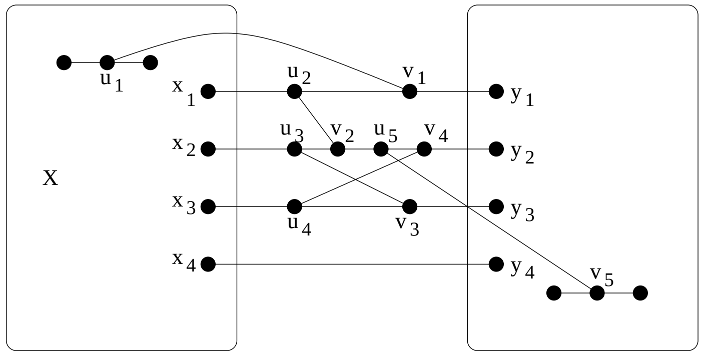


Figure 5: An X-augmentation of a graph, with $k = 4$ and $n = 5$.

(See Figure 5.)

In these circumstances we call G_n an X-*augmentation* of G (modulo F), and $(e_1, f_1), \ldots, (e_n, f_n)$ an X-*augmenting sequence* of G (modulo F). Note that we permit $n = 1$. The following is proved in lemma 3.4 of [3], applied to applied to F, $H \setminus E(F)$ and X.

8.1 *Let G be cubic and let F be a subgraph of G with minimum degree at least two. Let $X \subseteq V(G)$ with $\delta_G(X) \cap E(F) = \emptyset$, so that the edges in $\delta_G(X)$ pairwise have no common end. Let H be cubic so that F is a subgraph of H, and let η be a homeomorphic embedding of G in H fixing F. Then either*

- *there exists $X' \subseteq V(H)$ with $|\delta_H(X')| = |\delta_G(X)|$, such that for $v \in V(G)$, $v \in X$ if and only if $\eta(v) \in X'$, or*
- *there is an X-augmentation G' of G modulo F, and a homeomorphic embedding of G' in H fixing F.*

9 Jumps on a dodecahedron

Now we begin the second part of the paper. First we prove the following variant of 1.5 (equivalent to 1.6).

9.1 *Let H be cyclically five-connected and cubic. Then H is non-planar if and only if H contains one of Petersen, Triplex, Box and Ruby.*

Proof. “If” is clear. For “only if”, let H be cyclically five-connected and cubic, and contain none of the four graphs. By 1.5 it follows that H contains Dodecahedron. Let $G=$ Dodecahedron, let F and η_F be null, and let $\mathcal{C}$ be the set of circuits of G that bound regions in the drawing in Figure 3; then $(G, F, \mathcal{C})$ is a framework. We claim that (E1)–(E7) are satisfied. Most are trivial, because there are no twinned edges and F is null, and no paths in $\mathcal{C}$. Also, (E6) is vacuously true because no member of $\mathcal{C}$ has length ≥ 6; so the only axiom that needs work is (E2)(ii).

Let $e, f \in E(G)$ so that no member of $\mathcal{C}$ contains both e and f; we claim that $G + (e, f)$ contains one of Petersen, Triplex, Box, Ruby. Up to isomorphism of G there are five possibilities for e, f, namely (setting $e = ab$ and $f = cd$) $(a, b, c, d) = (1, 2, 6, 15)$, $(1, 2, 10, 15), (1, 2, 15, 20)$, $(1, 2, 18, 19), (1, 2, 19, 20)$. In the first three cases $G + (e, f)$ contains Ruby, and in the last two it contains Box.

Thus, (E2)(ii) holds; and so H is planar, by 7.1. This proves 9.1. ∎

Next, a small repair job. The definition of "dodecahedrally connected" in [3] differs from the definition in this paper, and our objective of the remainder of this section is to prove them equivalent. To do so, we essentially have to repeat the proof of 9.1 with slightly different hypotheses.

In this section we fix a graph F, and we need to look at several graphs such that F is a subgraph of all of them. If G, H are cubic, and F is a subgraph of them both, and there is a homeomorphic embedding of G in H fixing F, we say that H *F-contains* G.

Let G be cubic, and let F be a subgraph of G, so that every vertex in F has degree ≥ 2 in F. Let C be a circuit of G of length four, with vertices a_1, a_2, a_3, a_4 in order, none of them in $V(F)$. Let a_i be adjacent to $b_i \notin V(C)$ for $1 \leq i \leq 4$, where $b_1, \ldots, b_4$ are all distinct, and not in $V(F)$, and are pairwise non-adjacent. A *C-leap* of G means a graph $G + (e, f)$, where $e \in E(C)$ and $f \in E(G) \setminus E(F)$, with no vertex in $V(C)$.

9.2 *Let G be cubic and cyclically 4-connected, with $|V(G)| \geq 8$. Let F be a subgraph of G so that every vertex in F has degree ≥ 2 in F. Let C be a circuit of G of length 4, disjoint from F. Let $\mathcal{L}$ be a set of cubic graphs so that F is a subgraph of each of them. Suppose that every C-leap of G F-contains a member of $\mathcal{L}$. Let H be a cyclically 5-connected cubic graph containing F as a subgraph, that does not F-contain any member of $\mathcal{L}$. Then H does not F-contain G.*

Proof. Let $X = V(C)$. Then $\delta_G(X) \cap E(F) = \emptyset$ since $X \cap V(F) = \emptyset$. Since G is cyclically four-connected and $|V(G)| \geq 8$ it follows that no two members of $\delta_G(X)$ have a common end.

Suppose that H F-contains G. Let us apply 8.1. Since H is cyclically five-connected, 8.1(i) does not hold, and so 8.1(ii) holds. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be an X-augmenting sequence of G, so that there is a homeomorphic embedding of the corresponding X-augmentation G' in H fixing F. From condition (iii) in the definition of "X-augmenting sequence", it follows that $n = 1$, and so $G' = G + (e_1, f_1)$. Thus G' is a C-leap of G, and therefore F-contains a member of $\mathcal{L}$. But H F-contains G', and so H F-contains a member of $\mathcal{L}$, a contradiction. This proves 10.1. ∎

It is convenient from now on to make the following convention. When we speak of a graph $G + (e, f)$ and the vertices of G are numbered $1, \ldots, n$, the new vertices of $G + (e, f)$ will be assumed to be numbered $n + 1$ and $n + 2$ (in order), unless we specify otherwise.

Let G be Dodecahedron, and let F be a circuit of G of length five. If $e, f \in E(G) \setminus E(F)$, and at most one of e, f has an end in $V(F)$, and e, f are not incident with the same region of G, we call $G + (e, f)$ a *hop extension* of (G, F); and if in addition e, f are diverse, we call $G + (e, f)$ a *jump extension* of (G, F). We begin with the following lemma.

9.3 *Let G be Dodecahedron, and let F be a circuit of G of length five. Let H be a cyclically five-connected cubic graph, such that F is a subgraph of H. Suppose that*

- *H F-contains no jump extension of (G, F)*
- *for every $X \subseteq V(H) \setminus V(F)$ with $|\delta_H(X)| = 5$ and $X \neq V(H) \setminus V(F)$, there is no homeomorphic embedding η of G in H fixing F such that $\eta(v) \in X$ for all $v \in V(G) \setminus V(F)$.*

Let e, f be diverse edges of G not in $E(F)$; then H does not F-contain $G + (e, f)$.

Proof. Suppose it does. Hence $G + (e, f)$ is not a jump extension of (G, F), and so both e, f have ends in $V(F)$. Let us number the vertices of Dodecahedron as in Figure 3, and from the symmetry we may assume that e is 2-7 and f is 5-10. Let $G' = G + (e, f)$ with new vertices 21, 22 say. Let $X = \{6, 7, \dots, 20\}$. From the second bullet and 8.1, there is an X-augmenting sequence of G' modulo F, say $(e_1, f_1), \dots, (e_n, f_n)$, and a homeomorphic embedding η'' of the corresponding X-augmentation G'' in H fixing F. Now e_1 ($= a_1b_1$ say) has both ends in X, but f_1 does not, so f_1 is one of 1-6, 2-21, 7-21, 3-8, 4-9, 5-22, 10-22, 21-22; and from the symmetry we may assume that f_1 is one of 1-6, 2-21, 7-21, 3-8, 21-22.

Suppose that f_1 is one of 1-6, 3-8. Then e_1, f_1 are diverse, from the third condition in the definition of X-augmenting sequence; but then $G + (e_1, f_1)$ is a jump extension of (G, F) F-contained in $G' + (e_1, f_1)$ and hence in H, a contradiction. Similarly if f_1 is 7-21 then $G + (e_1, 2 - 7)$ is a jump extension F-contained in H. Thus f_1 is one of 21-22, 2-21, and in particular $n = 1$. Assume f_1 is 21-22. Then we may assume that e_1, 2-7 are not diverse in G (for otherwise $G + (e_1, 2 - 7)$ is a jump extension F-contained in H), and similarly e_1, 5-10 are not diverse in G. But this is impossible. Finally, assume that f_1 is 2-21. we may assume that $e_1, 2 - 7$ are not diverse in G, and so e_1 is one of

$$7 - 11, 7 - 12, 6 - 11, 11 - 16, 8 - 12, 12 - 17.$$

If e_1 is one of 7-12, 8-12, 12-17, rerouting 7-12 along 21-22 gives a jump extension of (G, F) F-contained in H; and if e_1 is one of 7-11, 6-11, 11-16, rerouting 7-11 along 21-22 gives a jump extension of (G, F) F-contained in H, again a contradiction. This proves 9.3. ∎

9.4 *Let G, F, H be as in 9.3. Then H F-contains no hop extension of (G, F).*

Proof. Let $\mathcal{L}$ be the set of all graphs $G + (e, f)$ where e, f are diverse edges of G not in $E(F)$. By 9.3, H F-contains no member of $\mathcal{L}$. Let G be labelled as in Figure 3. (We do not specify the circuit F at this stage; it is better to preserve the symmetry.) Let $G_1 = G + (a, b)$ be a hop extension of G, and suppose that H F-contains G_1. Thus $G_1 \notin \mathcal{L}$. From the symmetry of G, we may therefore assume that a is 15-20 and b is 16-17. Thus the edges 16-17 and 15-20 are not in $E(F)$. Since F is a circuit of length five, it follows that 16-20 is not in $E(F)$, and hence 16,20 are not in $V(F)$. Let C be the circuit 16-20-21-22-16 of G_1. Then no vertex of C is in $V(F)$, and H is cyclically five-connected, so we can apply 9.2. We deduce that H F-contains some C-leap $G_2 = G_1 + (e, f)$.

Now e is one of 16-20, 20-21, 21-22, 16-22. Since F is not yet specified, there is a symmetry of G_1 exchanging the edges 16-20 and 21-22; and one exchanging 20-11 and 16-22. Thus we may assume that e is one of 21-22, 20-21.

Now f is an edge of G not incident with either of 16,20. Since e is one of 21-22, 20-21, and $f \notin E(F)$, H F-contains $G + (15 - 20, f)$ in G_2, and so $G + (15 - 20, f) \notin \mathcal{L}$. Consequently f, 15-20 are not diverse, so f is one of

$$6 - 15, 10 - 15, 1 - 6, 6 - 11, 5 - 10, 10 - 14, 14 - 19, 18 - 19.$$

Suppose first that e is 21-22. Then by the same argument, f and 16-17 are not diverse in G, and so f is one of 6-11, 18-19. If f is 6-11, rerouting 6-15 along 24-23-21 gives a member of $\mathcal{L}$ F-contained in H (in future we just say "works") and if f is 18-19, rerouting 17-18 along 22-23-24 works. Thus the claim holds if e is 21-22.

Now we assume that e is 20-21. If f is one of 1-6,6-11,6-15 then rerouting 6-15 along 23-24 works; if f is one of 10-15,5-10,10-14, rerouting 10-15 along 23-24 works; and if f is 14-19 or 18-19 then rerouting 19-20 along 23-24 works. Thus us each case we have a contradiction. This proves 9.4. ∎

Next we need another similar lemma. Let G be Dodecahedron, labelled as in Figure 3, and let G_1 be $G+(1-6,2-7)$. Let $G_2 = G_1+(6-21,2-22)$. Thus the edge 1-6 of G has been subdivided to become a path 1-21-23-6 of G_2, and 2-7 has become 2-24-22-7.

