Excluding pairs of graphs

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Abstract

For a graph G and a set of graphs $\mathcal{H}$, we say that G is *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Given an integer $P > 0$, a graph G, and a set of graphs $\mathcal{F}$, we say that G *admits an $(\mathcal{F}, P)$-partition* if the vertex set of G can be partitioned into P subsets $X_1, \dots, X_P$, so that for every $i \in \{1, \dots, P\}$, either $|X_i| = 1$, or the subgraph of G induced by X_i is $\{F\}$-free for some $F \in \mathcal{F}$.

Our first result is the following. For every pair (H, J) of graphs such that H is the disjoint union of two graphs H_1 and H_2, and the complement J^c of J is the disjoint union of two graphs J_1^c and J_2^c, there exists an integer $P > 0$ such that every $\{H, J\}$-free graph has an $(\{H_1, H_2, J_1, J_2\}, P)$-partition. A similar result holds for tournaments, and this yields a short proof of one of the results of [1].

A *cograph* is a graph obtained from single vertices by repeatedly taking disjoint unions and disjoint unions in the complement. For every cograph there is a parameter measuring its complexity, called its *height*. Given a graph G and a pair of graphs H_1, H_2, we say that G is *$\{H_1, H_2\}$-split* if $V(G) = X_1 \cup X_2$, where the subgraph of G induced by X_i is $\{H_i\}$-free for $i = 1, 2$. Our second result is that for every integer $k > 0$ and pair $\{H, J\}$ of cographs each of height $k + 1$, where neither of H, J^c is connected, there exists a pair of cographs $(\tilde{H}, \tilde{J})$, each of height k, where neither of $\tilde{H}^c, \tilde{J}$ is connected, such that every $\{H, J\}$-free graph is $\{\tilde{H}, \tilde{J}\}$-split.

Our final result is a construction showing that if $\{H, J\}$ are graphs each with at least one edge, then for every pair of integers r, k there exists a graph G such that every r-vertex induced subgraph of G is $\{H, J\}$-split, but G does not admit an $(\{H, J\}, k)$-partition.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. For $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced by X. The complement of G, denoted by G^c, is the graph with vertex set $V(G)$ such that two vertices are adjacent in G if and only if they are non-adjacent in G^c. A *clique* in G is a set of vertices all pairwise adjacent; and a *stable set* in G is a set of vertices all pairwise non-adjacent. For disjoint $X, Y \subseteq V(G)$, we say that X is *complete* (*anticomplete*) to Y if every vertex of X is adjacent (non-adjacent) to every vertex of Y. If $|X| = 1$, say $X = \{x\}$, we say "x is complete (anticomplete) to Y" instead of "$\{x\}$ is complete (anticomplete) to Y".

We denote by K_n the complete graph on n vertices, and by S_n the complement of K_n. A graph is *complete multipartite* if its vertex set can be partitioned into stable sets, all pairwise complete to each other. For graphs H and G we say that G *contains* H if some induced subgraph of G is isomorphic to H. Let $\mathcal{F}$ be a set of graphs. We say that G is *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$. If $|\mathcal{F}| = 1$, say $\mathcal{F} = \{F\}$, we write "G is F-free" instead of "G is $\{F\}$-free". For a pair of graphs $\{H_1, H_2\}$, we say that G is *$\{H_1, H_2\}$-split* if $V(G) = X_1 \cup X_2$, and $G|X_i$ is H_i-free for $i = 1, 2$. We remind the reader that a *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set; thus in our language split graphs are precisely the graphs that are $\{K_2, S_2\}$-split.

Ramsey's theorem can be restated in the following way: for every pair of integers $m, n > 0$ there exists an integer P, such that for every $\{S_m, K_n\}$-free graph G, $V(G)$ can be partitioned into at most P well-understood parts (in fact, each part is a single vertex). One might ask whether a similar statement holds for more general pairs of graphs than just $\{S_m, K_n\}$ (adjusting the definition of "well-understood").

For instance, a result of [2] implies that if G is $\{C_4, C_4^c\}$-free (where C_4 is a cycle on four vertices), then $V(G)$ can be partitioned into three parts, each of which induces either a complete graph, or a graph with no edges, or a cycle of length five.

In [3] two of us made progress on this question, but to state the result we first need a definition. Given an integer $P > 0$, we say that a graph G *admits an $(\mathcal{F}, P)$-partition* if $V(X) = X_1 \cup \ldots \cup X_P$ such that for every $i \in \{1, \ldots, P\}$, either $|X_i| = 1$ or $G|X_i$ is $\{F\}$-free for some $F \in \mathcal{F}$. Please note that the first alternative in the definition of an $(\mathcal{F}, P)$-partition (the condition that $|X_i| = 1$) is only necessary when no graph in $\mathcal{F}$ has more than one vertex. We proved the following:

1.1. *For every pair of graphs (H, J) such that H^c and J are complete multipartite, there exist integers $k, P > 0$ such that every $\{H, J\}$-free graph admits a $(\{K_k, S_k\}, P)$-partition.*

and its immediate corollary:

1.2. *For every pair of graphs (H, J) such that H^c and J are complete multipartite, there exists an integer $k > 0$ such that every $\{H, J\}$-free graph is $\{K_k, S_k\}$-split.*

The first goal of this paper is to generalize 1.1 further. Let H be a graph. A *component* of H is a maximal connected subgraph of H. A graph is *anticonnected* if its complement is connected. An *anticomponent* of H is a maximal anticonnected induced subgraph of H. We denote by $c(H)$ the set of components of H, and by $ac(H)$ the set of anticomponents of H. We remark that for every non-null graph G, at least one of $c(G)$ or $ac(G)$ equals $\{G\}$. We prove the following generalization of 1.1 (please note that 1.3 is trivial whenever H is connected or J is anticonnected):

1.3. *For every pair of graphs (H, J) there exists an integer P such that every $\{H, J\}$-free graph admits a $(c(H) \cup ac(J), P)$-partition.*

Please note that applying 1.3 with J a complete graph and H a graph with no edges gives Ramsey's theorem. Using ideas similar to those of our proof of 1.3, we also give a short proof of one of the results of [1].

