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Coloring Locally Bipartite Graphs on Surfaces

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It is proved that there is a function $f: \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a graph embedded in a surface of Euler genus g with all faces of even size and with edge-width $\geqslant f(g)$. Then

(i) If every contractible 4-cycle of G is facial and there is a face of size > 4, then G is 3-colorable.

(ii) If G is a quadrangulation, then G is not 3-colorable if and only if there exist disjoint surface separating cycles $C_1, \ldots, C_g$ such that, after cutting along $C_1, \ldots, C_g$, we obtain a sphere with g holes and g Möbius strips, an odd number of which is nonbipartite.

If embeddings of graphs are represented combinatorially by rotation systems and signatures [5], then the condition in (ii) is satisfied if and only if the geometric dual of G has an odd number of edges with negative signature.

1. INTRODUCTION

Hutchinson [3] proved that if G is embedded in an orientable surface with large edge-width such that all facial walks have even length, then G is 3-colorable. The condition on large width is necessary since there are quadrangulations of surfaces whose underlying graph is the complete graph K_n (and n can be arbitrarily large). See also Section 4 for examples with arbitrarily large edge-width. On the other hand, the result of Hutchinson does not extend to nonorientable surfaces. For example, Youngs [8] proved that every nonbipartite quadrangulation of the projective plane has

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chromatic number 4. Similarly, Klavžar and Mohar [4, Theorem 2.4] proved that certain quadrangulations of the Klein bottle with arbitrarily large edge-width have chromatic number 4.

It is known [2] that graphs embedded in a surface with all faces of even size and with sufficiently large edge-width are 4-colorable. In this paper we completely characterize those which are not 3-colorable. It turns out that the only obstruction to 3-colorability can be expressed by means of nonbipartite projective quadrangulations; cf. Theorem 4.1. Similar results for quadrangulations were obtained independently by Archdeacon, Hutchinson, Nakamoto, Negami, and Ota [1, 6].

All embeddings of graphs in surfaces considered in this paper are 2-cell embeddings. Generally, we follow terminology in [5]. If G is embedded in a surface S with f faces, then the number $g = 2 - |V(G)| + |E(G)| - f$ is called the *Euler genus* of S. By $\chi(G)$ we denote the chromatic number of G.

2. 3-COLORING AND THE WINDING NUMBER

Let c be a fixed 3-coloring of the graph G with colors 1, 2, and 3. If $W = v_1 v_2 \cdots v_k v_1$ is a closed walk in G, then the coloring of $V(W)$ can be viewed as a mapping onto the 3-cycle C_3 and we may speak of the *winding number* $w_c(W)$, which is equal to the number of indices i such that $c(v_i) = 1$ and $c(v_{i+1}) = 2$ minus the number of indices i such that $c(v_i) = 2$ and $c(v_{i+1}) = 1$, $i = 1, \ldots, k$. An obvious fact that we shall use in the following is that $w_c(W)$ is odd (and hence nonzero) if the length of W is odd. If $v_{i+1} = v_{i-1}$, then $W' = v_1 \cdots v_{i-1} v_{i+2} \cdots v_k v_1$ is a closed walk, and $w_c(W') = w_c(W)$. We say that W' is obtained from W by an *edge-reduction* and that W is obtained from W' by an *edge-expansion*.

Suppose that W can be expressed as a *concatenation* of two closed walks W_1, W_2. Then, clearly,

$$w_c(W) = w_c(W_1) + w_c(W_2). \tag{1}$$

Suppose that G is embedded in some surface, $W_1 = v_1 \cdots v_k v_1$ is a closed walk in G, and $W_2 = v_1 v_k u_1 \cdots u_r v_1$ is a facial walk which traverses the edge $v_1 v_k$ in the opposite direction than W_1. Then $W = v_1 \cdots v_k u_1 \cdots u_r v_1$ is a closed walk which is obtained by a concatenation and an edge-reduction. We say that W has been obtained from W_1 by a *homotopic shift over a face*. Note that W is homotopic to W_1 on the surface. It is well known that every closed walk homotopic to W_1 can be obtained from W_1 by a sequence of edge-reductions, edge-expansions, and homotopic shifts over faces. Also, observe that if W_2 is of length 4, then $w_c(W_2) = 0$, so $w_c(W) = w_c(W_1)$ by (1). This implies:

LEMMA 2.1. *Let G be a quadrangulation of some surface and let c be a 3-coloring of G. If W and W' are homotopic closed walks of G, then $w_c(W) = w_c(W')$.*

Lemma 2.1 does not hold if G is not a quadrangulation but its conclusion is correct if we consider homotopy in the surface after we remove a point from the interior of each face whose size is different from 4.

