Coarse tree-width

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Abstract

We prove two theorems about tree-decompositions in the setting of coarse graph theory. First, we show that a graph G admits a tree-decomposition in which each bag is contained in the union of a bounded number of balls of bounded radius, if and only if G admits a quasi-isometry to a graph with bounded tree-width. (The "if" half is easy, but the "only if" half is challenging.) This generalizes a recent result of Berger and Seymour, concerning tree-decompositions when each bag has bounded radius.

Second, we show that if G admits a quasi-isometry ϕ to a graph H of bounded path-width, then G admits such a quasi-isometry that has error only an additive constant. Indeed, we will show a much stronger statement: that we can assign a non-negative integer length to each edge of H, such that the same function ϕ is a quasi-isometry to this weighted version of H, with error only an additive constant.

1 Introduction

Coarse graph theory is a new area that is filled with interesting open questions, and what is known so far consists mostly of special cases of statements that might be much more generally true. In this paper we make some unifying inroads. But we need to begin with some definitions.

Graphs in this paper may be infinite. (Our research was motivated by interest in finite graphs, but all the proofs work equally well for infinite graphs.) If X is a vertex of a graph G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\operatorname{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\operatorname{dist}_G(X, Y) = \infty$.)

Let G, H be graphs, and let $\phi : V(G) \to V(H)$ be a map. Let $L \geq 1$ and $C \geq 0$; we say that ϕ is an (L, C)-*quasi-isometry* if:

- for all u, v in $V(G)$, if $\operatorname{dist}_G(u, v)$ is finite then $\operatorname{dist}_H(\phi(u), \phi(v)) \leq L \operatorname{dist}_G(u, v) + C$;
- for all u, v in $V(G)$, if $\operatorname{dist}_H(\phi(u), \phi(v))$ is finite then $\operatorname{dist}_G(u, v) \leq L \operatorname{dist}_H(\phi(u), \phi(v)) + C$; and
- for every $y \in V(H)$ there exists $v \in V(G)$ such that $\operatorname{dist}_H(\phi(v), y) \leq C$.

If $X \subseteq V(G)$, let us say the *diameter of X in G* is the maximum of $\operatorname{dist}_G(u, v)$ over all $u, v \in X$. A *tree-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a tree, and B_t is a subset of $V(G)$ for each $t \in V(T)$ (called a *bag*), such that:

- $V(G)$ is the union of the sets B_t ($t \in V(T)$);
- for every edge $e = uv$ of G, there exists $t \in V(T)$ with $u, v \in B_t$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

(T might be infinite.) The *width* of a tree-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t| - 1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *tree-width* of G is the minimum width of a tree-decomposition of G. If T is a path, we call $(T, (B_t : t \in V(T)))$ a *path-decomposition*, and the *path-width* of G is defined analogously.

Our first result is an extension of a result of Berger and Seymour [1] (this result can also be derived from a combination of results of Chepoi et al. [3]). They proved:

1.1 *For all r, if G is connected and admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t has diameter at most r in G, then G admits a $(1, 6r + 1)$-quasi-isometry to a tree.*

This has a sort of converse, also proved in [1]: if G is connected and (L, C)-quasi-isometric to a tree then it admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t has diameter at most $L(L + C + 1) + C$ in G.

We will extend 1.1 from trees to graphs of bounded tree-width, as follows (although saying that this extends 1.1 is something of a stretch, because we do not know whether 1.2 holds with $L = 1$):

1.2 *For all k, r, there exist $L, C \geq 1$ such that if G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G, then G admits an (L, C)-quasi-isometry to a graph with tree-width at most k.*

A similar result (with weaker constants) was obtained independently by R. Hickingbotham [7], by applying a result of Dvořák and Norin [5].

In 1.2, we start with a tree-decomposition in which each bag is the union of k bounded-radius balls, and we obtain a tree-decomposition in which each bag has size at most $k+1$: and one might hope that the final k in the statement of 1.2 should be $k-1$. Obviously not for $k = 1$; but not when $k \geq 2$ either. To see this when $k = 2$, let G be a cycle, with vertices $v_1\text{-}\cdots\text{-}v_n\text{-}v_1$ in order. For $1 \leq i \leq n-1$, let $B_{v_i} = \{v_i, v_{i+1}, v_n\}$, and let T be the tree $G \setminus \{v_n\}$. Then $(T, (B_t : t \in V(T)))$ is a tree-decomposition of G, and each of its bags is the union of two balls of bounded radius (one the singleton $\{v_n\}$ and the other consisting of two adjacent vertices). On the other hand, for all (L, C), if n is large enough then there is no (L, C)-quasi-isometry from G to a graph with tree-width at most 1. A similar example works for each value of $k \geq 2$ (take a $k \times k$ grid and subdivide each of its edges many times).

Again, 1.2 has a sort of converse, because if G admits an (L, C)-quasi-isometry to a graph with tree-width at most k, then G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most $k+1$ sets each of bounded diameter — we will prove this in the next section. But if we start with a graph G that admits a quasi-isometry to a graph with tree-width at most k, and apply this converse, we obtain a tree-decomposition in which each bag is a union of $k+1$ sets of bounded diameter; and if we then apply 1.2, we obtain a quasi-isometry to a graph with tree-width at most $k+1$. Somewhere we went from tree-width k to tree-width $k+1$, and this is unsatisfying, at least on aesthetic grounds.

A way to get rid of it is to make a small tweak in the definition of tree-decomposition; say a *pseudo-tree-decomposition* $(T, (B_t : t \in V(T)))$ is the same as a tree-decomposition, except we relax the condition that every edge has both ends in some bag. Instead, we insist that for every edge uv, either some bag contains both u, v, or there is an edge st of T such that $B_s \setminus B_t = \{u\}$ and $B_t \setminus B_s = \{v\}$. Define *pseudo-tree-width* correspondingly (it differs from tree-width by at most one). We will prove a version of 1.2 with "tree-width at most k" replaced by "pseudo-tree-width at most $k-1$", and a version of 2.1 with "tree-width at most k" replaced by "pseudo-tree-width at most k", and the anomalous error of one is gone. More exactly, we will prove:

1.3 *For all k, r, there exist L, C such that if G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G, then G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$.*

Conversely, for all $L, C \geq 1$, if G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$, then G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each of diameter at most $2L(L+C)+C$.

For our second result, let us return to the definition of an (L, C)-quasi-isometry. What if we want $L = 1$? There is a remarkable theorem of V. Chepoi, F. Dragan, I. Newman, Y. Rabinovich, and Y. Vaxès [4], also proved by Alice Kerr [8]:

1.4 *For all L, C there exists C' such that if there is an (L, C)-quasi-isometry from a graph G to a tree, then there is a $(1, C')$-quasi-isometry from G to a tree.*

Is this special to trees, or can it be made much more general? For instance, Agelos Georgakopoulos asked (in private communication) whether the same statement was true if we (twice) replace "tree" by "planar graph". Let $\mathcal{C}$ be a class of graphs. Under what conditions on $\mathcal{C}$ can we say the following?

"For all L, C there exists C' such that if there is an (L, C)-quasi-isometry from a graph G to a member of $\mathcal{C}$, then there is a $(1, C')$-quasi-isometry from G to a member of $\mathcal{C}$."

For this to be true, $\mathcal{C}$ must have some closure properties: for instance, if $H \in \mathcal{C}$ and G is obtained from H by subdividing every edge once, there is a $(2, 0)$-quasi-isometry from G to H, but if we want there to be a $(1, C')$-quasi-isometry from G to a member of $\mathcal{C}$ then we need $\mathcal{C}$ to contain a graph much like G; and this is close to asking that $\mathcal{C}$ be closed under edge-subdivision. Similarly, if $H \in \mathcal{C}$ and G is obtained from H by contracting the edges in some matching of H, there is a $(3, 0)$-quasi-isometry from G to H, and so we need $\mathcal{C}$ to be more-or-less closed under edge-contraction. Is that enough, could the following be true?

1.5 Conjecture: *Let $\mathcal{C}$ be a class of connected graphs, closed under contracting edges and subdividing edges. For all L, C there exists C' such that if there is an (L, C)-quasi-isometry from a graph G to a member of $\mathcal{C}$, then there is a $(1, C')$-quasi-isometry from G to a member of $\mathcal{C}$.*

For instance, if G, H are respectively the infinite square lattice and the infinite triangular lattice, there is a quasi-isometry between them, but no $(1, C)$-quasi-isometry (for any constant C); but there is a $(1, 2)$-quasi-isometry from G to a graph obtained by subdividing edges of H, and a $(1, 100)$-quasi-isometry from H to a graph obtained by subdividing and contracting edges of G (we omit the proofs of all these statements).

We are far from proving the conjecture 1.5 in general, but we will prove a special case, which we will explain next. We will prove:

1.6 *For all L, C, k there exists C' such that if there is an (L, C)-quasi-isometry from a graph G to a graph H with path-width at most k, then there is a $(1, C')$-quasi-isometry from G to a graph H' obtained from H by subdividing and contracting edges.*

Let $\mathbb{N}$ denotes the set of nonnegative integers. Let H be a graph and let $w : E(G) \to \mathbb{N}$; we call (H, w) a *weighted graph.* One can define quasi-isometry for weighted graphs in the natural way, defining $\text{dist}_{H,w}(u, v)$ to be the minimum of $w(P)$ over all paths of H between u, v, where $w(P)$ means $\sum_{e \in E(P)} w(e)$. Subdividing and contracting edges of H is closely related to moving from H to (H, w) for an appropriate w, so we could express 1.6 in terms of weighted graphs. In this modified form of 1.6, rather than replacing H by H', we keep H and just put weights on its edges. But something much stronger is true: we don't need to change the quasi-isometry either.

1.7 *For all L, C, k there exists C' such that if ϕ is an (L, C)-quasi-isometry from a graph G to a graph H with path-width at most k, then there is a function $w : E(H) \to \mathbb{N}$ such that the same function ϕ is a $(1, C')$-quasi-isometry from G to the weighted graph (H, w).*

Indeed, the conjecture 1.5 suggests something even stronger, that we could omit the path-width condition from this:

1.8 Conjecture: *For all L, C there exists C' such that if ϕ is an (L, C)-quasi-isometry from a graph G to a graph H, then there is a function $w : E(H) \to \mathbb{N}$ such that the same function ϕ is a $(1, C')$-quasi-isometry from G to the weighted graph (H, w).*

We feel this is much too strong to be true, but have no counterexample.

1.6 has some applications. First, let $\mathcal{C}$ be the class of all subdivisions of graphs of path-width at most k. (Subdividing the edges of a graph might increase its path-width, but only by one — see [2]). Then 1.6 tells us that 1.5 holds for $\mathcal{C}$.

Here is another application. A. Georgakopoulos in private communication showed that for all L, C there exists C' such that if a finite graph G is (L, C)-quasi-isometric to a cycle, then G is $(1, C')$-quasi-isometric to a cycle. This immediately follows from 1.6. Similarly, we (unpublished) proved some time ago the following result about fat minors (we omit the definitions of fat minor, since we will not need them any more in this paper): for all k, C, there exists C' such that if G does not contain $K_{1,k}$ as a C-fat minor, then there is a $(1, C')$-quasi-isometry from G to a graph not containing $K_{1,k}$ as a minor. This strengthened a result of Georgakopoulos and Papasoglu [6] that all k, C, there exist L, C' such that if G does not contain $K_{1,k}$ as a C-fat minor, then there is a (L, C')-quasi-isometry from G to a graph not containing $K_{1,k}$ as a minor. Our proof was complicated, but graphs with no $K_{1,k}$ minor have path-width at most $k-1$ and are closed under taking subdivisions, and so our result follows via 1.6 from that of of Georgakopoulos and Papasoglu.