An anonymous referee asked whether the conclusion of 1.3 can be strengthened to say that $V(G) = X_1 \cup \ldots \cup X_P$ where for each $i \in \{1, \ldots, P\}$ either $|X_i| = 1$ or $G|X_i$ is (H', J')-free for some $H' \in c(H)$ and $J' \in ac(J)$. The answer to this question is "no", because of the following example: let n be a positive integer, G be the star $K_{1,n}$, J be a cycle of length four, and $H = J^c$. Then G is $\{H, J\}$-free, and the proposed strengthening would say that $V|(G)| \leq P$ for some fixed integer P, which is false, since n can be made arbitrarily large.

Next let us generalize the notion of a complete multipartite graph. A *cograph* is a graph obtained from 1-vertex graphs by repeatedly taking disjoint unions and disjoint unions in the complement. In particular, G is either not connected or not anticonnected for every cograph G with at least two vertices, and therefore for every cograph G with at least two vertices, exactly one of G, G^c is connected. It follows from [4] that cographs are precisely the graphs that have no induced three-edge paths. We recursively define a parameter, called the *height* of a cograph, that measures its complexity, as follows. The height of a one vertex cograph is zero. If G is a cograph that is not connected, let m be the maximum height of a component of G; then the height of G is $m+1$. If G is a cograph that is not anticonnected, let m be the maximum height of an anticomponent of G; then the height of G is $m+1$. We denote the height of G by $h(G)$.

We use 1.3 to prove the following:

1.4. *Let $k > 0$ be an integer, and let H and J be cographs, each of height $k+1$, such that H is anticonnected, and J is connected. Then there exist cographs $\tilde{H}$ and $\tilde{J}$, each of height k, such that $\tilde{H}$ is connected, and $\tilde{J}$ is anticonnected, and every $\{H, J\}$-free graph is $\{\tilde{H}, \tilde{J}\}$-split.*

The proof of 1.1 in [3] relies on the following lemma:

1.5. *Let $p > 0$ be an integer. There exists an integer $r > 0$ such that for every graph G, if every induced subgraph of G with at most r vertices is $\{K_p, S_p\}$-split, then G is $\{K_p, S_p\}$-split.*

Here is a weaker statement that would still imply the results of [3]:

1.6. *Let $p > 0$ be an integer. There exist integers $r, k > 0$ such that for every graph G, if every induced subgraph of G with at most r vertices is $\{K_p, S_p\}$-split, then G admits a $(\{K_p, S_p\}, k)$-partition.*

Originally we hoped that 1.4 could be proved along the same lines, and that a result similar to 1.6 might exist when the pairs $\{K_p, S_p\}$ were replaced by pairs of more general graphs. However, this turns out not to be the case, because of the following:

1.7. *Let H, J be graphs each with at least one edge. Then for any choice of integers r, k there is a graph G such that*

- *for every $S \subseteq V(G)$ with $|S| \leq r$, the graph $G|S$ is $\{H, J\}$-split, and*
- *G has no $(\{H, J\}, k)$-partition.*

By taking complements, the conclusion of 1.7 also holds if each of H and J has a non-edge. Thus 1.6 is in a sense the strongest result of this form possible.

This paper is organized as follows. In Section 2 we prove 1.3. In Section 3 we use 1.3 to prove 1.4. In Section 4 we reprove a result of [1]. Finally, Section 5 contains the proof of 1.7.

2 The proof of 1.3

The goal of this section is to prove 1.3. Let us start with some definitions. For a graph G and a set $X \subseteq V(G)$, we denote by $G \setminus X$ the graph $G|(V(G) \setminus X)$. If $|X| = 1$, say $X = \{v\}$, we write $G \setminus v$ instead of $G \setminus \{v\}$. Let S, T be induced subgraphs of a graph H^*. We say that S is an H^*-*extension* of T if $T = S \setminus u$ for some $u \in V(S)$.

Clearly, 1.3 follows from repeated applications of the following:

2.1. *Let H, J, H_1, H_2, J_1 and J_2 be non-null graphs such that H is the disjoint union of H_1 and H_2, and J^c is the disjoint union of J_1^c and J_2^c. Let $m = \max(|V(H_1)|, |V(H_2)|, |V(J_1)|, |V(J_2)|)$, and $\mathcal{F} = \{H_1, H_2, J_1, J_2\}$. Then every $\{H, J\}$-free graph G admits an $(\mathcal{F}, 2(m+1)^m)$-partition.*

Proof. We may assume without loss of generality that $|V(H_1)| = m$. We may also assume that G is not H_1-free, for otherwise the theorem holds. Choose some vertex $h^* \in V(H_2)$, and let H^* be the subgraph of H induced on $V(H_1) \cup \{h^*\}$. For an induced subgraph T of H^*, a T-*piece* is an isomorphism g from T to an induced subgraph of G, that we denote by $g(T)$. For a T-piece g, and an H^*-extension S of T, let $V(S) \setminus V(T) = \{s\}$; we say that $v \in V(G) \setminus V(g(T))$ g-*corresponds to* S if the map sending x to $g(x)$ for each $x \in V(T)$, and sending s to v, is an S-piece. We denote by $Y_S(g)$ the set of all vertices in G that g-correspond to S. Let $Y(g)$ be the union of the sets $Y_S(g)$. Thus $Y(g)$ is the set of all vertices in $V(G)$ that g-correspond to an H^*-extension of T.