3. EDGE-WIDTH AND LOCALLY BIPARTITE EMBEDDINGS

An embedding of a graph G in some surface is *locally bipartite* if all facial walks are of even length. It is easy to see that, in a locally bipartite embedding, every surface separating cycle (or a closed walk) of G is also of even length and that the parity of the length of a closed walk is a homotopy invariant.

The *edge-width* $\mathbf{ew}(G)$ of a graph G embedded in a nonsimply connected surface is defined as the length of a shortest noncontractible cycle in G. Similarly, the *face-width or representativeness*, denoted by $\mathbf{fw}(G)$, is the minimum k such that every noncontractible simple closed curve on the surface intersects G in at least k points.

LEMMA 3.1. *Let G be a graph with a locally bipartite embedding in some surface. Then G can be extended to a locally bipartite graph $\tilde{G} \supseteq G$ embedded in the same surface such that*

(a) $\mathbf{ew}(G) = \mathbf{ew}(\tilde{G}) = \mathbf{fw}(\tilde{G})$, *and*

(b) $\chi(\tilde{G}) = \chi(G)$.

Proof. If C is a facial walk in G of size $2r$, then add into the face of C a $2r$-cycle C' and join the ith vertex on C with the ith vertex on C'. Now, perform the same procedure with C' instead of C, then with the new cycle, etc., all together $r-1$ times. After doing this for all facial walks of G, the resulting locally bipartite embedding $\tilde{G}$ satisfies (a).

If c is a k-coloring of G, then the coloring of the facial walk C can be extended onto C' (and from there to all subsequent cycles) as follows. If $c(v) \in \{1, \ldots, k\}$ is the color of the ith vertex on C, then color the ith vertex of C' by $c(v)+1$ modulo k. This implies (b). ∎

Suppose that G is locally bipartite. We say that G is 4-*reduced* if every contractible 4-cycle of G is facial. If G is not 4-reduced and C is a contractible nonfacial 4-cycle, let G' be the graph obtained from G by deleting the edges and vertices in the interior of (the disk bounded by) C. Since the subgraph of G in the interior of C is bipartite, every k-coloring of G' can be extended to a k-coloring of G. Therefore, $\chi(G') = \chi(G)$. Because of this fact, we may only consider 4-reduced embeddings.

4. LARGE EDGE-WIDTH AND COLORING WITH FEW COLORS

Let w_0 and k be arbitrary integers. It is well known that there exists a connected graph G_0 of girth $\geqslant w_0$ and with chromatic number $\geqslant k$. Take an embedding of G_0 with only one facial walk. (Such embeddings, usually nonorientable, always exist; cf., e.g., [5].) Since every edge appears precisely twice on the facial walk, this embedding is locally bipartite. This example shows that the graph $\tilde{G}_0$ (cf. Lemma 3.1) has edge- and face-width $\geqslant w_0$ and chromatic number $\geqslant k$. Therefore, no fixed lower bound on the width of locally bipartite graphs implies a bounded chromatic number. However, a bound on the width depending on the genus of the embedding works. For instance, Fisk and Mohar [2] proved the following result. Let G be a graph of girth $\geqslant 4$ embedded in a surface of Euler genus g. If $\mathbf{ew}(G) \geqslant c \log g$ (where c is some constant), then $\chi(G) \geqslant 4$. In this paper we show that for locally bipartite embeddings we may usually save another color, and we determine when this is not possible.