Is 1.5 true at least when $\mathcal{C}$ is the class of graphs with tree-width at most k? Yes when $k = 1$, by 1.4, and indeed one can show that 1.6 also holds in this case (see the proof of 1.4 in [1]). What about tree-width two? A special case is when $\mathcal{C}$ is the class of all outer-planar graphs, and we can prove 1.5 in that case. (A hint for the proof: every outerplanar graph is quasi-isometric to a graph in which every non-trivial block is a cycle.) But for tree-width two in general, the result is open, as is the following weaker statement:

1.9 Conjecture: *For all L, C there exist C', k such that if there is an (L, C)-quasi-isometry from a graph G to a graph of tree-width at most two, then there is a $(1, C')$-quasi-isometry from G to a graph of tree-width at most k.*

2 The proof of 1.3

Let us state the definition of pseudo-tree-width more formally. A *pseudo-tree-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a tree, and B_t is a subset of $V(G)$ for each $t \in V(T)$ (called a *bag*), such that:

- $V(G)$ is the union of the sets B_t ($t \in V(T)$);
- for every edge $e = uv$ of G, either there exists $t \in V(T)$ with $u, v \in B_t$, or there is an edge $st \in E(T)$ such that $B_s \setminus B_t = \{u\}$ and $B_t \setminus B_s = \{v\}$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a pseudo-tree-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t| - 1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *pseudo-tree-width* of G is the minimum width of a pseudo-tree-decomposition of G. If T is a path, we call $(T, (B_t : t \in V(T)))$ a *pseudo-path-decomposition*, and the *pseudo-path-width* of G is defined analogously.

Before we prove the main part of 1.3, let us prove its (much easier) second part, the converse:

2.1 *If G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$, then G admits a tree-decomposition $(T, (D_t : t \in V(T)))$ such that for each $t \in V(T)$, D_t is the union of at most k sets each of diameter at most $2L(L + C) + C$.*

Proof. Let H be a graph with pseudo-tree-width at most $k-1$, and let $(T,(B_t : t \in V(T)))$ be a pseudo-tree-decomposition of H with width at most $k-1$. Let ϕ be an (L,C)-quasi-isometry from a graph G to H. For each $h \in V(H)$, let X_h be the set of vertices $i \in V(H)$ such that $\text{dist}_H(h,i) \le L+C$. For each $t \in V(T)$, let D_t be the set of all vertices $v \in V(G)$ such that $\phi(v) \in X_h$ for some $h \in B_t$. We claim that $(T,(D_t : t \in V(T)))$ is a tree-decomposition of G satisfying the theorem. So we must check that:

- $\bigcup_{t \in V(T)} D_t = V(G)$;
- for every edge uv of G there exists $t \in V(T)$ with $\{u,v\} \in D_t$;
- for all $t_1,t_2,t_3 \in V(T)$, if t_2 lies on the path of T between t_1,t_3, then $D_{t_1} \cap D_{t_3} \subseteq D_{t_2}$; and
- for each $t \in V(T)$, D_t is the union of at most k sets each of diameter (in G) at most $2L(L+C)+C$.

For the first statement, let $v \in V(G)$; then $\phi(v) \in V(H)$, and so $\phi(v) \in B_t$ for some $t \in V(T)$. In particular, since $\phi(v) \in X_{\phi(v)}$, it follows that $v \in D_t$. This proves the first statement.

For the second statement, let $uv \in E(G)$, and choose $t \in V(T)$ with $\phi(v) \in B_t$. Since ϕ is an (L,C)-quasi-isometry, $\text{dist}_H(\phi(u),\phi(v)) \le L+C$, and so $\phi(u) \in X_{\phi(v)}$. It follows that $u,v \in D_t$. This proves the second statement.

For the third statement, let $t_1,t_2,t_3 \in V(T)$, such that t_2 lies on the path of T between t_1,t_3, and let $v \in D_{t_1} \cap D_{t_3}$. Hence for $i=1,3$, there exists $h_i \in B_{t_i}$ with $\phi(v) \in X_{h_i}$; let P_i be a path of H between $\phi(v),h_i$ of length at most $L+C$. Since $P_1 \cup P_3$ is a connected graph with vertices in B_{t_1} and in B_{t_3}, it also has a vertex in B_{t_2}, say h_2. Thus h_2 belongs to one of $V(P_1),V(P_3)$, and so $\text{dist}_H(h_2,\phi(v)) \le L+C$; and hence $\phi(v) \in X_{h_2}$, and therefore $v \in D_{t_2}$. This proves the third statement.

Finally, for the fourth statement, let $t \in V(T)$. For each $h \in B(t)$, let F_h be the set of all $v \in V(G)$ such that $\phi(v) \in X_h$. Thus D_t is the union of the sets F_h $(h \in B_t)$, and there are $|B_t| \le k$ such sets. We claim that each F_h has diameter at most $2L(L+C)+C$ in G. If $u,v \in F_h$, then each of $\phi(u),\phi(v)$ has distance at most $L+C$ from h, and so $\text{dist}_H(\phi(u),\phi(v)) \le 2(L+C)$. Since ϕ is an (L,C)-quasi-isometry, it follows that $\text{dist}_H(u,v) \le 2L(L+C)+C$. This proves the fourth statement, and so proves 2.1. ∎

To prove 1.3, we need the following lemma:

2.2 *Let G be a graph, and let A,B be disjoint subsets of $V(G)$ with union $V(G)$. Let $|A|,|B| \le k$, and suppose that there are at most k edges between A,B. Then there is a pseudo-path-decomposition $(B_1,\dots,B_n)$ of G with width at most $k-1$ and with $A \subseteq B_1$ and $B \subseteq B_n$.*

Proof. We proceed by induction on $k+|A|+|B|$. If some vertex $a \in A$ has no neighbours in B, then from the inductive hypothesis, applied to $G \setminus \{a\}$, there is a pseudo-path-decomposition $(B_1,\dots,B_n)$ of $G \setminus \{a\}$ with width at most $k-1$ and with $A \setminus \{a\} \subseteq B_1$ and $B \subseteq B_n$. But then $(A,B_1,\dots,B_n)$ satisfies the theorem. Thus we may assume that each vertex in A has a neighbour in B, and vice versa.

If every vertex in A has exactly one neighbour in B and vice versa, the result is true; so we assume that some vertex in A has at least two neighbours in B, and hence $|A| \le k-1$. Let $b \in B$ with a neighbour in A, and let G' be obtained by deleting b. In G', there are at most $k-1$ edges between A and $B \setminus \{b\}$, and these two sets both have size at most $k-1$. From the inductive hypothesis applied to G', there is a pseudo-path-decomposition $(C_1, \ldots, C_n)$ of G' with width at most $k-2$ and with $A \subseteq C_1$ and $B \setminus \{b\} \subseteq C_n$. Define $B_i = C_i \cup \{b\}$ for $1 \le i \le n$; then $(B_1, \ldots, B_n)$ is a pseudo-path-decomposition of G satisfying the theorem. This proves 2.2. ∎

To prove the first part of 1.3, it suffices to prove it when G is connected (by working with each component of G separately); and it suffices to prove it when $r = 1$. To see the latter, let G be a connected graph that admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G. For each $t \in V(T)$, and each pair u, v of nonadjacent vertices of $G[B_t]$ with $\text{dist}_G(u, v) \le r$, add an edge joining u, v, and let G' be the resultant graph. Then $(T, (B_t : t \in V(T)))$ is a tree-decomposition of G', and for each $t \in V(T)$, B_t is the union of at most k cliques of G'. Moreover, the identity map is an $(r, 0)$-quasi-isometry between G, G'; and so if G' admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$, then G admits an (rL, rC)-quasi-isometry to the same graph. Consequently, for given k, if L, C satisfy the theorem when $r = 1$, then rL, rC satisfy the theorem for general r. Hence it suffices to prove the following:

2.3 *For all k, if G is connected and admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that B_t is the union of at most k cliques for each $t \in V(T)$, then G admits a $(2k+2, 2k-1)$-quasi-isometry to a graph with pseudo-tree-width at most $k-1$.*

Proof. Let $(T, (B_t : t \in V(T)))$ be a tree-decomposition of G such that for each $t \in V(T)$, B_t is the union of at most k cliques. Fix a root $r \in V(T)$ (arbitrarily). For each $t \in V(T)$, its *ancestors* are the vertices of the path of T between r, t, and its *strict ancestors* are its ancestors different from t. If s is an ancestor of t then t is a *descendant* of s, and descendants of t different from t are *strict descendants* of t. For $t \in V(T)$, its *height* is the length of the path of T between r, t.

We will recursively define a set of pairs, called "cores". Each core will be a pair (t, C) where $t \in V(T)$ and C is a subset of B_t inducing a non-null connected subgraph, and we will call t its *birthplace*. The set of all cores with the same birthplace will be given an arbitrary linear order called the "birth order", and if (t, C) precedes (t, C') in the birth order then we will say that (t, C) is an *elder sibling* of (t, C'), and (t, C') is a *younger sibling* of (t, C). Each core (t, C) will have a *spread* $S(t, C)$, which is the vertex set of a certain subtree of T with root t, defined below.

Here is the inductive definition. If there exists $t \in V(T)$ such that we have not yet defined the set of cores with birthplace t, choose some such t with minimum height. Let Z be the set of vertices $v \in B_t$ such that $v \notin C$ and v has no neighbour in C, for every strict ancestor s of t and every core (s, C) with $t \in S(s, C)$. We define the set of cores with birthplace t to be the set of all pairs (t, C) where C is a component of $G[Z]$. Choose an arbitrary linear order, called the birth order, of the set of cores with birthplace t. For each core (t, C), its spread $S(t, C)$ is the set of all $t' \in V(T)$ such that

- t' is a descendant of t;
- $C \cap B_{t'} \neq \emptyset$;
- $t' \in S(s, C')$ for every core (s, C') such that s is a strict ancestor t and $t \in S(s, C')$; and
- $t' \in S(t, C')$ for every elder sibling (t, C') of (t, C).

This completes the inductive definition of the set of all cores.

Two subsets $X, Y \subseteq V(G)$ are *anticomplete* if they are disjoint and there are no edges of G between them. We need, first:

(1) *If $(t_1, C_1), (t_2, C_2)$ are distinct cores and their spreads intersect, then C_1, C_2 are anticomplete.*

We may assume that $t_1 \neq t_2$. Since the spreads of $(t_1, C_1), (t_2, C_2)$ intersect, t_1, t_2 have a common descendant t_0 say, so one of t_1, t_2 is a strict ancestor of the other. Hence we may assume that t_1 is a strict ancestor of t_2, and therefore $t_2 \in S(t_1, C_1)$ since the spreads intersect. Since (t_2, C_2) is a core, it follows that for each $v \in C_2$, $v \notin C_1$ and v has no neighbour in C_1. Consequently, C_1, C_2 are anticomplete. This proves (1).