(1) *If g is an H_1-piece, then $Y(g)$ is anticomplete to $V(g(H_1))$, and the graph $G|Y(g)$ is H_2-free.*

Since $V(H_1)$ is anticomplete to $V(H_2)$ in H, and in particular anticomplete to h^*, it follows that $Y(g)$ is anticomplete to $V(g(H_1))$. But then, since G is H-free, it follows that $G|Y(g)$ is H_2-free. This proves (1).

Let

$$\phi(t) = \begin{cases} 2(m+1)^{m-t} & \text{for } 0 \le t \le m-1 \\ 1 & \text{for } t = m. \end{cases}$$

(2) *Let T be an induced subgraph of H_1, and let g be a T-piece. Write $t = |V(T)|$. Then for some H^*-extension S of T, the graph $G|Y_S(g)$ admits an $(\mathcal{F}, \phi(t))$-partition.*

The proof is by induction on $m - t$. If $t = m$, then (as $|H_1| = m$) it follows from (1) that $G|Y(g)$ admits an $(\mathcal{F}, 1)$-partition, so we may assume that $t \le m - 1$.

Choose an H^*-extension S of T, with S an induced subgraph of H_1. If the graph $G|Y_S(g)$ admits an $(\mathcal{F}, \phi(t))$-partition then we are done; so we may assume that for every partition of $Y_S(g)$ into at most $\phi(t)$ classes, some class contains each of H_1, H_2, J_1, J_2.

Let $V(S) \setminus V(T) = \{s\}$. Let $v \in Y_S(g)$, and let h be the S-piece mapping s to v and mapping x to $g(x)$ for each $x \in V(T)$. Inductively, there exists an H^*-extension Q of S such that $G|Y_Q(h)$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Let $V(Q) \setminus V(S) = \{r\}$, and let $R = Q \setminus s$. Thus R is an H^*-extension of T different from S. (Nevertheless, possibly $Y_R(g) = Y_S(g)$, if there is an isomorphism from S to R fixing T pointwise.) We say that v is *of type R*. From the definition of $Y_Q(h)$, it follows that $Y_Q(h) \subseteq Y_R(g)$, and either

- r, s are non-adjacent in H, and $Y_Q(h)$ is the set of vertices in $Y_R(g)$ that are different from and non-adjacent to v or
- r, s are adjacent in H, and $Y_Q(h)$ is the set of vertices in $Y_R(g)$ that are different from and adjacent to v.

For each H^*-extension R of T different from S, let Z_R be the set of vertices in $Y_S(g)$ that are of type R. Thus the sets Z_R have union $Y_S(g)$. Since $|V(H^*)| - |V(T)| = m + 1 - t$ and R must be different from S, it follows that there are at most $m - t$ different types of vertices in $Y_S(g)$. Since

$$m - t \leq 2(m+1)^{m-t} = \phi(t),$$

there is an H^*-extension R of T different from S, such that $G|Z_R$ contains each of H_1, H_2, J_1, J_2. Write $A = Y_R(g)$. Let $V(R) \setminus V(T) = \{r\}$.

Assume first that r, s are non-adjacent, and choose $B \subseteq Z_R$ such that $G|B$ is isomorphic to J_1. For each $b \in B$, let A_b be the set of vertices of $A \setminus B$ that are non-adjacent to b. Let A_0 be the set of vertices of $A \setminus B$ that are complete to B. Then $A \subseteq B \cup A_0 \cup \bigcup_{b \in B} A_b$ (We remind the reader that for two distinct H^*-extensions R and S of T, it is possible that $Y_R(g) = Y_S(g)$.)

Let $b \in B$; since $b \in Z_R$, since $A_b \subseteq Y_R(g)$, and since A_b is anticomplete to b, it follows from the definition of Z_R that $G|A_b$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Since G is J-free, it follows that $G|A_0$ is J_2-free. Since $|B| \leq m$, this implies that $G|A$ admits an $(\mathcal{F}, K)$-partition, where

$$K = m\phi(t+1) + m + 1 \leq 2(m+1)^{m-t} = \phi(t),$$

(even when $t = m - 1$), and so (2) holds.

Now we assume that r, s are adjacent. Choose $B \subseteq Z_R$ such that $G|B$ is isomorphic to H_1. For $b \in B$, let A_b be the set of vertices of $A \setminus B$ that are adjacent to b. Let A_0 be the set of vertices of $A \setminus B$ that are anticomplete to B. Then $A \subseteq B \cup A_0 \cup \bigcup_{b \in B} A_b$.

Let $b \in B$; since $b \in Z_R$, since $A_b \subseteq Y_R(g)$, and since A_b is complete to b, it follows from the definition of Z_R that $G|A_b$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Since G is H-free, it follows that $G|A_0$ is H_2-free. Since $|B| \leq m$, this implies that $G|A$ admits an $(\mathcal{F}, K)$-partition, where

$$K = m\phi(t+1) + m + 1 \leq 2(m+1)^{m-t} = \phi(t).$$

This proves (2).

