A counterexample to the coarse Menger conjecture

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November 19, 2023; revised January 12, 2024

\textsuperscript{1}Supported by AFOSR grants A9550-19-1-0187 and FA9550-22-1-0234, and by NSF grants DMS-1800053 and DMS-2154169.

\textsuperscript{2}Supported by EPSRC grant EP/X013642/1

\textsuperscript{3}Supported by AFOSR grants A9550-19-1-0187 and FA9550-22-1-0234, and by NSF grants DMS-1800053 and DMS-2154169.
Abstract

It was conjectured, independently by two sets of authors, that for all integers \( k, d \geq 1 \) there exists \( \ell > 0 \), such that if \( S, T \) are subsets of vertices of a graph \( G \), then either there are \( k \) paths between \( S, T \), pairwise at distance at least \( d \), or there is a set \( X \subseteq V(G) \) with \( |X| \leq k - 1 \) such that every path between \( S, T \) contains a vertex with distance at most \( \ell \) from some member of \( X \). The result is known for \( k \leq 2 \), but we will show that it is false for all \( k \geq 3 \), even if \( G \) is constrained to have maximum degree at most three. We also give a proof of the result when \( k = 2 \) that is simpler than the previous proofs.
1 Introduction

If $X$ is a subset of the vertex set of a graph $G$, or a subgraph of $G$, and the same for $Y$, then $d(X, Y)$ denotes the distance between $X, Y$, that is, the number of edges in the shortest path of $G$ with one end in $X$ and the other in $Y$. We are interested in when there are $k$ paths in $G$ between two given sets that are pairwise at distance at least $d$. The following conjecture was proposed by Albrechtsen, Huynh, Jacobs, Knappe, and Wollan [1], and independently by Georgakopoulos and Papasoglu [3]:

1.1 False conjecture: For all integers $k, d \geq 1$ there exists $\ell > 0$ with the following property. Let $G$ be a graph and let $S, T \subseteq V(G)$; then either

- there are $k$ paths between $S, T$, pairwise at distance at least $\ell$; or
- there is a set $X \subseteq V(G)$ with $|X| \leq k - 1$ such that every path between $S, T$ contains a vertex with distance at most $\ell$ from some member of $X$.

Both sets of authors proved the result for $k = 2$, but the $k \geq 3$ case remained open. Here we give a counterexample with $d = k = 3$, and with maximum degree at most three. We also give a simpler proof of the $k = 2$ case.

The case $k = 3$ was of special interest, because it is easy to see that if the result is true when $d = 3$ then it is true for general $k$ (apply the result when $d = 3$ to the $d$th power of $G$).

The case $d = 2$ and general $k$ is still open. Another related conjecture, still open, is 1.1 with the bound on $|X|$ relaxed:

1.2 Conjecture: For all $k, d$ there exist $\ell, n$ such that, with $G, S, T$ as before, either the first bullet of 1.1 holds, or there is a set $X \subseteq V(G)$ with $|X| \leq n$ such that every path between $S, T$ contains a vertex with distance at most $\ell$ from some member of $X$.

When $d = 2$, this last conjecture was shown to be true for graphs of bounded maximum degree by Hendrey, Norin, Steiner and Turcotte [4], and independently by Gartland, Koorhonen and Lokshtanov [2].

Agelos Georgakopoulos brought to our attention the following variant of 1.1, which also remains open:

1.3 Conjecture: For all integers $k, d \geq 1$ there exists $\ell > 0$ with the following property. Let $G$ be a graph and let $S, T \subseteq V(G)$; then either

- there are $k$ paths between $S, T$, pairwise at distance at least $\ell$; or
- there is a vertex $x$ of $G$ such that there do not exist $k - 1$ paths between $S$ and $T$, pairwise at distance at least $\ell$ and each with distance at least $\ell$ from $x$.

2 The counterexample

In this section, for each value of $\ell > 0$, we give an example of a graph $G$, and two subsets $S, T$ of $V(G)$, such that there do not exist three paths between $S, T$, pairwise at distance at least 3, and for every $X \subseteq V(G)$ with $|X| \leq 2$, there is a path $P$ between $S, T$ such that its distance from $X$ is more
than \( \ell \). It is illustrated in figure 1. The two sets \( S, T \) both have size three, and there is a vertex in \( S \cap T \).

To show that this is a counterexample, we need to check that for every \( X \subseteq V(G) \) with \( |X| \leq 2 \), there is a path \( P \) between \( S, T \) such that its distance from \( X \) is more than \( \ell \); and that there do not exist three paths between \( S, T \), pairwise at distance at least 3. The first is easy, so let us do it now.

**2.1 Let** \( \ell \geq 1 \) **be an integer, and let** \( G \) **be as in figure 1, where the binary tree has depth more than** \( 2\ell + 2 \), **and each of the dotted curves represents a path of length more than** \( 2\ell \). **Let** \( S, T \) **be as shown in the figure. If** \( X \subseteq V(G) \) **with** \( |X| \leq 2 \), **then there is a path** \( P \) **between** \( S, T \) **such that** \( d(X, P) > \ell \).**