(2) *For each $t \in V(T)$, there are at most k cores (s, C) such that $t \in S(s, C)$.*

Let $(s_1, C_1), \ldots, (s_n, C_n)$ be the set of all cores whose spread contains t, and let $D_1, \ldots, D_m$ be cliques with union B_t, with $m \leq k$. The sets $C_1 \cap B_t, \ldots, C_n \cap B_t$ are nonempty, and by (1) they are pairwise anticomplete. Consequently, for $1 \leq i \leq n$, there exists $j_i \in \{1, \ldots, m\}$ such that $C_i \cap B_t$ contains a vertex of D_{j_i}; and if $i, i' \in \{1, \ldots, n\}$ are distinct, then $j_i \neq j_{i'}$, because $C_i \cap B_t$ and $C_{i'} \cap B_t$ are anticomplete and D_{j_i} is a clique. Thus $n \leq m \leq k$. This proves (2).

For each $v \in V(G)$, there exists $t \in V(T)$ with $v \in B_t$, and the set of such vertices t induces a subtree of T. In particular, there is a unique $t \in V(T)$ of minimum height with $v \in B_t$, and we call t the *birth* of v. If t is the birth of v, there might or might not exist $C \subseteq B_t$ with $v \in C$ such that (t, C) is a core. If there exists such C we say v is *central*. If there exists a core (t', C') such that t' is a strict ancestor of t and $t \in S(t', C')$ and v has a neighbour in C', we say v is *peripheral*. (Note that v cannot belong to C', from the definition of t.)

(3) *Every vertex $v \in V(G)$ is central or peripheral, and not both.*

The first statement is clear from the definition of the set of cores with birthplace t. For the "not both" part, suppose that v is central and peripheral; choose $C \subseteq B_t$ with $v \in C$ such that (t, C) is a core, and choose a core (t', C') such that t' is a strict ancestor of t and $t \in S(t', C')$ and v has a neighbour in C'. Since $t \in S(t, C) \cap S(t', C')$, and $v \in C$ has a neighbour in C', this contradicts (1). This proves (3).

For each $v \in V(G)$, we define a core $\phi(v)$ as follows. Let $t_1 \in V(T)$ be the birth of v. If v is central, $\phi(v)$ is the core (t_1, C_1) with $v \in C_1$. Now assume v is peripheral. Hence there is a strict ancestor t_0 of t_1 and a core (t_0, C_0) such that $t_1 \in S(t_0, C_0)$, and v has a neighbour in C_0. Choose such t_0 of minimum height; and of all the cores (t_0, C_0) such that $t_1 \in S(t_0, C_0)$, and v has a neighbour in C_0, choose (t_0, C_0) with this property, as early as possible in the birth order. We define $\phi(v) = (t_0, C_0)$.

(4) *Let $v \in V(G)$, let $\phi(v) = (t_0, C_0)$, and let $t \in V(T)$, such that $v \in B_t$. Then exactly one of the following holds:*

- *v is peripheral, and $t \in S(t_0, C_0)$; or*

- *there is a core (t', C') with $t \in S(t', C')$ and $v \in C'$.*

If both statements hold, then since $t \in S(t_0, C_0)$ and $t \in S(t', C')$ and there is an edge between C_0, C' (because $v \in C'$ and has a neighbour in C_0), this contradicts (1). So not both hold. We prove that at least one holds by induction on the height of t. If there exists C with $v \in C$ such that (t, C) is a core, the claim is true, so we assume not. Hence, from the definition of cores, there is a core (t_2, C_2) with $t \in S(t_2, C_2)$, such that t_2 is a strict ancestor of t and v belongs to or has a neighbour in C_2. If $v \in C_2$, the claim holds, so we assume that $v \notin C_2$ and v has a neighbour in C_2.

Let t_1 be the birth of v. Thus, t_0, t_1, t_2 all belong to the path of T between r, t, and t_0 is an ancestor of t_1. Suppose that either t_2 is a strict ancestor of t_0, or (t_2, C_2) is an elder sibling of (t_0, C_0); and hence v is peripheral, in both cases. Since v has a neighbour in C_2, this contradicts the definition of $\phi(v)$. So we assume that either t_2 is a strict descendant of t_0 or (t_2, C_2) is a younger sibling of (t_0, C_0).

If $t = t_1$ the result is true, so we assume that $t \neq t_1$. Let s be the parent of t; so s lies in the path of T between t_1, t, and therefore $v \in B_s$. From the inductive hypothesis, either v is peripheral and $s \in S(t_0, C_0)$, or there is a core (t', C') with $s \in S(t', C')$ and $v \in C'$.

Suppose the first holds. Since either t_0 is a strict ancestor of t_2, or (t_0, C_0) is an elder sibling of (t_2, C_2), and since $S(t_2, C_2)$ contains t and $t_2 \in S(t_0, C_0)$, it follows (from the second half of the definition of cores) that $t \in S(t_0, C_0)$ and the claim is true.

So we assume the second holds, that is, there is a core (t', C') with $s \in S(t', C')$ and $v \in C'$. If $t \in S(t', C')$ the claim holds, so we assume not. Since t_2 is a strict ancestor of t and $t \in S(t_2, C_2)$, it follows that t_2 is an ancestor of s and $s \in S(t_2, C_2)$. But there is an edge between C_2, C', since $v \in C'$ and v has a neighbour in C_2; and so from (1), either $(t', C') = (t_2, C_2)$ or the spreads of (t', C') and (t_2, C_2) are disjoint. The first is impossible since $t \notin S(t', C')$ and $t \in S(t_2, C_2)$, and the second is impossible since s belongs to both spreads. This proves (4).

(5) *Let P be a path of T with one end r, and let $v \in V(G)$. Let $\phi(v) = (t_0, C_0)$. Let $\mathcal{C}(P, v)$ be the set of cores (t, C) such that $t \in V(P)$ and $v \in C$. Let the members of $\mathcal{C}(P, v)$ with birthplace different from t_0 be $(t_1, C_1), \dots, (t_n, C_n)$, numbered such that $t_0, t_1, \dots, t_n$ have strictly increasing height. Then:*

- *$t_i \notin S(t_h, C_h)$ for $0 \leq h < i \leq n$;*
- *for $1 \leq i \leq n$, let s_i be the parent of t_i: then $s_i \in S(t_{i-1}, C_{i-1})$;*
- *$n \leq k - 1$.*

The first bullet holds by (1), since $v \in C_i$ and either $v \in C_h$, or $h = 0$ and v has a neighbour in C_h. For the second bullet, let t_0' be the birth of v. Thus t_0 is an ancestor of t_0' (possibly $t_0' = t_0$), and $t_1, \dots, t_n$ are strict descendants of t_0' (to see that $t_1 \neq t_0'$, observe that this is trivially true if v is not central, and true if v is central since then $t_0 = t_0'$.)

Let $1 \leq i \leq n$. If $v \notin B_{s_i}$, then $i = 1$ and $t_i = t_0'$, which is impossible. So $v \in B_{s_i}$. If $s_i \in S(t_0, C_0)$, then $t_{i-1} \in S(t_0, C_0)$, and so $i = 1$ by the first bullet of (5) (because otherwise $t_{i-1} \notin S(t_0, C_0)$) and the claim is true. So we assume that $s_i \notin S(t_0, C_0)$. From (4), there is a core (t', C') with $s_i \in S(t', C')$ and $v \in C'$. Hence $(t', C') = (t_h, C_h)$ for some $h \in \{0, \dots, i-1\}$. If $h < i - 1$, then $(t_{i-1}, C_{i-1}) \notin S(t_h, C_h)$ by the first bullet of (5), contradicting that $t_i \in S(t', C')$. Thus $h = i - 1$ and the claim holds.

For the third bullet, we may assume that $n \geq 1$. For $0 \leq i \leq n$ define $g(i)$ to be the number of cores (t, C) such that t is a strict ancestor of t_i and $t_i \in S(t, C)$. We will prove by induction on i that $g(i) \leq k - i - 1$. Since there is a core (t_0, C_0), it follows that $g(0) \leq k - 1$ by (2). Inductively, suppose that $1 \leq i \leq n$, and $g(i-1) \leq k - (i-1) - 1$. Let A_{i-1} be the set of all cores (t, C) such that t is a strict ancestor of t_{i-1} and $t_{i-1} \in S(t, C)$; and let A_i be the set of all cores (t, C) such that t is a strict ancestor of t_i and $t_i \in S(t, C)$. Thus $g(i-1) = |A_{i-1}|$ and $g(i) = |A_i|$. We claim that $A_i \subseteq A_{i-1}$. Let $(t, C) \in A_i$, and suppose that $(t, C) \notin A_{i-1}$. Thus t is a strict ancestor of t_i, and a descendant of t_{i-1}. Since $t_i \notin S(t_{i-1}, C_{i-1})$, and $C_{i-1} \cap B_{t_i} \neq \emptyset$ (because it contains v), the definition of $S(t_{i-1}, C_{i-1})$ implies that there is a core (d, D) such that d is a strict ancestor of t_{i-1}, and $t_{i-1} \in S(d, D)$, and $t_i \notin S(d, D)$. But this contradicts the definition of the spread of (t, C), since d is a strict ancestor of t_{i-1} and $t_i \in S(t, C)$.

Consequently $A_{i-1} \subseteq A_i$ for $1 \leq i \leq n$. But for $1 \leq i \leq n$, since $C_{i-1} \cap B_{t_i} \neq \emptyset$ and yet $t_i \notin S(t_{i-1}, C_{i-1})$, there is a core (d, D) such that d is a strict ancestor of t_{i-1}, and $t_{i-1} \in S(d, D)$, and $t_i \notin S(d, D)$. But then $(d, D) \in A_{i-1} \setminus A_i$, and so $g(i) < g(i-1) \leq k - (i-1) - 1 = k - i - 1$. This proves the third bullet and so proves (5).

Next we construct a graph J. Its vertex set is the set of all triples (s, t, C) where (t, C) is a core and s is in its spread. Consequently s is a descendant of t for all vertices (s, t, C) of J. If $(s_1, t_1, C_1), (s_2, t_2, C_2) \in V(J)$ are distinct, they are adjacent in J if either:

- $s_1 = s_2$ and $\text{dist}_G(C_1, C_2) \leq 3$, or
- s_1, s_2 are adjacent in T and $C_1 \cap C_2 \neq \emptyset$.

In particular, if $(s, t, C) \in V(J)$ and $s \neq t$, let s' be the parent of s; then $(s', t, C) \in V(J)$ is adjacent in J to $(s, t, C) \in V(J)$, and edges of this type are called *green* edges. All edges of J that are not green are called *red*. We will eventually show that there is a $(2k+2, 2k-1)$-quasi-isometry from G to the graph obtained from J by contracting all green edges. But first we prove some properties of J.