Now let T be the null graph, and g the isomorphism from T into G. Then T is an induced subgraph of H_1, and g is a T-piece. Also, $Y_S(g) = V(G)$ for every H^*-extension S of T. But then, by (2), G admits an $(\mathcal{F}, 2(m+1)^m)$-partition. This proves 2.1. ∎

3 Tournament heroes

A *tournament* is a digraph such that for every two distinct vertices u, v there is exactly one edge with ends $\{u, v\}$ (so, either the edge uv or vu but not both). Let G be a tournament. If uv is an edge of G we say that u is *adjacent to* v, and v is *adjacent from* u. For $X \subseteq V(G)$, we denote by

$G|X$ the subtournament of G induced by X. We write $G \setminus X$ to mean $G|(V(G) \setminus X)$; and if $|X| = 1$, say $X = \{x\}$, we write $G \setminus x$ instead of $G \setminus \{x\}$.

If X and Y are two disjoint subsets of $V(G)$, we say that X is *complete to* Y, and Y is *complete from* X, if every vertex in X is adjacent to every vertex in Y; if $|X| = 1$, say $X = \{x\}$, we say that x is complete to Y, and Y is complete from x. If H is a tournament, we say G *contains* H if H is isomorphic to a subtournament of G, and otherwise G is *H-free*. For a set $\mathcal{H}$ of tournaments, G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. A set $X \subseteq V(G)$ is *transitive* if $G|X$ has no directed cycles. The *chromatic number* of G is the smallest integer k for which $V(G)$ can be partitioned into k transitive subsets. Given tournaments H_1 and H_2 with disjoint vertex sets, we write $H_1 \Rightarrow H_2$ to mean the tournament H with $V(H) = V(H_1) \cup V(H_2)$, and such that $H|V(H_i) = H_i$ for $i = 1, 2$, and $V(H_1)$ is complete to $V(H_2)$.

A tournament H is a *hero* if there exists c (depending on H) such that every H-free tournament has chromatic number at most c. One of the results of [1] is a complete characterization of all heroes. An important and the most difficult step toward that is the following:

3.1. *If H_1 and H_2 are heroes, then so is $H_1 \Rightarrow H_2$.*

It turns out that translating the proof of 2.1 into the language of tournaments gives a proof of 3.1 that is much simpler than the one in [1], and we include it here.

Let us start with some definitions. Let H^* be a tournament, and let S, T be subtournaments of H^*. As with undirected graphs, we say that S is an *H^*-extension* of T if $T = S \setminus u$ for some $u \in V(S)$. For an integer $k > 0$ and a set $\mathcal{F}$ of tournaments, we say that a tournament G admits an $(\mathcal{F}, k)$-partition if $V(G) = X_1 \cup \ldots \cup X_k$, where for all $i \in \{1, \ldots, k\}$, either $|X_i| = 1$, or $G|X_i$ is F-free for some $F \in \mathcal{F}$. Please note that the condition $|X_i| = 1$ is significant only when all members of $\mathcal{F}$ have at most one vertex.

First we prove the tournament analogue of 2.1.

3.2. *Let H_1, H_2 be non-null tournaments, and let H be $H_1 \Rightarrow H_2$. Let $m = \max(|V(H_1)|, |V(H_2)|)$, and $\mathcal{F} = \{H_1, H_2\}$. Then every H-free tournament G admits an $(\mathcal{F}, 2(m+1)^m)$-partition.*

Proof. By reversing all edges of T, if necessary, we may assume that $|V(H_1)| = m$. We may assume that G is not H_1-free, for otherwise the theorem holds. Choose $h^* \in V(H_2)$, and let H^* be the subtournament of H with vertex set $V(H_1) \cup \{h^*\}$. For a subtournament T of H^*, a *T-piece* is an isomorphism g from T to some subtournament of G that we denote by $g(T)$. For a T-piece g, and an H^*-extension S of T, let $V(S) \setminus V(T) = \{s\}$; we say that $v \in V(G) \setminus V(g(T))$ *g-corresponds to* S if the map sending x to $g(x)$ for each $x \in V(T)$, and sending s to v, is an S-piece. We denote by $Y_S(g)$ the set of all vertices in G that g-correspond to S. Let $Y(g)$ be the union of the sets $Y_S(g)$. Thus $Y(g)$ is the set of all vertices in $V(G)$ that g-correspond to an H^*-extension of T.

(1) *If g is an H_1-piece, then $Y(g)$ is complete from $V(g(H_1))$, and the graph $G|Y(g)$ is H_2-free.*

Since $V(H_1)$ is complete to $V(H_2)$ in H, and in particular complete to h^*, it follows that $Y(g)$ is complete from $V(g(H_1))$. But then, since G is H-free, it follows that $G|Y(g)$ is H_2-free. This proves (1).

Let

$$\phi(t) = \begin{cases} 2(m+1)^{m-t} & \text{for } 0 \le t \le m-1 \\ 1 & \text{for } t = m. \end{cases}$$

(2) *Let T be a subtournament of H_1, and let g be a T-piece. Write $t = |V(T)|$. Then for some H^*-extension S of T, the tournament $G|Y_S(g)$ admits an $(\mathcal{F}, \phi(t))$-partition.*

The proof is by induction on $m - t$. If $t = m$, then (as $|H_1| = m$) it follows from (1) that $G|Y(g)$ admits an $(\mathcal{F}, 1)$-partition, so we may assume that $t \le m - 1$.

Choose an H^*-extension S of T, with S a subtournament of H_1. If $G|Y_S(g)$ admits an $(\mathcal{F}, \phi(t))$-partition then we are done; so we may assume that for every partition of $Y_S(g)$ into at most $\phi(t)$ classes, some class contains both of H_1, H_2.