9.5 *Let G, F, H be as in opposite. Then H does not F-contain G_2.*

Proof. Let $X = \{6, 7, \ldots, 20\}$. By the second bullet hypothesis about H, and 8.1,

there is an X-augmenting sequence of G_2 modulo F, say $(e_1, f_1), \ldots, (e_n, f_n)$, and a homeomorphic embedding η' of the corresponding X-augmentation G' in H fixing F. Now e_1 $(= a_1b_1$ say) has both ends in X, but f_1 does not, so f_1 is one of

$$1-21, 21-23, 6-23, 2-24, 22-24, 7-22, 3-8, 4-9, 5-10, 21-22, 23-24,$$

and from the symmetry we may assume that f_1 is one of

$$1-21, 21-23, 6-23, 5-10, 4-9, 21-22.$$

If f_1 is one of 5-10, 4-9 then by the third condition in the definition of X-augmenting sequence, it follows that e_1, f_1 are diverse in G, and H contains the jump extension $G+(e_1, f_1)$, a contradiction. Similarly if f_1 is 6-23 then $e_1, 1-6$ are diverse in G, again a contradiction. Thus f_1 is one of 1-21, 21-23, 21-22. Hence H F-contains $G+(1-6, e_1)$, and so by 9.4, $G+(1-6, e_1)$ is not a hop extension of (G, F). Consequently f_1 is one of 10-15, 6-15, 6-11, 7-11. If f_1 is one of 6-11, 7-11, then rerouting 1-6 along 25-26 gives a jump extension of (G, F) F-contained in H; while if f_1 is one of 6-15, 10-15, rerouting 6-15 along 25-26, and then rerouting 7-11 along 23-24, give the desired jump extension. This proves 9.5. ∎

From these lemmas we deduce a kind of variant of 9.1:

9.6 *Let G be Dodecahedron, and let F be a circuit of G of length five. Let H be a cyclically five-connected cubic graph, such that F is a subgraph of H. Suppose that*

- *H F-contains no jump extension of (G, F)*
- *for every $X \subseteq V(H)\backslash V(F)$ with $|\delta_H(X)| = 5$ and $X \neq V(H)\backslash V(F)$, there is no homeomorphic embedding η of G in H fixing F such that $\eta(v) \in X$ for all $v \in V(G) \setminus V(F)$.*

Then H is planar, and can be drawn in the plane so that F bounds the infinite region.

Proof. Let $\mathcal{C}$ be the set of the following eleven subgraphs of $G =$ Dodecahedron; the six circuits that bound regions (in the drawing in Figure 3) that contain no edge incident with the infinite region, and for each $e \in E(F)$, the path $C\backslash e$ where $C \neq F$ is the boundary of a region incident with e. Let η_F be the identity homeomorphic embedding on F. By hypothesis there is a homeomorphic embedding of G in H extending η_F. We apply 7.1 to $(G, F, \mathcal{C})$ and H, η_F. There are no twinned edges and all members of $\mathcal{C}$ have ≤ 5 edges; so we have to check only (E2)(ii) and (E6). (Note that in this case, the paths in $\mathcal{C}$ are not induced subgraphs of G; this is the only one of our applications when this is so.) But the truth of (E2)(ii) and (E6) follows from the three lemmas above 9.3, 9.4, 9.5; and so by 7.1, the result follows. This proves 9.6. ∎

As we said earlier, we need this to prove the equivalence of the definitions of dodecahedral connectivity given in this paper and in [3], and now we turn to that. Let G be Dodecahedron, and let F be a circuit of G of length five. Let H be a cubic graph, and let $X \subseteq V(H)$. We say that H is *placid* on X if

- $|V(H) \setminus X| \geq 7$, and $\delta_H(X)$ is a matching of cardinality five
- $\{x_i y_i : 1 \leq i \leq 5\}$ is an enumeration of $\delta_H(X)$, with $x_i \in X$ $(1 \leq i \leq 5)$
- there is a homeomorphic embedding of G in H' mapping F to the circuit y_1-y_2-y_3-y_4-y_5-y_1, and
- there is no homeomorphic embedding of any jump extension of (G, F) in H' mapping F to y_1-y_2-y_3-y_4-y_5-y_1,

where H' is obtained from $H|(X \cup \{y_1, y_2, y_3, y_4, y_5\})$ by deleting all edges with both ends in $\{y_1, y_2, y_3, y_4, y_5\}$, and adding new edges $y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_1y_5$.

We say that a graph H is *strangely connected* if H is cubic and cyclically five-connected, and there is no $X \subseteq V(H)$ such that H is placid on X. (This is the definition of "dodecahedrally connected" in [3].)

9.7 *A graph H is dodecahedrally connected if and only if it is strangely connected.*

Proof. We may assume that H is cubic and cyclically five-connected. Suppose first that it is not dodecahedrally connected. Let $X \subseteq V(H)$ with $|X|, |V(H) \setminus X| \geq 7$ and $|\delta_H(X)| = 5$, $\delta_H(X) = \{x_1y_1, \ldots, x_5y_5\}$ say where $x_1, \ldots, x_5 \in V(H)$, so that $H|X$ can be drawn in a disc with $x_1, \ldots, x_5$ on the boundary in order. Let us choose such X with $|X|$ minimum. Since H is cyclically five-connected it follows that $x_1, \ldots, x_5$ are all distinct and so are $y_1, \ldots, y_5$. Also, from the planarity of $H|X$ it follows that $|X| \geq 9$, and so from the minimality of X, no two of $x_1, \ldots, x_5$ are adjacent. Let H' be obtained from H as in the definition of "placid", and let F' be the circuit made by the five new edges. It follows easily that H' is cyclically five-connected, and hence from 1.6 contains $G =$ Dodecahedron. Take a planar drawing of H', and choose a homeomorphic embedding η of G in H' so that the region of $\eta(G)$ including r is minimal, where r is the region of H' bounded by F'. It follows easily that $F' \subseteq \eta(G)$, and so from the symmetry of G we may choose η mapping F to F'. Hence H is placid on X (the final condition in the definition of "placid" holds because of the planarity of H') and so H is not strangely connected, as required.

For the converse, suppose that H is not strangely connected, and let X, x_iy_i $(1 \leq i \leq 5)$, F and H' be as in the definition of "strangely connected", such that H is placid on X via $x_1y_1, \ldots, x_5y_5$. Choose X minimal. By 9.6, $H|X$ can be drawn in a disc with $x_1, \ldots, x_5$ on the boundary in order; and so H is not dodecahedrally connected. This proves 9.7. ∎

10 Adding jumps to repair connectivity

Now that we have reconciled the two definitions of "dodecahedrally connected", we can apply results of [3] about this kind of connectivity.

The idea behind 9.2 is that cyclic five-connectivity is better than cyclic four-connectivity, and we begin with a graph G that is almost cyclically five-connected, except for the circuit C. We use the cyclic five-connectivity of H to prove that if H contains G then H also contains a slightly larger graph where the circuit C has been expanded to a circuit of length five by adding an edge to G. This can be useful, as we saw in the previous section. However, it has the defect that the edge we add to G to expand the circuit C might create a new circuit of length four, with its own problems. We can apply 9.2 again to this new circuit, but the process can go on forever. In fact, there is a stronger theorem; one can expand the circuit C to a circuit of length five, without adding any new circuits of length four, just by adding a bounded number of edges. That is the content of the next result, proved in [3]. But first we need some definitions.

Let $\mathcal{L}$ be a set of cubic graphs so that F is a subgraph of each of them. If F is a subgraph of some H, we say that H is *killed by* $\mathcal{L}$ if there is a homeomorphic embedding of some $G' \in \mathcal{L}$ in H fixing F. (We leave the dependence on F implicit.) Let G be cubic, and let F be a subgraph of G, so that every vertex in F has degree ≥ 2 in F. Let C be a circuit of G of length four, with vertices a_1, a_2, a_3, a_4 in order, none of them in $V(F)$. Let a_i be adjacent to $b_i \notin V(C)$ for $1 \leq i \leq 4$, where $b_1, \ldots, b_4$ are all distinct, and not in $V(F)$, and are pairwise non-adjacent. We denote by $\mathcal{P}(C, \mathcal{L})$ the set of all pairs (e, f) such that $f \in E(G)$ is incident with one of $b_1, \ldots, b_4$, say b_i, $f \neq a_i b_i, e \in E(C)$ is incident with a_i, and $G + (e, f)$ is not killed by $\mathcal{L}$.

Let $e = uv$ and $f = wx$ be edges of a cubic graph G. If $u, v \neq w, x$, and u is adjacent to w, and no other edge has one end in $\{u, v\}$ and the other in $\{w, x\}$, we denote by $(e, f)^*$ the pair of edges (e', f'), where e' $(\neq e, uw)$ is incident with u and f' $(\neq f, uw)$ is incident with w.

We shall frequently have to list the members of some set $\mathcal{P}(C, \mathcal{L})$ explicitly, and we can save some writing as follows. Clearly $(e, f) \in \mathcal{P}(C, \mathcal{L})$ if and only if $(e, f)^* \in \mathcal{P}(C, \mathcal{L})$, and so we really need only to list half the members of $\mathcal{P}(C, \mathcal{L})$. If X is a set of pairs of edges for which $(e, f)^*$ is defined for each $(e, f) \in X$, we denote by X^* the set $X \cup \{(e, f)^* : (e, f) \in X\}$.

If $e \in E(C)$, $f \notin E(F)$ and e, f are diverse in G, we call $G + (e, f)$ an *A-extension* of G. Now let $e \in E(C)$ and $f \in E(G) \setminus E(C)$ with $f \notin E(F)$, so that e, f are not diverse in G but have no common end. Let $G' = G + (e, f)$ with new vertices x_1, y_1. Label the vertices of C as $a_1, \ldots, a_4$ in order, and their neighbours not in $V(C)$ as $b_1, \ldots, b_4$ respectively, as before, such that $e = a_1 a_2$ and f is incident with b_1, $f = b_1 c_1$ say. If $g \in E(G)$, not incident in G with a_1, b_1, c_1, d_1 (where b_1 is adjacent in G to a_1, c_1, d_1) we call $G' + (b_1 y_1, g)$ a *B-extension* (*of* G) *via* (e, f). If $g \in E(G)$ incident with b_2 and not with c_1 or a_2, we call $G' + (x_1 y_1, g)$ a *C-extension via* (e, f) *onto* g. We call $G' + (a_1 x_1, a_3 b_3)$ a *D-extension via* (e, f). Finally, we say (e, f) and (e', f') are *C-opposite* if $e, e' \in E(C)$ and the labelling can be chosen as before with $e = a_1 a_2$, $f = b_1 c_1$, $e' = a_3 a_4$, and $f' = b_3 c_3$. Let $(e, f), (e', f')$ be C-opposite, with labels as above. Let $G'' = G' + (e', f')$ with new vertices x_2, y_2; then we call $G'' + (a_1 x_1, a_3 x_2)$ an *E-extension via* $(e, f), (e', f')$.

The following is a restatement of 9.2 in this language.

10.1 *Let G be cubic and cyclically 4-connected, with $|V(G)| \geq 8$. Let F be a subgraph of G so that every vertex in F has degree ≥ 2 in F. Let C be a circuit of G of length 4, disjoint from F. Let $\mathcal{L}$ be a set of cubic graphs so that F is a subgraph of each of them. Suppose that every A-extension of*

G is killed by $\mathcal{L}$, and $\mathcal{P}(C,\mathcal{L}) = \emptyset$. Let H be a cyclically 5-connected cubic graph containing F as a subgraph, that is not killed by $\mathcal{L}$. Then there is no homeomorphic embedding of G in H fixing F.

Here is the strengthening, proved in [3].

10.2 *Let G be cubic and three-connected with girth at least four, and let F be a subgraph of G with minimum degree at least two. Let C be a circuit of G of length four. Suppose that for every $X \subseteq V(G)$ with $|\delta_G(X)| \leq 4$, either $|X| \leq 2$ or $|V(G) \setminus X| \leq 2$ or $X = V(C)$ or $X = V(G) \setminus V(C)$. Let $\mathcal{L}$ be a set of cubic graphs so that F is a subgraph of each of them. Suppose that*

- *every A-extension of G is killed by $\mathcal{L}$*
- *for every $(e, f) \in \mathcal{P}(C,\mathcal{L})$, every B-extension via (e, f) is killed by $\mathcal{L}$, and so is the D-extension via (e, f)*
- *for all $(e, f_1), (e, f_2) \in \mathcal{P}(C,\mathcal{L})$ so that f_1, f_2 have no common end, the C-extension via (e, f_1) onto f_2 is killed by $\mathcal{L}$, and*
- *for all C-opposite $(e_1, f_1), (e_2, f_2) \in \mathcal{P}(C,\mathcal{L})$, the E-extension via $(e_1, f_1), (e_2, f_2)$ is killed by $\mathcal{L}$.*

Let H be a dodecahedrally connected cubic graph so that F is a subgraph of H, and H is not killed by $\mathcal{L}$. Then there is no homeomorphic embedding of G in H fixing F.