(6) *J has pseudo-tree-width at most $k - 1$.*

For each $s \in V(T)$, let A_s be the set of all $(s, t, C) \in V(J)$. Thus the sets A_s $(s \in V(T))$ are pairwise disjoint and have union $V(J)$. Let $s, t \in V(T)$ where s is the parent of t. There may be edges of J between A_s and A_t, but we claim that there are at most k such edges. Choose a set $\mathcal{F}$ of at most k cliques with union B_s. For each edge $e \in E(J)$ between A_s, A_t, we define $F_e \in \mathcal{F}$ as follows. Let the ends of e be $(s, s_1, C_1) \in V(J)$ and (t, t_1, D_1). Then $C_1 \cap D_1 \neq \emptyset$; choose $F_e \in \mathcal{F}$ that contains a vertex in $C_1 \cap D_1$. We claim that $F_{e_1} \neq F_{e_2}$ for all distinct edges e_1, e_2 between A_s, A_t. To see this, let e_i have ends $(s, s_i, C_i) \in V(J)$ and (t, t_i, D_i) for $i = 1, 2$. Either $(s_1, C_1) \neq (s_2, C_2)$ or $(t_1, D_1) \neq (t_2, D_2)$. In the first case, C_1, C_2 are anticomplete by (1); so no clique intersects both C_1, C_2; and so $F_{e_1} \neq F_{e_2}$. In the second case, D_1, D_2 are anticomplete by (1); so no clique intersects both D_1, D_2; and so $F_{e_1} \neq F_{e_2}$. Since $|\mathcal{F}| \leq k$, this proves that there are at most k edges of J between A_s, A_t.

Let $f = st$ be an edge of T, where s is the parent of t. From 2.2, since $|A_s|, |A_t| \leq k$ by (2), there is a pseudo-path-decomposition $(B_1^f, \ldots, B_{n(f)}^f)$ of $J[A_s \cup A_t]$ with width at most $k-1$ and with $A_s \subseteq B_1^f$ and $A_t \subseteq B_{n(f)}^f$. This defines $n(f)$, for each edge f of T. Subdivide each edge $f \in E(T)$

$n(f)$ times, making a tree T'. Define $C_t = B_t$ for each $t \in V(T)$. For each $f = st \in E(T)$ where s is the parent of t, let $s, u_1, \ldots, u_{n(f)}, t$ be the vertices in order of the path formed by subdividing f, and define $C_{u_i} = B_i^f$ for $1 \leq i \leq n(f)$. This defines a pseudo-tree-decomposition of J with width at most $k-1$, and so proves (6).

The function ϕ does not map into $V(J)$, since $\phi(v)$ is a pair, not a triple. For each $v \in V(G)$, define $\psi(v) = (t, t, C)$ where $\phi(v) = (t, C)$.

(7) *Let $v \in V(G)$, and let (t, C) be a core with $v \in C$. Then there is a path of J between $\psi(v)$ and (t, t, C) with at most $k-1$ red edges.*

Let P be the path of T between r, t, and define $(t_0, C_0), \ldots, (t_n, C_n)$ as in (5). By the second bullet of (5), for $0 \leq i < n$, there is a path of J from $(t_{i-1}, t_{i-1}, C_{i-1})$ to (t_i, t_i, C_i) in which all edges are green except the last; and since $n \leq k-1$ (again by (5)), and $(t, C) = (t_n, C_n)$, this proves (7).

(8) *Let $v_1, v_2 \in V(G)$ be adjacent. Then there is a path of J between $\psi(v_1), \psi(v_2)$ using at most k red edges.*

Let $\psi(v_i) = (t_i, t_i, C_i)$ for $i = 1, 2$, and let t'_i be the birth of v_i for $i = 1, 2$. Since v_i belongs to or has a neighbour in C_i, for $i = 1, 2$, and $v_1 v_2 \in E(G)$, it follows that $\text{dist}_G(C_1, C_2) \leq 3$. There exists $s \in V(T)$ with $v_1 v_2 \in B_s$, since $v_1 v_2$ is an edge; and by choosing s of minimum height we may assume that s is the birth of one of v_1, v_2, say v_2, and so $s = t'_2$.

A *green path* of J means a path of J containing only green edges. Suppose that $t_2 \in S(t_1, C_1)$. Conequently there is a green path of J between (t_1, t_1, C_1) and (t_2, t_1, C_1). with vertex set all the triples (t, t_1, C) such that t is in the path of T between t_1, t_2, in order. Since there is a (red) edge of J between (t_2, t_1, C_1) and (t_2, t_2, C_2) (from the definition of J, since $\text{dist}_G(C_1, C_2) \leq 3$), the claim is true. Thus we may assume that $t_2 \notin S(t_1, C_1)$. In particular, t_2 is a strict descendant of t'_1.

Since t_2 is in the path of T between t'_1, t'_2, and $v_1 \in B_{t'_1} \cap B_{t'_2}$, it follows that $v_1 \in B_{t_2}$. Since $t_2 \notin S(t_1, C_1)$, (4) implies that there is a core (d, D) with $t_2 \in S(d, D)$ and $v_1 \in D$. Thus (t_1, t_1, C_1) is joined to (d, d, D) be a path of J with only $k-1$ red edges, by (7); (d, d, D) is joined to (t_2, d, D) by a green path; and (t_2, d, D) is adjacent to (t_2, t_2, C_2) via a red edge, since $\text{dist}_G(C_2, D) \leq 2$ (because v_2 has a neighbour in both). This proves (8).

(9) *For each core (t, C), $G[C]$ has diameter at most $2k-1$.*

$G[C_1]$ has no stable set of size $k+1$ (because C can be partitioned into at most k cliques), and therefore $G[C]$ has no induced path with $2k+1$ vertices. Since it is connected, it has diameter at most $2k-1$, This proves (9).

(10) *If (s_1, t_1, C_1) and $(s_, t_2, C_2)$ are joined by a green path of J, and $v_1 \in C_1$ and $v_2 \in C_2$, then* $\text{dist}_G(v_1, v_2) \leq 2k-1$.

Any two vertices of J joined by a green edge have the same second and third coordinates, and so $t_1 = t_2$ and $C_1 = C_2$. Consequently $v_1, v_2 \in C_1$, and the result follows from (9). This proves (10).

(11) *Let* $v_1, v_2 \in V(G)$*, and suppose* P *is a path of* J *between* $\psi(v_1), \psi(v_2)$ *containing at most* n *red edges. Then* $\text{dist}_G(v_1, v_2) \le (2k+2)n + 2k - 1$.

If $n = 0$ the result follows from (10), so we assume that $n \ge 1$. Let P have ends b_0 and a_{n+1}, and let the red edges of P be $a_1b_1, a_2b_2, \dots, a_nb_n$ in order, numbered such that there there is a green subpath of P between b_i, a_{i+1} for $0 \le i \le n$. For $1 \le i \le n$, define α_i, β_i as follows: let $a_i = (s, t, C)$ and $b_i = (s', t', C')$ say; choose $\alpha_i \in C$ and $\beta_i \in C'$ with distance at most three in G. (This is possible from the definition of red edges.) Let $\beta_0 = v_1$ and $\alpha_{n+1} = v_2$. Thus $\text{dist}_G(\alpha_i, \beta_i) \le 3$ for $1 \le i \le n$; and $\text{dist}_G(\beta_i, \alpha_{i+1}) \le 2k - 1$ by (10). Consequently $\text{dist}_G(v_1, v_2) \le (2k+2)n + 2k - 1$.

(12) *For each* $j \in J$*, there exists* $v \in V(G)$ *such that there is a path of* J *between* j *and* $\psi(v)$ *using at most* $k - 1$ *red edges.*

Let $j = (s, t, C)$, and choose $v \in C \cap B_s$. There is a green path between j and (t, t, C); and by (7), since $v \in C \subseteq B_t$, there is a path between (t, t, C) and $\psi(v)$ containing at most $k - 1$ red edges. This proves (12).

Let H be obtained from J by contracting all green edges. Thus each vertex of H is formed by indentifying all the vertices (s, t, C) for a fixed core (t, C), and so we can identify $V(H)$ with the set of all cores in the natural way. From (6), and since contraction does not increase pseudo-tree-width, H has pseudo-tree-width at most $k - 1$, and from (8), (11), (12), the function ψ is a $(2k + 2, 2k - 1)$-quasi-isometry from G to H. This proves 2.3 and hence (with 2.1) proves 1.3. ∎

3 The proof of 1.7, part 1

Let (H, w) be a weighted graph. For each e with $w(e) > 0$, let us subdivide e $w(e) - 1$ times, that is, replace e by a path joining the ends of e of length $w(e)$, the internal vertices of which are new vertices. For each edge $e \in E(H)$ with $w(e) = 0$, let us contract e. This produces a multigraph, possibly with loops or parallel edges; delete all loops created and all except one of each parallel class of parallel edges, and let H' be a graph obtained. Each vertex $s \in V(H)$ is taken to a vertex of H' in the natural sense, that we call the *w-image* of s. We say H' is a *w-rescaling* of H.

Let G be a graph and let (H, w) be a weighted graph. A map ϕ from $V(G)$ to $V(H)$ is an *(L, C)-quasi-isometry from G to (H, w)* if:

- for all u, v in $V(G)$, if $\text{dist}_G(u, v)$ is finite then $\text{dist}_{(H,w)}(\phi(u), \phi(v)) \le L\,\text{dist}_G(u, v) + C$;
- for all u, v in $V(G)$, if $\text{dist}_{(H,w)}(\phi(u), \phi(v))$ is finite then $\text{dist}_G(u, v) \le L\,\text{dist}_{(H,w)}(\phi(u), \phi(v)) + C$; and
- for every $y \in V(H)$ there exists $v \in V(G)$ such that $\text{dist}_{(H,w)}(\phi(v), y) \le C$.

Let G, H, w, ϕ, L, C be as above, and let H' a w-rescaling of H. Define $\phi'(v)$ to be the w-image of v, for each $v \in V(H)$ (and call ϕ' the *w-rescaling of* ϕ). One might expect that ϕ' would be an (L, C)-quasi-isometry from G to H', but this is not correct: the third condition in the definition of an (L, C)-quasi-isometry might be violated by the new vertices introduced in the subdivision process. Let us say the *weight* of w is the maximum of $w(e)$ over all $e \in E(G)$. Then ϕ' is an $(L, C + \lceil (W-1)/2 \rceil)$-quasi-isometry from G to H', where W is the weight of w (we omit the proof, which is clear).

In the reverse direction, suppose that G is a graph, (H, w) is a weighted graph, and ϕ is an (L, C)-quasi-isometry from G to H. If w has weight W, one might expect that ϕ is a (WL, WC)-quasi-isometry from G to (H, w). Again this is wrong, but now it is the *second* condition in the definition that breaks, because there might be far-apart vertices in G that are joined by a path in which all edges e satisfy $w(e) = 0$. Let us say that (H, w) has *depth* D if D is the maximum of $\operatorname{dist}_H(u, v)$ over all $u, v \in V(H)$ such that $\operatorname{dist}_{(H,w)}(u, v) = 0$. It is easy to check (again, we omit the proof) that ϕ is an $(L \max(W, D), C \max(W, D))$-quasi-isometry from G to (H, w).

In order to prove 1.7, we start with an (L, C)-quasi-isometry from G to a graph H with path-width at most k, and we will find an appropriate w such that ϕ becomes a $(1, C')$-quasi-isometry from G to (H, w). But we really want that the w-rescaling of ϕ is a $(1, C'')$-quasi-isometry from G to the w-rescaling of H, so that we can deduce 1.6, and so we have to keep the weight of w under control.