**Proof.** Let \( r \) be the root of the binary tree (the vertex at the top of the figure). Since \( r \in S \cap T \), we may assume there exists \( x_1 \in X \) such that its distance from \( r \) is at most \( \ell \); and since all of the paths represented by the dotted curves (let us call them “dotted paths”) have distance at least \( 2\ell + 1 \) from \( r \) (because the binary tree has depth more than \( 2\ell + 2 \)), they all have distance more than \( \ell \) from \( x_1 \). Since the path \( (M \text{ say}) \) at the bottom of the figure is between \( S, T \), we may assume that there exists \( x_2 \in X \) with distance at most \( \ell \) from \( M \). The set of vertices of \( G \) with distance at most \( \ell \) from \( x_2 \) is either a subset of the vertex set of one of the dotted paths in \( M \), or it contains exactly one end \( v \) of a dotted path of \( M \), and consists of \( v \) together with subsets of the interiors of the (at most three) dotted paths incident with \( v \). In either case, there is an end \( v \) of one of the dotted paths in \( M \), such that the set of vertices of \( G \) with distance at most \( \ell \) from \( x_2 \) is a subset of the set consisting of \( v \) together with the interiors of the dotted paths incident with \( v \). But for every choice of \( v \), there is a path between \( S, T \) made by a union of dotted paths, none of them incident with \( v \); and consequently this path has distance more than \( \ell \) from \( X \). This proves 2.1. \( \square \)

To prove the second statement, that there do not exist three paths between \( S, T \), pairwise at distance at least \( d \), we will prove something stronger, by induction on the depth of the binary tree; and for that we need to allow the binary tree to have small depth, and we need to set up some notation, so let us define the graph more carefully.

Let \( k \geq 2 \) be an integer. Take a uniform binary tree \( B \) with depth \( k \). Thus, \( B \) has \( 2^k - 1 \) vertices, and a root, and every path from the root to one of the leaves has exactly \( k - 1 \) edges. Now add two more vertices, and let \( Z \) consist of the set of leaves of \( B \) together with the two new vertices; and add a path \( M \) with vertex set \( Z \), as shown in figure 2. We call the resulting graph \( G_k \). (To get the graph of the counterexample, we need to replace the edges of \( G_{2\ell + 3} \) incident with vertices in \( Z \) by long paths, but let us not do that yet.) For each vertex \( v \) of \( B \) that is not a leaf, there is a
copy of some $G_h$ with root $v$, formed by $v$ and its descendents in $B$, and two extra vertices; we will apply the inductive hypothesis to these smaller graphs. Let $M$ have ends $s_1, t_2$, and let $s_2, t_1$ be the neighbours in $M$ of $s_1, t_2$ respectively.

![Diagram](image.png)

Figure 2: The graph $G_k$. (In this figure, $k = 6$.) The vertex $b$ is some vertex on the path between $a_1$ and $s_1$; it need not be adjacent to $a_1$, and it might equal $a_1$. Its child on that path is $c$.

We will show:

2.2 *For each integer $k \geq 2$, if $P, Q$ are vertex-disjoint paths of $G_k$ between $\{s_1, s_2\}$ and $\{t_1, t_2\}$, then either*

- there is a path of length at most two between $V(P)$ and $V(Q)$ such that none of its edges have ends in $Z$, or
- one of $P, Q$ is the path of $B$ between $s_1, t_2$.

**Proof.** We proceed by induction on $k$. If $k = 1$ the result is clear, so we assume that $k \geq 2$ and the result holds for all smaller values of $k$. Suppose that $P, Q$ are vertex-disjoint paths of $G_k$ between $\{s_1, s_2\}$ and $\{t_1, t_2\}$, and the first bullet of the theorem is false. Let $r$ be the root of $B$, let $a_1$ be the neighbour of $r$ in the path of $B$ between $r, s_1$, and define $a_2$ similarly. Suppose first that neither of $P, Q$ contains $r$. Then we may assume that $P$ contains $q_1$ and $Q$ contains $q_2$, where $q_1, q_2$ are as shown in the figure; and so from the inductive hypothesis applied to the two copies of $G_{k-1}$ with roots $a_1$ and $a_2$, $P$ contains the path of $B$ between $s_1, q_2$, and $Q$ contains the path of $B$ between $q_1, t_2$. So $a - 1 \in V(P)$ and $a_2 \in V(Q)$; but $r$ is adjacent to both $a_1, a_2$, a contradiction.

So $r$ belongs to one of $P, Q$, say to $P$. We need to show that $P$ contains the two subpaths of $B$ between $r, s_1$ and between $r, t_2$. Suppose it does not contain the first, say; and let us call this path $A_1$. Choose a maximal common subpath of $P, A_1$, with ends $a_1, b$ say. (Possibly $b = a_1$.) We see that $b \neq s_1$, since $A \not\subseteq P$; and $b$ is not adjacent to $s_1$, since one of $P, Q$ must use the edge of $B$ incident with $s_1$. Thus the subpath of $A$ between $b, s_1$ has length $h$ say, where $h \geq 2$. Let $c$ be the child of $b$ in $V(A)$. It follows that neither of $P, Q$ contains $c$; and this contradicts the inductive hypothesis, applied to the copy of $G_{h-1}$ with root $c$. This proves 2.2.

From 2.2, it follows easily that the graph of figure 1, with $S, T$ as shown, has the property that there do not exist three paths between $S, T$, pairwise at distance at least 3. Because if $P, Q, R$ are three such paths, one of them contains the root $r$, and so consists just of the vertex $r$; and so the other two cannot use $r$, and so contradict 2.2.
3 A simpler proof

In this section we give a proof of 1.1 when \( k = 2 \), that is simpler than the proofs previously announced. We need some lemmas about intervals. An interval is a pair \((a, b)\) of integers with \( a \leq b \), and its length is \( b - a \). If \( \mathcal{H} \) is a set of intervals that can be written \( \mathcal{H} = \{(a_i, b_i) : 1 \leq i \leq t\} \) for some \( t \geq 1 \), such that \( 0 \leq a_1 < a_2 < \cdots < a_t \leq n \), and \( 0 \leq b_1 < \cdots < b_t \leq n \), we call this the standard form for \( \mathcal{H} \). Thus \( \mathcal{H} \) has a standard form if and only there do not exist distinct \((a, b), (c, d) \in \mathcal{H}\) with \( a \leq c \leq d \leq b \). (See figure 3.)