The other result of [3] that we need is the following. Let $n \geq 5$ be an integer, with $n \geq 10$ if n is even. The *n-biladder* is the graph with vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, where for $1 \leq i \leq n$, a_i is adjacent to a_{i+1} and to b_i, and b_i is adjacent to b_{i+2} (where $a_{n+1}, b_{n+1}, b_{n+2}$ mean a_1, b_1, b_2). Thus, Petersen is isomorphic to the 5-biladder, and Dodedahedron to the 10-biladder. The following follows from theorem 1.4 of [3].

10.3 *Let G be cubic and cyclically 5-connected. Let there be a homeomorphic embedding of G in H, where H is dodecahedrally connected. Then either*

- *there exist $e, f \in E(G)$, diverse in G, such that there is a homeomorphic embedding of $G+(e, f)$ in H, or*
- *G is isomorphic to an n-biladder for some n, and there is a homeomorphic embedding of the $(n + 2)$-biladder in H, or*
- *G is isomorphic to H.*

11 Graphs with crossing number at least two

At the end of the proof of 9.1, there were five statements left to the reader to verify, that five particular graphs contain either Ruby or Box. In the remainder of the paper there will be many more similar statements left to the reader; unfortunately, we see no way of avoiding this, since there are simply too many of them to include full details of each. But perhaps 95% of them are of the form that "Graph G contains Petersen", where G is cubic and cyclically five-connected; and here is a quick method for checking such a statement. Choose a circuit C of G with $|E(C)| = 5$, arbitrarily (there always is one, in this paper). Let C have vertices $v_1, \ldots, v_5$ in order. Let $u_1, \ldots, u_5$ be vertices of a 5-circuit of Petersen, in order. Check if there is a homeomorphic embedding η of Petersen in G with $\eta(u_i) = v_i \, (1 \leq i \leq 5)$. (This is easy to do by hand.) It is proved in [5] that such a homeomorphic embedding exists if and only if G contains Petersen.

This makes checking for containment of Petersen much easier. But even so, there are too many cases to reasonably do them all by hand, and we found it very helpful to write a simple computer programme to check containment for us. We suggest that the reader who wants to check these cases should do the same thing. There is a computer file available online with all the details of the case-checking [4].

In this section, we prove 1.7, which we restate as:

11.1 *Let H be dodecahedrally connected. Then H has crossing number ≥ 2 if and only if it contains one of Petersen, Triplex or Box.*

Dodecahedral connectivity cannot be replaced by cyclic 5-connectivity, because the graph of Figure 6 is a counterexample.

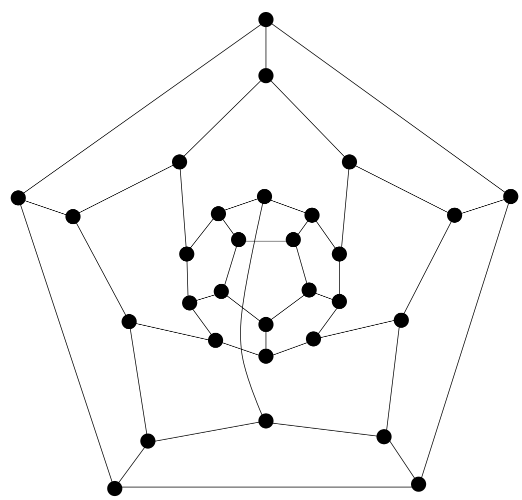


Figure 6: A counterexample to a strengthening of 11.1.

The graphs Window, Antibox, and Drape are defined in Figure 7. We prove 11.1 in three steps, as follows.

11.2 *Let H be a dodecahedrally connected graph containing Antibox; then H contains Petersen, Triplex or Box.*

11.3 *Let H be a cyclically 5-connected cubic graph containing Drape; then H contains Petersen, Triplex, Box or Antibox.*

11.4 *Let H be a cyclically 5-connected cubic graph containing Window, but not Petersen, Triplex, Box, Antibox or Drape. Then H has crossing number ≤ 1.*

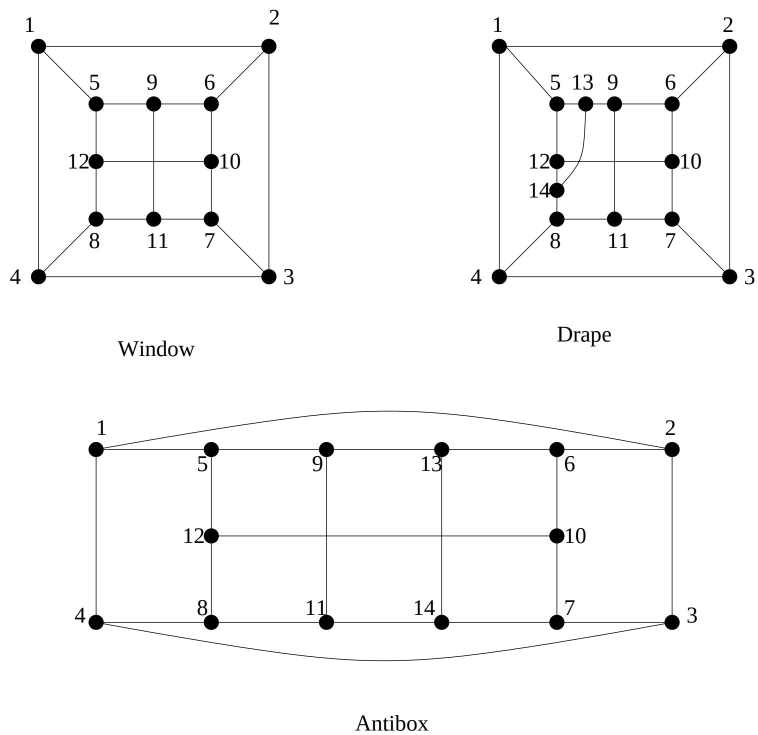


Figure 7: Window, Drape and Antibox.

Proof of 11.1, assuming 11.2, 11.3, 11.4. "If" is clear and we omit it. For "only if", let H be dodecahedrally connected, and contain none of Petersen, Triplex or Box. By 11.2 it does not contain Antibox, and by 11.3 it does not contain Drape. We may assume from 9.1 that it contains Ruby (in fact it must, for no dodecahedrally connected graph is planar), and hence Window, since Ruby contains Window. From 11.4, this proves 11.1. ∎

Proof of 11.2. We shall apply 10.2, with G = Antibox, F null, C the quadrangle of G, and $\mathcal{L} = \{$Petersen, Triplex, Box$\}$. Thus, $V(C) = \{1, 2, 3, 4\}$. We find that every A-expansion is killed by $\mathcal{L}$. In detail, let G' be $G + (ab, cd)$, where (a, b, c, d) is as follows; in each case G' contains the specified member of $\mathcal{L}$.

Petersen: (1, 2, 7, 10), (1, 2, 7, 14), (1, 2, 8, 11), (1, 2, 8, 12), (1, 2, 9, 11), (1, 2, 11,14), (1, 2, 13, 14), (1, 4, 6, 10), (1, 4, 6, 13), (1, 4, 7, 10), (1, 4, 7, 14), (1, 4, 9, 13), (1, 4, 11, 14), (1, 4, 13, 14).

Triplex: (1, 2, 5, 12), (1, 2, 6, 10), (1, 2, 10, 12), (1, 4, 5, 9), (1, 4, 8, 11), (1, 4, 9, 11).

Box: (1, 2, 9, 13), (1, 4, 10, 12).

In future we shall omit this kind of detail (because in the future it will get worse).

We find that $\mathcal{P}(C, \mathcal{L}) = \{(1-2, 5-9), (1-2, 6-13), (3-4, 8-11), (3-4, 7-14)\}^*$. Then we verify the hypotheses (ii)-(iv) of 10.2. This proves 11.2. ∎

Proof of 11.3. We apply 10.1, with G = Drape, F null, C the quadrangle of G with vertex set {5, 12, 13, 14}, and $\mathcal{L}$ = {Petersen, Triplex, Box, Antibox}. We find that every A-extension of G is killed by $\mathcal{L}$, and $\mathcal{P}(C, \mathcal{L}) = \emptyset$, so from 10.1, this proves 11.3. ∎

Proof of 11.4. Let G be Window, let F and η_F be null, and let $\mathcal{C}$ be the subgraphs of G induced on the following nine sets:

$$1, 2, 3, 4;$$
$$1, 2, 5, 6, 9;$$
$$2, 3, 6, 7, 10;$$
$$3, 4, 7, 8, 11;$$
$$1, 4, 5, 8, 12;$$
$$5, 9, 10, 11, 12;$$
$$6, 9, 10, 11, 12;$$
$$7, 9, 10, 11, 12;$$
$$8, 9, 10, 11, 12.$$

Then $(G, F, \mathcal{C})$ is a framework. We claim that (E1)–(E7) hold. The only twinned edges are $9 - 11$ and $10 - 12$, and again the only axiom that needs work is (E2). But if $e, f \in E(G)$ are not both in some member of $\mathcal{C}$, then $G + (e, f)$ contains one of Petersen, Triplex, Box, Antibox, Drape, and so (E2) holds. From 7.1, this proves 11.4. ∎

12 Non-projective-planar graphs

Now we digress, to prove a result that we shall not need; but it is pretty, and follows easily from the machinery we have already set up. The graph *Twinplex* is defined in Figure 8. We shall show the following.

12.1 *Let H be dodecahedrally connected. Then H cannot be drawn in the projective plane if and only if H contains one of Triplex, Twinplex, Box.*

Proof. "If" is easy and we omit it. For "only if", suppose that H contains none of Triplex, Twinplex, Box; we shall show that it can be drawn in the projective plane. If H has crossing number ≤ 1 this is true, so by 11.1 we may assume that H contains Petersen.

Let G_0 = Petersen. We may assume that H is not isomorphic to G_0, so by 10.3 either there are edges ab, cd of G_0 diverse in G_0 and a homeomorphic embedding of $G_0 + (ab, cd)$ in H, or H contains the 7-biladder. The former is impossible, because from the symmetry of G_0 we may assume that $(a, b, c, d) = (4, 5, 6, 8)$, and then $G_0 + (ab, cd)$ is isomorphic to Twinplex, a contradiction. Hence there is a homeomorphic embedding of G in H, where G is the 7-biladder. Let $V(G) =$

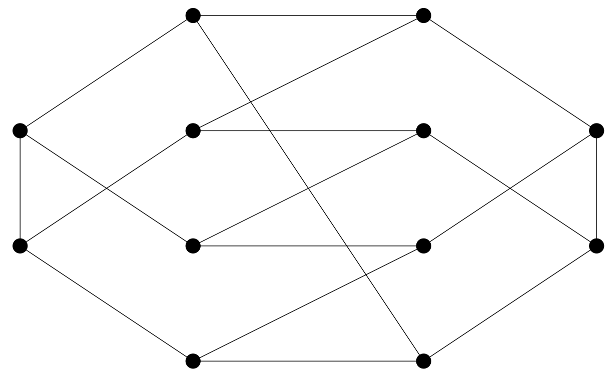


Figure 8: Twinplex.

$\{a_1, \ldots, a_7, b_1, \ldots, b_7\}$, as in the definition of "biladder". Let $\mathcal{C}$ be the subgraphs of G induced on the following vertex sets:

$$b_1, b_2, \ldots, b_7;$$
$$a_1, a_2, a_3, b_3, b_1;$$
$$a_2, a_3, a_4, b_4, b_2;$$
$$a_3, a_4, a_5, b_5, b_3;$$
$$a_4, a_5, a_6, b_6, b_4;$$
$$a_5, a_6, a_7, b_7, b_5;$$
$$a_6, a_7, a_1, b_1, b_6;$$
$$a_7, a_1, a_2, b_2, b_7.$$

(These are the face-boundaries of an embedding of G in the projective plane.) Let F and η_F be null; then $(G, F, \mathcal{C})$ is a framework, and we claim that (E1)–(E7) hold. All except (E2)(ii), (E3) and (E6) are obvious. To check (E2)(ii), let $G' = G + (ab, cd)$ where $ab, cd \in E(G)$ are not both in any member of $\mathcal{C}$. There are 12 possibilities for (a, b, c, d) up to isomorphism of G; in one case G' contain Box, in three others it contains Twinplex, and in the other eight it contains Triplex. Thus, (E2) holds. For (E3), the only diverse trinity (up to isomorphism of G) is $\{a_1a_2, b_1b_3, b_2b_7\}$, and $G + (a_1a_2, b_1b_3, b_2b_7)$ contains Twinplex. Hence (E3) holds. For (E10), we need only check cross extensions over the circuit with vertex set $\{b_1, \ldots, b_7\}$, since all other members of $\mathcal{C}$ have only five edges. There are four possibilities (up to isomorphism of G). Let $G' = G + (b_1b_3, b_2b_4)$ with new vertices x, y; then the possibilities are $G' + (ab, cd)$ where (a, b, c, d) is (b_1, x, b_2, y), (b_1, x, b_2, b_7), (b_1, b_6, b_2, b_7), (b_1, b_6, b_5, b_7). The first contains Box, and the other three contain Triplex. Hence (E10) holds, and from 7.1, this proves 12.1. ∎

13 Arched graphs

We say a graph H is *arched* if $H \backslash e$ is planar for some edge e. In this section we prove 1.8, which we restate as:

13.1 *Let H be dodecahedrally connected. Then H is arched if and only if it does not contain Petersen or Triplex.*

We start with the following lemma.