If $s, t \in V(H)$, an (s, t)-*geodesic in* H means a path between s, t of minimum length. If (H, w) is a weighted graph, an (s, t)-*geodesic in* (H, w) means a path between s, t with $w(P)$ minimum. A *geodesic* (in H or (H, w)) means an (s, t)-geodesic for some s, t.

3.1 *Let $C \geq 1000$, and let ϕ be a (C, C)-quasi-isometry from a graph G to a graph H. Let P be a geodesic in G. Let the vertices of P be $p_1, \dots, p_m$ in order. Then there is a function $w : E(H) \to \mathbb{N}$, with weight at most $25C^7 + 1$ and depth at most $12C^6$, such that*

$$|\operatorname{dist}_{(H,w)}(\phi(p_i), \phi(p_j)) - (j - i)| \leq 12C^6 + 1$$

for $1 \leq i < j \leq m$.

Proof. We may assume that $\phi(p_1) \neq \phi(p_m)$, since otherwise the result is clear. For $1 \leq i \leq m-1$, there is a path T_i of H between $\phi(p_i), \phi(p_{i+1})$ of length at most $2C$.

(1) *There is an induced path Q of H between $\phi(p_1), \phi(p_m)$, such that each vertex of Q belongs to one of the paths T_i $(1 \leq i \leq m-1)$; and so for each $q \in V(Q)$, there exists $i \in \{1, \dots, m\}$ such that $\operatorname{dist}_H(q, \phi(p_i)) \leq C$. Moreover, for all $u, v \in V(Q)$, the subpath of Q between u, v has length at most $2C^2 \operatorname{dist}_H(u, v) + 9C^3$.*

Choose an increasing sequence $i_1 < i_2 < \cdots < i_k$ with k minimal such that $\phi(p_1) \in V(T_{i_1})$, and $\phi(p_m) \in V(T_{i_k})$, and $V(T_{i_j}) \cap V(T_{i_{j+1}}) \neq \emptyset$ for $1 \leq j < k$. Thus, consecutive terms in the sequence $T_{i_1}, \dots, T_{i_k}$ share a vertex, and nonconsecutive terms are disjoint. It follows that there is a path Q' from $\phi(p_1)$ to $\phi(p_m)$ formed by concatenating subpaths of $T_{i_1}, \dots, T_{i_k}$ in order. If $u, v \in V(Q')$, let $u \in V(T_{i_a})$ and $v \in V(T_{i_b})$ say, with $i_a \leq i_b$; then the subpath of Q' between u, v contains only edges from $T_{i_a}, T_{i_{a+1}}, \dots, T_{i_b}$, and so has length at most the sum of the lengths of these paths, and so at most $2C(b + 1 - a)$. But

$$b - a = \operatorname{dist}_G(p_a, p_b) \leq C \operatorname{dist}_H(\phi(p_a), \phi(p_b)) + C,$$

and $\operatorname{dist}_H(\phi(p_a), \phi(p_b)) \leq \operatorname{dist}_H(u, v) + 4C$, and so the subpath of Q' between u, v has length at most

$$2C(b + 1 - a) \leq 2C + 2C^2(\operatorname{dist}_H(u, v) + 4C + 1) \leq 2C^2 \operatorname{dist}_H(u, v) + 9C^3.$$

Now Q' might not be induced, but there is an induced path Q between $\phi(p_1), \phi(p_m)$ using only vertices of Q', and keeping them in the same order, and so Q satisfies (1).

Let the vertices of Q in order be $\phi(p_1) = q_1\text{-}\cdots\text{-}q_n = \phi(p_m)$.

(2) *For $1 \le i \le m$ there exists $g(i) \in \{1, \dots, n\}$ such that $\mathrm{dist}_H(\phi(p_i), q_{g(i)}) \le C^3 + C^2 + 2C$. Moreover, $g(1) = 1$ and $g(m) = n$.*

For $1 \le j \le n$, choose $f(j) \in \{1, \dots, m\}$ such that $\mathrm{dist}_H(q_j, \phi(p_{f(j)})) \le C$, taking $f(1) = 1$ and $f(n) = m$. Now let $1 \le i \le m$. Taking $g(1) = 1$ and $g(m) = n$ satisfies the claim if $i \in \{1, m\}$, so we assume that $2 \le i \le m - 1$. Choose $j \in \{1, \dots, n\}$ maximal such that $f(j) \le i$. Since $i < m$ and $f(j) \le i$, it follows that $j < n$, and the maximality of j implies that $f(j + 1) > i$. Now $\mathrm{dist}_H(\phi(p_{f(j)}), \phi(p_{f(j+1)})) \le 2C+1$, because $\mathrm{dist}_H(q_j, \phi(p_{f(j)})) \le C$ and $\mathrm{dist}_H(q_{j+1}, \phi(p_{f(j+1)})) \le C$ and q_j, q_{j+1} are adjacent. Since ϕ is a (C, C)-quasi-isometry and $\mathrm{dist}_H(\phi(p_{f(j)}), \phi(p_{f(j+1)})) \le 2C+1$, it follows that

$$\mathrm{dist}_G(p_{f(j)}, p_{f(j+1)}) \le C(2C + 1) + C = 2C(C + 1).$$

But $\mathrm{dist}_G(p_{f(j)}, p_{f(j+1)}) = f(j+1)-f(j)$ since P is a geodesic of G and $f(j+1) > f(j)$. Consequently, since $f(j) \le i \le f(j + 1)$, one of $i - f(j), f(j + 1) - i$ is at most $C(C + 1)$. Choose $k \in \{j, j + 1\}$ with $\mathrm{dist}_G(p_i, p_{f(k)}) \le C(C + 1)$. Since ϕ is a (C, C)-quasi-isometry, it follows that

$$\mathrm{dist}_H(\phi(p_i), \phi(p_{f(k)})) \le C^2(C + 1) + C.$$

Since $\mathrm{dist}_H(\phi(p_{f(k)}), q_k) \le C$, it follows that $\mathrm{dist}_H(\phi(p_i), q_k) \le C^3 + C^2 + 2C$. Choose $g(i) = k$; then the claim is true. This proves (2).

(3) *Let $1 \le i_1 \le i_2 \le m$. If $g(i_2) = g(i_1)$, then $i_2 - i_1 \le 3C^4$. If $g(i_2) < g(i_1)$, then $i_2 - i_1 \le 6C^6$. Consequently, if $g(i_2) \le g(i_1)$ then $\mathrm{dist}_H(\phi(p_{i_2}), q_{g(i_1)}) \le 7C^7$, and $\mathrm{dist}_H(q_{g(i_1)}, q_{g(i_2)}) \le 7C^7$.*

If $g(i_2) = g(i_1)$, then $\mathrm{dist}_H(\phi(p_{i_1}), \phi(p_{i_2})) \le 2(C^3 + C^2 + 2C)$ by (2), and so

$$i_2 - i_1 = \mathrm{dist}_G(p_{i_1}, p_{i_2}) \le 2C(C^3 + C^2 + 2C) + C \le 3C^4.$$

Now suppose that $g(i_2) < g(i_1)$. Choose $i_3 \in \{i_2, \dots, m\}$ maximal such that $g(i_3) < g(i_1)$ (and thus $i_3 \ne m$). From the maximality of i_3, $g(i_3 + 1) \ge g(i_1)$. But $\mathrm{dist}_H(\phi(p_{i_3}), \phi(p_{i_3+1})) \le 2C$, since ϕ is a (C, C)-quasi-isometry; and so

$$\mathrm{dist}_H(q_{g(i_3)}, q_{g(i_3+1)}) \le 2C + 2(C^3 + C^2 + 2C) = 2C^3 + 2C^2 + 6C.$$

By (1), the subpath of Q between $q_{g(i_3)}, q_{g(i_3+1)}$ has length at most

$$2C^2(2C^3 + 2C^2 + 6C) + 9C^3 \le 5C^5.$$

This subpath contains $q_{g(i_1)}$, and so $\mathrm{dist}_H(q_{g(i_1)}, q_{g(i_3)}) \le 5C^5$. Consequently,

$$\mathrm{dist}_H(\phi(p_{i_1}), \phi(p_{i_3})) \le 5C^5 + 2(C^3 + C^2 + 2C);$$

and so

$$\text{dist}_G(p_{i_1}, p_{i_3}) \le C(5C^5 + 2(C^3 + C^2 + 2C)) + C \le 6C^6.$$

Since P is a geodesic of G, it follows that $i_3 - i_1 \le 6C^6$, and therefore $i_2 - i_1 \le 6C^6$. This also holds if $g(i_2) = g(i_1)$, and so in either case, $\text{dist}_H(\phi(p_{i_1}), \phi(p_{i_2})) \le 6C^7 + C$; and since

$$\text{dist}_H(\phi(p_1), q_{g(i_1)}) \le C^3 + C^2 + 2C,$$

it follows that

$$\text{dist}_H(\phi(p_{i_2}), q_{g(i_1)}) \le 6C^7 + C + C^3 + C^2 + 2C \le 7C^7,$$

and similarly,

$$\text{dist}_H(q_{g(i_1)}, q_{g(i_2)}) \le 6C^7 + C + 2(C^3 + C^2 + 2C) \le 7C^7.$$

This proves (3).

For $1 \le i \le m$, define r_i to be q_j, where $j = \max(g(h) : 1 \le h \le i)$. We see that $r_1 = q_1$, and $r_m = q_n$. From (3), $\text{dist}_H(q_{g(i)}, r_i) \le 7C^7$ (because $r_i = q_{g(h)}$ for some $h \le i$ with $g(h) \ge g(i)$). For $1 \le i \le m$, let R_i be the subpath of Q between q_1, r_i. Thus R_i is a subpath of R_j for all i, j with $i < j$ (although possibly $r_i = r_j$ and hence $R_i = R_j$).

Let $R = \{r_i : 1 \le i \le m\}$. Let I' be the set of all $i \in \{1, \dots, m\}$ such that $q_{g(i)} = r_i$, and choose $I \subseteq I'$ maximal such that the vertices r_i ($i \in I$) are all different, with $1, m \in I$. Define $K = 25C^7 + 1$. Choose a function $w : E(H) \to \mathbb{N}$ such that

- $w(R_i) = i - 1$ for each $i \in I$; and
- for each $q \in V(Q)$, if $q \notin \{r_i : i \in I\}$ then $w(e) > 0$ for at most one edge e of Q incident with q.
- $w(e) = K$ for every edge e of H not in $E(Q)$.

Thus, (H, w) is a weighted graph, and we will show it satisfies the theorem. We see that for $1 \le j \le n$, there exists $i \in I$ such that $\text{dist}_{(H,w)}(q_j, r_i) = 0$, from the second condition.

(4) *For $1 \le i \le j \le m$, if $r_i = r_j$ then $j - i \le 6C^6$. If $r_j > r_i$ and no vertex strictly between r_i, r_j in Q belongs to R, then $j - i \le 12C^6 + 1$.*

For the first claim, let $h \in \{1, \dots, m\}$ be minimum with $r_h = r_j$; then $h \le i \le j$, and $q_{g(h)} = r_h = r_j$. Since $g(h) \ge g(j)$, (3) implies that $j - h \le 6C^6$, and hence $j - i \le 6C^6$. This proves the first claim. For the second, choose $h \in \{1, \dots, m\}$ maximal such that $r_h = r_i$. Thus $i \le h < j$. Since $r_h = r_i$, and $r_{h+1} = r_j$, the first claim implies that $h - i \le 6C^6$, and $j - (h + 1) \le 6C^6$; and so this proves (4).