![Figure 3: Intervals in standard form (the curves join the ends of the intervals).](image)

If \((a, b), (c, d)\) are intervals, we say \((a, b)\) captures \((c, d)\) if \( a \leq c \leq d \leq b \); and a set \( \mathcal{H} \) of intervals captures \((c, d)\) if there exists \((a, b) \in \mathcal{H}\) that captures \((c, d)\). For integers \( \ell, n \) with \( 0 < \ell \leq n \), we say a set \( \mathcal{H} \) of intervals is \( \ell \)-powerful in \((0, n)\) if

- \( 0 \leq a \leq b \leq n \) for all \((a, b) \in \mathcal{H}\); and
- \( \mathcal{H} \) captures every interval of length \( \ell \) captured by \((0, n)\).

Our objective is, starting with an \( \ell \)-powerful set \( \mathcal{H} \) of intervals, to find a subset \( \mathcal{H}' \) of \( \mathcal{H} \) which is still \( \ell' \)-powerful for some \( \ell' \) not much smaller than \( \ell \), such that all the ends of all the intervals in \( \mathcal{H}' \) are far apart. (Not exactly: in standard form, we cannot arrange that \( a_{i+2} - b_i \) is large for each \( i \), and it might even be negative, and we cannot arrange that \( a_2 - a_1 \) and \( b_t - b_{t-1} \) are large, but we will make all the other differences large.) We begin with:

**3.1** If \( 0 < \ell \leq n \) and \( \mathcal{H} \) is a set of intervals that is \( \ell \)-powerful in \((0, n)\), and minimal with this property, then in standard form \( a_j \geq b_i - \ell + 2 \) for all \( i, j \in \{1, \ldots, t\} \) with \( j \geq i + 2 \).

**Proof.** (See figure 4.)

![Figure 4: For the proof of 3.1.](image)
If \( \mathcal{H} \) can be written in standard form, and \( \mathcal{H}' \subseteq \mathcal{H} \), then \( \mathcal{H}' \) can also be written in standard form. Some inequalities about the corresponding numbers \( a_i, b_i \) are hereditary; that is, if they hold for \( \mathcal{H} \) then they also hold for all its subsets \( \mathcal{H}' \). For instance, the inequalities of 3.1 are hereditary. Let \( \mathcal{H}' \subseteq \mathcal{H} \), and \( \mathcal{H} = \{(a_i, b_i) : 1 \leq i \leq t \} \) and \( \mathcal{H}' = \{(a_i', b_i') : 1 \leq i \leq t' \} \), in standard form. If we know that \( a_j \geq b_i - \ell + 2 \) for all \( i, j \in \{1, \ldots, t\} \) with \( j \geq i + 2 \), then it follows that \( a_j' \geq b_i' - \ell + 2 \) for all \( i, j \in \{1, \ldots, t'\} \) with \( j \geq i + 2 \). Most of the other inequalities we will prove are hereditary, and we leave verifying this to the reader.

3.2 If \( 0 < 2\ell \leq n \) and \( \mathcal{H} \) is a set of intervals that is \( 2\ell \)-powerful in \((0, n)\), then there exists \( \mathcal{H}' \subseteq \mathcal{H} \), minimally \( \ell \)-powerful in \((0, n)\), that can be written \( \mathcal{H}' = \{(a_i, b_i) : 1 \leq i \leq t \} \) in standard form, such that

- \( a_j \geq b_i - \ell + 2 \) for \( 1 \leq i, j \leq t \) with \( j \geq i + 2 \); and
- \( b_i - b_{i-1} \geq \ell \) for \( 1 < i < t \), and \( b_{i-1} - a_i \geq \ell \) for \( 1 < i \leq t \).

**Proof.** Choose \( \mathcal{H}_1 \subseteq \mathcal{H} \), minimal such that it is \( 2\ell \)-powerful in \((0, n)\). Let us choose \( \mathcal{H}_2 \subseteq \mathcal{H}_1 \), such that

- \( \mathcal{H}_2 \) captures the intervals \((0, 2\ell)\) and \((n - 2\ell, n)\);
- if \((a, b) \in \mathcal{H}_2\) and \( b + \ell < n \) then some member of \( \mathcal{H}_2 \) captures the interval \((b - \ell, b + \ell)\);
- \( \mathcal{H}_2 \) is minimal with these properties.

It follows that we can write \( \mathcal{H}_2 \) in standard form, \( \mathcal{H}_2 = \{(a_i, b_i) : 1 \leq i \leq t \} \) for some \( t \geq 1 \); and \( a_1 = 0 \) and \( b_t = n \); and the first bullet of the theorem holds, by 3.1. We claim that:

(1) For \( 1 < i < t \), \( b_{i-1} \leq n - \ell \) and \((a_i, b_i)\) captures \((b_{i-1} - \ell, b_{i-1} + \ell)\), and there is no \( j \in \{1, \ldots, t\} \) with \( j \neq i \) such that \((a_j, b_j)\) captures \((b_{i-1} - \ell, b_{i-1} + \ell)\).