13.2 *Let G be Box, let G' be obtained by deleting the edge 13-14, and let $\mathcal{C}$ be the set of circuits of G' that bound regions in the drawing in Figure 3. Let $e, f \in E(G)$, with no common end, and not both in any member of $\mathcal{C}$. Then either $G + (e, f)$ has a Petersen or Triplex minor, or (up to exchanging e and f, and automorphisms of G) e is $13 - 14$ and f is $1 - 2$ or $1 - 4$.*

We leave the proof to the reader (the details are in the Appendix [4]).

13.3 *Let G be Box, and let H be cyclically 5-connected, and not contain Petersen or Triplex. Let η be a homeomorphic embedding of G in H so that $\eta(13 - 14)$ has only one edge, g say. Then $H \backslash g$ is planar, and so H is arched.*

Proof. We apply 7.1, taking F to be the subgraph of G consisting of $13 - 14$ and its ends, and η_F the restriction of η to F. Let $\mathcal{C}$ be as in 13.2. Then $(G, F, \mathcal{C})$ is a framework, and we claim that (E1)–(E7) hold. (E2) follows from 13.2, and (E5) and (E6) are vacuously true, because all members of $\mathcal{C}$ have five edges. Also, (E3) and (E7) are vacuously true. For (E4), it suffices from symmetry to check

$$G + (1 - 2, 13 - 14) + (3 - 6, 13 - 16)$$
$$G + (1 - 2, 13 - 14) + (3 - 6, 14 - 16)$$
$$G + (1 - 2, 13 - 14) + (5 - 6, 13 - 16)$$
$$G + (1 - 4, 13 - 14) + (3 - 6, 13 - 16),$$

but all four contain Triplex. Hence (E4) holds. The other axioms are easy, so from 7.1, this proves 13.3. ∎

The graph Superbox is defined in Figure 9. (It is isomorphic to Box + $(1 - 4, 13 - 14)$.)

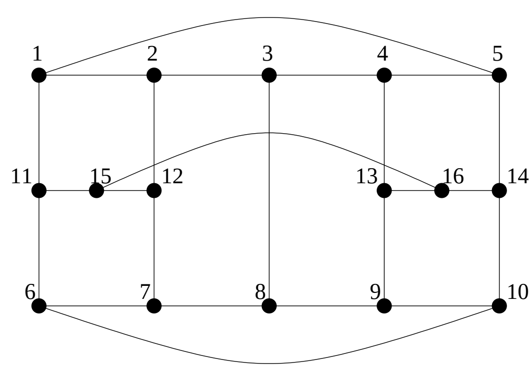


Figure 9: Superbox.

13.4 *Let G be Superbox, let G' be obtained by deleting the edge 15-16, and let $\mathcal{C}$ be the set of circuits of G' that bound regions in the drawing in Figure 9. Let $e, f \in E(G)$ with no common end, and not both in any member of $\mathcal{C}$. Then either $G + (e, f)$ has a Petersen or Triplex minor, or (up to exchanging e, f and automorphisms of G) e is $15 - 16$ and f is $1 - 2$ or $1 - 11$.*

We leave the proof to the reader. (Actually, it follows quite easily from 13.2.)

13.5 *Let G be Superbox, and let H be cyclically 5-connected, and not contain Petersen or Triplex. Let η be a homeomorphic embedding of G in H such that $\eta(15 - 16)$ has only one edge, g say. Then $H\backslash g$ is planar, and so H is arched.*

Proof. We apply 7.1 to $(G, F, \mathcal{C})$, where F consists of $15-16$ and its ends, and η_F is the restriction of η to F, and $\mathcal{C}$ is as in 13.4. Because of 13.4, it remains to verify (E4), (E5) and (E6), because (E3), (E7) are vacuous. Checking (E4) is exactly like in 13.3 (indeed, by deleting $14-16$ from G we obtain Box, so actually we could deduce that (E4) holds now from the fact that it held in the proof of 13.3). For (E5), we must check

$$G + (1 - 11, 15 - 16) + (6 - 11, 15 - 18) + (ab, cd)$$

where (ab, cd) is either $(11 - 17, 10 - 14)$ or $(11 - 19, 5 - 14)$; and both contain Triplex. Thus (E5) holds. For (E6), we need only check cross extensions over the circuit bounding the infinite region, since all other members of $\mathcal{C}$ have length five; and from symmetry, it suffices to check

$$G + (1 - 11, 10 - 14) + (1 - 17, 10 - 18)$$
$$G + (1 - 11, 10 - 14) + (6 - 11, 5 - 14)$$
$$G + (1 - 11, 10 - 14) + (1 - 5, 6 - 10)$$
$$G + (1 - 5, 6 - 10) + (1 - 17, 10 - 18).$$

All four contain Petersen. Hence (E6) holds, and from 7.1, this proves 13.5. ∎

Proof of 13.1. "Only if" is easy and we omit it. For "if", let H be dodecahedrally connected, and not contain Petersen or Triplex. Since graphs of crossing number ≤ 1 are arched, we may assume from 11.1 that G contains Box. Choose a homeomorphic embedding of G in H, where G is either Box or Superbox, so that $|E(S)|$ is minimum, where $S = \eta(15 - 16)$ if G is Box, and $S = \eta(17 - 18)$ if G is Superbox. We claim that $|E(S)| = 1$. For suppose not. Since H is three-connected, there is an η-path P with one end in $V(S)$ and the other, t, in $V(\eta(G)) \setminus V(S)$. Let $t \in \eta(f)$ say, and let $e = 15 - 16$ if G is Box, and $e = 17 - 18$ if G is Superbox. If e, f have a common end in G, then by rerouting f along P we contradict the minimality of $|E(S)|$. If some edge g of G joins an end of e to an end of f, then by rerouting g along P we contradict the minimality of $|E(S)|$. Hence e, f are diverse in G. By the symmetry we may therefore assume, by 13.2 and 13.4, that either G is Box and $f = 1 - 4$, or G is Superbox and $f = 1 - 2$. In the first case, by adding P to $\eta(G)$ we obtain a homeomorphic embedding of Superbox contradicting the minimality of $|E(S)|$. In the second case, by adding P to $\eta(G\backslash\{3 - 8, 6 - 7\})$ we obtain a homeomorphic embedding of Box contradicting the minimality of $|E(S)|$.

This proves our claim that $|E(S)| = 1$. From 13.3 and 13.5, H is arched. This proves 13.1. ∎

14 The children of Drum

The graph *Drum* is defined in Figure 10.

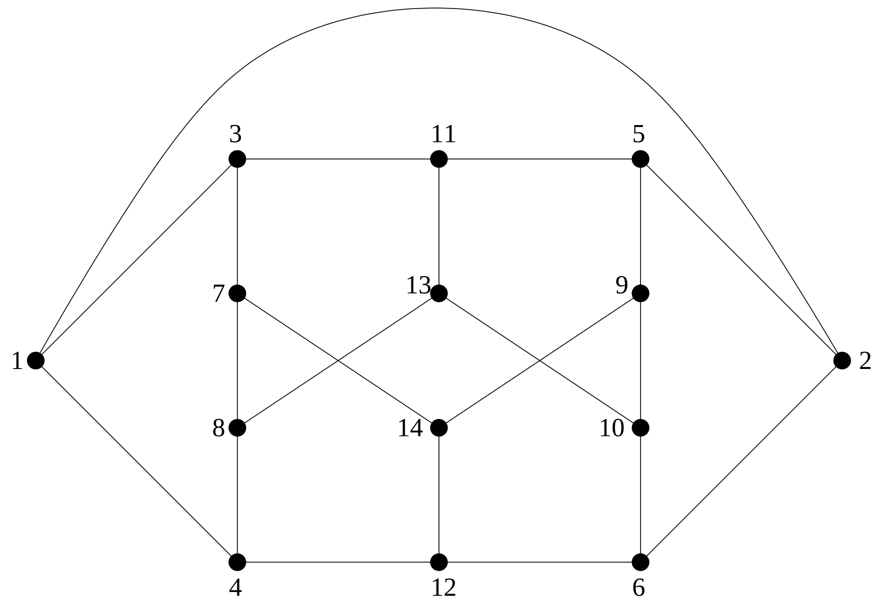


Figure 10: Drum.

14.1 *Let H be dodecahedrally connected, and not isomorphic to Triplex. Then H is arched if and only it contains none of Petersen, Drum.*

Proof. Since Drum contains Triplex (delete $9-10$) "only if" follows from 13.1. For "if", let H be dodecahedrally connected, not isomorphic to Triplex, and not arched, and suppose that H does not contain Petersen. We must show that H contains Drum. By 13.1, H contains Triplex; and so by 10.3, since Triplex is not a biladder, it follows that H contains Triplex + (e, f), where e, f are diverse edges of Triplex. But for all such choices of e, f, Triplex + (e, f) either contains Petersen or is isomorphic to Drum. This proves 14.1. ∎

In Figure 11 we define the graphs *Firstapex, Secondapex, Thirdapex, Fourthapex*, and *Sailboat*. They all contain Drum. We call the first four of them *Apex-selectors*.

14.2 *Let H be dodecahedrally connected, and not isomorphic to Triplex or Drum. Then H is arched if and only if it contains none of Petersen, an Apex-selector, or Sailboat.*

Proof. As in 14.1, "only if" is easy, and for "if" we may assume that H contains Drum, by 14.1. By 10.3 H contains Drum + (e, f) where e, f are diverse edges of Drum. There are (up to isomorphism of Drum) 26 possibilities for $\{e, f\}$; let $e = ab$, $f = cd$, and $G' =$ Drum + (ab, cd). If (a, b, c, d) is one of

$$(1, 2, 11, 13), (1, 3, 8, 13), (3, 7, 5, 9), (3, 11, 9, 14), (7, 14, 11, 13),$$

G is isomorphic to Firstapex, Secondapex, Thirdapex, Fourthapex and Sailboat respectively, and in all other cases G contains Petersen. This proves 14.2. ∎

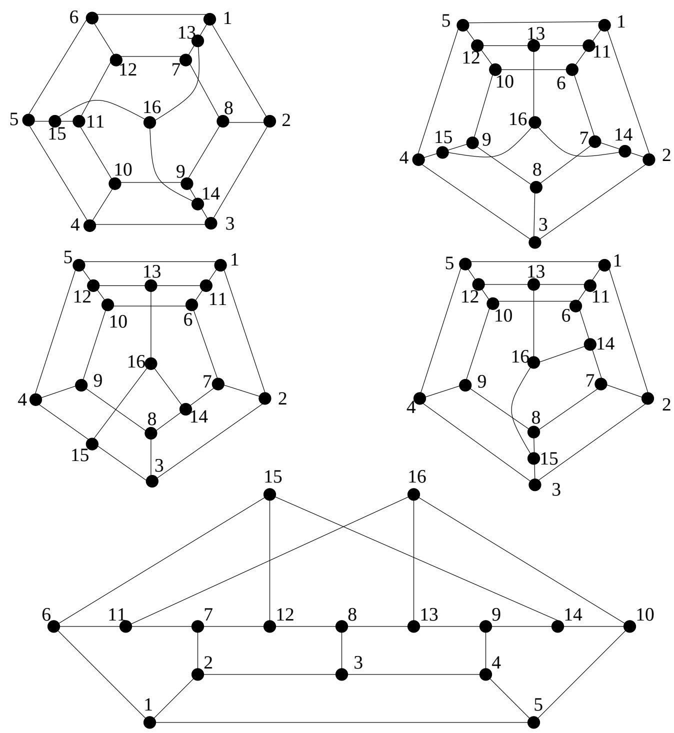


Figure 11: Firstapex, Secondapex, Thirdapex, Fourthapex and Sailboat.