Let $V(S) \setminus V(T) = \{s\}$. Let $v \in Y_S(g)$, and let h be the S-piece mapping s to v and mapping x to $g(x)$ for each $x \in V(T)$. Inductively, there exists an H^*-extension Q of S such that $G|Y_Q(h)$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Let $V(Q) \setminus V(S) = \{r\}$, and let $R = Q \setminus s$. We say that v is *of type R*. From the definition of $Y_Q(h)$, it follows that $Y_Q(h) \subseteq Y_R(g)$, and either

- r is adjacent to s in H, and $Y_Q(h)$ is the set of vertices in $Y_R(g)$ that are different from and adjacent from v, or
- s is adjacent to r in H, and $Y_Q(h)$ is the set of vertices in $Y_R(g)$ that are different from and adjacent to v.

For each H^*-extension R of T different from S, let Z_R be the set of vertices in $Y_S(g)$ that are of type R. Thus the sets Z_R have union $Y_S(g)$. Since $|V(H^*)| - |V(T)| = m + 1 - t$ and R must be different from S, it follows that there are at most $m - t$ different types of vertices in $Y_S(g)$. Since

$$m - t \le 2(m+1)^{m-t} = \phi(t),$$

there is an H^*-extension R of T different from S, such that $G|Z_R$ contains both of H_1, H_2. Write $A = Y_R(g)$. Let $V(R) \setminus V(T) = \{r\}$.

Assume first that r is adjacent from s, and choose $B \subseteq Z_R$ such that $G|B$ is isomorphic to H_2. For each $b \in B$, let A_b be the set of vertices of $A \setminus B$ that are adjacent from b. Let A_0 be the set of vertices of $A \setminus B$ that are complete to B. Then $A \subseteq B \cup A_0 \cup \bigcup_{b \in B} A_b$.

Let $b \in B$; since $b \in Z_R$, since $A_b \subseteq Y_R(g)$, and since A_b is complete from b, it follows from the definition of Z_R that $G|A_b$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Since G is H-free, it follows that $G|A_0$ is H_1-free. Since $|B| \le m$, this implies that $G|A$ admits an $(\mathcal{F}, K)$-partition, where

$$K = m\phi(t+1) + m + 1 \le 2(m+1)^{m-t} = \phi(t),$$

and so (2) holds.

Now we assume that r is adjacent to s. Choose $B \subseteq Z_R$ such that $G|B$ is isomorphic to H_1. For $b \in B$, let A_b be the set of vertices of $A \setminus B$ that are adjacent to b. Let A_0 be the set of vertices of $A \setminus B$ that are complete from B. Then $A \subseteq B \cup A_0 \cup \bigcup_{b \in B} A_b$.

Let $b \in B$; since $b \in Z_R$, since $A_b \subseteq Y_R(g)$, and since A_b is complete to b, it follows from the definition of Z_R that $G|A_b$ admits an $(\mathcal{F}, \phi(t+1))$-partition. Since G is H-free, it follows that $G|A_0$ is H_2-free. Since $|B| \leq m$, this implies that $G|A$ admits an $(\mathcal{F}, K)$-partition, where

$$K = m\phi(t+1) + m + 1 \leq 2(m+1)^{m-t} = \phi(t).$$

This proves (2).

Now let T be the null tournament, and g the isomorphism from T into G. Then T is an subtournament of H_1, and g is a T-piece. Also, $Y_S(g) = V(G)$ for every H^*-extension S of T. But then, by (2), G admits an $(\mathcal{F}, 2(m+1)^m)$-partition. This proves 3.2. ∎

Now 3.1 follows easily:

Proof of 3.1. Since H_1 and H_2 are heroes, there exists an integer $c > 0$ such that every H_i-free tournament has chromatic number at most c for $i = 1, 2$. By 3.2, every H-free tournament G has an $(\{H_1, H_2\}, 2(m+1)^m)$-partition, where $m = \max(|V(H_1)|, |V(H_2)|)$; and therefore $V(G)$ can be partitioned into $2(m+1)^m c$ transitive subsets. Thus every H-free tournament has chromatic number at most $2(m+1)^m c$, and consequently H is a hero. This proves 3.1. ∎

4 Cographs

In this section we prove 1.4, which is the cograph analogue of 1.2. Let $\mathcal{F}$ be a set of graphs, where $k \geq 1$ is an integer, and let $P > 0$ be an integer. We say that a graph C is *$(\mathcal{F}, P)$-universal* if for every partition $X_1, \ldots, X_P$ of $V(C)$ (by a *partition* we mean that the sets $X_1, \ldots, X_P$ are pairwise disjoint, and have union $V(C)$), there exists $i \in \{1, \ldots, P\}$ such that $C|X_i$ contains every member of $\mathcal{F}$ (in other words, C does not admit an $(\mathcal{F}, P)$-partition.

We start with a lemma that establishes the existence of universal cographs.

4.1. *Let P, k be positive integers. Let $\mathcal{F}$ be a finite set of connected cographs, all of height at most k. Then there exists a connected cograph of height k that is $(\mathcal{F}, P)$-universal.*

Proof. The proof is by induction on k. Suppose first that $k = 1$. Then the members of $\mathcal{F}$ are complete graphs; let $m = \max_{F \in \mathcal{F}} |V(F)|$. Now the complete graph on mP vertices is $(\mathcal{F}, P)$-universal.

Next we consider a general $k > 1$. For every $F \in \mathcal{F}$, the members of the set $ac(F)$ are anticonnected cographs of height at most $k - 1$. Let $A = \bigcup_{F \in \mathcal{F}} ac(F)$. Inductively, passing to the complement, there exists an (A, P)-universal anticonnected cograph C of height $k - 1$. Denote by s the maximum number of anticomponents of a member of $\mathcal{F}$, and write $K = (s-1)P + 2$. Then $K \geq 2$. Let U be the cograph obtained from K vertex-disjoint copies $C_1, \ldots, C_K$ of C by making $V(C_i)$ complete to $V(C_j)$ for all $1 \leq i < j \leq K$. Then U is a connected cograph of height k.