We prove this by induction on \( i \), so we may assume that either \( i = 2 \), or the claim holds for \( i - 1 \). Since \( a_i > a_1 \geq 0 \) and \( b_i < b_t = n \), \((a_i, b_i)\) captures neither of \((0, 2\ell)\), \((n - 2\ell, n)\); and so from the minimality of \( \mathcal{H}_2 \), there exists \( h \in \{1, \ldots, t\} \) such that \( b_h \leq n - \ell \) and \((a_i, b_i)\) captures \((b_h - \ell, b_h + \ell)\), and there is no \( j \in \{1, \ldots, t\} \) with \( j \neq i \) such that \((a_j, b_j)\) captures \((b_h - \ell, b_h + \ell)\). It follows that \( h < i \), since \( b_h < b_i \); and \( h > i - 2 \), since if \( h \leq i - 2 \) then from the inductive hypothesis, there is no \( j \neq h + 1 \) such that \((a_j, b_j)\) captures \((b_h - \ell, b_h + \ell)\). Hence \( h = i - 1 \). This completes the induction, and so proves (1).

Consequently \( b_i - b_{i-1} \geq \ell \) and \( b_{i-1} - a_i \geq \ell \) for \( 1 < i < t \). For the second bullet we still need to check that \( b_{i-1} - a_i \geq \ell \). This is true if \( b_{i-1} \geq n - \ell \), since \((a_i, b_i)\) captures \((n - 2\ell, n)\). If \( b_{i-1} < n - \ell \), then some \((a_j, b_j)\) captures \((b_{i-1} - \ell, b_{i-1} + \ell)\); and \( j = t \) since \( b_j > b_{i-1} \). This proves that \( \mathcal{H}_2 \) satisfies the second bullet.

We claim that \( \mathcal{H}_2 \) is \( \ell \)-powerful in \((0, n)\). To see this, let \( 0 \leq h \leq n - \ell \); we must show that some member of \( \mathcal{H}_2 \) captures \((h, h + \ell)\). We may assume that \( h < n - 2\ell \), since some member of \( \mathcal{H}_2 \) captures \((n - 2\ell, n)\). Choose \( i \in \{1, \ldots, t\} \) maximal such that \( a_i \leq h \). We may assume that \( b_i < h + \ell \), since otherwise \((a_i, b_i)\) captures \((h, h + \ell)\). Hence \( b_i < n - \ell \), and so \((a_{i+1}, b_{i+1})\) captures
(b_i - \ell, b_i + \ell), by (1). From the choice of i, \(a_{i+1} > h\); but \(a_{i+1} \leq b_i - \ell\), so \(b_i \geq a_{i+1} + \ell > h + \ell\), a contradiction. This proves that \(H_2\) is \(\ell\)-powerful in \((0, n)\). Choose \(H' \subseteq H_2\), minimal such that it is \(\ell\)-powerful in \((0, n)\). Then by 3.1, the first bullet of the theorem holds; and the second still holds, since it is hereditary. This proves 3.2.

\[\begin{array}{c}
\text{Figure 5: The output of 3.3.} \quad \text{\(a_3, b_1\) have been drawn in the same place because we do not know which is larger.} \\
\end{array}\]

3.3 If \(0 < 4\ell \leq n\) and \(H\) is a set of intervals that is \(4\ell\)-powerful in \((0, n)\), then there exists \(H' \subseteq H\), \(\ell\)-powerful in \((0, n)\), which can be written \(H' = \{(a_i, b_i) : 1 \leq i \leq t\}\) in standard form, such that the order of the numbers \(a_1, \ldots, a_t\) and \(b_1, \ldots, b_t\) is:

\[
0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \cdots < \min(a_t, b_{t-2}) < \max(a_t, b_{t-2}) < b_{t-1} < b_t = n.
\]

(See figure 5.) Every two of \(a_1, \ldots, a_t, b_1, \ldots, b_t\) differ by at least \(\ell\), except possibly the pairs \((a_1, a_2), (b_{t-1}, b_t)\), and \((a_i, b_{i-2})\) for \(3 \leq i \leq t\).

**Proof.** From 3.2 with \(\ell\) replaced by \(2\ell\), there exists \(H_1 \subseteq H\), \(2\ell\)-powerful in \((0, n)\), that satisfies the inequalities of 3.2. By 3.2 again, applied to \(H_1\), with the order of \(\{0, \ldots, n\}\) reversed, there exists \(H_2 \subseteq H_1\), minimally \(\ell\)-powerful in \((0, n)\), such that in standard form,

- \(b_i - b_{i-1} \geq 2\ell\) for \(1 < i < t\), and \(b_{i-1} - a_i \geq 2\ell\) for \(1 < i \leq t\) (since these inequalities are true for \(H_1\) and are hereditary);
- \(a_{i+1} - a_i \geq \ell\) for \(1 < i < t\); and
- \(a_j \geq b_i - \ell + 2\) for all \(i, j \in \{1, \ldots, t\}\) with \(j \geq i + 2\).

It follows that \(b_j - b_i \geq \ell\) for all \(i, j \in \{1, \ldots, t\}\) with \(j > i\) and \((i, j) \neq (t - 1, t)\); that \(a_j - a_i \geq \ell\) for all \(i, j \in \{1, \ldots, t\}\) with \(j > i\) and \((i, j) \neq (1, 2)\). Now let \(i, j \in \{1, \ldots, t\}\); we want to show that \(|a_j - b_i| \geq \ell\), unless \(i = j - 2\). If \(i > j\), then \(b_i \geq b_j + \ell \geq a_j + \ell\) as required. If \(j = i\), the minimality of \(H_2\) implies that \(b_i - a_i \geq \ell\), as required. If \(i = j - 1\), we have \(b_i - a_j \geq 2\ell\) from the first bullet above, as required. Now suppose that \(i \leq j - 3\). Hence \(b_{j-2} \leq a_j + \ell - 2\), and \(b_{j-2} \geq b_{j-3} + 2\ell\); so \(a_j + \ell - 2 \geq b_{j-3} + 2\ell\), that is, \(a_j \geq b_{j-3} + \ell + 2 \geq b_i + \ell\) as required. This proves 3.3.