Let us say H is *doubly-apex* if it has two vertices u, v so that the graph obtained from H by identifying u and v is planar. Sailboat is doubly-apex (identify 15 and 16) but the Apex-selectors are not, and Petersen is not. The main result of this section is the following.

14.3 *Let H be dodecahedrally connected. Then H is either arched or doubly-apex if and only if it does not contain Petersen or an Apex-selector.*

14.3 follows from the following.

14.4 *Let H be dodecahedrally connected, and contain Sailboat but not Petersen or any Apex-selector. Then H is doubly-apex.*

Proof of 14.3 assuming 14.4.

"If" is easy, and we omit it. For "only if", let H not contain Petersen or an Apex-selector. If H is isomorphic to Triplex or Drum it is doubly-apex as required. Otherwise, by 14.2 either it is arched or it contains Sailboat; and in the latter case by 14.4 it is doubly-apex. This proves 14.3. ∎

It remains to prove 14.4. That will require several lemmas. Let $\mathcal{C}$ be the set of the subgraphs of Sailboat induced on the following vertex sets:

$$1, 2, 3, 4, 5;$$
$$1, 2, 7, 11, 6;$$
$$2, 3, 8, 12, 7;$$
$$3, 4, 9, 13, 8;$$
$$4, 5, 10, 14, 9;$$
$$15, 6, 1, 5, 10, 16;$$
$$15, 6, 11, 16;$$
$$16, 11, 7, 12, 15;$$
$$15, 12, 8, 13, 16;$$
$$16, 13, 9, 14, 15;$$
$$15, 14, 10, 16.$$

These bound the regions when Sailboat is drawn in the plane with 15 and 16 identified.

Let Boat(1),...,Boat(7) be Sailboat + (ab, cd) where respectively (a, b, c, d) is

$$(2, 7, 12, 15), (7, 12, 6, 15), (1, 6, 11, 16), (2, 7, 11, 16), (6, 11, 12, 15), (9, 14, 12, 15), (6, 15, 12, 15).$$

14.5 *Let G be Sailboat, and let ab and cd be edges of G so that no member of $\mathcal{C}$ contains them both. Then $G + (ab, cd)$ contains Petersen or an Apex-selector or one of Boat(1),...,Boat(7).*

Proof. If $a = c$ then since no member of $\mathcal{C}$ contains ab and cd it follows that $a = 15$ or 16, and then $G + (ab, cd)$ is isomorphic to Boat(7). We assume therefore that $a, b \neq c, d$.

Up to the symmetry of Sailboat and exchanging ab with cd, there are 88 cases to be checked. Let $G' = G + (ab, cd)$. If (a, b, c, d) is (1, 6, 11, 16) or (6, 15, 7, 11), G' is (isomorphic to) Boat(3). If (a, b, c, d) is (7,12,6,11) or (2,7,11,16), G' is Boat(4). If (a, b, c, d) is (2, 7, 12, 15) or (7, 11, 8, 12), G' is Boat(1). If (a, b, c, d) is (1, 6, 14, 15) or (6, 11, 12, 15), G' is Boat(5). If (a, b, c, d) is (7, 12, 6, 15) or (8, 12, 14, 15), G' is Boat(2). If (a, b, c, d) is (9, 14, 12, 15) or (10, 14, 6, 15), G' is Boat(6). If $(a, b, c, d) =$ (2, 3, 12, 15), G' contains Firstapex; if $(a, b, c, d) =$ (1, 6, 10, 14), (3, 8, 12, 15) or (7, 11, 8, 13) it contains Secondapex; if (a, b, c, d) is one of

$$(1, 5, 6, 11), (1, 5, 14, 15), (1, 2, 6, 15), (1, 2, 11, 16), (2, 7, 6, 15), (8, 12, 11, 16)$$

G' contains Thirdapex; and in the remaining 66 cases, G' contains Petersen. This proves 14.5. ∎

14.6 *Let H be dodecahedrally connected, and not contain Petersen or an Apex-selector. Then H contains none of Boat(1),...,Boat(7).*

Proof.

(1) *H does not contain Boat(1).*

Subproof. Let $\mathcal{L}_1$ consist of Petersen and the four Apex-Selectors, and let C be the quadrangle of Boat(1). Then every A-extension of Boat(1) is killed by $\mathcal{L}_1$, and $\mathcal{P}(C, \mathcal{L}_1) = \emptyset$, so the claim follows from 10.1, taking F null. This proves (1).

(2) *H does not contain Boat(2).*

Subproof. Let C be the quadrangle of Boat(2). Then every A-extension of Boat(2) is killed by $\mathcal{L}_1$, and

$$\mathcal{P}(C, \mathcal{L}_1) = \{(17-18, 6-11), (17-18, 7-11)\}^*.$$

We apply 10.2 with F null, and the result follows from 10.2. This proves (2).

(3) *H does not contain Boat(3) or Boat(4).*

Subproof. Let G be Boat(3) or Boat(4), and $\mathcal{L}_3 = \mathcal{L}_1 \cup \{Boat(2)\}$. Let C be the quadrangle of G. Then every A-extension of G is killed by $\mathcal{L}_3$, and $\mathcal{P}(C, \mathcal{L}_3) = \emptyset$, so the result follows from (2) and 10.1. This proves (3).

(4) *H does not contain Boat(5) or Boat(6).*

Subproof. Let G be Boat(5) or Boat(6), and let

$$\mathcal{L}_4 = \mathcal{L}_3 \cup \{Boat(3), Boat(4)\}.$$

Let C be the quadrangle of G. Then every A-extension of G is killed by $\mathcal{L}_4$, and $\mathcal{P}(C, \mathcal{L}_4) = \emptyset$, so the result follows from (2), (3) and 10.1. This proves (4).

(5) *H does not contain Boat(7).*

Subproof. Let G be Boat(7), and let C be its circuit of length 3. Let $X = V(C)$. Suppose that there is a homeomorphic embedding of G in H; then by 8.1, there is a X-augmenting sequence $(e_1, f_1), \dots, (e_n, f_n)$ of G so that H contains $G + (e_1, f_1) + \dots + (e_n, f_n)$. From the definition of "X-augmentation" it follows that $n = 1$ since $|E(C)| = 3$; and so H contains $G(e_1, f_1)$ for some $e_1 \in E(C)$ and $f_1 \in E(G \backslash X)$. But for all such $e_1, f_1, G + (e_1, f_1)$ contains a member of $\mathcal{L}_1$ or one of Boat(2), Boat(5), Boat(6), a contradiction by (2) and (4). This proves (5).

From (1)–(5), this proves 14.6. ∎

Proof of 14.4.

Let H be dodecahedrally connected and not contain Petersen or an Apex-selector. Let η be a homeomorphic embedding of G in H, where G is Sailboat. Let $V(F) = \{15, 16\}$ and $E(F) = \emptyset$; and let η_F be the restriction of η to F. Let $\mathcal{C}$ be as before. Then $(G, F, \mathcal{C})$ is a framework, and we claim that (E1)–(E7) hold. By 14.6 H contains none of Boat(1),..., Boat(7), so by 14.5 (E2)(ii) holds. All the others are clear except for (E6), and for (E6) we need only consider cross-extensions of G on some of the paths in $\mathcal{C}$, namely the ones with vertex sets

$$\{15, 6, 1, 5, 10, 16\}, \{16, 11, 7, 12, 15\}, \{15, 12, 8, 13, 16\}$$

(and two more, that from symmetry we need not consider). We need to examine

$$G + (6-15, 10-16) + (1-6, 16-18)$$
$$G + (6-15, 10-16) + (6-17, 16-18)$$
$$G + (6-15, 5-10) + (6-17, 10-18)$$
$$G + (6-15, 5-10) + (1-6, 10-16)$$
$$G + (11-16, 12-15) + (11-17, 15-18)$$
$$G + (12-15, 13-16) + (12-17, 16-18);$$

they contain Thirdapex, Boat(3), Boat(3), Petersen, Boat(3) and Boat(3) respectively. Hence (E6) holds, and from 7.1, this proves 14.4. ∎

15 Dodecahedrally connected non-apex graphs

The graphs *Diamond, Concertina* and *Bigdrum* are defined in Figure 12.

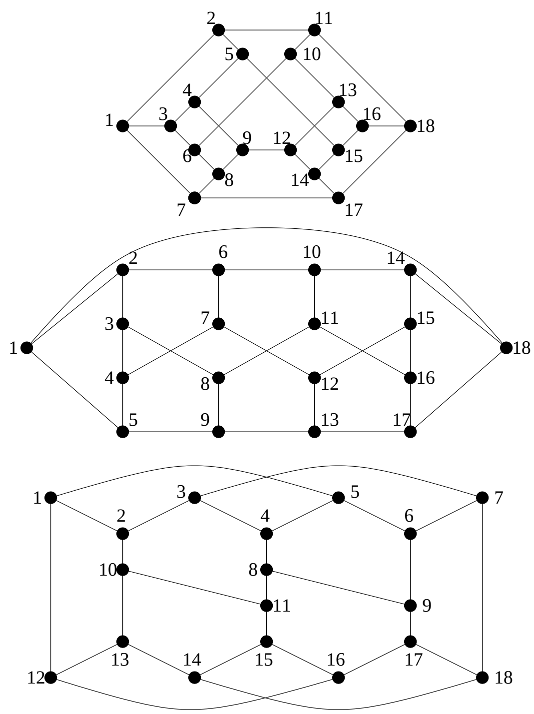


Figure 12: Diamond, Bigdrum and Concertina.

In this section we prove the following.

15.1 *Let H be dodecahedrally connected. Then H is apex if and only if it contains none of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum.*

Let Square(1) be Secondapex + $(14-16, 11-13)$. Let Square(2),..., Square(5) be Fourthapex + (ab, cd) where (a, b, c, d) is

$$(1, 5, 10, 12), (1, 11, 6, 10), (6, 14, 13, 16), (12, 13, 15, 16)$$

respectively. Let Square(6) and Square(7) be Thirdapex + (ab, cd) where (a, b, c, d) is $(3, 15, 14, 16)$ and $(2, 3, 8, 9)$ respectively.

15.2 *Let H be dodecahedrally connected, and not contain any of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum. Then it contains none of Square(1),..., Square(7).*

Proof.

(1) *H does not contain Square(1).*

Subproof. Let G be Square(1), let C be the quadrangle of G, and let

$$\mathcal{L}_1 = \{\text{Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum}\}.$$

Every A-extension of G is killed by $\mathcal{L}_1$ (indeed, by {Petersen, Jaws, Starfish}), and

$$\mathcal{P}(C, \mathcal{L}_1) = \{(13-18, 5-12), (13-18, 10-12), (13-18, 1-11), (13-18, 6-11)\}^*.$$

(Note that $G + (13-18, 1-5)$ is isomorphic to Jaws, and $G + (16-17, 3-8)$ to Starfish.) Then we verify the hypotheses of 10.2; and find that all the various extensions listed in 10.2 contain Petersen, except for the B-extensions

$$G + (13-18, 12-5) + (12-20, 4-15)$$
$$G + (13-18, 12-5) + (12-20, 11-18)$$
$$G + (13-18, 12-5) + (12-20, 15-16).$$

(which contain Jaws, Diamond, and Concertina respectively) and the C-extension

$$G + (13-18, 12-5) + (19-20, 1-11)$$

(which contains Jaws), and isomorphic extensions. Hence, from 10.2, this proves (1).

Now let

$$\mathcal{L}_2 = \{\text{Petersen, Square(1), Diamond, Concertina, Bigdrum}\}$$

(Jaws and Starfish are no longer necessary, since they both contain Square(1).)

(2) *H does not contain Square(2).*

Subproof. We apply 10.1 to the quadrangle C of Square(2), with $\mathcal{L} = \mathcal{L}_2$. All A-extensions are killed by $\mathcal{L}_2$, and $\mathcal{P}(C, \mathcal{L}_2) = \emptyset$, so the result follows from 10.1. This proves (2).

(3) *H does not contain Square(3).*

Subproof. Let C be the quadrangle of $G =$ Square(3); we apply 10.2, with $\mathcal{L} = \mathcal{L}_2$. All A-extensions are killed by $\mathcal{L}_2$, and

$$\mathcal{P}(C, \mathcal{L}_2) = \{(6-11, 13-16), (6-11, 14-16)\}^*.$$

We verify the hypotheses of 10.2. This proves (3).