THEOREM 4.1. *There is a function $f: \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a locally bipartite graph embedded in a surface of Euler genus g with edge-width $\geqslant f(g)$. Then*

(a) *G is 4-colorable.*

(b) *If the embedding is 4-reduced and there is face of size >4, then G is 3-colorable.*

(c) *If G is a quadrangulation, then G is not 3-colorable if and only if there exist disjoint surface separating cycles $C_1, \ldots, C_g$ such that, after cutting along $C_1, \ldots, C_g$, we obtain a sphere with g holes and g Möbius strips, an odd number of which is nonbipartite.*

Proof. Part (a) follows from the aforementioned result of Fisk and Mohar [2]. Let us now prove (c). We assume that G is a quadrangulation, and we may assume that it is 4-reduced. By the result of Hutchinson [3], we may assume that the surface of the embedding is $\mathbb{N}_g$, the nonorientable surface of Euler genus g.

Let H_g be the graph embedded in $\mathbb{N}_g$ as shown in Fig. 1. More precisely, H_g is composed of 8 outer cycles $Q_1, \ldots, Q_8$ and the inner cycle Q_0. These cycles are contractible in $\mathbb{N}_g$ and joined by $2g$ paths R_i, R'_i, $i = 1, \ldots, g$. Between R_i, R'_i, Q_0, and Q_1, there is a copy of the graph $K_{3,3}$ embedded so

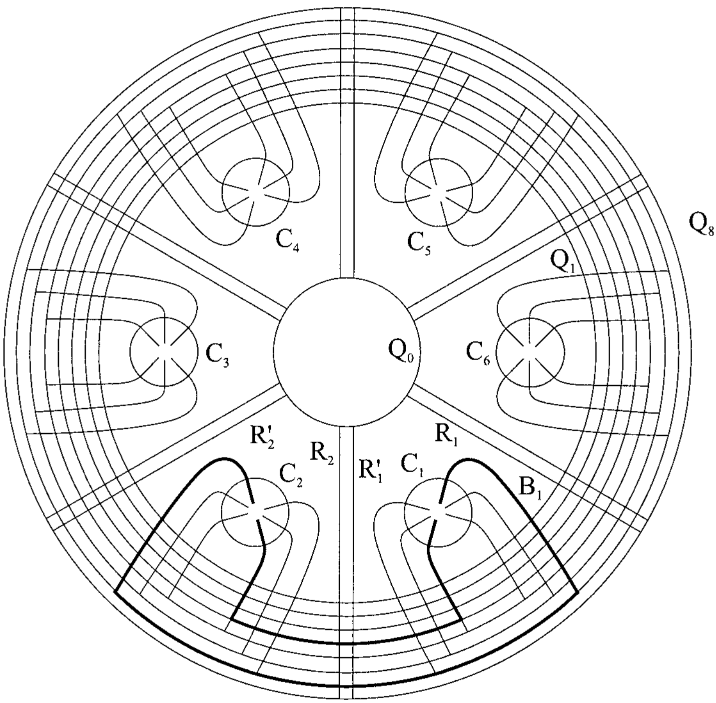

FIG. 1. The graph H_6 in $\mathbb{N}_6$.

that its 6-cycle C_i bounds a Möbius strip. The cycle C_i is joined to the paths Q_5, Q_6, and Q_7 by six disjoint paths as shown in Fig. 1 ($i = 1, ..., g$). Robertson and Seymour [7] proved that, if the face-width of G in $\mathbb{N}_g$ is sufficiently large (which we may assume by choosing $f(g)$ large enough), then G has the graph H_g as a surface minor.

If H_g is a surface minor of G and C is a cycle of H_g, then G contains a cycle $\tilde{C}$ that contracts to C when edges are contracted to obtain the minor. By this correspondence, we shall consider cycles of H_g as being contained in G. Let us observe that C and $\tilde{C}$ have the same homotopy properties on the surface.

Let $1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant g$ be the indices i for which C_i bounds a nonbipartite Möbius strip in G. Suppose first that q is odd. Suppose that G has a 3-coloring c. If $i \in \{i_1, i_2, ..., i_g\}$, then the projective plane graph R determined by C_i is nonbipartite. Let C be an odd cycle in R. Then the concatenation $C + C$ is homotopic to C_i in $\mathbb{N}_g$. By the results of Section 2, $w_c(C)$ is odd (since C is of odd length) and $w_c(C_i) = 2w_c(C)$ is congruent to 2 modulo 4. Similarly we see that $w_c(C_i) \equiv 0 \pmod 4$ for $i \notin \{i_1, i_2, ..., i_q\}$. Now, consider the 3-coloring of the sphere D obtained after cutting the surface $\mathbb{N}_g$ along $C_1, \ldots, C_g$. Clearly, the cycles Q_0 and Q_1 are contractible and hence homotopic in $\mathbb{N}_g$. However, in D, Q_1 can be obtained from Q_0 by a sequence of homotopic shifts over faces (plus some edge-reductions). All of the faces used in homotopic shifts, except $C_1, \ldots, C_g$, are 4-cycles. Therefore, (1) implies