Now let us deduce 1.1 when \(k = 2\). As we said earlier, it suffices to handle the case when \(d = 3\), so we will prove:

3.4 Let \(G\) be a non-null graph and let \(S, T \subseteq V(G)\); then either
• there are two paths between $S, T$ with distance at least 3; or
• there exists $x \in V(G)$ such that every path between $S, T$ contains a vertex with distance at most 161 from $x$.

**Proof.** We assume the second outcome is false. Let $c \geq 7$ and $\ell \geq 2c+5$ be integers: we will prove that the theorem holds with 161 replaced by $8\ell + c+2$. We may assume that $G$ is connected, and there is a path between $S, T$; let $R$ be such a path with minimum length, and let its vertices be $r_1, \ldots, r_{n-1}$ in order, where $r_1 \in S$ and $r_{n-1} \in T$. There is a path $P$ between $S, T$ with $d(r_1, P) > 8\ell + c + 2$, and we can assume that $d(P, R) \leq 2$, so $n - 2 \geq 8\ell + c + 1$. In particular, $S \cap T = \emptyset$ and there are no edges between $S, T$.

Let $W$ be the set of all vertices with distance at most $c$ from $R$. We call the set of all vertices $v$ with $d(v, R) = c$ the surface. For each $v \in V(G)$, if $v \in S$ and $d(v, R) \geq c + 4$, let $a(v) = 0$, and otherwise let $a(v)$ be the smallest $i \in \{1, \ldots, n - 1\}$ such that there is a path between $v$ and $r_i$ of length $d(v, R)$. Similarly, if $v \in T$ and $d(v, R) \geq c + 4$ let $b(v) = n$, and otherwise let $b(v)$ be the largest $i \in \{1, \ldots, n - 1\}$ such that there is a path between $v$ and $r_i$ of length $d(v, R)$. For each $w \in W$, it follows from the choice of $R$ that $0 \leq b(w) - a(w) \leq 2d(w, R) \leq 2c$.

Let $C$ be the set of components of $G \setminus W$, and for each $C \in C$, let $N(C)$ be the set of vertices in $W$ with a neighbour in $V(C)$. Thus $N(C) \neq \emptyset$, and $N(C)$ is a subset of the surface. Let $a(C)$ be the smallest value of $a(v)$ for $v \in V(C)$, and define $b(C)$ similarly. Thus $0 \leq a(C) \leq b(C) \leq n$ for each $C \in C$.

(1) The set of intervals $\{(a(C), b(C)) : C \in \mathcal{C}\}$ is $16\ell$-powerful in $(0, n)$.

Let $8\ell \leq m \leq n - 8\ell$: we must show that there exists $C \in \mathcal{C}$ such that $(a(C), b(C))$ captures $(m - 8\ell, m + 8\ell)$. Let $B$ be the set of all vertices of $G$ with distance at most $8\ell + c + 2$ from $r_m$. Let $W_1$ be the set of all $w \in W \setminus B$ with $a(w) \leq m$, and let $W_2$ be the set of all $w \in W \setminus B$ with $b(w) \geq m$. Thus $W_1 \cup W_2 = W \setminus B$. If $w \in W_1$, then since $d(w, r_m) > 8\ell + c + 2$ and $d(w, r_{a(w)}) \leq c$, it follows that $d(r_{a(w)}, r_m) > 8\ell + 2$, and so $a(w) \leq m - 8\ell - 3$; and similarly if $w \in W_2$ then $b(w) \geq m + 8\ell + 3$. Thus, if $w_1 \in W_1$ and $w_2 \in W_2$, then

$$b(w_2) - a(w_1) \geq (m + 8\ell + 3) - (m - 8\ell - 3) = 16\ell + 6.$$  

There is a path between $w_1$ and $S$ of length at most $a(w_1) + c - 1$, and a path between $w_2$ and $T$ of length at most $|E(R)| - b(w_2) + c - 1$, and, since there is no path between $S, T$ of length less than $|E(R)|$, it follows that

$$d(w_1, w_2) \geq |E(R)| - (a(w_1) + c - 1) - (|E(R)| - b(w_2) + c - 1)$$

$$= b(w_2) - a(w_1) - 2(c - 1) \geq (16\ell + 6) - 2(c - 1) \geq 2.$$

Consequently $W_1 \setminus B$ and $W_2 \setminus B$ are disjoint, and there are no edges between them. Moreover, if $w_1 \in S$ and $w_2 \in W_2$ then $b(w_2) \geq m + 8\ell + 3 \geq 16\ell + 3$, and similarly

$$d(w_1, w_2) \geq |E(R)| - (|E(R)| - b(w_2) + c - 1) = b(w_2) - (c - 1) \geq (16\ell + 3) - 2(c - 1) \geq 2.$$