(4) *H does not contain Square(4).*

Subproof. Now let $\mathcal{L}_4 = \mathcal{L}_2 \cup$ {Square(2), Square(3)}. The result follows from 10.1, applied to the quadrangle of Square(4) and $\mathcal{L}_4$, using (2) and (3). This proves (4).

(5) *H does not contain Square(5).*

Subproof. Let $\mathcal{L}_5 = \mathcal{L}_4 \cup$ {Square(4)}, and C the quadrangle of $G =$ Square(5). Then all A-extensions are killed by $\mathcal{L}_5$, and

$$\mathcal{P}(C, \mathcal{L}_5) = \{(13-17, 6-11)\}^*;$$

and we verify the hypotheses of 10.2 to prove (5).

(6) *H does not contain Square(6).*

Subproof. Let $\mathcal{L}_6 = \mathcal{L}_5 \cup$ {Square(5)}, and C, G as usual. All A-extensions are killed by $\mathcal{L}_6$, and

$$\mathcal{P}(C, \mathcal{L}_6) = \{(17-18, 3-8), (17-18, 8-14)\}^*;$$

and again the result follows from 10.2. This proves (6).

(7) *H does not contain Square(7).*

Subproof. Let $\mathcal{L}_7 = \mathcal{L}_6 \cup$ {Square(6)}, and C, G as usual. Then all A-extensions are killed by $\mathcal{L}_7$, and $\mathcal{P}(C, \mathcal{L}_7) = \emptyset$, so (7) follows from 10.1.

From (1)–(7), this proves 15.2. ∎

The graph *Extrapex* is defined in Figure 13. We say that G is an *Apex-forcer* if either it is an Apex-selector or it is Extrapex. By the *Non-apex family* we mean

{Petersen, Diamond, Concertina, Bigdrum, Square(1),..., Square(7)}.

15.3 *Let G be an Apex-forcer. Let $\mathcal{C}$ be the set of circuits that bound regions in the planar drawing of $G\backslash 16$. If ab and cd are edges of G with $a, b \neq c, d$, and no member of $\mathcal{C}$ contains them both, then either $G + (ab, cd)$ contains a member of the Non-apex family, or one of a, b, c, d is 16 and the other three belong to some member of $\mathcal{C}$.*

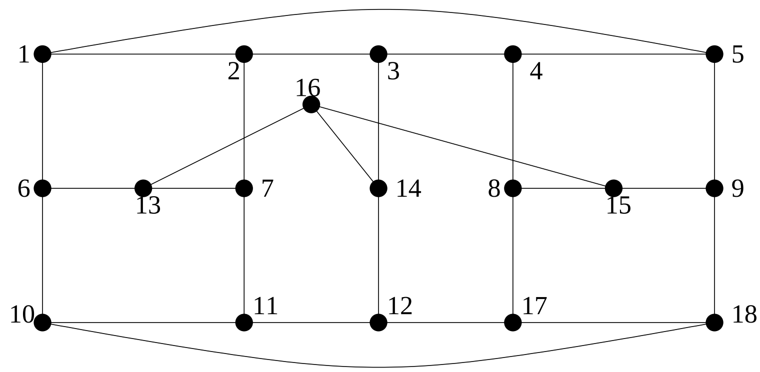


Figure 13: Extrapex.

We leave the proof to the reader (the details are in the Appendix [4]).

If G is an Apex-forcer, and η is a homeomorphic embedding of G in H, we define the *spine* of η to be $\eta(13-16) \cup \eta(14-16) \cup \eta(15-16)$.

15.4 *Let H be cubic and cyclically 4-connected, and contain no member of the Non-apex family. Let H contain some Apex-forcer. Then there is a homeomorphic embedding η of some Apex-forcer in H such that its spine has only three edges.*

Proof. Choose an Apex-forcer G and a homeomorphic embedding η of G in H, such that its spine is minimal. Suppose its spine has more than three edges; then since H is cyclically four-connected, there is an η-path P with one end in $\eta(e)$ and the other in $\eta(f)$, where f is one of $13-16, 14-16, 15-16$ and e is not incident with 16. If e and f have a common end then by rerouting e along P we obtain a new homeomorphic embedding with smaller spine, a contradiction. Similarly, it follows that e and f are diverse in G. By 15.3 there exists $C \in \mathcal{C}$ so that $e \in E(C)$ and f has an end in $V(C)$. Now we must examine cases. Let $e = ab$ and $f = c - 16$.

If G is Firstapex, we may assume that $(a, b, c) = (2, 8, 13)$ from the symmetry. Then $\eta(G \backslash 6 - 12) \cup P$ yields a homeomorphic embedding of Secondapex with smaller spine, a contradiction.

If G is Secondapex, there are three possibilities for (a, b, c) : $(1, 5, 13)$ (when $\eta(G \backslash 6-10) \cup P$ yields a homeomorphic embedding of Firstapex), (1, 11, 14) (when $\eta(G \backslash 1-5) \cup P$ yields a homeomorphic embedding of Fourthapex), and (3, 8, 14) (when $\eta(G) \cup P$ yields a homeomorphic embedding of Extrapex), in each case contradicting the minimality of the spine.

If G is Thirdapex, the possibilities for (a, b, c) are: (1, 5, 13) or (2, 3, 14) (when $\eta(G \backslash 8-9) \cup P$ yields a homeomorphic embedding of Fourthapex), (6, 10, 14) (when $\eta(G \backslash 1-11) \cup P$ yields a homeomorphic embedding of Thirdapex), and (9, 10, 14) (when $\eta(G \backslash 2-7) \cup P$ yields a homeomorphic embedding of Firstapex), in each case a contradiction.

If G is Fourthapex, the possibilities are: (1, 5, 13) (when $\eta(G \backslash 4-9) \cup P$ yields a homeomorphic embedding of Thirdapex), (6, 10, 13) (when $\eta(G) \cup P$ yields a homeomorphic embedding of Extrapex), (1, 2, 14) (when $\eta(G \backslash 4-9) \cup P$ yields a homeomorphic embedding of Secondapex), and (1, 11, 14) (when $\eta(G \backslash 10-12) \cup P$ yields a homeomorphic embedding of Thirdapex), in each case a contradiction.

If G is Extrapex, the possibilities are: (1, 2, 13) (when $\eta(G \backslash \{7-13, 1-6\}) \cup P$ yields a homeomorphic embedding of Secondapex) and (2, 7, 14) (when $\eta(G \backslash \{2-3, 10-11\}) \cup P$ yields a homeomorphic embedding of Thirdapex), in each case a contradiction.

Hence the spine has only three edges. This proves 15.4. ∎

Proof of 15.1.
"Only if" is easy, and we omit it. For "if", let H be dodecahedrally connected, and not contain any of Petersen, Jaws, Starfish, Diamond, Concertina, Bigdrum. By 15.2 it contains none of Square(1),..., Square(7). We may assume that H is not arched or doubly-apex, for such graphs are apex; and so by 14.3 H contains an Apex-selector. By 15.4, there is a homeomorphic embedding η of some Apex-forcer G in H so that its spine has only three edges. Let F be the subgraph of G induced on $\{13, 14, 15, 16\}$, and let η_F be the restriction of η to F. Let $\mathcal{C}$ be as in 15.3; then $(G, F, \mathcal{C})$ is a framework, and H, η_F satisfies (E1). We claim they satisfy (E2)–(E7). (E2) follows from 15.3, and (E3), (E7) are vacuously true. For (E4), (E5) and (E6) a large amount of case-checking is required, for G = Firstapex, Secondapex, Thirdapex, Fourthapex and Extrapex, separately. The details are in the Appendix [4]. From 7.1, this proves 15.1. ∎

16 Die-connected non-apex graphs

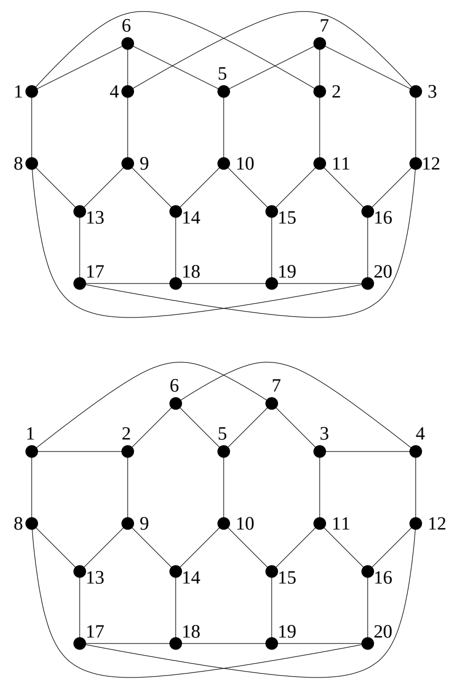


Figure 14: Antilog and Log.

Our next real objective in this paper is modify 15.1 to find all the cubic graphs G minimal with the properties that they are non-apex and dodecahedrally connected, and $|\delta(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$. (There are only three such graphs, namely Petersen, Jaws and Starfish, as we shall see in the next section.) Diamond, Concertina and Bigdrum all have subsets X with $|\delta(X)| = 5$ and $|X|, |V(G) \setminus X| \geq 9$, so they are rather far from having the property we require; and a convenient half-way stage is afforded by "die-connectivity". We recall that a graph G is *die-connected* if it is dodecahedrally connected (and hence cubic and cyclically five-connected) and $|\delta(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 9$. In this section we find all minimal graphs that are non-apex and die-connected. The graphs Log, Antilog, and Dice(1),..., Dice(4) are defined

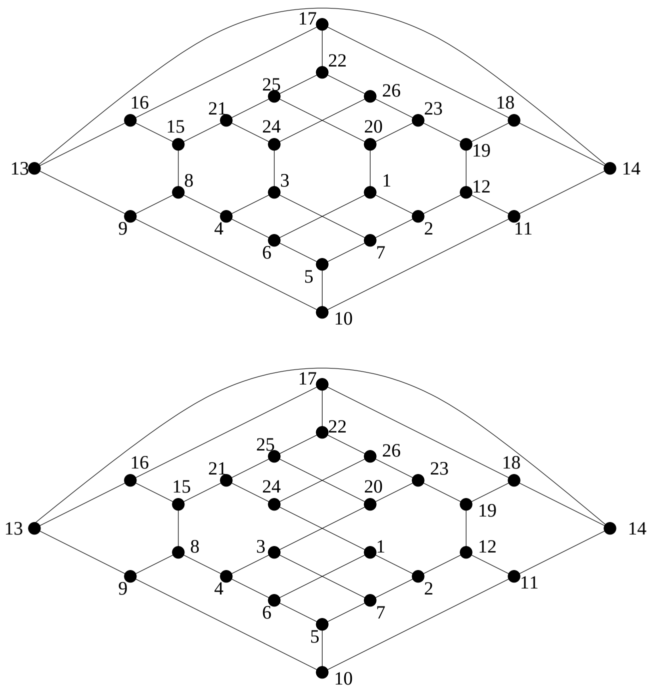


Figure 15: Dice(1) and Dice(3).

in Figures 14, 15 and 16. We shall show the following.

16.1 *Let H be die-connected. Then H is apex if and only if H contains none of Petersen, Jaws, Starfish, Log, Antilog, Dice(1), Dice(2), Dice(3), Dice(4).*

We begin with the following.

16.2 *Any die-connected graph that contains Diamond also contains one of Petersen, Antilog, Dice(4).*

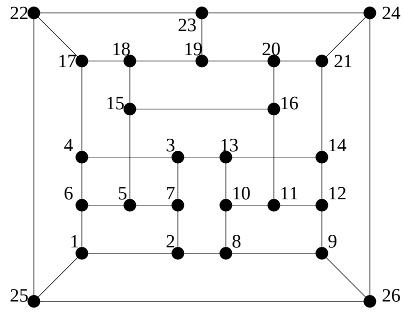
Proof. Let H be die-connected, and contain no member of $\mathcal{L} = \{$Petersen, Antilog, Dice(4)$\}$. We claim first that

(1) *H does not contain Diamond $+(1-2, 10-11)$.*

Subproof. Let C be the quadrangle of $G =$ Diamond $+(1-2, 10-11)$. Then all A-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(2-19, 4-5), (11-20, 10-13)\}^*.$$

We verify the hypotheses of 10.2 (the E-extension is isomorphic to Dice(4)). This proves (1).



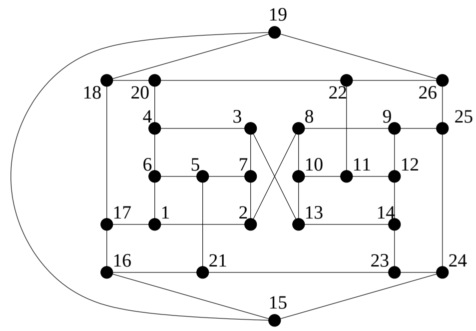


Figure 16: Dice(2) and Dice(4).