We claim that U is $(\mathcal{F}, P)$-universal. Let $X_1, \ldots, X_P$ be a partition of $V(U)$. We need to prove that $U|X_j$ contains every member of $\mathcal{F}$ for some $j \in \{1, \ldots, P\}$.

For $i \in \{1, \ldots, K\}$ and $j \in \{1, \ldots, P\}$ write $C_i^j = U|(V(C_i) \cap X_j)$. Since C is (A, P)-universal, it follows that for every $i \in \{1, \ldots, K\}$ there exists $j \in \{1, \ldots, P\}$ such that C_i^j contains every member of A. For every $j \in \{1, \ldots, P\}$, let

$$I_j = \{i \in \{1, \dots, K\} \text{ such that } C_i^j \text{ contains every member of } A\}.$$

Since $K > P(s-1)$, there exists $j \in \{1, \dots, P\}$ such that $|I_j| \geq s$. Let D be the graph obtained from the graphs $\{C_i^j\}_{i \in I_j}$ by making $V(C_i^j)$ complete to $V(C_h^j)$ for all distinct $i, h \in I_j$. Then each anticomponent of D contains every member of A, and D has at least s anticomponents. But since each member of $\mathcal{F}$ has at most s anticomponents, it follows that D contains every member of $\mathcal{F}$. Since D is an induced subgraph of $U|X_j$, it follows that $U|X_j$ contains every member of $\mathcal{F}$. This proves the claim that U is $(\mathcal{F}, P)$-universal, and completes the proof of 4.1. ∎

We are now ready to prove 1.4. First we note that 1.3 immediately implies the following:

4.2. *Let $k \geq 0$ be an integer, and let H and J be cographs, each of height $k+1$, such that H is anticonnected, and J is connected. Then there exists an integer P such that every $\{H, J\}$-free graph admits a $(c(H) \cup ac(J), P)$-partition.*

We repeat 1.4:

4.3. *Let $k \geq 1$ be an integer, and let H and J be cographs, each of height $k+1$, such that H is anticonnected, and J is connected. Then there exist cographs $\tilde{H}$ and $\tilde{J}$, each of height k, such that $\tilde{H}$ is connected, and $\tilde{J}$ is anticonnected, and every $\{H, J\}$-free graph G is $\{\tilde{H}, \tilde{J}\}$-split.*

Proof. Write $A = c(H)$ and $B = ac(J)$. Then the members of A are connected cographs of height at most k, and the members of B are anticonnected cographs of height at most k. By 4.2, there exists an integer P such that G admits an $(A \cup B, P)$-partition. We may assume that there exists $j \in \{1, \dots, P\}$ such that for $1 \leq i \leq j$ either $|X_i| = 1$, or the subgraph of G induced by X_i is $\{F\}$-free for some $F \in A$, and for $j < i \leq P$ either $|X_i| = 1$, or the subgraph of G induced by X_i is $\{F\}$-free for some $F \in B$. Let $X = \bigcup_{1 \leq i \leq j} X_i$, and $Y = \bigcup_{j < i \leq P} X_i$. By 4.1, there exists a connected cograph $\tilde{H}$, of height k, such that $\tilde{H}$ is (A, P)-universal. By 4.1 (complemented) there exists an anticonnected cograph $\tilde{J}$, of height k, such that $\tilde{J}$ is (B, P)-universal. From the definition of X and Y, it follows that $G|X$ is $\tilde{H}$-free, and $G|Y$ is $\tilde{J}$-free, and so G is $\{\tilde{H}, \tilde{J}\}$-split, as required. This proves 4.3. ∎

5 A construction

Let G be a graph (not necessarily connected). A *block* of G is a maximal subgraph of G that is either 2-connected or isomorphic to K_2 so in this paper isolated vertices do not belong to a block). Please note that two distinct blocks cannot share more than one vertex. In this section we prove 1.7, which we restate:

5.1. *Let L, M be graphs each with at least one edge. Then for every choice of non-negative integers r, k there is a graph G such that*

- *for every $S \subseteq V(G)$ with $|S| \leq r$, the graph $G|S$ is $\{L, M\}$-split, and*
- *G has no $(\{L, M\}, k)$-partition.*

Proof. Let $L_1, \ldots, L_l$ be the blocks of L and let $M_1, \ldots, M_m$ be the blocks of M. We may assume that $L_1, \ldots, L_l, M_1, \ldots, M_m$ each have at most $|V(L_1)|$ vertices, and $l, m \geq 1$ (since L, M each have at least one edge by hypothesis). Note that L, M may not be connected, and there may be isolated vertices that are not contained in any block. Fix r, k; we may assume that $r \geq \max\{3|L|, 3|M|\}$. Choose a small constant $\epsilon > 0$ ($\epsilon = 1/(r+2)$ will do).

Let V be a set of size $n \geq 1$. By a *hypergraph* with vertex set V we mean in this paper a set of subsets of V (all different), and we call these subsets *hyperedges*. We generate $l + m$ independent random hypergraphs with vertex set V as follows. For $i = 1, \ldots, l$, we let H_i^L be a random $|V(L_i)|$-uniform hypergraph, where each possible hyperedge is present independently with probability $p_i = n^{-(|V(L_i)|-1)+\epsilon}$; for $j = 1, \ldots, m$, we let H_j^M be a random $|V(M_j)|$-uniform hypergraph, where each possible hyperedge is present independently with probability $q_j = n^{-(|V(M_j)|-1)+\epsilon}$. We then set H to be the union of these $l + m$ hypergraphs, labeling each hyperedge of H with the name of the block (L_i or M_j) it came from; we refer to the L_i and M_j as *pieces of* H. Note that at this point a hyperedge might have more than one label.