From (4), it follows that $w(e) \le 12C^6 + 1$ for each edge $e \in E(Q)$, and so w has weight K.

(5) $\text{dist}_{(H,w)}(\phi(p_i), r_i) \le 7C^7K$ *for $1 \le i \le m$.*

There exists $h \le i$ such that $q_{g(h)} = r_i$ and so $g(h) \ge g(i)$. By (3), $i - h \le 6C^6$, and so

$\operatorname{dist}_H(\phi(p_h), \phi(p_i)) \le 6C^7+C$. Since $\operatorname{dist}_H(\phi(p_h), q_{g(h)}) \le C^3+C^2+2C$, it follows that $\operatorname{dist}_H(\phi(p_i), r_i) \le 6C^7 + C + C^3 + C^2 + 2C \le 7C^7$. Since w has weight K, this proves (5).

(6) *For* $1 \le i < j \le m$, $|\operatorname{dist}_{(H,w)}(r_i, r_j) - (j - i)| \le 12C^6$.

Choose $i_1 \in I$ with $r_{i_1} = r_i$, and $i_2 \in I$ with $r_{i_2} = r_j$. Thus

$$\operatorname{dist}_{(H,w)}(r_i, r_j) = \operatorname{dist}_{(H,w)}(r_{i_1}, r_{i_2}) = |i_2 - i_1|.$$

But by (4), $|i_1 - i| \le 6C^6$ and $|i_2 - j| \le 6C^6$; and so $|(|i_2 - i_1|) - (j - i)| \le 12C^6$. This proves (6).

(7) *Q is a $(\phi(p_1), \phi(p_m))$-geodesic in (H, w), and $w(Q) = m - 1$.*

Since $1, m \in I$, it follows that $w(Q) = m - 1$. Suppose that Q is not a $(\phi(p_1), \phi(p_m))$-geodesic in (H, w); then there is a path R of H with distinct ends both in $V(Q)$ and with no internal vertices in $V(Q)$, such that $w(R) < w(S)$, where S is the subpath of Q joining the ends of R. Let R have ends $q_j, q_{j'}$ say, and choose $i, i' \in I$ such that $\operatorname{dist}_{(H,w)}(q_j, r_i) = 0$, and $\operatorname{dist}_{(H,w)}(q_{j'}, r_{i'}) = 0$. Thus $w(S) = i' - i$, and since $w(e) = K$ for each $e \in R$, and $|E(R)| \ge \max(1, \operatorname{dist}_H(q_j, q_{j'}))$, we deduce that

$$i' - i > w(R) = K|E(R)| \ge \max\left(K \operatorname{dist}_H(q_j, q_{j'}), K\right).$$

By (4), $\operatorname{dist}_H(q_j, r_i) \le 12C^6 + 1$, and $\operatorname{dist}_H(q_{j'}, r_{i'}) \le 12C^6 + 1$, and consequently $i' - i > K$ and

$$i' - i > K(-2(12C^6 + 1) + \operatorname{dist}_H(r_i, r_{i'})).$$

But $\operatorname{dist}_H(r_i, r_{i'}) = \operatorname{dist}_H(q_{g(i)}, q_{g(i')})$; and the latter is at least

$$-\operatorname{dist}_H(q_{g(i)}, \phi(p_i)) + \operatorname{dist}_H(\phi(p_i), \phi(p_{i'})) - \operatorname{dist}_H(\phi(p_{i'}), q_{g(i')}).$$

The first and third terms here are each at most $C^3 + C^2 + 2C$, in absolute value, by (2); and the second is at least $(i' - i - C)/C$, since ϕ is a (C, C)-quasi-isometry. Combining these facts, we deduce that:

$$i' - i > K(-2(12C^6 + 1) + (i' - i - C)/C - 2(C^3 + C^2 + 2C)) > K((i' - i)/C - 25C^6).$$

Consequently $(K/C - 1)(i' - i) < 25KC^6$ and $i' - i > K$, and so $(K/C - 1)K < 25KC^6$, a contradiction, since $K > 25C^7$. This proves (7).

Since $w(e) = 0$ only for some edges e of Q, (4) implies that (H, w) has depth at most $12C^6$. Since w has weight K, this completes the proof of 3.1. ∎

4 The proof of 1.7, part 2

Now we turn to the second part of the proof of 1.7. Let us say a function $\kappa : \mathbb{N} \to \mathbb{N}$ is an *additive bounder* for a class $\mathcal{C}$ of graphs if or all $C \ge 1$, and every (C, C)-quasi-isometry ϕ from a graph G to a graph $H \in \mathcal{C}$, there is a function $w : E(H) \to \mathbb{N}$ with weight at most $\kappa(C)$ such that ϕ is a $(1, \kappa(C))$-quasi-isometry from G to (H, w).

A class $\mathcal{C}$ of graphs is *hereditary* if for every $H \in \mathcal{C}$, all induced subgraphs of H also belong to $\mathcal{C}$.

4.1 *Let $\mathcal{C}$ be a hereditary class of graphs, with an additive bounder κ. For all $c \ge 1000$ there exists c_0 with the following property. Suppose that:*

- *ϕ is a (c, c)-quasi-isometry from a graph G to a graph H;*
- *P is a geodesic in G, with vertices $p_1, \dots, p_m$ in order;*
- *$|\operatorname{dist}_H(\phi(p_i), \phi(p_j)) - (j - i)| \le c$ for $1 \le i < j \le m$; and*
- *the subgraph of H induced on the set of all $v \in V(H)$ with $\operatorname{dist}_H(v, \phi(P)) > c$ belongs to $\mathcal{C}$, where $\phi(P) = \{\phi(p_1), \dots, \phi(p_m)\}$.*

Then there is a function $w : E(H) \to \mathbb{N}$ with weight at most c_0, such that ϕ is a $(1, c_0)$-quasi-isometry from G to (H, w).

Proof. Let $r = 2c(c+1)$, and $c' = \max(\kappa(c), 1)$. Let $c_2 = \max(2c + c', (2r + 7)c + 2(r + 2)c^2)$. Define

$$c_3 = c_2 + c(2(r + 2)c + 2) + (r + 2)cc' + (r + 2)c,$$

and

$$c_0 = \max(2(c' + 1 + 2cc') + 2r + c + 2(cr + c)c_3, 2c + c', (2r + 7)c + 2(r + 2)c^2).$$

We will show that c_0 satisfies the theorem.

Let G, H, ϕ, P and so on be as in the hypothesis of the theorem. Let A be the set of all $v \in V(G)$ such that $\operatorname{dist}_G(v, P) \le r$. Let $B = V(G) \setminus A$. Let $X = \{\phi(v) : v \in B\}$.

(1) $\operatorname{dist}_H(X, \phi(P)) \ge r/c - 1$.

Let $b \in B$ and $i \in \{1, \dots, m\}$. Then

$$\operatorname{dist}_H(\phi(b), \phi(p_i)) \ge (\operatorname{dist}_G(b, p_i) - c)/c \ge r/c - 1.$$

This proves (1).

(2) *There is a partition (Y, Z) of $V(H) \setminus X$, such that*

- *for every $y \in Y$ there is a path of $H[X \cup Y]$ from y to X, of length at most $(r + 2)c$, and $\operatorname{dist}_H(y, \phi(P)) \ge (r/c - 1)/2 > c$;*
- *for every $z \in Z$, there is a path of $G[Z]$ from z to $\phi(P)$, of length at most $(r + 2)c$, and $\operatorname{dist}_H(z, X) > (r/c - 1)/2$.*

Let Y be the set of all $h \in V(H) \setminus X$ such that $\operatorname{dist}_H(h, X) \le \operatorname{dist}_H(h, \phi(P))$, and let $Z = V(H) \setminus (X \cup Y)$. We claim that (2) is satisfied. Let $h \in V(H) \setminus X$. We claim first that either $\operatorname{dist}_H(h, X) \le c$, or $\operatorname{dist}_H(h, \phi(P)) \le cr + 2c$. To see this, choose $v \in V(G)$ with $\operatorname{dist}_H(\phi(v), h) \le c$. If $v \in B$ then $\phi(v) \in X$ and the claim holds, so we assume that $v \in A$. Hence $\operatorname{dist}_G(v, P) \le r$, and so $\operatorname{dist}_H(\phi(v), \phi(P)) \le cr + c$. Consequently $\operatorname{dist}_H(h, \phi(P)) \le cr + 2c$, and again the claim holds. Hence

$$\min(\operatorname{dist}_H(h, X), \operatorname{dist}_H(h, \phi(P))) \le (r + 2)c,$$

and so the first assertion of each bullet of (2) holds. For the second assertion, from (1), if $\mathrm{dist}_H(h, X) \le (r/c - 1)/2$ then $\mathrm{dist}_H(h, X) \le \mathrm{dist}(h, \phi(P))$ and therefore $h \in Y$; and similarly if $\mathrm{dist}_H(h, \phi(P)) < (r/c - 1)/2$ then $h \in Z$. This proves (2).

Let $H' = H[X \cup Y]$. From (1) and (2), $\mathrm{dist}_H(y, \phi(P)) > c$ for each $y \in X \cup Y$. Since the subgraph of H induced on the set of all $v \in V(H)$ with $\mathrm{dist}_H(v, \phi(P))) > c$ belongs to $\mathcal{C}$, by hypothesis, and $\mathcal{C}$ is hereditary, it follows that $H' \in \mathcal{C}$. For each pair $b, b' \in B$, if $\mathrm{dist}_{H'}(\phi(b), \phi(b')) \le 2(r+2)c + 1$, let $F_{b,b'} = F_{b',b}$ be a path between b, b' of length $\mathrm{dist}_G(b, b')$, where all its internal vertices are new vertices. Let F be the union of $G[B]$ and all the paths $F_{b,b'}$. Define $\psi : V(F) \to V(H)$ as follows. For each $v \in B$, $\psi(v) = \phi(v)$. For all $b, b' \in B$ and every internal vertex v of $F_{b,b'}$, let $\psi(v)$ be one of b, b', chosen arbitrarily.

(3) *If* $u, v \in V(F)$*, then* $\mathrm{dist}_{H'}(\psi(u), \psi(v)) \le (2(r+2)c + 1)\,\mathrm{dist}_F(u, v)$.

It suffices to show that $\mathrm{dist}_{H'}(\psi(u), \psi(v)) \le 2(r+2)c+1$ for every edge uv of F (and then sum over all edges of a (u, v)-geodesic in F). Thus, let $uv \in E(F)$. If uv is an edge of one of the paths $F_{b,b'}$, then

$$\mathrm{dist}_{H'}(\psi(u), \psi(v)) \le \mathrm{dist}_{H'}(\phi(b), \phi(b')) \le 2(r+2)c + 1,$$

as required. If $uv \in E(G[B])$, then $\mathrm{dist}_H(\phi(u), \phi(v)) \le 2c$ since ϕ is a (c, c)-quasi-isometry from G to H. Let S be a path of H between $\phi(u), \phi(v)$ of length at most $2c$; so each of its vertices has distance at most c from one of $\phi(u), \phi(v) \in X$, and so $V(S) \subseteq X \cup Y$, since $c \le (r/c - 1)/2$. Consequently,

$$\mathrm{dist}_{H'}(\psi(u), \psi(v)) \le 2c \le 2(r+2)c + 1.$$

This proves (3).