Now let $\mathcal{L}' = \{$Petersen, Antilog, Diamond + $(1-2, 10-11)\}$, and $X = \{1, \ldots, 9\}$.

(2) *Every X-augmentation of Diamond contains a member of $\mathcal{L}'$.*

Subproof. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be an X-augmenting sequence, and suppose the corresponding X-augmentation contains no member of $\mathcal{L}'$. In particular, Diamond + (e_1, f_1) contains no member of $\mathcal{L}'$, and so (by checking all possibilities) it follows that f_1 is $6-10$ and e_1 is one of $1-2, 1-7, 4-9$. In particular, $n \geq 2$. Since $f_1 = 6-10$ it follows that $e_2 = 6-20$. If e_1 is $1-7$ or $4-9$ there is no possibility for f_2. Thus $e_1 = 1-2$, and then f_2 is $9-12$, and $n \geq 3$, and $e_3 = 9-22$. Again by checking cases it follows that $f_3 = 7-17$, and hence $n \geq 4$ and $e_4 = 7-24$; and there is no possibility for f_4, a contradiction. This proves (2).

From (1), (2) and 8.1, the result follows since H is die-connected. This proves 16.2. ∎

16.3 *Every die-connected graph that contains Bigdrum also contains one of Petersen, Diamond or Dice*(2)*.*

Proof. Let H be die-connected, and contain no member of $\mathcal{L} = \{$Petersen, Diamond, Dice(2)$\}$. We claim first

(1) *H does not contain Bigdrum* $+(3-8, 10-11)$.

Subproof. Let G = Bigdrum + $(3-8, 10-11)$, and let C be the quadrangle of G. Then all A-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(8-11, 9-13), (19-20, 10-14)\}^*.$$

The result follows from 10.2 by checking all the various extensions (in particular,

$$G + (8-19, 5-9) + (11-20, 10-14) + (8-21, 20-23)$$

is isomorphic to Dice(2)). This proves (1).

Now let $\mathcal{L}' = \{$Petersen, Diamond, Bigdrum + $(3-8, 10-11)\}$ and $X = \{1, \ldots, 9\}$. We claim that

(2) *Every X-augmentation of Bigdrum contains a member of $\mathcal{L}'$.*

Subproof. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be an X-augmenting sequence, so that the corresponding X-augmentation contains no member of $\mathcal{L}'$. Then by checking cases it follows that (e_1, f_1) is one of $(3-8, 6-10), (4-7, 9-13)$, and by the symmetry we may assume the first. Then $n \geq 2$, and $e_2 = 6-20$; and there is no possibility for f_2, a contradiction. This proves (2).

From (1), (2) and 8.1, this proves 16.3. ∎

16.4 *Any die-connected graph that contains Concertina also contains one of Petersen, Log, Diamond, Bigdrum, Dice*(1)*, Dice*(3)*.*

Proof. Let H be a die-connected graph that contains no member of $\mathcal{L} = \{$Petersen, Log, Diamond, Bigdrum, Dice(1), Dice(3)$\}$. Let Conc(1), Conc(2), Conc(3) be Concertina + (e, f) where (e, f) is $(4-8, 10-11)$, $(6-7, 17-18)$, $(8-9, 16-17)$; and let Conc(4) be Concertina + $(2-3, 8-11)+$ $(8-20, 16-17)$.

(1) *H does not contain Conc(1).*

Subproof. Let C be the quadrangle of G = Conc(1). All A-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(8-11, 9-17), (19-20, 2-10)\}^*;$$

and the result follows by verifying the other hypotheses of 10.2. (The E-extension is isomorphic to Dice(1).)

Let Conc(21) be Conc(2) + $(7-19, 1-5)$, let Conc(211) be Conc(21) + $(1-2, 3-4)$, and let Conc(212) be Conc(21) + $(1-2, 3-7)$. This proves (1).

(2) *H does not contain Conc*(211) *or Conc*(212)*.*

Subproof. Let G = Conc(211) and let C be its quadrangle. Then all A-extensions are killed by $\mathcal{L}$, and

$$\mathcal{P}(C, \mathcal{L}) = \{(2-23, 1-12)\}^*,$$

and the result for Conc(211) follows by verifying the other hypotheses of 10.2.

Now let G = Conc(212) and let C be its quadrangle. Again all A-extensions are killed by $\mathcal{L}$, and again

$$\mathcal{P}(C, \mathcal{L}) = \{(2-23, 1-12)\}^*$$

and again the result follows from 10.2. (Conc(212) + $(3-24, 1-22)$ is isomorphic to Dice(3).) This proves (2).

(3) *H does not contain Conc(21).*

Subproof. Let $\mathcal{L}_1 = \mathcal{L} \cup$ {Conc(211), Conc(212)}. Let $X = \{1, 2, 10, 11, 12, 13, 14, 15, 16\}$; we claim that every X-augmentation of Conc(21) contains a member of $\mathcal{L}_1$. For suppose not, and let the corresponding sequence be $(e_1, f_1), \ldots, (e_n, f_n)$. By checking cases, $e_1 = 12-16$ and $f_1 = 14-18$; and so $n \geq 2$, and $e_2 = 14-20$, and there is no possibility for f_2. Hence (3) follows from 8.1 and (2).

(4) *H does not contain Conc(2).*

Subproof. Let $\mathcal{L}_2 = \mathcal{L} \cup$ {Conc(21)}, G = Conc(2), and C the quadrangle of G. Then all A-extensions are killed by $\mathcal{L}_2$, and

$$\mathcal{P}(C, \mathcal{L}_2) = \{(19-20, 6-9), (19-20, 9-17)\}^*$$

and the result follows by verifying the hypotheses of 10.2. This proves (4).

(5) *H does not contain Conc(3).*

Subproof. Let $\mathcal{L}_3 = \mathcal{L} \cup$ {Conc(2)}, G = Conc(3), and C the quadrangle of G. Then all A-extensions are killed by $\mathcal{L}_3$, and

$$\mathcal{P}(C, \mathcal{L}_3) = \{(9-19, 4-8)\}^*,$$

and the result follows by verifying the hypotheses of 10.2. This proves (5).

(6) *H does not contain Conc(4).*

Subproof. Let $\mathcal{L}_4 = \mathcal{L} \cup$ {Conc(2), Conc(3)}, and X = {3, 4, 5, 6, 7, 8, 9, 17, 18}. We claim that every X-augmentation of G = Conc(4) contains a member of $\mathcal{L}_4$. Suppose not, and let the corresponding sequence be $(e_1, f_1), \ldots, (e_n, f_n)$. By checking cases, $e_1 = 3-7$ and f_1 is $1-5$; so $n \geq 2$, and e_2 is $5-24$, and there is no possibility for f_2, a contradiction. Hence (6) follows from 8.1.

Let $\mathcal{L}_5 = \mathcal{L}_4 \cup$ {Conc(1), Conc(4)}, and X = {1,. . . ,9}. We claim that every X-augmentation of G = Concertina contains a member of $\mathcal{L}$. Suppose not, and let $(e_1, f_1), \ldots, (e_n, f_n)$ be the corresponding sequence. By checking cases (e_1, f_1) is one of $(2-3, 8-11)$, $(4-8, 2-10)$; so $n \geq 2$, and in either case there is no possibility for f_2. Hence the result follows from (1), (4), (5), (6) and 8.1. This proves 16.4. ∎

Proof of 16.1. "Only if" is easy, and we omit it. For "if", let H contain none of the given graphs. By 16.2, 16.3, 16.4 it contains none of Diamond, Bigdrum, Concertina; and so by 15.1 it is apex. This proves 16.1. ∎

17 Theta-connected non-apex graphs

We recall that G is *theta-connected* if it is cubic and cyclically five-connected, and $|\delta(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 7$ (and hence it is dodecahedrally connected). None of the graphs of Figures 14 - 16 are theta-connected, and our next objective is to make a version of 16.1 for theta-connected graphs. It becomes much simpler:

17.1 *Let H be theta-connected. Then H is apex if and only if it contains none of Petersen, Jaws and Starfish.*

For the proof we use 17.2 below. A *domino* in a cubic graph H is a subgraph D with $|V(D)| = 7$, consisting of the union of three paths P_1, P_2, P_3 of lengths two, three and three respectively, which have common ends and otherwise are disjoint. The middle vertex of P_1 is called the *centre* of the domino, and the other four vertices of degree two are its *corners*; an *attachment sequence*

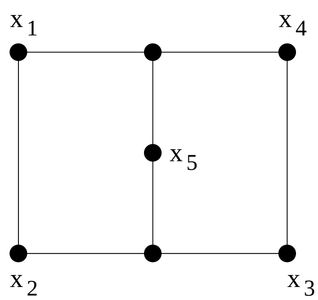


Figure 17: A domino.

is some sequence $(x_1, \ldots, x_5)$ where $x_1, \ldots, x_4$ are the corners, x_5 is the centre, x_1x_2 is an edge, and x_2, x_3 have a common neighbour. (See Figure 17.)

A domino D in G with attachment sequence $(x_1, \ldots, x_5)$ is said to be *crossed* if

- there are two disjoint connected subgraphs P, Q of G, both edge-disjoint from D, with $V(P \cap D) = \{x_1, x_3\}$ and $V(Q \cap D) = \{x_2, x_4, x_5\}$, and
- there are two disjoint connected subgraphs P, Q of G, both edge-disjoint from D, with $V(P \cap D) = \{x_1, x_3, x_5\}$ and $V(Q \cap D) = \{x_2, x_4\}$.

17.2 *Let D be a crossed domino with attachment sequence $x_1, \ldots, x_5$, in a cyclically 5-connected cubic graph G with $|V(G)| \geq 14$. Let x_5 be incident with $g \notin E(D)$. Let F be a subgraph of G with minimum degree ≥ 2, with $F \cap D$ null. Let H be a cubic graph, cyclically 5-connected, such that F is a subgraph of H; and let η be a homeomorphic embedding of G in H fixing F. Then either*

- *there exists $X \subseteq V(H)$ with $|\delta_H(X)| = 5$, so that for all $v \in V(G)$, $\eta(v) \in X$ if and only if $v \in V(D)$, or*

- *H contains Petersen, or*
- *for some $e \in E(D)$ and $f \in E(G \backslash V(D)) \setminus E(F)$ there is a homeomorphic embedding η' of $G + (e, f)$ in H fixing F, or*
- *for some $e \in \{x_1x_2, x_3x_4\}$, and for some $f \in E(G \backslash V(D)) \setminus E(F)$ such that f, g are diverse in G, there is a homeomorphic embedding η' of*

$$G + (e, g) + (yx_5, f)$$

in H fixing F, where x, y are the new vertices of $G + (e, g)$.

Proof. Let $X = V(D)$. We assume that (i) and (ii) are false. Since $|V(G)| \geq 14$ and $|\delta_G(X)| = 5$, and since (i) is false, it follows from 8.1 that there is an X-augmentation G' of G, and a homeomorphic embedding η' of G' in G fixing F. Let $(e_1, f_1), \ldots, (e_n, f_n)$ be the corresponding sequence. If $n = 1$ then (iii) is true, so we assume that $n \geq 2$. For $1 \leq i \leq 5$, let x_i be adjacent in G to $y_i \in V(G) \backslash V(D)$. Let the neighbours of x_5 in G be y_5, x_6, x_7, where x_6 is adjacent to x_1. Let $G_1 = G + (e_1, f_1)$ with new vertices s_1, t_1, and let D_1 be the subgraph of G_1 induced on $V(D) \cup \{s_1, t_1\}$.

Suppose first that $f_1 = x_1y_1$. Then since e_1 and f_1 are diverse in G, it follows that $e_1 = a_1b_1$ say where $a_1, b_1 \in \{x_3, x_4, x_5, x_7\}$, that is, e_1 is one of x_3x_4, x_3x_7, x_5x_7. If $f_1 = 3-4$ or $3-7$, let P, Q be disjoint paths of G_1 from x_2 to x_4 and from t_1 to x_5, with no vertices or edges in D_1 except their ends; and let R be a path of $G \backslash V(D)$ between $V(P)$ and $V(Q)$ with no internal vertex or edge in P or Q. Then $D_1 \cup P \cup Q \cup R$ is homeomorphic to Petersen, and so G_1 and hence H contains Petersen, and (ii) is true, a contradiction. So $e_1 = x_5x_7$. Let P, Q be disjoint paths of G_1 from t_1 to x_3 and from x_2 to x_5, with no vertices or edges in D_1 except their ends, and let R be as before. Then $D_1 \cup P \cup Q \cup R$ again is homeomorphic to Petersen, a contradiction.