$$w_c(Q_1) = w_c(Q_0) + \sum_{i=1}^{g} w_c(C_i). \tag{2}$$

Consequently, $\sum_{i=1}^{g} w_c(C_i) = 0$. On the other hand, the above discussion shows that $\sum_{i=1}^{g} w_c(C_i) \equiv 2 \pmod 4$. This contradiction proves that c does not exist.

Suppose now that q is even. Let $D(1, 2)$ be the cycle in G which separates $C_{i_1} \cup C_{i_2}$ from the rest of the surface and which corresponds to the following cycle in H_g: First, follow R_{i_1} from Q_0 to Q_8, continue clockwise on Q_8 until reaching the path R'_{i_2}, follow R'_{i_2} to Q_0, go anticlockwise on Q_0 until R_{i_2}, descend on R_{i_2} to Q_1, use Q_1 anticlockwise back to R'_{i_1}, return on R'_{i_1} to Q_0, and close up on Q_0 in the anticlockwise direction. Similarly we define $D(3, 4), \ldots, D(q-1, q)$. After cutting along the cycles $D(1, 2), \ldots, D(q-1, q)$, we obtain a surface S of Euler genus $g-q$ and $q/2$ surfaces homeomorphic to the Klein bottle in which the face corresponding to $D(j, j+1)$ is special. The subgraph G' of G on S is bipartite. Fix a 2-coloring (using colors 1 and 2) of G'. This 2-coloring induces a 2-coloring on each of the special faces in $q/2$ Klein bottles. It suffices to see that, in each case, the 2-coloring of the special face D can be extended to a 3-coloring of the whole subgraph G'' of G in the corresponding Klein bottle K.

Observe that H_g and hence also G'' contain three pairwise disjoint cycles B_1, B_2, B_3 which are two-sided noncontractible in K and are disjoint from D. Each of them passes through crosscaps bounded by C_{i_j} and $C_{i_{j+1}}$ in K, and B_r closes up by using paths on the cycles Q_r and Q_{r+3} (if $r = 2, 3$), while B_1 uses paths on Q_4 and Q_7. In Fig. 1, the cycle B_1 is shown. The cycles B_1, B_2, and B_3 are homotopic in K and partition K into three cylinders B_{12}, B_{23}, and B_{31}, where B_{ij} is bounded by B_i and B_j. The cylinder B_{12} contains D. The cycle B_1 is the sum of a cycle which is homotopic to D (hence of even length) and two cycles (each of odd length) passing through crosscaps. Therefore, B_1 and the corresponding homotopic cycles B_2 and B_3 are also of even length. Hence, each B_{ij} has a locally bipartite embedding in the plane. Consequently, B_{ij} is a bipartite graph.

Let c_{12} be the 2-coloring of B_{12} with colors 1 and 2 which extends the coloring of D. Let c_{23} be the 2-coloring of B_{23} with colors 2 and 3 which coincides on B_2 with c_{12} on vertices of color 2. Let c_{31} be the 2-coloring of B_{31} with colors 3 and 1 which coincides on B_3 with c_{23} on vertices of color 3. Since K contains two nonbipartite crosscaps, it is not bipartite.

This implies that c_{31} coincides on B_1 with c_{12} on vertices of color 1. Consequently, by setting $c(v) = c_{ij}(v)$, if $v \in V(B_{ij} - B_j)(ij \in \{12, 23, 31\})$, we get the required 3-coloring of G''. This completes the proof of (c).