(4) *If* $u, v \in V(F)$*, then* $\mathrm{dist}_F(u, v) \le 2c(2(r+2)c+1)\,\mathrm{dist}_{H'}(\psi(u), \psi(v)) + 4c(2(r+2)c+1)$.

Choose $u' \in B$ with $\psi(u) = \phi(u')$, and choose v' similarly for v. Let T be the H'-geodesic between $\phi(u'), \phi(v')$, and let its vertices be $t_0, \dots, t_n$ in order, where $t_0 = \phi(u')$ and $t_n = \phi(v')$. For $0 \le i \le n$, since $t_i \in X \cup Y$, there is a path T_i of H' from t_i to X with length at most $(r+2)c$; let its end in X be x_i, and choose $b_i \in B$ with $\phi(b_i) = x_i$. For $1 \le i \le n$, there is a path from x_{i-1} to x_i with vertex set a subset of $V(T_{i-1}) \cup V(T_i)$, and its length is at most $2(r+2)c+1$; and consequently F_{b_{i-1},b_i} exists, and so

$$\mathrm{dist}_F(b_{i-1}, b_i) = \mathrm{dist}_G(b_{i-1}, b_i) \le 2c\,\mathrm{dist}_H(x_{i-1}, x_i) \le 2c(2(r+2)c + 1);$$

so $\mathrm{dist}_F(b_{i-1}, b_i) \le 2c(2(r+2)c+1)$. But $\mathrm{dist}_F(b_0, b_n)$ is at most $\sum_{1 \le i \le n} \mathrm{dist}_F(b_{i-1}, b_i)$ and consequently

$$\mathrm{dist}_F(u', v') \le 2c(2(r+2)c+1)n = 2c(2(r+2)c+1)\,\mathrm{dist}_{H'}(\psi(u), \psi(v)).$$

But $\mathrm{dist}_F(u, u') \le 2c(2(r+2)c+1)$, and the same for $\mathrm{dist}_F(u, u')$; so

$$\mathrm{dist}_F(u, v) \le 2c(2(r+2)c+1)\,\mathrm{dist}_{H'}(\psi(u), \psi(v)) + 4c(2(r+2)c+1).$$

This proves (4).

From the definition of Y, for each $y \in X \cup Y$ there exists $v \in V(F)$ such that $\text{dist}_{H'}(\psi(v), y) \leq (r+3)c$; and so ψ is a $(2c(2(r+2)c+1), 4c(2(r+2)c+1))$-quasi-isometry from F to H'. Since κ is an additive bounder for $\mathcal{C}$, and $H' \in \mathcal{C}$, there is a function $w' : E(H') \to \mathbb{N}$ with weight at most c', such that ψ is a $(1, c')$-quasi-isometry from G to (H', w'), where $c' = \max(\kappa(c), 1)$. Let Δ be the set of edges of H between $X \cup Y$ and Z. Define $w : E(H) \to \mathbb{N}$ by:

- If $e \in E(H')$ then $w(e) = w'(e)$;
- If $e \in E(G[Z])$ then $w(e) = 1$;
- If $e \in \Delta$ then $w(e) = c_3$.

Thus w has weight at most c_3, and we will show that ϕ is a $(1, c_0)$-quasi-isometry from G to (H, w).

(5) *Let $u, v \in V(G)$. Then*

$$\text{dist}_{(H,w)}(\phi(u), \phi(v)) \leq \text{dist}_G(u, v) + 2(c' + 1 + 2cc') + 2r + c + 2(cr + c)c_3.$$

Observe first that if T is a geodesic of G, with $V(T) \subseteq B$ and with ends b_1, b_2 say, then

$$\text{dist}_{(H,w)}(\phi(b_1), \phi(b_2)) \leq \text{dist}_{(H',w')}(\psi(b_1), \psi(b_2)) \leq \text{dist}_F(b_1, b_2) + c' = \text{dist}_G(b_1, b_2) + c',$$

from the choice of w'. Now let T be a (u, v)-geodesic T in G; and we may therefore assume that $V(T) \not\subseteq B$. Let a_1, a_2 be the first and last vertices of T that belong to A. If $a_1 \neq u$, let $b_1 \in V(T)$ be adjacent in T to a_1, and not between a_1, a_2; thus $b_1 \in B$ from the definition of a_1. If $a_1 = u$ then b_1, T_1 are undefined. Define b_2, T_2 similarly if $a_2 \neq v$.

If b_1, T_1 exist, then T_1 is a geodesic of G with vertex set in B, and so

$$\text{dist}_{(H,w)}(\phi(u), \phi(b_1)) \leq \text{dist}_G(u, b_1) + c',$$

as above. Since $a_1 b_1 \in E(G)$ and ϕ is a (c, c)-quasi-isometry from G to H, it follows that $\text{dist}_H(\phi(a_1), \phi(b_1)) \leq 2c$. Consequently $\text{dist}_{H'}(\phi(a_1), \phi(b_1)) \leq 2c$, as the corresponding path in H is contained in H'; and since w' has weight at most c', it follows that $\text{dist}_{(H,w)}(\phi(a_1), \phi(b_1)) \leq 2cc'$. Thus, if b_1, T_1 exist, then

$$\text{dist}_{(H,w)}(\phi(u), \phi(a_1)) \leq \text{dist}_G(u, b_1) + c' + 2cc' \leq \text{dist}_G(u, a_1) + c' + 1 + 2cc'.$$

This last is also trivially true if b_1, T_1 do not exist, since then $u = a_1$. A similar inequality holds for v, a_2.

Let T_0 be the subpath of T between a_1, a_2. Since $a_1 \in A$, there exists $i_1 \in \{1, \ldots, m\}$ such that $\text{dist}_G(a_1, p_{i_1}) \leq r$. Choose i_2 similarly for a_2. Thus $\text{dist}_G(p_{i_1}, p_{i_2}) \leq \text{dist}_G(a_1, a_2) + 2r$, and so $\text{dist}_G(a_1, a_2) \geq |i_2 - i_1| - 2r$. Now since $\text{dist}_G(a_1, p_{i_1}) \leq r$, and ϕ is a (c, c)-quasi-isometry from G to H, it follows that $\text{dist}_H(\phi(a_1), \phi(p_{i_1})) \leq cr + c$, and so $\text{dist}_{(H,w)}(\phi(a_1), \phi(p_{i_1})) \leq (cr + c)c_3$. The same holds for a_2, p_{i_2}; and so

$$\text{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) \leq \text{dist}_{(H,w)}(\phi(p_{i_1}), \phi(p_{i_2})) + 2(cr + c)c_3.$$

Since $\text{dist}_{(H,w)}(\phi(p_{i_1}), \phi(p_{i_2})) \leq |i_2 - i_1| + c$, we deduce that

$$\text{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) \leq |i_2 - i_1| + c + 2(cr + c)c_3.$$

But $\operatorname{dist}_G(a_1, a_2) \geq |i_2 - i_1| - 2r$, and so

$$\operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) \leq \operatorname{dist}_G(a_1, a_2) + 2r + c + 2(cr + c)c_3.$$

We deduce that

$$\begin{aligned}\operatorname{dist}_{(H,w)}(\phi(u), \phi(v)) &\leq \operatorname{dist}_{(H,w)}(\phi(u), \phi(a_1)) + \operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) + \operatorname{dist}_{(H,w)}(\phi(v), \phi(a_2)) \\ &\leq \operatorname{dist}_G(u, a_1) + c' + 1 + 2cc' + \operatorname{dist}_G(a_1, a_2) + 2r \\ &\quad + c + 2(cr + c)c_3 + \operatorname{dist}_G(v, a_2) + c' + 1 + 2cc' \\ &= \operatorname{dist}_G(u, v) + 2(c' + 1 + 2cc') + 2r + c + 2(cr + c)c_3.\end{aligned}$$

This proves (5).

(6) *Let $a_1, a_2 \in V(G)$, with $\phi(a_1), \phi(a_2) \in Z$. Then*

$$|\operatorname{dist}_G(a_1, a_2) - \operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2))| \leq (2r + 7)c + 2(r + 2)c^2.$$

For $j = 1, 2$, since $\phi(a_j) \in Z$, there exists $i_j \in \{1, \ldots, m\}$ such that there is a path of $H[Z]$ between $\phi(a_j), \phi(p_{i_j})$ of length at most $(r + 2)c$. We may assume that $i_1 \leq i_2$ with loss of generality. Since $|\operatorname{dist}_{(H,w)}(\phi(p_{i_1}), \phi(p_{i_2})) - (i_2 - i_1)| \leq c$, it follows that

$$|\operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) - (i_2 - i_1)| \leq (2r + 5)c,$$

and so

$$|\operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2)) - \operatorname{dist}_G(p_{i_1}, p_{i_2})| \leq (2r + 5)c.$$

But $\operatorname{dist}_H(\phi(a_j), \phi(p_{i_j})) \leq (r+2)c$, and so $\operatorname{dist}_G(a_j, p_{i_j}) \leq (r+2)c^2+c$. Consequently $|\operatorname{dist}_G(a_1, a_2) - \operatorname{dist}_G(p_{i_1}, p_{i_2})| \leq 2((r + 2)c^2 + c)$, and therefore

$$|\operatorname{dist}_G(a_1, a_2) - \operatorname{dist}_{(H,w)}(\phi(a_1), \phi(a_2))| \leq (2r + 5)c + 2((r + 2)c^2 + c) = (2r + 7)c + 2(r + 2)c^2.$$

This proves (6).

(7) *Let $u, v \in V(G)$, and let T be a path of H between $\phi(u), \phi(v)$. Then $\operatorname{dist}_G(u, v) \leq w(T) + c_2$.*

We proceed by induction on $|\Delta \cap E(T)|$. Suppose first that $\Delta \cap E(T) = \emptyset$, and so T is a path of one of H', $H[Z]$. We assume first that T is a path of H'. Thus there exist $b_1, b_2 \in B$ with $\phi(b_1) = \phi(u)$ and $\phi(b_2) = \phi(v)$. Since ϕ is a (c, c)-quasi-isometry from G to H, it follows that $\operatorname{dist}_G(u, b_1), \operatorname{dist}_G(v, b_2) \leq c$. Moreover, $\operatorname{dist}_{H'}(\phi(u), \phi(v)) \leq w(T)$, and so $\operatorname{dist}_G(b_1, b_2) \leq \operatorname{dist}_F(b_1, b_2) \leq w(T) + c'$, since ϕ is a $(1, c')$-quasi-isometry from H' to $H[X \cup Y]$. It follows that in this case, $\operatorname{dist}_G(u, v) \leq w(T) + 2c + c'$, and so the result holds.

Now we assume that T is a path of $H[Z]$. Then by (6),

$$\operatorname{dist}_G(u, v) \leq \operatorname{dist}_{(H,w)}(\phi(u), \phi(v)) + (2r + 7)c + 2(r + 2)c^2 \leq w(T) + (2r + 7)c + 2(r + 2)c^2,$$

and again the result holds.