So $S \cap W_2 = \emptyset$ and there are no edges between them, and the same for $W_1, T$. Thus $W_1 \cup S$ is disjoint from $W_2 \cup T$ and there are no edges between them.
Since the second outcome of the theorem is false, there is an $S - T$ path $P$ in $G$ disjoint from $B$. Consequently, there is a subpath $P'$ of $P$ with first vertex in $W_1 \cup S$, last vertex in $W_2 \cup T$, and with no other vertex in $W \cup B$. Since $W_1 \cup S$ is disjoint from $W_2 \cup T$ and there are no edges between them, $P'$ has an internal vertex, which is therefore not in $W$. But the only vertices of $P'$ in $W$ are ends of $P'$, and so there exists $C \in \mathcal{C}$ such that $V(P') \subseteq V(C) \cup W$. We will show that $a(C) \leq m - 8\ell$ and $b(C) \geq m + 8\ell$. Let $w_1 \in W_1 \cup S$ and $w_2 \in W_2 \cup T$ be the ends of $P'$. Either $w_1 \in W_1$, or $w_1 \in S \setminus W$ and $d(w_1, R) \geq c + 4$, or $w_1 \in S \setminus W$ and $d(w_1, R) \leq c + 3$, and we claim that $a(w_1) \leq m - 8\ell$ in each case. In the first case, $w_1$ has a neighbour $v \in V(C)$; and since $d(v, R) = c + 1 = d(w_1, R) + 1$, it follows that $a(C) \leq a(v) \leq a(w_1) \leq m - 8\ell - 3 \leq m - 8\ell$ as required.

In the other two cases, $w_1 \in V(C)$, and so $a(C) \leq a(w_1)$. In the second case, $a(w_1) = 0 \leq m - 8\ell$ as required. In the third case, when $w_1 \in S$ and $d(w_1, R) \leq c + 3$, it follows that $d(w_1, r_{a(w_1)}) \leq c + 3$, and since $d(w_1, r_m) > 8\ell + c + 2$, this implies that $a(w_1) \leq m - 8\ell$ as required. This proves that $a(C) \leq m - 8\ell$ in all three cases. Similarly $b(C) \geq m + 8\ell$. This proves (1).

From 3.3, with $\ell$ replaced by $4\ell$, there exists $\mathcal{D} \subseteq \mathcal{C}$ such that the set of intervals $\{(a(C), b(C)) : C \in \mathcal{D}\}$ is $4\ell$-powerful in $(0, n)$, and the members of $\mathcal{D}$ can be numbered $D_1, \ldots, D_t$ such that:

- $0 = a(D_1) < a(D_2) < \cdots < a(D_t) \leq n$, and $0 \leq b(D_1) < b(D_2) < \cdots < b(D_t) = n$; and

- $a(D_i) - b(D_{i-3}) \geq 4\ell$ for $4 \leq i \leq t$.

(We are not using the full strength of 3.3 here; that will come later.)

(2) For $1 \leq i, j \leq t$, if $j \geq i + 3$ then $d(D_i, D_j) \geq 4\ell - 2c + 2$.

Suppose not. Hence there exist $w \in N(D_i)$ and $w' \in N(D_j)$ such that $d(w, w') < 4\ell - 2c$; and there is a path between $w, V(R)$ of length $c$, and its end in $R$ is some $r_k$ where $k \leq b(D_i)$. Similarly there is a path between $w', V(R)$ of length $c$, and its end in $R$ is some $r_{k'}$ where $k \geq a(D_j)$. Thus there exist $k \leq b(D_i)$ and $k' \geq a(D_j)$ such that there is a path between $r_k, r_{k'}$ of length less than $4\ell$. Since $R$ is a shortest path, it follows that $4\ell - 1 \geq k' - k \geq a(D_j) - b(D_i) \geq 4\ell$, a contradiction. This proves (2).

Let $\Delta$ be the union of the vertex sets of the members of $\mathcal{D}$. If $X \subseteq V(G)$, let $\Delta(X)$ be the set of $D \in \mathcal{D}$ with $X \cap N(D) \neq \emptyset$. Let us say a joint is a subset $X \subseteq V(G)$, inducing a connected subgraph, such that

- $|\Delta(X)| \geq 2$;
- if $|\Delta(X)| = 2$ then $|X| \leq 3$;
- if $|\Delta(X)| \leq 3$ then $|X| \leq 8$;
- $|X| \leq 29$; and
- every vertex of $X \cap W$ belongs to a path of $G[X]$ with both ends in the surface.

(3) For every joint $X$, $|\Delta(X)| \leq 3$, and hence $|X| \leq 8$; and $d(X, R) \geq c - (|X| - 1)/2$. 

8
Suppose that $|\Delta(X)| \geq 4$; then we can choose $i, j$ with $1 \leq i, j \leq t$ and $j \geq i + 3$ such that $D_i, D_j \in \Delta(X)$. Since $G[X]$ is connected, there is a path between $D_i, D_j$ of length at most $|X| + 1 \leq 30$, contrary to (2), since $4\ell - 2c + 2 > 30$. Thus $|\Delta(X)| \leq 3$. Moreover, if $v \in X \cap W$, then since $v$ belongs to a path of $G[X]$ with both ends in the surface, there is a path from $v$ to the surface of length at most $(|X| - 1)/2$; and since every path from the surface to $R$ has length at least $c$, it follows that $d(v, R) \geq c - (|X| - 1)/2$. This proves (3).

Let $Z$ be the union of all the joints. Each component of $G[Z \cup \Delta]$ is called a supercomponent. Each supercomponent includes at least one member of $D$, but may be composed of many joints and members of $D$.

(4) Let $F_1, F_2$ be distinct supercomponents, and let $Q$ be a path with ends $f_1 \in V(F_1)$ and $f_2 \in V(F_2)$. If both $f_1, f_2$ belong to joints then $|Q| \geq 16$; if exactly one of $f_1, f_2$ belongs to a joint $X$ then $|Q| \geq 8$, and $|Q| \geq 24$ if $|\Delta(X)| \geq 3$; and if neither of $f_1, f_2$ belong to joints then $|Q| \geq 6$. In any case, $d(F_1, F_2) \geq 5$.