Hence $f_1 \neq x_1y_1$, and so by symmetry $f_1 \neq x_2y_2, x_3y_3, x_4y_4$; and hence $f = x_5y_5$. Hence $e_1 = 1-2$ or $3-4$, and by symmetry we may assume the first. Also, $e_2 = x_5t_1$, and there are (up to the symmetry) three possibilities for f_2, namely $f_2 = x_1y_1, f_2 = x_4y_4$, and $f_4 \in E(G \backslash V(D))$. In the third case the theorem is true, so we assume for a contradiction that one of the first two cases hold. Let $G_2 = G_1 + (e_2, f_2)$, with new vertices s_2, t_2, and let D_2 be the subgraph of G_2 induced on $V(D) \cup \{s_1, t_1, s_2, t_2\}$.

If $f_2 = x_1y_1$, let P, Q be disjoint paths of G_2 from t_2 to x_3 and from t_1 to x_4 with no vertices or edges in D_2 except their ends; then $D_2 \cup P \cup Q$ is homeomorphic to Petersen, a contradiction. But if $f_2 = x_4y_4$, let P, Q be disjoint paths of G_2 from x_2 to t_2 and t_1 to x_3, with no vertices or edges in D_2 except their ends; then $D_2 \cup P \cup Q$ is homeomorphic to Petersen, a contradiction. This proves 17.2. ∎

Proof of 17.1. "Only if" is easy and we omit it. For "if", let H be theta-connected and not contain Petersen, Jaws or Starfish.

(1) *H does not contain Antilog.*

Subproof. Let G be Antilog, let $X = \{1, \ldots, 7\}$, and let $D = G|X$. Then D is a crossed domino of G. But the following all contain Petersen:

- $G + (e, f)$ for all $e \in E(D)$ and $f \in E(G\backslash X)$
- $G + (1-6, 5-10) + (5-22, xy)$ for all $xy \in E(G\backslash X)$ with $x, y \neq 10, 14, 15$.

From 17.2 (taking F null), this proves (1).

Let $\mathcal{L} = \{$Petersen, Jaws$\}$.

(2) *H does not contain Log.*

Subproof. Let Log(1) be Log + $(1-2, 8-13)$, let C be its quadrangle, and let $\mathcal{L}_1 = \mathcal{L} \cup \{$Antilog$\}$. All A-extensions are killed by $\mathcal{L}_1$, and

$$\mathcal{P}(C, \mathcal{L}_1) = \{(21-22, 2-9), (21-22, 13-9)\}^*,$$

and it follows by verifying the hypotheses of 10.2 that H does not contain Log(1).

Let Log(2) be Log + $(1-2, 9-13)$, let C be its quadrangle, and $\mathcal{L}_2 = \mathcal{L}_1 \cup \{$Log(1)$\}$. All A-extensions are killed by $\mathcal{L}_2$, and $\mathcal{P}(C, \mathcal{L}_2) = \emptyset$, and so by 10.1 H does not contain Log(2).

Now let $G =$ Log, $X = 1, \ldots, 7$, and $\mathcal{L}_3 = \mathcal{L}_2 \cup \{$Log(2)$\}$. For any edge e of $G|X$ and edge f of G not in $G|X$ (we permit f to have one end in X), if e, f are diverse then $G + (e, f)$ contains a member of $\mathcal{L}_3$; and so H does not contain Log, by (1) and 8.1. This proves (2).

(3) *H does not contain Dice(1).*

Subproof. Let Dice(11) = Dice(1) + $(1-2, 20-23)$, let C be its quadrangle, and $\mathcal{L}_4 = \{$Petersen, Jaws, Log, Antilog$\}$. All A-extensions are killed by $\mathcal{L}_4$, and $\mathcal{P}(C, \mathcal{L}_4) = \emptyset$, so by 10.1 H does not contain Dice(11).

Now let $\mathcal{L}_5 = \mathcal{L}_4 \cup \{$Dice(11)$\}$, let $G =$ Dice(1), $X = \{1, \ldots, 7\}$ and $D = G|X$; then D is a crossed domino in G. For all $e \in E(D)$ and $f \in E(G\backslash X)$, $G + (e, f)$ contains a member of $\mathcal{L}_4$; and for all $xy \in E(G\backslash X)$ with $x, y \neq 9, 10, 11$, $G + (1-2, 5-10) + (5-28, xy)$ contains Petersen. Hence the result follows from 17.2. This proves (3).

(4) *H does not contain Dice(2).*

Subproof. Let $G =$ Dice(2), $X = \{1, \ldots, 7\}$ and $\mathcal{L}_6 = \{$Petersen, Antilog, Dice(1)$\}$. For all $e \in E(G|X)$ and $f \in E(G) \setminus E(G|X)$, if e, f have no common end then $G + (e, f)$ contains a member of $\mathcal{L}_6$; so (4) follows from (1), (3) and 8.1.

(5) *H does not contain Dice(3).*

Subproof. Let Dice(31) = Dice(3) + $(3-4, 13-14)$, let C be its quadrangle, and $\mathcal{L}_4$ as before. All A-extensions are killed by $\mathcal{L}_4$, and $\mathcal{P}(C, \mathcal{L}_4) = \emptyset$, so by 10.1 H does not contain Dice(31).

Let $\mathcal{L}_7 = \mathcal{L}_4 \cup \{$Dice(31)$\}$. Let $G =$ Dice(3), $X = \{1, \ldots, 7\}$, and $D = G|X$. Then D is a crossed domino in G. For all $e \in E(D)$ and $f \in E(G\backslash X)$, $G + (e, f)$ contains a member of $\mathcal{L}_7$. Moreover, for all $xy \in E(G\backslash X)$ with $x, y \neq 15, 16, 18$,

$$G + (1-2, 5-15) + (5-28, xy)$$

$$G + (3-4, 5-15) + (5-28, xy)$$

both contain Petersen or Log. From (1)-(3) and 17.2, this proves (5).

(6) *H does not contain Dice(4).*

Subproof. Let G = Dice(4), $X = \{1,\ldots,7\}$ and $D = G|X$. Then D is a crossed domino in G. But for all $e \in E(D)$ and $f \in E(G\backslash X)$, $G+(e,f)$ contains Petersen or Log; and for all $xy \in E(G\backslash X)$ with $x, y \neq 16, 21, 23$,

$$G + (1-2, 5-21) + (5-28, xy)$$

$$G + (3-4, 5-21) + (5-28, xy)$$

both contain Petersen or Log. The result follows from (2) and 17.2. This proves (6).

From (1)-(6) and 16.2, this proves 17.1. ∎

The reader may have noticed that Starfish hardly ever is needed for anything. There is an explanation, the following (previously stated as 1.2).

17.3 *Every dodecahedrally connected graph H containing Starfish either is isomorphic to Starfish or contains Petersen.*

Proof. If H "properly" contains G = Starfish, then by 10.3 H contains a graph G' = Starfish + (e, f) for some choice of diverse edges e, f of G. But every such graph G' contains Petersen. This proves 17.3. ∎

From 17.3 we obtain a slightly stronger reformulation of 17.1, previously stated as 1.3.

17.4 *Let H be theta-connected, and not isomorphic to Starfish. Then H is apex if and only if it contains none of Petersen, Jaws.*

The proof is clear.

18 Excluding Petersen

In this section we prove 1.3, thereby completing the proof of 1.1. We restate it:

18.1 *Let H be theta-connected, and contain Jaws but not Petersen. Then H is doublecross.*

Proof. Let Jaws(1) be Jaws $+(1-2, 3-4)$, let Jaws(11) be Jaws(1) $+(3-22, 1-6)$, and let Jaws(12) be Jaws(1) $+(21-22, 1-6)$.

(1) *H does not contain Jaws(11) or Jaws(12).*

Subproof. Let G be Jaws(11), and let $X = V(G)\backslash$ {1, 2, 3, 21, 22, 23, 24}. If $ab \in E(G|X)$ and $cd \in E(G) \setminus E(G|X)$, with $a, b \neq c, d$ and with a, b non-adjacent to any of c, d that are in X, then $G + (ab, cd)$ contains Petersen. Hence the result follows from 8.1 when G is Jaws(11).

When G is Jaws(12), the argument is not so simple. Again we apply 8.1 to the same set X. Let $(e_1, f_1), \ldots, (e_k, f_k)$ be an augmenting sequence. By checking cases, we find that f_1 is not an edge of $G \setminus X$ (because every choice of $e_1 \in E(G|X)$ and $f_1 \in E(G \setminus X)$ gives a Petersen), and so $k \geq 2$; and having fixed (e_1, f_1), we try all the possibilities for (e_2, f_2). Again, there is no case with $f_2 \in E(G \setminus X)$, and so $k \geq 3$, and for each surviving choice of (e_2, f_2) we try the possibilities for (e_3, f_3). We find in every case that there is no choice of (e_3, f_3). (See the Appendix [4].) This proves (1).

(2) *H does not contain Jaws(1).*

Subproof. Let C be the quadrangle of $G =$ Jaws(1), and let $\mathcal{L} = \{$Petersen, Jaws(11), Jaws(12)$\}$. Then all A-extensions are killed by $\mathcal{L}$, and $\mathcal{P}(C, \mathcal{L}) = \emptyset$, so (2) follows from 10.1.

Let Jaws(2) be Jaws $+(8, 3, 5, 6) + (21, 3, 22, 6)$, let Jaws(21) be Jaws(2) $+(6, 7, 11, 12)$, and let Jaws(22) be Jaws(2) $+(7, 8, 19, 10)$.

(3) *H does not contain Jaws(21).*

We apply 10.2 to the quadrangle $\{25, 26, 12, 7\}$, taking $\mathcal{L}$ to be $\{$Petersen, Jaws1$\}$. Again, see the Appendix for details.

(4) *H does not contain Jaws(22).*

This is easier; we apply 10.1 to the quadrangle $\{8, 20, 26, 25\}$, taking $\mathcal{L}$ to be $\{$Petersen, Jaws1, Jaws(21)$\}$.

(5) *H does not contain Jaws(2).*

Let $X = \{6, 7, 8, 21, 22, 23, 24\}$. We apply 8.1 to X, and try all possibilities for the first three terms of the augmenting sequence; and find in each case contains one of Petersen, Jaws(1), Jaws(21), Jaws(22). (See the Appendix.)

Now let $\mathcal{C}_1$ be the set of the seven circuits of Jaws that bound regions in the drawing in Figure 2, not containing $1-6, 3-8, 13-18$ or $15-20$. Let $\mathcal{C}_2$ be the set of paths of Jaws induced on the following sets:

$$6, 1, 2, 3, 8;$$
$$8, 3, 4, 5, 6, 1;$$
$$1, 6, 7, 8, 3;$$
$$3, 8, 20, 15;$$
$$15, 20, 19, 18, 13;$$
$$13, 18, 17, 16, 15, 20;$$
$$20, 15, 14, 13, 18;$$
$$18, 13, 1, 6.$$

Let $G =$ Jaws, let F and η_F be null, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$; then $(G, F, \mathcal{C})$ is a framework. By hypotheses, there is a homeomorphic embedding η of G in H. We claim that (E1)–(E7) hold.

Since F is null, (E2)(i), (E4), (E5) are vacuously true, and (E1), (E3) are obvious. It remains to check (E2)(ii), (E6) and (E7). For (E2)(ii) we check that if $e, f \in E(G)$, not both in some member of $\mathcal{C}$, then $G + (e, f)$ contains either Petersen or Jaws(1); so (E2)(ii) follows from (2). For (E6) it is only necessary to check cross extensions on the circuit with vertex set {4, 5, 11, 17, 16, 10} and the path with vertex set {1, 6, 5, 4, 3, 8}, since all the other circuits and paths are too short or are equivalent by symmetry. Hence we must check

$$G + (4-5, 16-17) + (4-21, 17-22)$$
$$G + (4-5, 16-17) + (4-10, 11-17)$$
$$G + (4-10, 11-17) + (4-21, 17-22)$$
$$G + (4-10, 11-17) + (10-16, 5-11)$$
$$G + (3-8, 5-6) + (3-4, 1-6)$$
$$G + (3-8, 5-6) + (3-21, 6-22);$$

but they all contain Petersen, except the last which contains Jaws(2). Hence (E6) holds.

For (E7) we must check

$$G + (3-8, 5-6) + (3-21, 1-6) + (8-21, 1-24);$$

but this contains Petersen. Hence (E7) holds. From 7.1, this proves 18.1. ∎

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