For $t \geq 3$, a *cycle of length* t in H (or t*-cycle*) is a sequence of distinct vertices $v_1, \ldots, v_t$ such that for $1 \leq i \leq t$, some hyperedge A satisfies $A \cap \{v_1, \ldots, v_t\} = \{v_i, v_{i+1}\}$, where subscripts are taken modulo t. A *cycle of length* 2 in H (or 2*-cycle*) is a pair of distinct vertices v_1, v_2 such that there are at least two distinct hyperedges containing v_1, v_2. If $R \subseteq V$, we denote by $H \setminus R$ the hypergraph of all hyperedges of H that are disjoint from R.

(1) *If n is sufficiently large, then with high probability, there is a set R of size $o(n/\log n)$ such that $H \setminus R$ has no cycles of length at most r and no hyperedges with multiple labels.*

Let us deal first with multiple labels. If a hyperedge of size k has at least two labels, then it has been chosen in (at least) two distinct pieces of H. This has probability at most $\binom{l+m}{2}(n^{-(k-1)+\epsilon})^2 = O(n^{-2(k-1)+2\epsilon})$. The expected number of such hyperedges is $O(n^{-2(k-1)+2\epsilon}\binom{n}{k}) = O(n^{-k+2+2\epsilon}) = O(n^{2\epsilon})$, so by Markov's Inequality there are with high probability at most $o(n/\log n)$ such hyperedges.

For vertices v, w in H, let X_{vw} be the number of hyperedges of H that contain both v and w. Then

$$\begin{aligned}
\mathbb{E}X_{vw} &= \sum_i \binom{n-2}{|V(L_i)|-2} p_i + \sum_j \binom{n-2}{|V(M_j)|-2} q_j \\
&\leq \sum_i n^{|V(L_i)|-2} n^{-(|V(L_i)|-1)+\epsilon} + \sum_j n^{|V(M_j)|-2} n^{-(|V(M_j)|-1)+\epsilon} \\
&= O(n^{-1+\epsilon}).
\end{aligned}$$

The probability that there is a pair of hyperedges both containing the pair $\{v, w\}$ is at most $\mathbb{E}\binom{X_{vw}}{2}$, the expected number of pairs of hyperedges containing $\{v, w\}$. Since X_{vw} is a sum of independent indicator variables, we have $\mathbb{E}X_{vw}(X_{vw}-1) \leq (\mathbb{E}X_{vw})^2$ and so $\mathbb{E}\binom{X_{vw}}{2} = O(n^{-2+2\epsilon})$. Summing over all v, w, we see that the expected number of pairs $\{v, w\}$ that lie in two or more hyperedges is $O(n^{2\epsilon})$ and so by Markov's Inequality is with high probability $O(n^{3\epsilon})$. Consequently with high probability the number of 2-cycles is $O(n^{3\epsilon})$.

For fixed $t \geq 3$, we now bound the number of t-cycles. Let $v_1, \ldots, v_t$ be a sequence of distinct vertices, and for $i = 1, \ldots t$, let Y_i be the number of hyperedges that meet $\{v_1, \ldots, v_t\}$ in exactly

$\{v_i, v_{i+1}\}$ (subscripts taken modulo t). Note that the random variables $Y_1, \ldots, Y_t$ are independent, as they depend on disjoint sets of hyperedges; also, for each i, we have $Y_i \leq X_{v_i v_{i+1}}$. Then the probability that the sequence $v_1, \ldots, v_t$ forms a t-cycle is

$$\mathbb{P}[Y_1 \cdots Y_t > 0] \leq \mathbb{E}[Y_1 \cdots Y_t] = \prod_{i=1}^{t} \mathbb{E}Y_i \leq \prod_{i=1}^{t} \mathbb{E}X_{v_i v_{i+1}} = O(n^{-t+t\epsilon}).$$

Summing over all $O(n^t)$ choices of $v_1, \ldots, v_t$, we see that the expected number of t-cycles is $O(n^{t\epsilon})$. So by Markov's Inequality, with high probability the number of t-cycles is $O(n^{(t+1)\epsilon}) = O(n^{(r+1)\epsilon})$. Consequently, with high probability the number of cycles of length at most r is at most $O(n^{(r+1)\epsilon}) = o(n/\log n)$.

Finally, let R consist of one vertex from each hyperedge with multiple labels, and one vertex from each cycle of length at most r. By the argument above, this gives with high probability a total of at most $o(n/\log n)$ vertices. This proves (1).

(2) *If n is sufficiently large, then with high probability, every set of at least $n/\log n$ vertices contains hyperedges from every H_i^L and H_j^M.*

Let S be a set of at least $n/\log n$ vertices. Then, for any i, the probability that S contains no hyperedge from H_i^L is at most

$$(1-p_i)^{\binom{|S|}{|V(L_i)|}} \leq (1-p_i)^{n^{|V(L_i)|-\epsilon/2}} \leq \exp(-p_i n^{|V(L_i)|-\epsilon/2}) = \exp(-n^{1+\epsilon/2}),$$

provided n is sufficiently large. The same bound holds for hyperedges from H_j^M. There are fewer than 2^n choices for S, and $O(1)$ choices of i or j, so with high probability every set of at least $n/\log n$ vertices contains hyperedges from every H_i^L and H_j^M. This proves (2).