Let $Q^*$ be the set of vertices in the interior of $Q$. Suppose first that $f_i$ belongs to a joint $X_i$ for $i = 1, 2$. Then $X_1 \cup X_2 \cup Q^*$ is not a joint (since $F_1, F_2$ are distinct supercomponents). But $|\Delta(X_1 \cup X_2 \cup Q^*)| \geq 4$ (since $|\Delta(X_i)| \geq 2$ for $i = 1, 2$, and $\Delta(X_1) \cap \Delta(X_2) = \emptyset$), and every vertex of $(X_1 \cup X_2 \cup Q^*) \cap W$ belongs to a path of $G[X_1 \cup X_2 \cup Q^*]$ with both ends in the surface; and so $|X_1 \cup X_2 \cup Q^*| \geq 30$. Since $|X_1|, |X_2| \leq 8$ by (3), it follows that $|Q^*| \geq 14$, and so $|Q| \geq 16$ as claimed. Thus we may assume that $f_2 \in D'$ for some $D' \in D$ included in $F_2$, and so $V(Q) \cap N(D') \neq \emptyset$, and hence $D' \in \Delta(Q^*)$. Next, suppose that $f_1 \in X_1$ for some joint $X_1$. Then $|\Delta(X_1 \cup Q^*)| \geq 3$, and since $X_1 \cup Q^*$ is not a joint (because $F_1, F_2$ are distinct supercomponents), it follows that either $|\Delta(X_1 \cup Q^*)| = 3$ and $|X_1 \cup Q^*| \geq 9$, or $|X_1 \cup Q^*| \geq 30$. In the first case, $|\Delta(X_1)| = 2$, so $|X_1| \leq 3$, and therefore $|Q^*| \geq 6$ and $|Q| \geq 8$; and in the second case, $|X_1| \leq 8$ by (3), and so $|Q^*| \geq 22$ and $|Q| \geq 24$, as claimed. Finally, we may assume that $f_1 \in D$ for some $D \in D$ included in $F_1$, and so $D, D' \in \Delta(Q^*)$, and since $Q^*$ is not a joint, $|Q^*| \geq 4$ and so $|Q| \geq 6$. This proves (4).

(5) If $F_1, F_2$ are distinct supercomponents, and $A$ is a path of length at most $c + 1$ between $F_2$ and $R$, then $d(F_1, A) \geq 3$.

Suppose that $f_1 \in V(F_1)$ and $v \in V(A)$ have distance at most two. Let $f_2$ be the end of $A$ in $V(F_2)$. Suppose first that $f_1 \in \Delta$. Consequently every path from $f_1$ to $V(R)$ has length more than $c$, and so the subpath of $A$ between $v$ and $R$ has length at least $c - 1$; and therefore the subpath from $v$ to $f_2$ has length at most two. Hence there is a path $Q$ of length at most four between $f_1, f_2$; so $|Q| \leq 5$, contrary to (4). This proves that $f_1 \notin \Delta$, and so $f_1 \in X_1$ for some joint $X_1$ of $F_1$. By (3), $d(f_1, R) \geq c - (|X_1| - 1)/2$, and so the subpath of $A$ between $v, R$ has length at least $c - (2 + (|X_1| - 1)/2)$. Hence the subpath of $A$ between $v, f_2$ has length at most $3 + ((|X_1| - 1)/2)$. Consequently there is a path $Q$ between $f_1, f_2$ of length at most $5 + (|X_1| - 1)/2$, and so $|Q| \leq 6 + (|X_1| - 1)/2$. Since $|X_1| \leq 8$, and so $|Q| \leq 9$, it follows from (4) that $f_2$ is not in a joint and $|\Delta(X_1)| = 2$. But then $|X_1| \leq 3$, so $|Q| \leq 7$, contrary to (4). This proves (5).

Let $F$ be the set of all supercomponents. For each $F \in F$, let $a(F)$ be the minimum of $a(v)$ over all $v \in V(F)$, and define $b(F)$ similarly. Since for every $D \in \mathcal{D}$, there exists $F \in F$ with $D \subseteq F$, and
hence with $a(F) \leq a(D) \leq b(D) \leq b(F)$, it follows that the set of intervals $\{(a(F), b(F)) : F \in \mathcal{F}\}$ is 4$\ell$-powerful. By 3.3, there exists $\mathcal{H} \subseteq \mathcal{F}$ such that $\{(a(F), b(F)) : F \in \mathcal{H}\}$ is $\ell$-powerful, and it can be numbered as $\{H_1, \ldots, H_s\}$, where, writing $a(H_i) = a_i$ and $b(H_i) = b_i$ for each $i$:

(6) The order of the numbers $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ is:

$$0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \cdots$$

$$\cdots < \min(a_s, b_{s-2}) < \max(a_s, b_{s-2}) < b_{s-1} < b_s = n.$$  

Every two of $a_1, \ldots, a_s, b_1, \ldots, b_s$ differ by at least $\ell$, except possibly the pairs $(a_1, a_2)$, $(b_{s-1}, b_s)$, and $(a_i, b_{i-2})$ for $3 \leq i \leq s$.

Figure 6: Various paths. The dotted line is the boundary of $W$. All the dashed lines represent paths, and the one at the bottom is $R$. The thick dashed lines are the paths we will use to construct a pair of $S - T$ paths with distance three. The paths $Q_i$ have been drawn as though their interiors are disjoint from $W$, but they might not be; each is a path of a supercomponent, but the supercomponent might penetrate into $W$ a short distance, via joints. In particular the ends of each $Q_i$ have been drawn exactly on the surface, but they might be slightly deeper in $W$, or just outside of $W$.