It remains to prove (b). After filling up the faces of size $\geqslant 6$ in the same way as in the proof of Lemma 3.1, we can produce a 4-reduced graph which contains G, is embedded in the same surface, and has face-width $\geqslant \frac{1}{2}\,\mathbf{ew}(G)$. Moreover, by adding some additional edges if necessary, we may assume that all faces except one of the resulting graph $G' \supseteq G$ are 4-cycles and that the exceptional face F_0 is a 6-cycle. Define the graph H'_g in a similar way as H_g except that now we replace each of the cycles $Q_0, ..., Q_8, C_1, ..., C_g$, the paths $R_1, R'_1, ..., R_g, R'_g$, and the paths connecting the crosscaps with the Q_j's by five disjoint homotopic copies of that cycle or path. (We shall use the same notation as before for any of the five disjoint copies of each of these cycles or paths.) Now, we take the same steps as in the proof of (c), working in G' and assuming the face-width is large enough so that H'_g is a surface minor of G'.

We may assume that q is odd. Denote by M_i the Möbius strip bounded by a cycle composed of R_i, R'_i and the appropriate segments of Q_0 and $Q_1, i = 1, ..., g$. We may assume that $i_1 = 1$ and that if the 6-face F_0 is in some $M_i(1 \leqslant i \leqslant g)$, then $i_q \leqslant i \leqslant g$. Then the cycles $D(1, 2), ..., D(q-2, q-1)$ can be selected so that F_0 is not contained in any of the Klein bottles bounded by these cycles. For $j \in \{1, 3, ..., q-2\}$, let K be the Klein bottle bounded by $D(j, j+1)$. Since the cycles and paths of H_g are replaced by five disjoint homotopic copies in H'_g, the cycles B_1, B_2, B_3 in K can be chosen so that they are disjoint from and not adjacent to $D(j, j+1)$. We say that a 3-coloring of an even cycle C is *almost a 2-coloring* (and that C is *almost 2-colored*) if one of the color classes is equal to one of the bipartite classes of C. The proof of (c) shows that any almost 2-coloring of $D(j, j+1)$ can be extended to a 3-coloring of K.

Now we cut out the Klein bottles bounded by the cycles $D(1, 2), ...,$ $D(q-2, q-1)$ and cut out all projective planes $M_i, i \notin \{i_1, i_2, ..., i_q\}$, so that F_0 does not intersect any of the $r = (g-1) - (q-1)/2$ cycles $F_1, ..., F_r$ used in the cutting. The resulting surface S is the projective plane (since C_{i_q} is in S) with special faces $F_1, ..., F_r$. Since all cycles of H_g have been replaced in H'_g by five disjoint homotopic copies, we can choose the cycles $F_1, ..., F_r$ such that for every $i, 1 \leqslant i \leqslant r$, there are disjoint cycles F'_i, F''_i which are disjoint from F_i such that each of them bounds a disk in S with F_i in the interior but with all other cycles $F_j(j \in \{0, 1, ..., r\} \setminus \{i\})$ in its exterior.

Suppose that F_0 is not in S. Then it is in some $M_i, i_q < i \leqslant g$. In such a case we add M_i back to S and cut out the same crosscap along a different cycle C' so that F_0 remains in S. To achieve this, we can take $C' = C_i$ (the "innermost" of the five copies) unless F_0 is inside M_i in one of the shaded

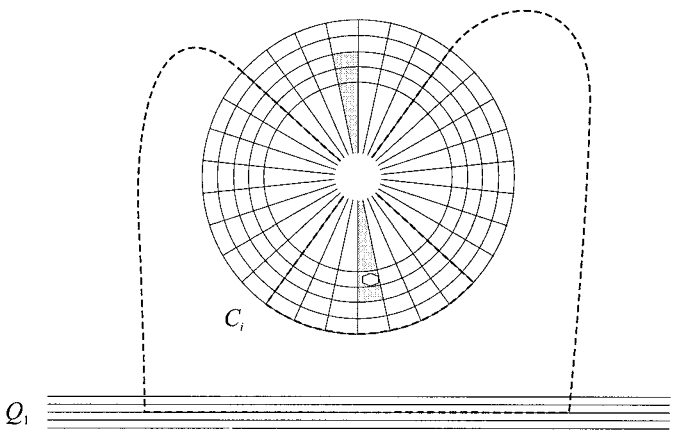

FIG. 2. The cycle C' in M_i.

regions as represented in Fig. 2. In that case we take for C' the dotted cycle shown in Fig. 2. Hence, we may assume that F_0 is contained in S.