Thus we may assume that there exists $yz \in \Delta \cap E(T)$, where $y \in X \cup Y$ and $z \in Z$. By exchanging u, v if necessary we may assume that $\phi(u), y, z, \phi(v)$ are in order in T. Since $y \in X \cup Y$, there exists

$b \in B$ such that $\text{dist}_{H'}(\phi(b), y) \leq (r+2)c$; and since $z \in Z$, there exists $i \in \{1, \ldots, m\}$ such that $\text{dist}_{H[Z]}(z, \phi(p_i)) \leq (r+2)c$, as before. Hence there are paths R_1, R_2 of H, where R_1 is between $\phi(u), \phi(b)$, and R_2 is between $\phi(p_i), \phi(v)$, such that

$$w(R_1) + w(R_2) \leq w(T) + 2(r+2)c \leq w(T) + (r+2)cc' + (r+2)c - c_3,$$

and R_1, R_2 both have fewer than $|\Delta \cap E(T)|$ edges in Δ. From the inductive hypothesis, $\text{dist}_G(u, b) \leq w(R_1) + c_2$, and $\text{dist}_G(p_i, v) \leq w(R_2) + c_2$. But

$$\text{dist}_G(u, v) \leq \text{dist}_G(u, b) + \text{dist}_G(b, p_i) + \text{dist}_G(p_i, v),$$

and

$$\text{dist}_G(b, p_i) \leq c\,\text{dist}_H(\phi(b), \phi(p_i)) + c \leq c(2(r+2)c+1) + c;$$

so

$$\begin{aligned}\text{dist}_G(u, v) &\leq \text{dist}_G(u, b) + \text{dist}_G(p_i, v) + c(2(r+2)c+2) \\ &\leq w(R_1) + c_2 + w(R_2) + c_2 + c(2(r+2)c+2) \\ &\leq w(T) + 2c_2 + c(2(r+2)c+2) + (r+2)cc' + (r+2)c - c_3 \\ &\leq w(T) + c_2.\end{aligned}$$

This proves (7).

(8) *For each $v \in V(H)$, there exists $u \in V(G)$ such that* $\text{dist}_{(H,w)}(\phi(u), v) \leq (r+2)cc'$.

If $v \in Z$, then by (2), there is a path of $H[Z]$ from v to $\phi(P)$, of length at most $(r+2)c$, and hence $\text{dist}_{(H,w)}(\phi(u), v) \leq (r+2)c$. If $v \in X \cup Y$, by (2) there is a path of H' from v to X, of length at most $(r+2)c$, and hence $\text{dist}_{(H,w)}(v, X) \leq (r+2)cc'$. This proves (8).

By (5), (7) and (8), ϕ is a $(1, c_0)$-quasi-isometry from G to (H, w), and its weight is c_3. This proves 4.1. ∎

5 The proof of 1.7, part 3

There is an annoyance here. To prove 1.7, we work by induction on the path-width of H. We find a subgraph H' that has smaller path-width than H, but we want to apply the inductive hypothesis to a *subdivision* of H', and subdividing edges can increase path-width (for instance, the path-width of the complete bipartite graph $K_{2,3}$ is two, but if we subdivide its edges its path-width becomes three). The easiest fix seems to be a slight modification of the definition of path-width.

Let G be a graph. A *multisubset* of $V(G)$ is a map $\alpha : V(G) \to \mathbb{N}$; its *support* is the set of $v \in V(G)$ with $\alpha(v) > 0$, and its *size* is $\sum_{v \in V(G)} \alpha(v)$. Two multisubsets α, β are *1-close* if there exists $u \in V(G)$ such that $\alpha(v) = \beta(v)$ for all $v \in V(G) \setminus \{u\}$, and $|\alpha(u) - \beta(u)| \leq 1$. They are *2-close* if there exist $u, u' \in V(G)$ such that $\alpha(v) = \beta(v)$ for all $v \in V(G) \setminus \{u, u'\}$, and $\alpha(u) = \beta(u) + 1$ and $\alpha(u') = \beta(u') - 1$. When α, β are 2-close we say that $\{u, u'\}$ is their *difference*.

Let us say an *edge-search* of a graph G is a sequence $(\alpha_1, \ldots, \alpha_n)$ of multisubsets of $V(G)$, such that:

- for $1 \leq i < n$, α_i, α_{i+1} are 1-close or 2-close;
- $V(G)$ is the union of the supports of $\alpha_1, \ldots, \alpha_n$;
- for every edge uv of G, there exists $i \in \{1, \ldots, n-1\}$ such that α_i, α_{i+1} are 2-close with difference $\{u, v\}$; and
- for all i, j, k with $1 \leq i \leq j \leq k \leq n$, the support of α_j includes the intersection of the supports of α_i, α_k.

We define the *width* of the edge-search to be the maximum size of its terms, and the *edge-search-width* $\text{esw}(G)$ of G to be the minimum width of all edge-searches of G. Let us write $\text{pw}(G)$ for the path-width of G.

This is motivated by the method of graph searching studied by LaPaugh [9] and others, where the goal is to clean a contaminated graph by moving cleaners around the graph (any part of the graph that is connected to a contaminated part by a path containing no cleaners is instantly recontaminated). An edge is cleaned by moving a cleaner along it (and keeping it safe from recontamination); and they want to use as few cleaners as possible. Each multisubset in the edge-search records the position of the cleaners at a given time.

We need the following easy facts about edge-search-width, which we leave to the reader:

5.1 *For every graph G:*

- $\text{esw}(G) \in \{\text{pw}(G) + 1, \text{pw}(G) + 2\}$*;*
- *if H is a minor of G (that is, H is obtained from a subgraph of G by contracting edges) then* $\text{esw}(H) \leq \text{esw}(G)$*;*
- *if H is a subdivision of G then* $\text{esw}(H) = \text{esw}(G)$*.*

Now we prove 1.7, which we restate, with some slight changes, for convenience (we might as well assume that $L = C \geq 1000$; and the statement with edge-search-width is equivalent to the statement with path-width, because of 5.1):

5.2 *For all C, k there exists C' such that if ϕ is a (C, C)-quasi-isometry from a graph G to a graph H with edge-search-width at most k, then there is a function $w : E(H) \to \mathbb{N}$ with weight at most C', such that the same function ϕ is a $(1, C')$-quasi-isometry from G to the weighted graph (H, w).*

Proof. We proceed by induction on k. If $k = 0$ then H is null and the result is trivial, so we assume that $k \geq 1$ and the result holds for $k-1$. Thus, the class of all graphs with edge-search-width at most $k-1$ has an additive bounder κ. By increasing C we may assume that $C \geq 1000$. Let $c = 30C^8$, and let c_0 be as in 4.1. Let $C' = (25C^7 + 1)c_0$; we will show that C' satisfies the theorem.

Every vertex of H belongs to a component of H containing $\phi(v)$ for some $v \in V(G)$, from the third condition for an quasi-isometry; and for $u, v \in V(G)$, u, v belong to the same component of G if and only if $\phi(u), \phi(v)$ belong to the same component of H, by the first two conditions. Consequently we may assume that G, H are connected, without loss of generality.

Hence H admits an edge-search $(\alpha_1, \ldots, \alpha_n)$ of width at most k in which the support of each α_i is nonempty. For $1 \leq i \leq n$ let A_i be the support of α_i. So there exist $v_1, v_2 \in V(G)$ such that

$\operatorname{dist}_H(\phi(v_1), A_1), \operatorname{dist}_H(\phi(v_2), A_n) \le C$. Let P be a (v_1, v_2)-geodesic of G, and let the vertices of P be $v_1 = p_1, \dots, p_m = v_2$ in order. Let $\phi(P) = \{\phi(p_1), \dots, \phi(p_m)\}$. By 3.1, there is a function $w_1 : E(H) \to \mathbb{N}$, with weight at most $25C^7 + 1$ and depth at most $12C^6$, such that

$$|\operatorname{dist}_{(H,w_1)}(\phi(p_i), \phi(p_j)) - (j - i)| \le 12C^6 + 1$$

for $1 \le i < j \le m$. Define $D = 25C^7 + 1$.

(1) *For $1 \le j \le n$ there exists i with $1 \le i \le m$ such that $\operatorname{dist}_{(H,w_1)}(\phi(p_i), A_j) \le CD$.*

Suppose there is no such i. Choose $a_1 \in A_1$ with $\operatorname{dist}_H(\phi(v_1), a_1) \le C$, and choose $a_n \in A_n$ similarly. Since w_1 has weight at most D, it follows that $\operatorname{dist}_{(H,w_1)}(\phi(v_1), a_1) \le CD$, and $\operatorname{dist}_{(H,w_1)}(\phi(v_2), a_n) \le CD$. From the assumption, $\operatorname{dist}_{(H,w_1)}(\phi(p_1), A_j) > CD$; let X be the vertex set of the component of $H \setminus A_j$ that contains $\phi(p_1)$. Since $\operatorname{dist}_{(H,w_1)}(\phi(v_1), a_1) \le CD$, it follows that $a_1 \in X$. We claim that $\phi(p_1), \dots, \phi(p_m) \in X$. For suppose not, and choose $h \in \{1, \dots, m\}$ minimal with $\phi(p_h) \notin X$. Thus $h \ge 2$, and every path of H between $\phi(p_{h-1}), \phi(p_h)$ contains a vertex in A_j (since $\phi(p_{h-1}) \in X$ and $\phi(p_h) \notin X$). But $\operatorname{dist}_{(H,w_1)}(\phi(p_{h-1}), \phi(p_h)) \le 12C^6 + 2$, and so $\operatorname{dist}_{(H,w_1)}(\phi(p_{h-1}), A_j) \le 12C^6 + 2$, a contradiction. In particular, $\phi(p_m) \in X$; and since $\operatorname{dist}_{(H,w_1)}(\phi(p_m), a_n) \le CD$ and $\operatorname{dist}_{(H,w_1)}(\phi(p_m), A_j) > CD$, it follows that $a_n \in X$. But then there is a path of H between a_1, a_n, with no vertex in A_j, contradicting that $(\alpha_1, \dots, \alpha_n)$ is an edge-search and $a_1 \in A_1$ and $a_n \in A_n$. This proves (1).

Let H_1 be the w_1-rescaling of H, and let ϕ_1 be the w_1-rescaling of ϕ. Define $c = 30C^8$. It follows that ϕ_1 is a $(CD, CD + D)$-quasi-isometry (and hence a (c, c)-quasi-isometry) from G to H_1. From (1) and 5.1, the set of all vertices $v \in V(H_1)$ with $\operatorname{dist}_{H_1}(v, \phi(P)) \ge c$ induces a subgraph of H_1 with edge-search-width at most $k - 1$. Let c_0 be as in 4.1. From 4.1 applied to ϕ_1, H_1, there is a function $w_2 : E(H_1) \to \mathbb{N}$ with weight at most c_0, such that ϕ_1 is a $(1, c_0)$-quasi-isometry from G to (H_1, w_2). For each edge $e \in E(H)$, define $w(e)$ to be the sum of $w_2(f)$, over all edges f of the path of H_1 made by subdividing e (if $w_1(e) = 0$, then $w(e) = 0$). Thus w has weight at most the product of the weights of w_1, w_2, and so at most $(25C^7 + 1)c_0$. It follows that ϕ is a $(1, c_0)$-quasi-isometry from G to (H, w). This proves 5.2. ∎

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