Now we are ready to construct a pair of $S - T$ paths that are distance three apart. (See figure 6.) Since $a(H_1) = 0$, there is a vertex $v$ of $H_1$ with $a(v) = 0$, and hence with $v \in S$ and $d(v, R) \geq c + 4$; let $A_1$ be the one-vertex path consisting of this vertex. For $2 \leq i \leq s$, let $A_i$ be a shortest path with one end in $V(H_i)$ and the other equal to $r_{a_i}$. Similarly, define $B_s$ to be a one-vertex path with vertex some $v \in V(H_s) \cap T$ with $b(v) = 0$; and for $1 \leq i < s$ let $B_i$ be a shortest path with one end in $V(H_i)$ and the other equal to $r_{b_i}$. It follows that $A_i, B_i$ have length at most $c + 1$. Let $Q_i$ be a path of $H_i$ joining the ends of $A_i$ and $B_i$ in $H_i$. For $1 \leq i, j \leq n - 1$, let $R(i, j) = R(j, i)$ be the subpath of $R$ with ends $r_i, r_j$. Define $b_0 = 1$ and $a_{s+1} = n - 1$. It follows that the union of the paths

- $A_i \cup Q_i \cup B_i$ for $i \in \{1, \ldots, s\}$, odd; and
- $R(b_i, a_{i+2})$ for $i \in \{0, \ldots, s - 1\}$, odd

contains an $S - T$ path. (It might not itself be a path, because for instance the paths $B_1, A_3$ might intersect). Also, the union of the paths

10
• $A_j \cup Q_j \cup B_j$ for $j \in \{1, \ldots, s\}$, even; and
• $R(b_j, a_{j+2})$ for $j \in \{0, \ldots, s - 1\}$, even

contains an $S - T$ path. We must check that these paths have distance at least three. This is true if $s = 1$, so we assume that $s \geq 2$. We have to check that:

(i) $d(Q_i, Q_j) \geq 3$ for all $i, j \in \{1, \ldots, s\}$ with $i$ odd and $j$ even;
(ii) $d(A_i, A_j), d(A_i, B_j), d(B_i, A_j), d(B_i, B_j) \geq 3$ for all $i, j \in \{1, \ldots, s\}$ with $i$ odd and $j$ even;
(iii) $d(R(b_i, a_{i+2}), R(b_j, a_{j+2})) \geq 3$ for all $i, j \in \{0, \ldots, s - 1\}$ with $i$ odd and $j$ even;
(iv) $d(Q_i, A_j), d(Q_i, B_j), d(A_i, Q_j), d(B_i, Q_j) \geq 3$ for all $i, j \in \{1, \ldots, s\}$ with $i$ odd and $j$ even;
(v) $d(Q_i, R(b_j, a_{j+2})) \geq 3$ for all $i, j$ with $i \in \{1, \ldots, s\}$, odd and $j \in \{0, \ldots, s - 1\}$, even; and
$\quad d(R(b_i, a_{i+2}), Q_j) \geq 3$ for all $i, j$ with $i \in \{0, \ldots, s - 1\}$, odd and $j \in \{1, \ldots, s\}$, even; and
(vi) $d(A_i, R(b_j, a_{j+2})), d(B_i, R(b_j, a_{j+2})) \geq 3$ for all $i, j$ with $i \in \{1, \ldots, s\}$, odd and $j \in \{0, \ldots, s - 1\}$, even; and
$\quad d(R(b_i, a_{i+2}), A_j), d(R(b_i, a_{i+2}), B_j) \geq 3$ for all $i, j$ with $i \in \{0, \ldots, s - 1\}$, odd and $j \in \{1, \ldots, s\}$, even.

Statement (i) follows from (4). For statement (ii), suppose first that $i, j \neq 1, s$. Since every vertex in $A_i$ has distance at most $c + 1$ from $r_{a_i}$, and the same for $B_i, A_j, B_j$, it suffices to check that $d(r_{a_i}, r_{a_j}) \geq 2c + 5$ and so on. But these four distances are $|a_i - a_j|, |a_i - b_j|, |b_i - a_j|, |b_i - b_j|$ respectively, and so are all at least $\ell \geq 2c + 5$, by (6). If $i = 1$ and $j \neq s$, the same argument shows that $d(B_1, A_j), d(B_1, B_j) \geq 3$, but we need to check that $d(A_1, A_j), d(A_1, B_j) \geq 3$. In this case $V(A_1)$ is a vertex $v \in S$, with $d(v, R) \geq c + 4$. Since $A_j, B_j$ both have length at most $c + 1$, it follows that $d(v, A_j), d(v, B_j) \geq 3$. The argument is similar if $i = s$ or $j = s$. This proves (ii).

For statement (iii), note that the subpaths $R(b_i, a_{i+2}), R(b_j, a_{j+2})$ of $R$ are disjoint (by (6)), and the distance between them is at least $\ell \geq 3$ (again by (6)). Statement (iv) follows from (5). For statement (v), (3) implies that every path from a supercomponent to $R$ has length at least $c - (8 - 1)/2 \geq 3$. Finally, for statement (vi), we will prove that $d(A_i, R(b_j, a_{j+2})) \geq 3$ for all $i, j$ with $i \in \{1, \ldots, s\}$, odd and $j \in \{0, \ldots, s - 1\}$, even (the other statement is proved similarly). If $i = 1$ then the claim is true since $d(A_1, R) \geq c + 4 \geq 3$, so we assume that $i \geq 3$. The distance between $r_{a_i}$ and $R(b_j, a_{j+2})$ is at least $\ell$, by (6); and the distance between $r_{a_i}$ and each vertex of $A_i$ is at most $c + 1$, and so the distance between $A_i$ and $R(b_j, a_{j+2})$ is at least $\ell - c - 1 \geq 3$. This proves 3.4.

References