Moreover, we may assume that for each of the special faces $F_j (1 \leqslant j \leqslant r)$, there exist corresponding cycles F'_j, F''_j. Denote by H the subgraph of G' in S. As mentioned above, any almost 2-coloring of F_j can be extended to a 3-coloring of the corresponding Klein bottle if F_j corresponds to one of $D(1, 2), ..., D(q-2, q-1)$. Since the removed projective planes M_i are all bipartite, the same holds for the cycle F_j corresponding to M_i. Therefore it suffices to prove that H has a 3-coloring so that all special faces $F_1, ..., F_r$ are almost 2-colored.

Let $F_0 = v_1 v_2 \cdots v_6$. Let $\hat{H}$ be the graph in S obtained from H by adding a vertex of degree 4 in each 4-face of H, joining it to the vertices on that face. We claim that $\hat{H}$ contains disjoint paths P_1, P_2, P_3 where P_i connects v_i and v_{i+3}, $i = 1, 2, 3$. As proved by Robertson and Seymour in [7], such paths exist if and only if there is no contractible simple closed curve γ in S which intersects $\hat{H}$ in at most five points such that F_0 is contained in the disk bounded by γ. Suppose that such a curve γ exists. Because of the existence of F'_j, F''_j, the curve γ does not pass through $F_j, j = 1, ..., r$. Since all other faces of $\hat{H}$ are of size 3, γ determines a cycle in $\hat{H}$ of length $\leqslant 5$. This cycle then determines a contractible closed walk W in H of length $\leqslant 5$ such that F_0 is in the interior of W. Since G' and hence also H are locally bipartite, W is of even length, so it must be a 4-cycle. This contradicts the fact that G' is 4-reduced. Hence γ does not exist. This proves the claim.

Now, cut S along P_1, P_2, P_3 and use a 2-coloring on each of the three resulting disks. These colorings can be combined into a 3-coloring of H in the same way as in the proof of (c). Clearly, under such a 3-coloring, each of the special cycles $F_1, ..., F_r$ is almost 2-colored. This completes the proof. ∎

Theorem 4.1 implies, in particular, that for every nonorientable surface S, there are infinitely many 4-critical graphs of girth 4 on S. Examples of such graphs are 4-reduced non-3-colorable quadrangulations of large edge-width.

Suppose that C is a cycle of the embedded graph G such that, after cutting the surface along C, an orientable surface is obtained. Then C is said to be an *orientizing cycle*. If G is as in the proof of Theorem 4.1, then any cycle passing through all g Möbius strips bounded by $C_1, \ldots, C_g$ is orientizing. This yields another formulation of Theorem 4.1(c), whose "only if" part was discovered independently by Archdeacon *et al.* [1]. Later, Nakamoto *et al.* [6] proved also the "if" part.

Corollary 4.2. *If G is a quadrangulation of $\mathbb{N}_g$ and the edge-width of G is sufficiently large, then there is an orientizing cycle C, and G is* 3*-colorable if and only if C is of even length.*

To verify the condition on "odd number of nonbipartite Möbius strips" in Theorem 4.1(c), one may take any orientizing cycle and check if it is of odd length. An even easier criterion is the following. Suppose that the embeddings of graphs in surfaces are represented combinatorially by means of rotation systems and signatures [5].

Proposition 4.3. *Let G be a locally bipartite graph in a closed surface S. Then the length of every orientizing closed walk in G has the same parity as the number of edges with negative signature in the dual graph.*

Proof. We may assume that S is nonorientable. Consider a combinatorial description of the dual graph G^*. A *local change* (cf. [5, Section 4.1]) changes the signature of all edges incident with a vertex v and reverses the local rotation at v. Every local change gives rise to another description of the same embedding and preserves the parity of the number of edges with negative signature.

Let C be an orientizing closed walk in G. Since the embedding obtained after cutting along C is orientable, there is a sequence of local changes in G^* such that all edges of G^* that are not dual to $E(C)$ have positive signature. Moreover, since S is nonorientable, an edge e^* dual to $e \in E(C)$ has negative signature if and only if e occurs an odd number of times in the walk C. This completes the proof. ∎

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