From (1) and (2), if n is sufficiently large then there exists a hypergraph H and a subset $R \subseteq V$ such that

(3) *R has size $o(n/\log n)$, and $H \setminus R$ has no cycles of length at most r, and has no hyperedges with multiple labels, and every set of at least $n/\log n$ vertices of V contains hyperedges of H from every H_i^L and H_j^M.*

Let $H' = H \setminus R$. We construct a graph G, with $V(G) = V(H')$, by replacing each surviving hyperedge of form H_i^L by a copy of L_i, and each surviving hyperedge of form H_j^M by a copy of M_j (in each case, choosing an arbitrary ordering of the vertices). Note that this is well-defined, as H' has no hyperedges with multiple labels, and no pair of vertices belongs to two hyperedges (so the subgraphs we are inserting intersect pairwise in at most one vertex).

We claim that, provided n is sufficiently large, G satisfies the theorem.

(4) *For every subset S of $V(G)$ with size at most r, the graph $G|S$ is (L, M)-split.*

Fix an S. We show that S can be partitioned into two sets, X and Y, so that $G|X$ is L-free, and $G|Y$ is M-free. Let H_S be the hypergraph containing all sets $A \cap S$, where $A \in H'$ and $|A \cap S| \geq 2$.

We label each hyperedge of H_S with the label of the hyperedge that generated it (note that this is well defined: all hyperedges of H_S have size at least two, and so are contained in only one hyperedge of H').

By construction, the hypergraph H_S has no cycles, since any cycle in H_S is a cycle in H', and H' has no cycles of length at most r. It can be easily proved that there is a partition (X, Y) of S such that every hyperedge of H_S has exactly one vertex in Y.

Suppose that $G|X$ is not L-free. Then, in particular, there exists a subset B of X such that $G|B$ is isomorphic to L_1. Since L_1 is 2-connected, and H_S has no cycles, it follows that B is contained in some hyperedge E of H_S. But all hyperedges of H_S have size at most $|V(L_1)|$, and therefore $E \cap Y = \emptyset$, a contradiction. This proves that $G|X$ is L-free.

Next suppose that $G|Y$ is not M-free. Then, in particular, there exists a subset B of Y such that $G|B$ is isomorphic to M_1. Since M_1 is 2-connected, and H_S has no cycles, it follows that B is contained in some hyperedge E of H_S. But $|E \cap Y| = 1$, a contradiction since $|V(M_1)| \geq 2$. Thus $G|Y$ is M-free. This proves (4).

(5) *G has no $(\{L, M\}, k)$-partition.*

It is enough to show that for every subset S of $V(G)$ with $|S| \geq n/2k$, the graph $G|S$ is not L-free and not M-free. Let $\mathcal{B} = \{L_1, \ldots, L_l, M_1, \ldots, M_m\}$. Note that, by adding fewer than $2|L|$ additional blocks each isomorphic to L_1, we can generate a connected graph L' that has L as an induced subgraph, and such that all its blocks belong to $\mathcal{B}$; and similarly for M. Thus, since $r \geq \max\{3|L|, 3|M|\}$, it will be enough to prove the following.

(6) *For every integer $t \geq 1$ with $t < r$, there is an integer $K(t) \geq 0$ such that if n is sufficiently large then the following holds. Let F be a connected graph with exactly t blocks, such that all its blocks are isomorphic to members of $\mathcal{B}$; then for every set $W \subseteq V(G)$ of at least $K(t)n/\log n$ vertices, there is an induced subgraph F' of $G|W$, isomorphic to F, such that the vertex set of every block of F' is a hyperedge of H'.*

We argue by induction on t. For $t = 1$ this follows from (3), taking $K(1) = 1$. So suppose $t \geq 2$ and we have shown the existence of $K(t-1)$. Write $K = K(t-1)$. Let F be a graph with t blocks, all from $\mathcal{B}$. Since $t \geq 2$ and F is connected, and the bipartite graph of blocks versus cutpoints of F is a tree, it follows that there is a block B of F, and a vertex v_0 of B, such that no other block of F contains any vertex of B different from v_0. Write $F' = F \setminus (V(B) \setminus \{v_0\})$. Then F' has $t-1$ blocks. If P is an induced subgraph of G and $v \in V(P)$, and there is an isomorphism between P and F' mapping v to v_0, we call v an *anchor* of P.

Pick a large constant M, let W be any set of at least $Mn/\log n$ vertices in G, and let $\mathcal{F}'$ be a maximal collection of pairwise vertex-disjoint copies of F' with vertices from W. Then, by our inductive hypothesis, the union of the vertex sets of the members of $\mathcal{F}'$ contains all but at most $Kn/\log n$ vertices from W, and so

$$|\mathcal{F}'| \geq \frac{(M-K)n}{|V(F')| \log n} > \frac{n}{\log n},$$

provided M is sufficiently large. Let T be a set consisting of an anchor of each member of $\mathcal{F}'$. Then $|T| \geq n/\log n$, and since $K(1) = 1$, T contains an hyperedge E of H such that $G|E$ is isomorphic to B. Let $z \in E$ such that some such isomorphism takes z to v_0. Let $P \in \mathcal{F}'$ be such that $E \cap V(P) = \{z\}$. If some vertex in $E \setminus \{z\}$ is adjacent in G to some vertex in $V(F) \setminus \{z\}$, then since every edge of G is contained in a hyperedge of H', and E is a hyperedge of H', and P has at most $r-2$ blocks, each with vertex set some hyperedge of H', it follows that H' has a cycle of length at most r, a contradiction. Thus there is no such edge, and so $G|(V(P) \cup E)$ is isomorphic to F, and (6) holds taking $K(t) = M$. This proves (6) and completes the proof of 5.1. ∎

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